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Sturmian words, β-shifts, and transcendence

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Abstract

Consider the minimal β-shift containing the shift space generated by a given Sturmian word. In this paper we characterize such β and investigate their combinatorial, dynamical and measure-theoretical properties and prove that such β are transcendental numbers.

Keywords: Sturmian word; β-shift; Self-Sturmian number

1. Introduction

A Sturmian word is an infinite word s over a binary alphabet A, whose complexity function satisfies $P(s,n) = n+1$ for all $n \geqslant 0$, i.e., the number of factors of s with length n is exactly $n+1$. Sturmian words are aperiodic infinite words with minimal complexity [14,7].

Let $\beta > 1$ be a real number. We consider the β-transformation T_β on $[0,1]$ defined by $T_\beta : x \mapsto \beta x \bmod 1$. Then the β-expansion of $x \in [0,1]$, denoted by $d_\beta(x)$, is a sequence of integers determined by the following rule:

$$d_\beta(x) = (x_i)_{i \geqslant 1} \text{ if and only if } x_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor,$$

where $\lfloor t \rfloor$ is the largest integer not greater than t. The β-shift S_β is the closure of $\{d_\beta(x) | x \in [0,1)\}$ with respect to the topology of $A^{\mathbb{N}}$. In [15], Parry completely

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characterized S_β in terms of $d_\beta(1)$ and the lexicographic order on $A^{\mathbb{N}}$. From Parry's result we note that the collection of all β-shifts is totally ordered. The main concern of this article is about the minimal β-shift containing the shift space generated by a Sturmian word.

We call $\beta > 1$ a (maximal) self-Sturmian number if $d_\beta(1)$ is a Sturmian word (and $\alpha(d_\beta(1)) = \{\lfloor\beta\rfloor - 1, \lfloor\beta\rfloor\}$). We show that S_β minimally contains the shift space generated by some Sturmian word if and only if β is self-Sturmian. This gives a large class of specified β-transformations T_β and moreover for a maximal self-Sturmian β, the diameter of the closure of $\{T_\beta^n 1\}_{n \geqslant 0}$ is minimal in a certain sense. We also prove the transcendence of such β. This is a partial answer to the question posed by Blanchard [5].

2. Sturmian words and lexicographic order

Since $P(s,1) = 2$, Sturmian words are forced to be infinite words over the alphabet $A = \{0,1\}$ by renaming if necessary. Then the *height* $h(x)$ of a word x is the number of occurrences of 1 in x. We say a subset X of A^* is *balanced* if for any $x, y \in X$, $|h(x) - h(y)| \leqslant 1$ whenever x and y have the same lengths. An infinite word s is also called *balanced* if the factor set $F(s)$ is balanced. In [7], Coven and Hedlund described the balanced property in more detail.

Theorem 2.1. *Suppose $X \subset A^*$ and $x \in X$ implies $F(x) \subset X$. Then X is unbalanced if and only if there exists a palindrome word w such that both $0w0$ and $1w1$ lie in X.*

For a real number t, $\lceil t\rceil$ is the smallest integer not less than t, and $\{t\}$ the fractional part of t, i.e., $t = \lfloor t\rfloor + \{t\}$. Let α, ρ be two real numbers in $[0,1]$. We now define two infinite words over $\{0,1\}$. Consider, for $n \geqslant 0$,

$$s_{\alpha,\rho}(n) = \lfloor \alpha(n+1) + \rho \rfloor - \lfloor \alpha n + \rho \rfloor, \quad s'_{\alpha,\rho}(n) = \lceil \alpha(n+1) + \rho \rceil - \lceil \alpha n + \rho \rceil.$$

The infinite words $s_{\alpha,\rho}$, $s'_{\alpha,\rho}$ are termed *lower* and *upper mechanical words*, respectively, with *slope* α and *intercept* ρ. If α is irrational, we see $s_{\alpha,0} = 0c_\alpha$, $s'_{\alpha,0} = 1c_\alpha$ for some infinite word c_α. Here the word c_α is called the *characteristic word* of slope α. Morse and Hedlund [14] proved two alternative characterizations of Sturmian words.

Theorem 2.2. *For an infinite word s, the following are equivalent.*
- *s is Sturmian.*
- *s is aperiodic and balanced.*
- *s is irrationally mechanical, i.e., the slope is irrational.*

The following proposition prescribes factor sets of Sturmian words.

Proposition 2.1 (Mignosi [13]). *For two Sturmian words s, t, if they have the same slope, then $F(s) = F(t)$. And $F(s) \cap F(t)$ is finite otherwise.*

We denote by σ the shift map, and by $\overline{\mathcal{O}}(s)$ its orbit closure of s. Since Sturmian words are uniformly recurrent, $\overline{\mathcal{O}}(s)$ is minimal if s is Sturmian.

Proposition 2.2. *Let s be a Sturmian word with slope α. Then $\overline{\mathcal{O}}(s)$ is the set of all mechanical words of slope α.*

The proof is a consequence of a lemma.

Lemma 2.1. *For a fixed irrational $\alpha \in (0,1)$, $s_{\alpha,\rho}$ is continuous from the right and $s'_{\alpha,\rho}$ from the left as functions of ρ.*

Proof. Let $\varepsilon > 0$, s_{α,ρ_0}, s'_{α,ρ_0} be given. We choose an integer $N > 0$ such that $2^{-N} < \varepsilon$. Put $\delta_1 = \min\{1 - \{\alpha n + \rho_0\} \mid 0 \leqslant n \leqslant N+1\}$. Then $0 \leqslant \rho - \rho_0 < \delta_1/2$ implies $d(s_{\alpha,\rho}, s_{\alpha,\rho_0}) < \varepsilon$. For the upper mechanical word, we define δ_2 by the minimum of nonzero fractions $\{\alpha n + \rho_0\}$ for $0 \leqslant n \leqslant N+1$. Then $0 \leqslant \rho_0 - \rho < \delta_2/2$ implies $d(s'_{\alpha,\rho}, s'_{\alpha,\rho_0}) < \varepsilon$. If $\rho_0 = 0$, then we can assume $\rho_0 = 1$. □

Proof of Proposition 2.2. By the minimality of $\overline{\mathcal{O}}(s)$ we may assume $s = s'_{\alpha,0} = 1c_\alpha$. Since α is irrational, αn is never an integer for nonzero n. Hence $\sigma^n(s) = s_{\alpha,\{\alpha n\}} = s'_{\alpha,\{\alpha n\}}$ holds for any $n \geqslant 1$. Given $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$ we can pick two increasing sequences of integers $(p_n)_{n\geqslant 0}$, $(q_n)_{n\geqslant 0}$ such that $\{\alpha p_n\} \searrow \rho$ and $\{\alpha q_n\} \nearrow \rho$. Then one finds

$$\lim_{n\to\infty} \sigma^{p_n}(s) = s_{\alpha,\rho}, \quad \lim_{n\to\infty} \sigma^{q_n}(s) = s'_{\alpha,\rho}.$$

Conversely assume $t \in \overline{\mathcal{O}}(s)$. Then t is balanced since $F(t) \subset F(s)$, and the minimality implies the aperiodicity of t. t has the slope α. Otherwise $F(s) \cap F(t)$ would be finite by Proposition 2.1. But every factor of t also occurs in s. □

In the next section, what we need critically is the lexicographic order between Sturmian words. We have the following. For its proof, see [10].

Proposition 2.3. *Suppose $\alpha \in (0,1)$ is irrational and $\rho, \rho' \in [0,1)$ are real. Then*

$$s_{\alpha,\rho} < s_{\alpha,\rho'} \text{ if and only if } \rho < \rho'.$$

Corollary 2.3.1 (Borel and Laubie [6]). *Let α be an irrational number in $(0,1)$. Then $0c_\alpha < s_{\alpha,\rho} < 1c_\alpha$ for any $0 < \rho < 1$. In particular we have for all $n \geqslant 1$,*

$$1c_\alpha > \sigma^n(1c_\alpha) \quad \text{and} \quad 0c_\alpha < \sigma^n(0c_\alpha).$$

3. β-shifts and self-Sturmian numbers

Just as the number 1 dominates any number in $[0,1)$, so does $d_\beta(1)$ in S_β with respect to lexicographic order, which was shown by Parry [15]. Moreover, Parry also determined the sequences that can be β-expansions of 1 for some $\beta > 1$. These sequences obey the next rule.

Theorem 3.1. *A sequence $s \in \{0, 1, \ldots, \lfloor \beta \rfloor\}^{\mathbb{N}}$ is a β-expansion of 1 for some β if and only if $\sigma^n(s) < s$ for all $n \geqslant 1$. In this case, such a β is unique.*

From now on we replace the alphabet $A = \{0, 1\}$ by $\{a, b\}$ for any $a, b \in \mathbb{Z}$ and $0 \leqslant a < b$. For a Sturmian word s over $\{a, b\}$ with slope α, any $t \in \overline{\mathcal{O}}(s)$ lies between ac_α and bc_α, where c_α is the characteristic word over $\{a, b\}$. We write this as $ac_\alpha \leqslant \overline{\mathcal{O}}(s) \leqslant bc_\alpha$. This notation represents β-shift as $0^\infty \leqslant S_\beta \leqslant d_\beta(1)$. In both cases, the two inequalities are best possible. By Theorem 3.1 and Corollary 2.3.1, there exists a unique $\beta > 1$ such that $d_\beta(1) = bc_\alpha$. From the fact that $\gamma < \theta$ implies $S_\gamma \subsetneq S_\theta$, one can deduce that S_β is the minimal β-shift containing $\overline{\mathcal{O}}(s)$. Moreover, the closure of $\{\sigma^n(d_\beta(1))\}_{n \geqslant 0}$ is equal to $\overline{\mathcal{O}}(s)$ and ac_α, bc_α are accumulation points by Proposition 2.2. We state these as a theorem. For an infinite word x, $\alpha(x)$ is the set of letters involved in x.

Theorem 3.2. *Suppose s is Sturmian of slope α and $\alpha(s) = \{a, b\}$ with $0 \leqslant a < b$. If S_β is the smallest β-shift containing $\overline{\mathcal{O}}(s)$, then $\overline{\{\sigma^n(d_\beta(1))\}}_{n \geqslant 0} = \overline{\mathcal{O}}(s)$ and β is the unique positive solution of $1 = \sum_{n=0}^{\infty} ((b-a)s'_{\alpha,0}(n) + a)/x^{n+1}$.*

In [5], Blanchard classified β-shifts into five categories, and for each β contained in some classes the morphology of $d_\beta(1)$ was totally understood by Parry and Bertrand-Mathis [15,4]. The language theoretical terminology used in the next proposition is referred to [5] or the bibliography therein.

Proposition 3.1. *For $\beta > 1$, the following equivalences hold.*

- *$\beta \in \mathscr{C}_1$: S_β is a shift of finite type if and only if $d_\beta(1)$ is finite.*
- *$\beta \in \mathscr{C}_2$: S_β is sofic if and only if $d_\beta(1)$ is ultimately periodic.*
- *$\beta \in \mathscr{C}_3$: S_β is specified if and only if there exists $n \in \mathbb{N}$ such that the number of consecutive 0's in $d_\beta(1)$ is less than n.*
- *$\beta \in \mathscr{C}_4$: S_β is synchronizing if and only if some word of $F(S_\beta)$ does not appear in $d_\beta(1)$.*
- *$\beta \in \mathscr{C}_5$: S_β has none of the above properties if and only if all words of $F(S_\beta)$ appear at least once in $d_\beta(1)$.*

One sees immediately the inclusions:

$$\emptyset \neq \mathscr{C}_1 \subset \mathscr{C}_2 \subset \mathscr{C}_3 \subset \mathscr{C}_4 \subset (1, \infty), \quad \mathscr{C}_5 = (1, \infty) \backslash \mathscr{C}_4.$$

On the other hand, Schmeling [16] determined each size of the classes.

Proposition 3.2. *$\mathscr{C}_3$ has Hausdorff dimension 1 and $\mathscr{C}_5$ has full Lebesgue measure.*

Now we concentrate on a special class of real numbers that is contained in $\mathscr{C}_3$.

Definition. Let $\beta > 1$. We call β a *self-Sturmian number* if $d_\beta(1)$ is a Sturmian word over a binary alphabet $A = \{a, b\}$, $0 \leqslant a < b = \lfloor \beta \rfloor$. In particular, β is *maximally self-Sturmian* if it is self-Sturmian and $\alpha(d_\beta(1)) = \{\lfloor \beta \rfloor - 1, \lfloor \beta \rfloor\}$.

Remark. Not every Sturmian word is equal to $d_\beta(1)$ for some $\beta > 1$. In such cases, $d_\beta(1) = bc_\alpha$ and $\alpha(c_\alpha) = \{a, b\}$ for some irrational α and integers $0 \leqslant a < b$.

Definition. $\mathrm{diam} : (1, \infty) \to [0, 1]$ is the function that maps β to the diameter of T_β-orbit of 1, i.e., $\mathrm{diam}(\beta) := \mathrm{diam}\{T_\beta^n 1\}_{n \geqslant 0} = \sup\{|x - y| : x, y \in \{T_\beta^n 1\}_{n \geqslant 0}\}$.

One can note that if $\beta \in \mathscr{C}_1$ or $\beta \in (1, \infty) \setminus \mathscr{C}_3$, then $\mathrm{diam}(\beta) = 1$ since both 0 and 1 lie in the closure of $\{T_\beta^n 1\}_{n \geqslant 0}$. We get from the definitions,

Proposition 3.3. *Suppose β is self-Sturmian and $\alpha(d_\beta(1)) = \{a, b\}$ with $0 \leqslant a < b = \lfloor \beta \rfloor$. Then $\beta \in \mathscr{C}_3 \setminus \mathscr{C}_2$ and* $\mathrm{diam}(\beta) = (b - a)/\beta$.

Maximal self-Sturmian numbers are distinguished from the dynamical point of view.

Theorem 3.3. *$\beta > 1$ is maximally self-Sturmian if and only if $\beta \notin \mathscr{C}_2$ and $1 - 1/\beta \leqslant T_\beta^n 1 \leqslant 1$ for any $n \geqslant 0$.*

Proof. We prove the sufficiency. The hypothesis implies that $d_\beta(1)$ is aperiodic and $\alpha(d_\beta(1)) = \{\lfloor \beta \rfloor - 1, \lfloor \beta \rfloor\}$. Put $a = \lfloor \beta \rfloor - 1$ and $b = \lfloor \beta \rfloor$. If $d_\beta(1)$ is unbalanced, then Theorem 2.1 guarantees the existence of a palindrome word w such that both awa, bwb are factors of $d_\beta(1)$. For $d_\beta(1) = bd_1d_2\ldots$, one sees $d_\beta(1 - 1/\beta) = ad_1d_2\ldots$. We get

$$ad_1 \cdots d_n d_{n+1} \leqslant awa < bwb \leqslant bd_1 \cdots d_n d_{n+1},$$

where n is the length of w. This yields a contradiction. Hence $d_\beta(1)$ is a Sturmian word of some slope α and it dominates all its shifts, and therefore $d_\beta(1) = bc_\alpha$. $\square$

The diameter of a maximal self-Sturmian number is minimal in the following sense.

Corollary 3.3.1. *For any $\beta > 1$, either $\beta \in \mathscr{C}_2$ or* $\mathrm{diam}(\beta) \geqslant 1/\beta$.

Proposition 3.2 shows the set of self-Sturmian numbers is of Lebesgue measure zero. Then what about the size of $\overline{\{T_\beta^n 1\}}_{n \geqslant 0}$ for a fixed self-Sturmian number β? The last paragraph of this section is devoted to showing $\overline{\{T_\beta^n 1\}}_{n \geqslant 0}$ has Lebesgue measure zero, whereas the orbit closure of an irrational rotation has full Lebesgue measure even though Theorem 3.2 indicates that two orbit closures in full shift coincide.

A β-transformation T_β has an invariant ergodic measure v_β whose Radon–Nikodym derivative with respect to Lebesgue measure is given by

$$h_\beta(x) = \frac{1}{F(\beta)} \sum_{x < T_\beta^n 1} \frac{1}{\beta^n}, \quad x \in [0, 1].$$

Here $F(\beta)$ is the normalizing factor. Suppose $d_\beta(1) = \varepsilon_0\varepsilon_1\ldots$. Parry noted that

$$F(\beta) = \int_0^1 \sum_{x<T_\beta^n 1} \frac{1}{\beta^n}\,\mathrm{d}x = \int_0^1 \left(\sum_{n=0}^{\infty} \frac{a_n(x)}{\beta^n}\right)\mathrm{d}x = \sum_{n=0}^{\infty} \frac{T_\beta^n 1}{\beta^n} = \sum_{n=0}^{\infty} \frac{(n+1)\varepsilon_n}{\beta^{n+1}},$$

where

$$a_n(x) = \begin{cases} 1 \text{ if } x < T_\beta^n 1, \\ 0 \text{ otherwise.} \end{cases}$$

The frequency of $\lfloor\beta\rfloor$ in $d_\beta(x)$ is, if the limit exists, given by

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi(T_\beta^i(x)),$$

where χ is the characteristic function of $[\lfloor\beta\rfloor/\beta, 1]$. Owing to Birkhoff Ergodic Theorem, we can say more. For almost all x in $[0,1]$, the frequency of $\lfloor\beta\rfloor$ in $d_\beta(x)$ equals

$$\mu_\beta(\lfloor\beta\rfloor) := \frac{1}{F(\beta)} \int_{\lfloor\beta\rfloor/\beta}^1 \sum_{x<T_\beta^n 1} \frac{1}{\beta^n}\,\mathrm{d}x.$$

A similar reasoning also applies to the frequency of the other digit.

Lemma 3.1. *If β is self-Sturmian and $\alpha(d_\beta(1)) = \{a,b\}$ with $0 \leqslant a < b = \lfloor\beta\rfloor$, then for almost every x in $[0,1]$, the frequency of b in $d_\beta(x)$ is equal to*

$$\mu_\beta(b) = \frac{\mathscr{I}}{F(\beta)} = \frac{1}{F(\beta)} \sum_{n=0}^{\infty} \lceil \alpha n \rceil \frac{\varepsilon_n}{\beta^{n+1}},$$

and the frequency of a in $d_\beta(x)$ is equal to

$$\mu_\beta(a) = \frac{\mathscr{J}}{F(\beta)} = \frac{1}{F(\beta)} \left(\sum_{n\in J} \frac{1}{\beta^{n+1}} + \sum_{n=0}^{\infty} (n - \lceil \alpha n \rceil) \frac{\varepsilon_n}{\beta^{n+1}} \right),$$

where $d_\beta(1) = \varepsilon_0\varepsilon_1\varepsilon_2\ldots$ and $J = \{n \geqslant 0 \mid \varepsilon_n = b\}$, $K = \{n \geqslant 0 \mid \varepsilon_n = a\}$.

Proof. Let α be a number such that $d_\beta(1) = bc_\alpha = \varepsilon_0\varepsilon_1\varepsilon_2\ldots$, where c_α is the characteristic word of slope α over the alphabet $\{a,b\}$. First we compute the following integral:

$$\mathscr{I} := \int_{\lfloor\beta\rfloor/\beta}^1 \sum_{x<T_\beta^n 1} \frac{1}{\beta^n}\,\mathrm{d}x = \int_{\lfloor\beta\rfloor/\beta}^1 \sum_{n=0}^{\infty} \frac{a_n(x)}{\beta^n}\,\mathrm{d}x$$

$$= \sum_{n=0}^{\infty} \frac{1}{\beta^n} \int_{\lfloor\beta\rfloor/\beta}^1 a_n(x)\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{b_n}{\beta^n},$$

where

$$b_n = \begin{cases} T_\beta^n 1 - \lfloor\beta\rfloor/\beta & \text{if } \lfloor\beta\rfloor/\beta \leqslant T_\beta^n 1, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\lfloor\beta\rfloor/\beta \leqslant T_\beta^n 1$ is equivalent to $\varepsilon_n = b$, it follows from Fubini Theorem that

$$\mathscr{I} = \sum_{n\in J} \frac{1}{\beta^n}\left(T_\beta^n 1 - \frac{\lfloor\beta\rfloor}{\beta}\right) = \sum_{n\in J}\sum_{m=n+1}^{\infty} \frac{\varepsilon_m}{\beta^{m+1}} = \sum_{n=0}^{\infty} h_n \frac{\varepsilon_{n+1}}{\beta^{n+2}} = \sum_{n=0}^{\infty} h_{n-1} \frac{\varepsilon_n}{\beta^{n+1}},$$

where h_n is the number of b's in the word $\varepsilon_0\varepsilon_1\cdots\varepsilon_n$ and we put $h_{-1} = 0$ by convention. Noting that $h_n = \lceil\alpha(n+1)\rceil$, we finally get $\mathscr{I} = \sum_{n=0}^{\infty}\lceil\alpha n\rceil\varepsilon_n/\beta^{n+1}$.

For almost every x in $[0,1]$, the frequency of a in $d_\beta(x)$ is equal to

$$\mu_\beta(a) = \frac{\mathscr{I}}{F(\beta)} = \frac{1}{F(\beta)} \int_{a/\beta}^{(a+1)/\beta} \sum_{x<T_\beta^n 1} \frac{1}{\beta^n}\,\mathrm{d}x.$$

The integration is derived as follows:

$$\begin{aligned} \mathscr{I} &:= \int_{a/\beta}^{(a+1)/\beta} \sum_{x<T_\beta^n 1} \frac{1}{\beta^n}\,\mathrm{d}x = \int_{a/\beta}^{(a+1)/\beta} \sum_{n=0}^{\infty} \frac{a_n(x)}{\beta^n}\,\mathrm{d}x \\ &= \sum_{n=0}^{\infty} \frac{1}{\beta^n} \int_{a/\beta}^{(a+1)/\beta} a_n(x)\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{b_n}{\beta^n}, \end{aligned}$$

where

$$b_n = \begin{cases} 1/\beta & \text{if } (a+1)/\beta \leqslant T_\beta^n 1, \\ T_\beta^n 1 - a/\beta & \text{if } a/\beta \leqslant T_\beta^n 1 < (a+1)/\beta, \\ 0 & \text{if } T_\beta^n 1 < a/\beta. \end{cases}$$

Since only a and b appear in $d_\beta(1)$, the inequality $T_\beta^n 1 < a/\beta$ never occurs and $(a+1)/\beta \leqslant T_\beta^n 1$ is reduced to $\lfloor\beta\rfloor/\beta \leqslant T_\beta^n 1$. So the integration is expressed as

$$\mathscr{I} = \sum_{n\in J} \frac{1}{\beta^{n+1}} + \sum_{n\in K} \frac{1}{\beta^n}\left(T_\beta^n 1 - \frac{a}{\beta}\right) = \sum_{n\in J} \frac{1}{\beta^{n+1}} + \sum_{n\in K}\sum_{m=n+1}^{\infty} \frac{\varepsilon_m}{\beta^{m+1}}.$$

By changing the order of summation indices, we find

$$\mathscr{I} = \sum_{n\in J} \frac{1}{\beta^{n+1}} + \sum_{n=0}^{\infty} (n+1-h_n)\frac{\varepsilon_{n+1}}{\beta^{n+2}} = \sum_{n\in J} \frac{1}{\beta^{n+1}} + \sum_{n=0}^{\infty} (n-h_{n-1})\frac{\varepsilon_n}{\beta^{n+1}}. \qquad \square$$

Theorem 3.4. *If β is self-Sturmian, then $\overline{\{T_\beta^n 1\}}_{n\geqslant 0}$ is of Lebesgue measure zero.*

Proof. We adopt the notations used in Lemma 3.1 above. For any x in $\overline{\{T_\beta^n 1\}}_{n\geqslant 0}$, the infinite word $d_\beta(x)$ is Sturmian. The frequency of b in $d_\beta(x)$, therefore, has the value

α, while the frequency of a has $1-\alpha$. We will prove that at least one of these values is different from those given in Lemma 3.1.

At first we suppose $a=0$. Then the integration is given by $\mathscr{I}=\sum_{n\in J}\lceil\alpha n\rceil b/\beta^{n+1}$. Similarly, one sees $\alpha F(\beta)=\sum_{n\in J}(\alpha n+\alpha)b/\beta^{n+1}$. It holds that $n\in J$ if and only if $\lceil\alpha n\rceil<\alpha n+\alpha$ because $\varepsilon_n=b(\lceil\alpha(n+1)\rceil-\lceil\alpha n\rceil)$. Whence $\mu_\beta(b)=\mathscr{I}/F(\beta)<\alpha$.

Next, we suppose $1\leqslant a<b$ and, in addition, $\lfloor\beta\rfloor\alpha>1$. Noting the index set J contains 0, let k_0 be the smallest element of K. One sees

$$\alpha F(\beta)-\mathscr{I}=\sum_{n\in J}\frac{(\alpha n+\alpha)-\lceil\alpha n\rceil}{\beta^{n+1}}\,b-\sum_{n\in K}\frac{\lceil\alpha n\rceil-(\alpha n+\alpha)}{\beta^{n+1}}\,a.$$

We know that $(\alpha n+\alpha)>\lceil\alpha n\rceil$ if $n\in J$, and $\lceil\alpha n\rceil>(\alpha n+\alpha)$ if $n\in K$. The series can be bounded from below as

$$\begin{aligned}\alpha F(\beta)-\mathscr{I}&>\sum_{n\in J}\frac{(\alpha n+\alpha)-\lceil\alpha n\rceil}{\beta^{n+1}}\,b-\sum_{n\in K}\frac{a}{\beta^{n+1}}=\left(\frac{\alpha}{\beta}\,b+\cdots\right)-\left(\frac{a}{\beta^{k_0+1}}+\cdots\right)\\&>\frac{1}{\beta}-\left(\frac{a}{\beta^{k_0+1}}+\cdots\right)>0,\end{aligned}$$

because $\lfloor\beta\rfloor\alpha>1$ and $k_0\geqslant 1$. Hence we have $\mu_\beta(b)=\mathscr{I}/F(\beta)<\alpha$.

If $\lfloor\beta\rfloor\alpha<1$, then $\alpha<1/\lfloor\beta\rfloor\leqslant\frac{1}{2}$, which implies $k_0=1$. We observe

$$\begin{aligned}(1-\alpha)F(\beta)-\mathscr{I}&=\sum_{n\in J}\frac{1+\lceil\alpha n\rceil-(\alpha n+\alpha)}{\beta^{n+1}}\,b+\sum_{n\in K}\frac{1+\lceil\alpha n\rceil-(\alpha n+\alpha)}{\beta^{n+1}}\,a\\&\quad-\sum_{n\in J}\frac{1}{\beta^{n+1}}\\&=\left(\frac{1-\alpha}{\beta}\,b+\cdots\right)+\sum_{n\in K}\frac{1+\lceil\alpha n\rceil-(\alpha n+\alpha)}{\beta^{n+1}}\,a-\left(\frac{1}{\beta}+\cdots\right).\end{aligned}$$

By the assumption, the inequality $(1-\alpha)b>b-1\geqslant 1$ is true. Hence we have

$$\begin{aligned}(1-\alpha)F(\beta)-\mathscr{I}&>\sum_{n\in K}\frac{1+\lceil\alpha n\rceil-(\alpha n+\alpha)}{\beta^{n+1}}\,a-\sum_{n\in J\backslash\{0\}}\frac{1}{\beta^{n+1}}\\&>\frac{1}{\beta^2}-\sum_{n\in J\backslash\{0\}}\frac{1}{\beta^{n+1}}>0,\end{aligned}$$

since the least integer in $J\backslash\{0\}$ is greater than or equal to 2 and if $n\in K$, then $1+\lceil\alpha n\rceil-(\alpha n+\alpha)>1$. We have proved $\mu_\beta(a)=\mathscr{I}/F(\beta)<1-\alpha$. □

4. Transcendence of self-Sturmian numbers

We know that β is an algebraic integer for every $\beta\in\mathscr{C}_2$. Then are there transcendental numbers in $\mathscr{C}_3$, $\mathscr{C}_4$, and $\mathscr{C}_5$? This was questioned by Blanchard [5]. From Schmeling's results, $\mathscr{C}_5$ is abundant in transcendental numbers. But a transcendental number reported in $\mathscr{C}_3$ is, to the knowledge of the authors, only Komornik–Loreti constant $\delta = 1.787231650\ldots$. This constant is the smallest number in $(1,2)$, for which there is only one expansion of 1 as $1 = \sum_{n=1}^{\infty} \varepsilon_n \delta^{-n}$, $\varepsilon_n \in \{0,1\}$ [9]. Later it turned out to be transcendental [1,2]. This section contains the proof that all self-Sturmian numbers are transcendental. That enriches $\mathscr{C}_3$ with transcendental numbers of continuum cardinality. In fact, Sturmian words hitherto have given births to transcendental numbers in other manners, e.g. [8,3]. We need a classical result on transcendence.

Proposition 4.1. *Let the function f be defined by*

$$f(w,z) = \sum_{n=1}^{\infty} \lfloor nw \rfloor z^n,$$

where w is real and z is complex with $|z| < 1$. *Then* $f(\omega,\alpha)$ *is transcendental if* ω *is irrational and* α *is a nonzero algebraic number with* $|\alpha| < 1$.

Indeed Mahler [12] proved in 1929 the preceding result for quadratic irrational ω's, and Loxton and van der Poorten [11] extended the case to arbitrary irrational ω's. We are now in a position to state the main result of this section.

Theorem 4.1. *Every self-Sturmian number is transcendental.*

Proof. Since β is self-Sturmian, we have for some irrational $\alpha \in (0,1)$,

$$1 - \frac{b-a}{\beta} = \sum_{n=0}^{\infty} \frac{(b-a)s_{\alpha,0}(n) + a}{\beta^{n+1}} = \sum_{n=0}^{\infty} \frac{(b-a)(\lfloor \alpha(n+1) \rfloor - \lfloor \alpha n \rfloor) + a}{\beta^{n+1}}$$
$$= (b-a) \sum_{n=0}^{\infty} \frac{\lfloor \alpha(n+1) \rfloor - \lfloor \alpha n \rfloor}{\beta^{n+1}} + \frac{a}{\beta - 1}.$$

Thus the following equality holds:

$$\sum_{n=0}^{\infty} \frac{\lfloor \alpha(n+1) \rfloor - \lfloor \alpha n \rfloor}{\beta^{n+1}} = \frac{1}{b-a}\left(1 - \frac{b-a}{\beta} - \frac{a}{\beta-1}\right). \tag{1}$$

On the other hand,

$$\sum_{n=0}^{\infty} \frac{\lfloor \alpha(n+1) \rfloor - \lfloor \alpha n \rfloor}{\beta^{n+1}} = \sum_{n=1}^{\infty} \frac{\lfloor \alpha n \rfloor}{\beta^n} - \frac{1}{\beta} \sum_{n=1}^{\infty} \frac{\lfloor \alpha n \rfloor}{\beta^n} = \left(1 - \frac{1}{\beta}\right) \sum_{n=1}^{\infty} \frac{\lfloor \alpha n \rfloor}{\beta^n}.$$

If β were algebraic, the left-hand side of Eq. (1) would be transcendental by Proposition 4.1 whereas the right one algebraic. □

Example. For $0 \leqslant a < b$ let $f_0 = a$, $f_1 = ab$ and $f_{n+2} = f_{n+1} f_n$, $n \geqslant 0$. The Fibonacci word f, which is Sturmian, is defined by

$$f = \lim_{n\to\infty} f_n = abaababaabaababaababaabaababaabaab\cdots.$$

We assume $d_\beta(1) = bf$. Then such $\beta \in (b, b+1)$ exists and is transcendental. Furthermore, $\mathrm{diam}\{T_\beta^n 1\}_{n\geqslant 0} = (b-a)/\beta$ and $\overline{\{T_\beta^n 1\}}_{n\geqslant 0}$ is Lebesgue negligible.

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References

[1] J.-P. Allouche, M. Cosnard, The Komornik–Loreti constant is transcendental, Amer. Math. Monthly 107 (2000) 448–449.
[2] J.-P. Allouche, M. Cosnard, Non-integer bases, iteration of continuous real maps, and an arithmetic self-similar set, Acta Math. Hungar. 91 (2001) 325–332.
[3] J.-P. Allouche, J.L. Davison, M. Queffélec, L.Q. Zamboni, Transcendence of Sturmian or morphic continued fractions, J. Number Theory 91 (2001) 39–66.
[4] A. Bertrand-Mathis, Développement en base θ; répartition modulo un de la suite $(x\theta^n)_{n\geqslant 0}$; langages codés et θ-shift, Bull. Soc. Math. France 114 (1986) 271–323.
[5] F. Blanchard, β-expansions and symbolic dynamics, Theoret. Comput. Sci. 65 (1989) 131–141.
[6] J.-P. Borel, F. Laubie, Quelques mots sur la droite projective réelle, J. Théor. Nombres Bordeaux 5 (1993) 23–51.
[7] E.M. Coven, G.A. Hedlund, Sequences with minimal block growth, Math. Systems Theory 7 (1973) 138–153.
[8] S. Ferenczi, C. Mauduit, Transcendence of numbers with a low complexity expansion, J. Number Theory 67 (1997) 146–161.
[9] V. Komornik, P. Loreti, Unique developments in non-integer bases, Amer. Math. Monthly 105 (1998) 636–639.
[10] M. Lothaire, Algebraic Combinatorics on Words, Cambridge University Press, Cambridge, 2002.
[11] J.H. Loxton, A.J. van der Poorten, Arithmetic properties of certain functions in several variables III, Bull. Austral. Math. Soc. 16 (1977) 15–47.
[12] K. Mahler, Arithmetische Eigenschaften der Lösungen einer Klasse von Funktionalgleichungen, Math. Ann. 101 (1929) 342–366.
[13] F. Mignosi, Infinite words with linear subword complexity, Theoret. Comput. Sci. 65 (1989) 221–242.
[14] M. Morse, G.A. Hedlund, Symbolic dynamics II, Sturmian trajectories, Amer. J. Math. 62 (1940) 1–42.
[15] W. Parry, On the β-expansions of real numbers, Acta Math. Acad. Sci. Hungar. 11 (1960) 401–416.
[16] J. Schmeling, Symbolic dynamics for β-shifts and self-normal numbers, Ergodic Theory Dynam. Systems 17 (1997) 675–694.