Abstract

A Hilbert space operator $T$, $T \in B(\mathcal{H})$, is totally hereditarily normaloid, $T \in \mathcal{THN}$, if every part and (also) every invertible part of $T$ is normaloid. The class $\mathcal{THN}$ is large. It is proved that if a $T \in \mathcal{THN}$ is such that the isolated eigenvalues of $T$ are normal, then the Riesz projection $P_\lambda$ associated with a $\lambda \in \text{iso} \sigma(T)$ is self-adjoint and $P_\lambda \mathcal{H} = (T - \lambda)^{-1}(0) = (T^* - \lambda)^{-1}(0)$.

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1. Introduction and notation

If $T$ is a Hilbert space operator, $T \in B(\mathcal{H})$, and $\lambda$ is an isolated point of the spectrum of $T$, $\lambda \in \text{iso} \sigma(T)$, the Riesz projection $P_\lambda$ associated (via the Riesz functional calculus) with $\lambda$ is defined by the familiar Cauchy integral [10]
\[ P_\lambda := \frac{1}{2\pi i} \int_{\Gamma} (z - T)^{-1} \, dz, \]

where \( \Gamma \) is an open disk centered at \( \lambda \), \( z \in \Gamma \) and \( \Gamma \cap \sigma(T) = [\lambda] \). Evidently, \( P_\lambda^2 = P_\lambda \), \( P_\lambda T = T P_\lambda \), \( \sigma(T|_{P_\lambda H}) = [\lambda] \) and \( (T - \lambda)^{-n}(0) = (T - \lambda I)^{-n}(0) \subseteq P_\lambda H \) for every \( n = 0, 1, 2, \ldots \). Stampfli \[16\] proved that if an operator \( T \in B(H) \) satisfies the growth condition \( G_1 \) (in particular, if \( T \) is hyponormal), then \( P_\lambda \) is self-adjoint and \( P_\lambda H = (T - \lambda)^{-1}(0) \). This result has since been extended to \( p \)-hyponormal (and log-hyponormal) operators by Chō and Takahashi \[4\], \( w \)-hyponormal operators \[2\] by Han et al. \[9\], \((p, q)\)-quasihyponormal operators by Tanahashi et al. \[18\], and to paranormal operators by Uchiyama \[19\]. Recall that an operator \( T \in B(H) \) is said to be normaloid if \( \|T\| \) equals the spectral radius \( r(T) \) of \( T \). A part of an operator is its restriction to an invariant subspace. We say that \( T \in B(H) \) is totally hereditarily normaloid, denoted \( T \in \mathcal{FN} \), if every part of \( T \), and (also) invertible part of \( T \), is normaloid. The class \( \mathcal{FN} \) is large; it contains a number of the often considered classes of Hilbert space operators. In this paper we show that the Riesz projection \( P_\lambda \) associated with a \( \lambda \in \sigma(T) \) is self-adjoint with \( P_\lambda H = (T - \lambda)^{-1}(0) = (T^* - \overline{\lambda})^{-1}(0) \) for operators \( T \in \mathcal{FN} \) which satisfy the property that isolated eigenvalues of \( T \) are normal. (Recall that \( \lambda \) is a normal eigenvalue of \( T \) if the eigenspace corresponding to \( \lambda \) reduces \( T \).) More precisely, we prove:

**Theorem 1.1.** Suppose that an operator \( T \in B(H) \) has a representation

\[
T = \begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}
\]

such that \( T_3 \) is nilpotent and \( \sigma(T_1) \subset \sigma(T) \subset \sigma(T_1) \cup \{0\} \). If \( T_1 \in \mathcal{FN} \), non-zero isolated eigenvalues of \( T_1 \) are normal and \( (T_1 - \lambda)^{-1}(0) \oplus 0 \subseteq (T^* - \overline{\lambda})^{-1}(0) \), then the Riesz projection \( P_\lambda \) associated with \( \lambda \) is self-adjoint and \( P_\lambda H = (T - \lambda)^{-1}(0) = (T^* - \overline{\lambda})^{-1}(0) \) for every non-zero \( \lambda \in \sigma(T) \).

We shall prove Theorem 1.1 in Section 2, and discuss its applications to various classes of Hilbert space operators in Section 3. Meanwhile, we introduce our notation and terminology.

Let \( \mathbb{C} \) denote the set of complex numbers. A Banach space operator \( T \in B(\mathcal{H}) \), is said to be Fredholm if \( T(\mathcal{H}) \) is closed and both its deficiency indices \( \dim(T^{-1}(0)) \) and \( \dim(\mathcal{H}/T(\mathcal{H})) \) are finite, and then the index of \( T \), \( \text{ind}(T) \), is defined by \( \text{ind}(T) = \dim(T^{-1}(0)) - \dim(\mathcal{H}/T(\mathcal{H})) \). The ascent (descent) of \( T \), denoted \( \text{asc}(T) \) (resp., \( \text{dsc}(T) \)), is the least non-negative integer \( n \) such that \( T^{-n}(0) \) is closed (resp., \( T^{n}(\mathcal{H}) = T^{n+1}(\mathcal{H}) \)). \( T \) has the single-valued extension property (SVEP) at a point \( \lambda_0 \in \mathbb{C} \) if for every open disc \( D_{\lambda_0} \) centered at \( \lambda_0 \) the only analytic function \( f : D_{\lambda_0} \to \mathcal{H} \) which satisfies

\[(T - \lambda)f(\lambda) = 0 \quad \text{for all } \lambda \in D_{\lambda_0} \]
is the function \( f \equiv 0 \). Trivially, every operator \( T \) has SVEP at points of the resolvent \( \rho(T) = \mathbb{C} \setminus \sigma(T) \); also \( T \) has SVEP at \( \lambda \in \text{iso} \sigma(T) \). The quasi-nilpotent part \( H_0(T) \) and the analytic core \( K(T) \) of \( T \) are defined by

\[
H_0(T) = \left\{ v \in \mathcal{V} : \lim_{n \to \infty} \| T^n v \|^{1/n} = 0 \right\}
\]

and

\[
K(T) = \{ v \in \mathcal{V} : \text{there exists a sequence } \{ v_n \} \subset \mathcal{V} \text{ and } \delta > 0 \text{ for which } v = v_0, T v_{n+1} = v_n \text{ and } \| v_n \| \leq \delta^n \| v \| \text{ for all } n = 1, 2, \ldots \}.
\]

We note that \( H_0(T) \) and \( K(T) \) are (generally) non-closed hyperinvariant subspaces of \( T \) such that \( T^n(0) \subseteq H_0(T) \) for all \( n = 0, 1, 2, \ldots \), and \( TK(T) = K(T) \) [13].

Recall that if \( \lambda \in \text{iso} \sigma(T) \), then \( \mathcal{V} = H_0(T - \lambda) \oplus K(T - \lambda) \), where both \( H_0(T - \lambda) \) and \( K(T - \lambda) \) are closed (see [13,11]).

If \( M \) is a linear subspace of \( \mathcal{V} \), let \( M^\perp = \{ \phi \in \mathcal{V}^*: \phi(m) = 0 \text{ for all } m \in M \} \) denote the annihilator of \( M \) (in \( \mathcal{V}^* \)), and if \( N \) is a linear subspace of \( \mathcal{V}^* \), let \( ^\perp N = \{ v \in \mathcal{V} : \phi(v) = 0 \text{ for all } \phi \in N \} \) denote the pre-annihilator of \( N \) (in \( \mathcal{V} \)). By the bi-polar theorem, \( ^\perp (M^\perp) \) is the norm closure of \( M \) and \( ^\perp (N^\perp) \) is the weak*-closure of \( N \). For every \( T \in B(\mathcal{V}) \), \( T^{\perp^{-1}}(0) = T(\mathcal{V}^\perp) \) and \( T^{-1}(0) = T^*(\mathcal{V}^\perp) \). In the case in which \( \mathcal{V} = \mathcal{H} \) is a Hilbert space, it is well known that \( (T^* - \lambda)^{-1}(0)^\perp = (T - \lambda)\mathcal{H} \).

2. Proof of Theorem 1.1

The hypothesis \( (0 \neq) \lambda \in \text{iso} \sigma(T) \) implies \( \lambda \in \text{iso} \sigma(T_1) \). Hence,

\[
\mathcal{H}_1 = H_0(T_1 - \lambda) \oplus K(T_1 - \lambda)
\]

as a topological direct sum, where \( H_0(T_1 - \lambda) \) is non-trivial, both \( H_0(T_1 - \lambda) \) and \( K(T_1 - \lambda) \) are closed, \( \sigma(T_{10}) = \sigma(T_1|_{H_0(T_1 - \lambda)}) = \{ \lambda \} \) and \( \sigma(T_{11}) = \sigma(T_1|_{K(T_1 - \lambda)}) = \sigma(T_1) \setminus \{ \lambda \} \) (see [12, Theorem 1.6] and [13]). Since \( T_1 \in \mathcal{F} \mathcal{H} \mathcal{N} \), \( T_{10} \in \mathcal{F} \mathcal{H} \mathcal{N} \implies A_1 = \frac{1}{\lambda} T_{10} \in \mathcal{F} \mathcal{H} \mathcal{N} \). (\( \mathcal{F} \mathcal{H} \mathcal{N} \) operators are closed under multiplication by non-zero scalars.) Observe that \( \sigma(A_1) = [1] \), and \( \sup_k \| A_k^k \| \leq 1 \), where the supremum is taken over all integers \( k \). Applying [11, Theorem 1.5.14], it follows that \( A_1 = I|_{H_0(T_1 - \lambda)} \implies T_{10} = \lambda I|_{H_0(T_1 - \lambda)} \). Hence,
\[ \mathcal{H}_1 = (T_1 - \lambda)^{-1}(0) \oplus K(T_1 - \lambda) \]
\[ \implies (T_1 - \lambda)\mathcal{H}_1 = 0 \oplus (T_1 - \lambda)K(T_1 - \lambda) = K(T_1 - \lambda) \]
\[ \implies \mathcal{H}_1 = (T_1 - \lambda)^{-1}(0) \oplus (T_1 - \lambda)\mathcal{H}_1. \]

Setting \( T_{11} = T_{111} \), it follows that \( T_1 - \lambda \) has a triangulation
\[ T_1 - \lambda = \begin{bmatrix} 0 & A_0 \\ 0 & T_{11} - \lambda \end{bmatrix} \]
\[ \lambda \]

By hypothesis, isolated non-zero eigenvalues of \( T_1 \) are normal. Hence, if we let \( x_1 \in (T_{10} - \lambda)^{-1}(0) \), then \( x_1 \oplus 0 \in (T_{11} - \lambda)^{-1}(0) \implies A_0^nx_1 = 0 \) for every \( x_1 \in (T_{10} - \lambda)^{-1}(0) \). Hence, \( A_0^n \), and so also \( A_0 \), is the zero operator, which implies that \( T_1 - \lambda = 0 \oplus T_{11} - \lambda \).

If \( \lambda \in \sigma(T) \), then \( H_0(T - \lambda) \) is non-trivial and
\[ \mathcal{H} = H_0(T - \lambda) \oplus K(T - \lambda) \]
holds as a topological direct sum [12, Theorem 1.6]. Furthermore, both \( H_0(T - \lambda) \)
and \( K(T - \lambda) \) are closed [13]. Let \( \lambda \neq 0 \); then \( \lambda \) is a normal eigenvalue of \( T_1 \) (and \( T_3 - \lambda \) is invertible). Set \( (T_1 - \lambda)^{-1}(0) = \mathcal{H}'_1 \), \( \mathcal{H}_1 \oplus \mathcal{H}'_1 = \mathcal{H}_3 \) and \( \mathcal{H}_3 \oplus \mathcal{H}_2 = \mathcal{H}_2 \). Then it follows from the above that
\[ T_{12} - \lambda = \begin{bmatrix} 0 & A_0 & T_{21} \\ 0 & T_{11} - \lambda & T_{22} \\ 0 & 0 & T_{3} - \lambda \end{bmatrix} \]
\[ \lambda \]

where \( A = [0, T_{21}] \), and where
\[ B = \begin{bmatrix} T_{11} - \lambda & T_{22} \\ T_{12} - \lambda & T_{3} - \lambda \end{bmatrix} \]
is invertible. Since
\[ H_0(T - \lambda) = \{ x \in \mathcal{H} : \lim_{n \to \infty} \| (T - \lambda)^nx \|^{1/n} = 0 \} \]
\[ = \{ x = x_1 \oplus x_2 \in \mathcal{H} : \lim_{n \to \infty} \left\| \begin{bmatrix} AB^{n-1}x_1 \\ B^n x_2 \end{bmatrix} \right\|^{1/n} = 0 \}, \]
the invertibility of \( B \) implies that
\[ \| x_2 \|^{1/n} \leq \| B^{-1} \| \| B^n x_2 \|^{1/n} \to 0 \text{ as } n \to \infty. \]

Hence, \( x_2 = 0 \), and
\[ H_0(T - \lambda) = \{ x = x_1 \oplus 0 \in \mathcal{H}'_1 \oplus \mathcal{H}'_2 = \mathcal{H} \} = (T - \lambda)^{-1}(0). \]

By hypothesis, \( (T - \lambda)^{-1}(0) \subseteq (T^* - \lambda)^{-1}(0) \); hence
\[ (T^* - \lambda)(x_1 \oplus 0) = 0 \oplus A^*x_1 = 0 \]
Corollary 2.1. Let $T \in B(H)$ be a normal operator, and let $\mathcal{H} = H_0(T - \lambda)$ for some $\lambda \in \sigma(T)$. Then
\[
\mathcal{H} = H_0(T - \lambda) \oplus K(T - \lambda) = (T - \lambda)^{-1}(0) \oplus K(T - \lambda).
\]
In particular, $(T - \lambda)\mathcal{H}$ is closed. Observe that $(T^* - \overline{\lambda})^{-1}(0) = (T - \lambda)\mathcal{H}$; hence, $(T^* - \overline{\lambda})^{-1}(0) = (T - \lambda)^{-1}(0)$, $(T - \lambda)\mathcal{H} = (T - \lambda)^{-1}(0)$. Thus, $P_\lambda^{-1}(0) = P_\lambda\mathcal{H} \Rightarrow P_\lambda$ is self-adjoint. This completes the proof. \(\square\)

The following corollary is immediate from the proof above. (Observe that if $T \in \mathcal{K}$ and $0 \in \sigma(T)$, then $\lambda(T) = 0 \Rightarrow T|_{H_0(T)} = 0 \Rightarrow 0$ is an eigenvalue of $T$.)

**Corollary 2.1.** If $T \in \mathcal{K}$ and $0 \in \sigma(T)$, then $P_\lambda$ is self-adjoint and $P_\lambda\mathcal{H} = (T - \lambda)^{-1}(0) = (T^* - \overline{\lambda})^{-1}(0)$ for every $\lambda \in \sigma(T)$.

### 3. Applications

An operator $T \in B(H)$ is
- **hyponormal** if $|T^*|^2 \leq |T|^2$;
- **p-hyponormal**, $0 \leq p \leq 1$, if $|T^*|^{2p} \leq |T|^{2p}$;
- **w-hyponormal** if $|T^*| \leq |T| \leq |T|$, where, for $T = U|T|$, $\tilde{T}$ is the Aluthge transform $\tilde{T} = |T|^{1/2}U|T|^{1/2}$ of $T$;
- **M-hyponormal** if there exists a number $M \geq 1$ such that $|T^* - \overline{\lambda}|^2 \leq M|T - \lambda|^2$ for all $\lambda \in \mathcal{C}$;
- **k-quasihyponormal** for some integer $k \geq 1$ if $T^{*k}(|T|^2 - |T^*|^2)T^k \geq 0$;
- **(k, p)-quasihyponormal** for some integer $k \geq 1$, and $0 < p \leq 1$, if $T^{*k}(|T|^2 - |T^*|^2)T^k \geq 0$;
- **of class $\mathcal{A}$** if $|T|^2 \leq |T|^2$;
- **paranormal** if $\|T^*x\|^2 \leq \|T^*x\|$ for every unit vector $x \in \mathcal{H}$, and $T$ is totally $*$-paranormal if $\|\lambda(T - \lambda)^*x\|^2 \leq \|\lambda(T - \lambda)^*x\|$ for all $\lambda \in \mathcal{C}$ and every unit vector $x \in \mathcal{H}$.

The following inclusions are proper: hyponormal $\subset$ p-hyponormal $\subset$ w-hyponormal $\subset$ paranormal, and $\mathcal{A} \subset$ paranormal. (We refer the interested reader to the monograph [7] for information on (most of) these classes of operators.)

Let $\mathcal{C}_1$ denote the class of operators $T$ which are either hyponormal or p-hyponormal, and let $\mathcal{C}_2$ denote the class of $T$ which are either w-hyponormal or of class $\mathcal{A}$.
or paranormal. Then \( C_1 \cup C_2 \subseteq THN \). Points \( \lambda \in \text{iso } \sigma (T) \), \( T \in C_1 \), are normal eigenvalues of \( T \). Corollary 2.1 applies, and we have:

**Corollary 3.1** ([4]). If \( T \in C_1 \), then the Riesz projection \( P_\lambda \) associated with points \( \lambda \in \text{iso } \sigma (T) \) is self-adjoint and \( P_\lambda H = (T - \lambda)^{-1}(0) = (T^* - \lambda)^{-1}(0) \).

**Remark 3.2.** \( M \)-hyponormal operators are not, in general, normaloid; hence \{ \( M \)-hyponormal \} \( \not \subseteq THN \). It is however known that \( M \)-hyponormal operators satisfy the property that \( H_0(T - \lambda) = (T - \lambda)^{-1}(0) \) for every complex number \( \lambda \) [1, p. 176]; furthermore, points \( \lambda \in \text{iso } \sigma (T) \) are simple poles of the resolvent of \( T \), and the eigenvalues of \( T \) are normal. Hence, by (some of) the argument of the proof of Theorem 1.1, the Riesz projection \( P_\lambda \) associated with points \( \lambda \in \text{iso } \sigma (T) \) of an \( M \)-hyponormal operator is self-adjoint and \( P_\lambda H = (T - \lambda)^{-1}(0) = (T^* - \lambda)^{-1}(0) \).

Operators \( T \) which are \((k, p)\)-quasihyponormal (a \((k, 1)\)-quasihyponormal operator is \( k \)-quasihyponormal) have a triangulation

\[
T = \begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix} \begin{bmatrix} T_1 & \mathcal{H} \\ T_2 & T_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]

where \( T_1 \) is \( p \)-hyponormal, \( T_3^k = 0 \) and \( \sigma (T_1) \subseteq \sigma (T) \subseteq \sigma (T_1) \cup \{ 0 \} \) (see [3,18]). Furthermore, non-zero eigenvalues of \( T \) are normal (see [3] and [18, Lemma 3]). Theorem 1.1 applies, and we have:

**Corollary 3.3** [17,18]. If \( T \) is either \( k \)-quasihyponormal or \((k, p)\)-quasihyponormal, then the Riesz projection \( P_\lambda \) associated with points \( \lambda \in \text{iso } \sigma (T) \) is self-adjoint and \( P_\lambda H = (T - \lambda)^{-1}(0) = (T^* - \lambda)^{-1}(0) \).

Isolated points of the spectrum of a paranormal operator are (simple poles of the resolvent operator, and hence) eigenvalues of the operator [5]. Uchiyama [19] has observed that if \((0 \neq) \lambda \in \text{iso } \sigma (T) \), then the paranormal \( T \) has a triangulation

\[
T = \begin{bmatrix} \lambda & A \\ 0 & B \end{bmatrix},
\]

where \( B \) is paranormal and \( A(B - \lambda) = 0 \). From this he deduces that

\[
(T - \lambda)^{-1}(0) = (T^* - \lambda)^{-1}(0), \text{ i.e., } \lambda \text{ is a normal eigenvalue of } T.
\]

Corollary 2.1 applies and one has:

**Corollary 3.4** [9,19]. If \( T \in C_2 \), then the Riesz projection \( P_\lambda \) associated with points \( \lambda \in \text{iso } \sigma (T) \) is self-adjoint and \( P_\lambda H = (T - \lambda)^{-1}(0) = (T^* - \lambda)^{-1}(0) \).

The class of \(*\)-paranormal operators \( T \) (i.e., operators \( T \) such that \( \|T^*x\|^2 \leq \|T^2x\| \) for all unit vectors \( x \in \mathcal{H} \)) is independent of the class of paranormal operators. Recall that \(*\)-paranormal operators are normaloid [8]. If \( \lambda \) is an eigenvalue of \( T \), \( T \) \(*\)-paranormal, and \( x \in \mathcal{H} \) is an eigenvector corresponding to the
eigenvalue \( \lambda \), then a simple calculation (using the definition) shows that \( \| (T^* - \overline{\lambda})x \| = 0 \). Hence, eigenvalues of a \( * \)-paranormal operator \( T \) are normal eigenvalues of \( T \). It is known, see for example [8], that \( \text{asc}(T - \lambda) \leq 1 \) for all \( \lambda \in \mathbb{C} \) (\( \Rightarrow T \) has SVEP at all \( \lambda \in \mathbb{C} \)). Totally, \( * \)-paranormal operators \( T \) satisfy the property that \( H_0(T - \lambda) = (T - \lambda)^{-1}(0) \) for all \( \lambda \in \mathbb{C} \) [8, Lemma 2.2]. Thus, if \( \lambda \in \text{iso}(T) \) for a totally \( * \)-paranormal operator \( T \), then \( \mathcal{H} = (T - \lambda)^{-1}(0) \oplus (T - \lambda) \mathcal{H}^* \); in particular, isolated points are simple poles of the resolvent operator, and hence normal eigenvalues, of \( T \). Applying the argument of the proof of Theorem 1.1 one has:

**Corollary 3.5.** If \( T \) is a totally \( * \)-paranormal operator, then the Riesz projection \( P_\lambda \) associated with every \( \lambda \in \text{iso}(T) \) is self-adjoint and \( P_\lambda \mathcal{H} = (T - \lambda)^{-1}(0) = (T^* - \overline{\lambda})^{-1}(0) \).

We conclude this paper with the observation that operators \( T \) belonging to the class \( \mathcal{P} \), consisting of operators \( T \in B(\mathcal{H}) \) which belong to one of the classes defined (above) at the beginning of this section, satisfy Weyl’s theorem. If we let \( \sigma_w(T) = \{ \lambda \in \mathbb{C} : \text{either } T - \lambda \text{ is not Fredholm or } \text{ind}(T - \lambda) \neq 0 \} \) denote the Weyl spectrum of \( T \), and \( \pi_{00}(T) = \{ \lambda \in \mathbb{C} : \lambda \in \text{iso}(T) \text{ and } 0 < \text{dim}(T - \lambda)^{-1}(0) < \infty \} \), then \( T \) is said to satisfy Weyl’s theorem if \( \sigma_w(T) \setminus \sigma_w(T) = \pi_{00}(T) \). Recall from [6, Theorem 2.3] that \( T \) satisfies Weyl’s theorem if and only if \( T \) has SVEP at all points \( \lambda \in \sigma(T) \setminus \sigma_w(T) \) and \( \pi_{00}(T) = \pi_0(T) \), where \( \pi_0(T) \) is the set of Riesz points of \( T \) (i.e., points \( \lambda \) such that both \( \text{asc}(T - \lambda) \) and \( \text{asc}(T - \lambda) \) are finite). Operators \( T \in \mathcal{G}_1, M \)-hyponormal operators and \( * \)-paranormal operators have finite ascent, and hence SVEP, at all \( \lambda \in \mathbb{C} \). If \( T \in \mathcal{Q}_2 \), then \( T \) has SVEP at all \( \lambda \in \mathbb{C} \setminus \sigma_w(T) \) [6, Theorem 3.2]. If \( T \) is \((k, p)\)-quasihyponormal, then the non-zero eigenvalues of \( T \) being normal (see [3] and [18, Lemma 3]), \( \text{asc}(T - \lambda) \leq 1 \) for all \( 0 \neq \lambda \in \mathbb{C} \). For \( \lambda = 0 \), let \( T^{k+1}x = 0; x \in \mathcal{H} \). Then (using the Hölder–McCarthy inequality [14])

\[
\| T^kx \|^2 = \langle (T|^{2p}T^{k-1}x, T^{k-1}x \rangle \leq \langle (T|^{2p+1}T^{k-1}x, T^{k-1}x \rangle \frac{1}{p^2} \| T^{k-1}x \|^{2p} \| T^{k-1}x \|^{2p} \\
= \langle (T|^{2p}T^{k}x, T^{k}x \rangle \frac{1}{p^2} \| T^{k-1}x \|^{2p} \| T^{k-1}x \|^{2p} \\
\leq \langle (T|^{2p}T^{k}x, T^{k}x \rangle \frac{1}{p^2} \| T^{k-1}x \|^{2p} \| T^{k-1}x \|^{2p} \\
\leq \langle (T|^{2p}T^{k}x, T^{k}x \rangle \frac{1}{p^2} \| T^{k-1}x \|^{2p} \| T^{k-1}x \|^{2p} \rangle = 0,
\]

which implies that \( \text{asc}(T) \leq k \). Hence, \( \text{asc}(T - \lambda) \leq k \), which implies that \( T \) has SVEP at all \( \lambda \in \mathbb{C} \).

**Corollary 3.6.** Operators \( T \in \mathcal{P} \) satisfy Weyl’s theorem.

**Proof.** As seen above, \( T \) has SVEP at all \( \lambda \in \sigma(T) \setminus \sigma_w(T) \). Also, as remarked upon above, isolated points of the spectrum of \( T \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \{ M \text{-hyponormal} \} \) are
simple poles of the resolvent of $T$. Hence, $\pi_{00}(T) \subseteq \pi_0(T)$ for operators $T \in C_1 \cup C_2 \cup \{M\text{-hyponormal}\}$. If $T$ is a $(k, p)$-quasihyponormal operator and $\lambda \neq 0$, then $\pi_{00}(T) \subseteq \pi_0(T)$ (see the proof of Theorem 1.1); also, if $\lambda = 0$, then $H_0(T) = T^{k-1}(0)$ [18, Theorem 6(ii)]. Hence, $\pi_{00}(T) \subseteq \pi_0(T)$ for $T \in \mathcal{P}$. Since $\pi_0(T) \subseteq \pi_{00}(T)$ for every operator $T$, $\pi_0(T) = \pi_{00}(T)$ for $T \in \mathcal{P}$. This completes the proof. □

Recall that an operator $T$ is said to be isoloid if the isolated points of $\sigma(T)$ are eigenvalues of $T$. Operators $T \in \mathcal{P}$ are isoloid.

**Corollary 3.7.** If $T \in \mathcal{P}$, then $f(T)$ satisfies Weyl’s theorem for every function $f$ which is analytic on an open neighbourhood of $\sigma(T)$.

**Proof.** Apply [15, Theorem 1]. □

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**References**