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A Trotter–Kato type theorem in the weak topology and an application to a singular perturbed problem

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ABSTRACT

In this paper we prove a result of the Trotter–Kato type in the weak topology. Let $\{A^\varepsilon\}_{\varepsilon>0}$ be a family of quasi m-accretive linear operators on a Hilbert space X and let us denote by J_λ^ε the resolvent of A^ε. Under certain conditions, the result states that if for any $x \in X$ and $k = 1, 2, \ldots,$ the sequence $(J_\lambda^\varepsilon)^k x$ converges weakly to $(J_\lambda)^k x$ as $\varepsilon \to 0$, where J_λ is the resolvent of a linear quasi m-accretive operator A on X, then the sequence of the semigroups generated by $-A^\varepsilon$ tends weakly to the semigroup generated by $-A$, uniformly with respect to t on compact intervals. The result is different from other results of the same type (see e.g., Yosida (1980) [9, p. 269]) and gives an answer to an open problem put in Eisner and Serény (2010) [3]. It is finally applied to compare the asymptotic behavior of a singular perturbation problem associated to a first order hyperbolic problem with diffusion.

1. Introduction

In a Hilbert space X, with the scalar product and norm denoted by $(\cdot,\cdot)$ and $\|\cdot\|$ respectively, let $A^\varepsilon : D(A^\varepsilon) \subset X \to X$ be a family of linear quasi m-accretive operators and let $A : D(A) \subset X \to X$ be a quasi m-accretive operator, such that the sequence of the resolvents of A^ε tends weakly to the resolvent of the operator A. We are concerned with the proof of the weakly convergence of the sequence of semigroups generated by $-A^\varepsilon$ to the semigroup generated by $-A$. We recall that the Trotter–Kato theorem (see [9, p. 272]) amounts to saying that the strong convergence of the sequence of resolvents of A^ε to the resolvent of A ensures the strong semigroup convergence, i.e., $e^{-tA^\varepsilon}x \to e^{-tA}x$ as $\varepsilon \to 0$, for any $x \in X$, uniformly with respect to t on compact intervals. We also mention that a Trotter–Kato type convergence theorem in a weak topology of a Banach space follows by a general result given in [9] (see p. 269) requiring the equi-continuity of the family of operators $\{A^\varepsilon\}_{\varepsilon>0}$ in locally convex Banach spaces. The validity of Trotter–Kato theorem in the weak topology is also discussed in [3] for C_0-semigroups in Banach spaces and it is shown that the direct analogue (i.e., by replacing the strong convergence of the resolvent sequence by the weak convergence) fails.

We prove here a different weak version of this theorem in the context of quasi m-accretive operators in Hilbert spaces, specifying that as far as we know the result presented is new. Its necessity arises in the numerical study of concrete problems and this type of result constitutes a good tool for these applications. Moreover, we think that it may be interesting by itself because it gives an answer to the open problem put in the paper [3].

This result sets a functional framework for the study of the asymptotic behavior of parabolic boundary value problems with a small parameter intervening in the diffusion part (see examples in [7,6]) in the case when the (strong version of) Trotter–Kato theorem cannot be used. In such situations the strong convergence of the resolvents cannot be proved due to the absence of some estimates necessary for the compactness. The general result established in this paper is illustrated for a hyperbolic equation with diffusion and shows that the singular perturbed equation obtained by completing the deficient

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diffusion operator approximates in a certain sense the hyperbolic equation. This is of benefit especially in the perspective of numerical computations since the numerical solutions to the perturbed equation (which is of parabolic type) are more stable than those computed in the limit case.

2. Main results

We recall that a linear operator A on a Hilbert space is called quasi m-accretive if there exists $\omega > 0$ such that $I + \lambda A$ is positive semidefinite and surjective on X, for any λ, $0 < \lambda < \omega$.

For any $\lambda > 0$ we denote by $J_\lambda = (I + \lambda A)^{-1}$ and $A_\lambda = \frac{1}{\lambda}(I - J_\lambda)$ the resolvent and the Yosida approximation of A, respectively. Similarly, we set $J_\lambda^\varepsilon = (I + \lambda A^\varepsilon)^{-1}$ and $A_\lambda^\varepsilon = (A^\varepsilon)_\lambda$ the resolvent and the Yosida approximation of A^ε.

The main result of this paper resides in Theorem 2.1 below. We indicate the weak limit by $\rightharpoonup$ or by w-lim and by $(\cdot,\cdot)$ and $\|\cdot\|$ the scalar product and norm in X, respectively.

Theorem 2.1. *Let $\{A^\varepsilon\}_{\varepsilon>0}$ be a family of linear operators in a Hilbert space X such that $(I + \lambda A^\varepsilon)$ is m-accretive for $0 < \lambda < \lambda_0$ with λ_0 independent of ε and*

$$w\text{-}\lim_{\varepsilon\to 0}\left(I + \lambda A^\varepsilon\right)^{-k} x = (I + \lambda A)^{-k} x, \quad k = 1, 2, 3, \ldots \tag{2.1}$$

for any $x \in X$. Assume further that the space

$$\mathcal{X} = \left\{ x \in \bigcap_{0<\varepsilon\leqslant 1} D\left(A^\varepsilon\right);\ \sup_{0<\varepsilon\leqslant 1}\left\|A^\varepsilon x\right\| < \infty \right\} \tag{2.2}$$

is dense in X. Then,

$$w\text{-}\lim_{\varepsilon\to 0} e^{-tA^\varepsilon} x = e^{-tA} x, \tag{2.3}$$

for all $x \in X$, uniformly on any interval $[0, T]$, $T < \infty$.

We note that the hypothesis $\overline{\mathcal{X}} = X$ is in particular satisfied if there is a linear closed and densely defined operator G such that $D(G) \subset D(A^\varepsilon)$ for any $\varepsilon \in (0, 1]$, that is $\|A^\varepsilon x\| \leqslant C\|Gx\|$ for all $x \in D(G)$, where C is a positive constant. For instance this may happen if $A^\varepsilon = A_0 + \varepsilon A_1$ where A_0 and A_1 are quasi m-accretive operators and $\|A_1 x\| \leqslant C\|A_0 x\|$ for all $x \in D(A_0)$.

In Section 3 we give another example which is relevant for the parabolic regularization approach to a hyperbolic equation with diffusion arising in population dynamics.

Proof of Theorem 2.1. We shall prove the result in two steps and begin by setting $x \in \mathcal{X}$.

For any $t \geqslant 0$ and $x \in X$, in particular for $x \in \mathcal{X}$, we can write

$$\left|\left(e^{-tA^\varepsilon} x - e^{-tA} x, \varphi\right)\right| \leqslant \left|\left(e^{-tA^\varepsilon} x - e^{-tA_\lambda^\varepsilon} x, \varphi\right)\right| + \left|\left(e^{-tA_\lambda^\varepsilon} x - e^{-tA_\lambda} x, \varphi\right)\right| + \left|\left(e^{-tA_\lambda} x - e^{-tA} x, \varphi\right)\right|, \tag{2.4}$$

for each $\varphi \in X$.

We have, by Hille–Yosida's theorem (see [8]) that for any $x \in X$, in particular for $x \in \mathcal{X}$, we have

$$\lim_{\lambda\to 0}\left\|e^{-tA_\lambda} x - e^{-tA} x\right\| = 0, \tag{2.5}$$

uniformly with respect to t on compact intervals. Therefore,

$$\lim_{\lambda\to 0}\left|\left(e^{-tA_\lambda} x - e^{-tA} x, \varphi\right)\right| = 0, \quad \text{uniformly on } [0, T], \text{ for any } x \in \mathcal{X} \text{ and } \varphi \in X.$$

Also, we have for any $\varepsilon > 0$

$$\lim_{\lambda\to 0}\left(e^{-tA^\varepsilon} x - e^{-tA_\lambda^\varepsilon} x, \varphi\right) = 0, \quad \text{uniformly on } [0, T], \text{ for any } x \in \mathcal{X} \text{ and } \varphi \in X. \tag{2.6}$$

Let us show that (2.6) holds uniformly with respect to ε, for each $x \in \mathcal{X}$. Indeed, let

$$y_{\varepsilon,\lambda}(t) = e^{-tA_\lambda^\varepsilon} x, \qquad y_\varepsilon(t) = e^{-tA^\varepsilon} x$$

be the orbits of the semigroups generated by $-A_\lambda^\varepsilon$ and $-A^\varepsilon$, respectively. Then $y_{\varepsilon,\lambda}$ is the strong solution to

$$\begin{aligned} &\frac{dy_{\varepsilon,\lambda}}{dt}(t) + A_\lambda^\varepsilon y_{\varepsilon,\lambda}(t) = 0, \quad t \in [0, T], \\ &y_{\varepsilon,\lambda}(0) = x, \end{aligned} \tag{2.7}$$

with the properties

$$\|y_{\varepsilon,\lambda}(t)\| \leqslant e^{\frac{t}{\lambda_0-\lambda}}\|x\|, \qquad \left\|\frac{dy_{\varepsilon,\lambda}(t)}{dt}\right\| = \|A^{\varepsilon}_{\lambda}y_{\varepsilon,\lambda}(t)\| \leqslant e^{\frac{t}{\lambda_0-\lambda}}\|A^{\varepsilon}_{\lambda}x\| \tag{2.8}$$

for any $t \in [0,T]$, given by the proof of Hille–Yosida's theorem.

Also, we have $y_{\varepsilon,\lambda} \to y_{\varepsilon}$ in $C([0,T];X)$ as $\lambda \to 0$, as specified before.

We note down the following properties of the resolvent and the Yosida approximation,

$$\begin{aligned}
&\|J^{\varepsilon}_{\lambda}x\| \leqslant \frac{\|x\|}{1-\frac{\lambda}{\lambda_0}}, \qquad (A^{\varepsilon}_{\lambda}x,x) \geqslant -\frac{\|x\|^2}{\lambda_0(1-\frac{\lambda}{\lambda_0})}, \quad \forall x \in X,\\
&\|A^{\varepsilon}_{\lambda}x\| = \|J^{\varepsilon}_{\lambda}A^{\varepsilon}x\| \leqslant \frac{1}{1-\frac{\lambda}{\lambda_0}}\|A^{\varepsilon}x\|, \quad \forall x \in D(A^{\varepsilon}).
\end{aligned} \tag{2.9}$$

Subtracting the two equations in (2.7) corresponding to $y_{\varepsilon,\lambda}$ and $y_{\varepsilon,\mu}$, multiplying the difference by $(y_{\varepsilon,\lambda}(t) - y_{\varepsilon,\mu}(t))$ and performing some computation we successively have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|y_{\varepsilon,\lambda}(t) - y_{\varepsilon,\mu}(t)\|^2 &= -(A^{\varepsilon}_{\lambda}y_{\varepsilon,\lambda}(t) - A^{\varepsilon}_{\mu}y_{\varepsilon,\mu}(t), y_{\varepsilon,\lambda}(t) - y_{\varepsilon,\mu}(t))\\
&= -(A^{\varepsilon}_{\lambda}y_{\varepsilon,\lambda}(t) - A^{\varepsilon}_{\mu}y_{\varepsilon,\mu}(t), y_{\varepsilon,\lambda}(t) - J^{\varepsilon}_{\lambda}y_{\varepsilon,\lambda}(t)\\
&\quad + J^{\varepsilon}_{\lambda}y_{\varepsilon,\lambda}(t) - J^{\varepsilon}_{\mu}y_{\varepsilon,\mu}(t) + J^{\varepsilon}_{\mu}y_{\varepsilon,\mu}(t) - y_{\varepsilon,\mu}(t))\\
&= -(A^{\varepsilon}_{\lambda}y_{\varepsilon,\lambda}(t) - A^{\varepsilon}_{\mu}y_{\varepsilon,\mu}(t), \lambda A^{\varepsilon}_{\lambda}y_{\varepsilon,\lambda}(t) - \mu A^{\varepsilon}_{\mu}y_{\varepsilon,\mu}(t))\\
&\quad - (A^{\varepsilon}_{\lambda}y_{\varepsilon,\lambda}(t) - A^{\varepsilon}_{\mu}y_{\varepsilon,\mu}(t), J^{\varepsilon}_{\lambda}y_{\varepsilon,\lambda}(t) - J^{\varepsilon}_{\mu}y_{\varepsilon,\mu}(t))\\
&= -(A^{\varepsilon}_{\lambda}y_{\varepsilon,\lambda}(t) - A^{\varepsilon}_{\mu}y_{\varepsilon,\mu}(t), \lambda A^{\varepsilon}_{\lambda}y_{\varepsilon,\lambda}(t) - \mu A^{\varepsilon}_{\mu}y_{\varepsilon,\mu}(t))\\
&\quad - (A^{\varepsilon}(J^{\varepsilon}_{\lambda}y_{\varepsilon,\lambda}(t)) - A^{\varepsilon}(J^{\varepsilon}_{\mu}y_{\varepsilon,\mu}(t)), J^{\varepsilon}_{\lambda}y_{\varepsilon,\lambda}(t) - J^{\varepsilon}_{\mu}y_{\varepsilon,\mu}(t)).
\end{aligned}$$

Since A^{ε} is quasi m-accretive we obtain by (2.8)

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|y_{\varepsilon,\lambda}(t) - y_{\varepsilon,\mu}(t)\|^2 &\leqslant (\|A^{\varepsilon}_{\lambda}y_{\varepsilon,\lambda}(t)\| + \|A^{\varepsilon}_{\mu}y_{\varepsilon,\mu}(t)\|)(\lambda\|A^{\varepsilon}_{\lambda}y_{\varepsilon,\lambda}(t)\| + \mu\|A^{\varepsilon}_{\mu}y_{\varepsilon,\mu}(t)\|)\\
&\quad + \lambda_0^{-1}\|J^{\varepsilon}_{\lambda}y_{\varepsilon,\lambda}(t) - J^{\varepsilon}_{\mu}y_{\varepsilon,\mu}(t)\|^2\\
&\leqslant (e^{\frac{T}{\lambda_0-\lambda}}\|A^{\varepsilon}_{\lambda}x\| + e^{\frac{T}{\lambda_0-\mu}}\|A^{\varepsilon}_{\mu}x\|)(\lambda e^{\frac{T}{\lambda_0-\lambda}}\|A^{\varepsilon}_{\lambda}x\| + \mu e^{\frac{T}{\lambda_0-\mu}}\|A^{\varepsilon}_{\mu}x\|)\\
&\quad + \lambda_0^{-1}\|J^{\varepsilon}_{\lambda}y_{\varepsilon,\lambda}(t) - J^{\varepsilon}_{\mu}y_{\varepsilon,\mu}(t)\|^2
\end{aligned} \tag{2.10}$$

for all $t \in [0,T]$.

Now we estimate the last term on the right-hand side in (2.10).

Let $z_{\varepsilon,\lambda}, z_{\varepsilon,\mu} \in X$ and denote $J^{\varepsilon}_{\lambda}z_{\varepsilon,\lambda} = Y_{\varepsilon,\lambda}$, $J^{\varepsilon}_{\mu}z_{\varepsilon,\mu} = Y_{\varepsilon,\mu}$.

We subtract the equations

$$\begin{aligned}
&Y_{\varepsilon,\lambda} + \lambda A^{\varepsilon}Y_{\varepsilon,\lambda} = z_{\varepsilon,\lambda},\\
&Y_{\varepsilon,\mu} + \lambda A^{\varepsilon}Y_{\varepsilon,\mu} = z_{\varepsilon,\mu}
\end{aligned}$$

and multiply the difference by $(Y_{\varepsilon,\lambda} - Y_{\varepsilon,\mu})$. We get

$$\|Y_{\varepsilon,\lambda} - Y_{\varepsilon,\mu}\|^2 - \frac{\lambda}{\lambda_0}\|Y_{\varepsilon,\lambda} - Y_{\varepsilon,\mu}\|^2 \leqslant |\lambda-\mu|\|A^{\varepsilon}Y_{\varepsilon,\mu}\|\|Y_{\varepsilon,\lambda} - Y_{\varepsilon,\mu}\| + \|z_{\varepsilon,\lambda} - z_{\varepsilon,\mu}\|\|Y_{\varepsilon,\lambda} - Y_{\varepsilon,\mu}\|.$$

But $\|A^{\varepsilon}Y_{\varepsilon,\mu}\| = \|A^{\varepsilon}J^{\varepsilon}_{\mu}z_{\varepsilon,\mu}\| = \|A^{\varepsilon}_{\mu}z_{\varepsilon,\mu}\|$ and so

$$\|J^{\varepsilon}_{\lambda}z_{\varepsilon,\lambda} - J^{\varepsilon}_{\mu}z_{\varepsilon,\mu}\| \leqslant \frac{1}{1-\frac{\lambda}{\lambda_0}}|\lambda-\mu|\|A^{\varepsilon}_{\mu}z_{\varepsilon,\mu}\| + \frac{1}{1-\frac{\lambda}{\lambda_0}}\|z_{\varepsilon,\lambda} - z_{\varepsilon,\mu}\|.$$

We go back to (2.10) and write

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|y_{\varepsilon,\lambda}(t) - y_{\varepsilon,\mu}(t)\|^2 &\leqslant (e^{\frac{T}{\lambda_0-\lambda}}\|A^{\varepsilon}_{\lambda}x\| + e^{\frac{T}{\lambda_0-\mu}}\|A^{\varepsilon}_{\mu}x\|)(\lambda e^{\frac{T}{\lambda_0-\lambda}}\|A^{\varepsilon}_{\lambda}x\| + \mu e^{\frac{T}{\lambda_0-\mu}}\|A^{\varepsilon}_{\mu}x\|)\\
&\quad + \frac{2}{\lambda_0}\left\{\frac{1}{(1-\frac{\lambda}{\lambda_0})^2}|\lambda-\mu|^2\|A^{\varepsilon}_{\mu}y_{\varepsilon,\mu}(t)\|^2 + \frac{1}{(1-\frac{\lambda}{\lambda_0})^2}\|y_{\varepsilon,\lambda}(t) - y_{\varepsilon,\mu}(t)\|^2\right\}.
\end{aligned}$$

Using (2.8) and (2.9) and integrating with respect to t we obtain

$$\begin{aligned}\left\|y_{\varepsilon,\lambda}(t)-y_{\varepsilon,\mu}(t)\right\|^2 \leqslant{}& 2\left\|A^\varepsilon x\right\|^2\left(\frac{e^{\frac{T}{\lambda_0-\lambda}}}{1-\frac{\lambda}{\lambda_0}}+\frac{e^{\frac{T}{\lambda_0-\mu}}}{1-\frac{\mu}{\lambda_0}}\right)\left(\frac{\lambda e^{\frac{T}{\lambda_0-\lambda}}}{1-\frac{\lambda}{\lambda_0}}+\frac{\mu e^{\frac{T}{\lambda_0-\mu}}}{1-\frac{\mu}{\lambda_0}}\right)t \\ &+\frac{4}{\lambda_0}\frac{1}{(1-\frac{\lambda}{\lambda_0})^2}|\lambda-\mu|^2\frac{e^{\frac{2T}{\lambda_0-\mu}}}{(1-\frac{\mu}{\lambda_0})^2}\left\|A^\varepsilon x\right\|^2 t \\ &+\frac{4}{\lambda_0}\frac{1}{(1-\frac{\lambda}{\lambda_0})^2}\int_0^t\left\|y_{\varepsilon,\lambda}(\tau)-y_{\varepsilon,\mu}(\tau)\right\|^2 d\tau.\end{aligned}$$

By Gronwall's lemma we deduce that

$$\begin{aligned}\left\|y_{\varepsilon,\lambda}(t)-y_{\varepsilon,\mu}(t)\right\|^2 \leqslant{}& 2\left\|A^\varepsilon x\right\|^2\left\{\left(\frac{e^{\frac{T}{\lambda_0-\lambda}}}{1-\frac{\lambda}{\lambda_0}}+\frac{e^{\frac{T}{\lambda_0-\mu}}}{1-\frac{\mu}{\lambda_0}}\right)\left(\frac{\lambda e^{\frac{T}{\lambda_0-\lambda}}}{1-\frac{\lambda}{\lambda_0}}+\frac{\mu e^{\frac{T}{\lambda_0-\mu}}}{1-\frac{\mu}{\lambda_0}}\right)\right. \\ &\left.+\frac{2}{\lambda_0}\frac{1}{(1-\frac{\lambda}{\lambda_0})^2}|\lambda-\mu|^2\frac{e^{\frac{2T}{\lambda_0-\mu}}}{(1-\frac{\mu}{\lambda_0})^2}\right\}T\exp\left(\frac{4}{\lambda_0}\frac{1}{(1-\frac{\lambda}{\lambda_0})^2}t\right).\end{aligned}$$

We pass to the limit as $\mu\to 0$ and recall that $x\in\mathcal{X}$, i.e., $\sup_{\varepsilon\in(0,1]}\|A^\varepsilon x\|<\infty$. We obtain

$$\begin{aligned}\left\|y_{\varepsilon,\lambda}(t)-y_{\varepsilon}(t)\right\|^2 \leqslant{}& 2\Big(\sup_{\varepsilon\in(0,1]}\left\|A^\varepsilon x\right\|\Big)^2\lambda\left\{\left(\frac{e^{\frac{T}{\lambda_0-\lambda}}}{1-\frac{\lambda}{\lambda_0}}+e^{\frac{T}{\lambda_0}}\right)\frac{e^{\frac{T}{\lambda_0-\lambda}}}{1-\frac{\lambda}{\lambda_0}}+\frac{2}{\lambda_0}\frac{\lambda e^{\frac{2T}{\lambda_0-\mu}}}{(1-\frac{\lambda}{\lambda_0})^2}\right\}T \\ &\times\exp\left(\frac{4}{\lambda_0}\frac{1}{(1-\frac{\lambda}{\lambda_0})^2}t\right).\end{aligned}$$

Passing to the limit as $\lambda\to 0$ we finally deduce that

$$\lim_{\lambda\to 0}\left\|y_{\varepsilon,\lambda}(t)-y_\varepsilon(t)\right\|=0$$

uniformly with respect to ε. This implies that (2.6) takes place uniformly with respect to ε, as claimed.

Taking into account the convergences (2.5) and (2.6) we can fix $\lambda=\lambda_*$, sufficiently small such that

$$\left|\left(e^{-tA_{\lambda_*}}x-e^{-tA}x,\varphi\right)\right|\leqslant\frac{\delta}{3} \tag{2.11}$$

and

$$\left|\left(e^{-tA^\varepsilon_{\lambda_*}}x-e^{-tA^\varepsilon}x,\varphi\right)\right|\leqslant\frac{\delta}{3} \tag{2.12}$$

with δ arbitrary but fixed and independent of $\varepsilon\in(0,1]$.

Resuming (2.4) with $\lambda=\lambda_*$ it remains to estimate $|(e^{-tA^\varepsilon_{\lambda_*}}x-e^{-tA_{\lambda_*}}x,\varphi)|$.

Using the representation formula of the semigroup we can write

$$e^{-tA^\varepsilon_{\lambda_*}}x=\sum_{k=0}^\infty\frac{(-1)^k}{k!}\left(A^\varepsilon_{\lambda_*}\right)^k x \tag{2.13}$$

and we prove that

$$\left(A^\varepsilon_{\lambda_*}\right)^k x\rightharpoonup A^k_{\lambda_*}x\quad\text{as }\varepsilon\to 0. \tag{2.14}$$

Indeed, taking into account (2.1) we have that

$$\left(\left(A^\varepsilon_{\lambda_*}\right)^k x,\varphi\right)=\left(\frac{1}{\lambda_*^k}\left(I-J^\varepsilon_{\lambda_*}\right)^k x,\varphi\right)\to\left(\frac{1}{\lambda_*^k}(I-J_{\lambda_*})^k x,\varphi\right)=\left(A^k_{\lambda_*}x,\varphi\right),$$

as $\varepsilon\to 0$, for any $\varphi\in X$.

By (2.13) we get for each $x\in\mathcal{X}$ that

$$\lim_{\varepsilon\to 0}\left(e^{-tA^\varepsilon_{\lambda_*}}x-e^{-tA_{\lambda_*}}x,\varphi\right)=0 \tag{2.15}$$

and therefore we can write

$$\left|\left(e^{-tA^{\varepsilon}_{\lambda*}}x - e^{-tA_{\lambda*}}x, \varphi\right)\right| \leqslant \frac{\delta}{3},$$

for ε sufficiently small. Resuming (2.4) with $\lambda = \lambda_*$ and taking into account (2.11)–(2.12) and (2.15) we get that

$$\left|\left(e^{-tA^{\varepsilon}}x - e^{-tA}x, \varphi\right)\right| \leqslant \delta,$$

for ε sufficiently small. Since δ is arbitrary we obtain (2.3) for $x \in \mathcal{X}$.

Now, let us assume that $x \in X$. Since $\mathcal{X}$ is dense in X we can take a sequence $x_n \in \mathcal{X}$ such that

$$x_n \to x \quad \text{in } X, \text{ as } n \to \infty.$$

Let us denote

$$y_n^{\varepsilon}(t) = e^{-tA^{\varepsilon}}x_n, \qquad y^{\varepsilon}(t) = e^{-tA^{\varepsilon}}x, \qquad y(t) = e^{-tA}x.$$

We need to prove that

$$y^{\varepsilon}(t) \rightharpoonup y(t) \quad \text{in } X, \text{ uniformly with respect to } t \in [0, T]. \tag{2.16}$$

We compute

$$\begin{aligned}\left(y^{\varepsilon}(t) - y(t), \varphi\right) &= \left(e^{-tA^{\varepsilon}}x - e^{-tA}x, \varphi\right) \\ &= \left(e^{-tA^{\varepsilon}}x - e^{-tA^{\varepsilon}}x_n, \varphi\right) + \left(e^{-tA^{\varepsilon}}x_n - e^{-tA}x_n, \varphi\right) + \left(e^{-tA}x_n - e^{-tA}x, \varphi\right).\end{aligned}$$

We have

$$\left|\left(e^{-tA}x_n - e^{-tA}x, \varphi\right)\right| \leqslant \left\|e^{-tA}\right\| \|x_n - x\| \|\varphi\| \leqslant \frac{\delta}{3}, \quad \text{for } n = n_0$$

sufficiently large. (Here $\|e^{-tA}\|$ is the operatorial norm of e^{-tA} in X.) Then, for $n = n_0$

$$\left|\left(e^{-tA^{\varepsilon}}x_{n_0} - e^{-tA}x_{n_0}, \varphi\right)\right| = \left|\left(\left(e^{-tA^{\varepsilon}} - e^{-tA}\right)x_{n_0}, \varphi\right)\right|$$

converges to zero due to the results at the 1st step, because $x_{n_0} \in \mathcal{X}$ and so we can write

$$\left|\left(e^{-tA^{\varepsilon}}x_{n_0} - e^{-tA}x_{n_0}, \varphi\right)\right| \leqslant \frac{\delta}{3},$$

for ε sufficiently small. Finally,

$$\left|\left(e^{-tA^{\varepsilon}}x - e^{-tA^{\varepsilon}}x_n, \varphi\right)\right| \leqslant \left\|e^{-tA^{\varepsilon}}\right\| \|x_n - x\| \|\varphi\| \leqslant \frac{\delta}{3} \quad \text{for } n = n_0, \ t \in [0, T].$$

Summing up these three results we get (2.16) as claimed and end the proof. □

Theorem 2.1 applies as well to the nonhomogeneous Cauchy problem

$$\begin{aligned}&\frac{dy}{dt}(t) + Ay(t) = f(t) \quad \text{a.e. } t \in (0, T), \\ &y(0) = x \in X,\end{aligned} \tag{2.17}$$

with A a quasi m-accretive operator.

Corollary 2.2. *Let $f \in L^1(0, T; X)$ and let $\{A^{\varepsilon}\}_{\varepsilon > 0}$ be a family of linear quasi m-accretive operators in a Hilbert space X satisfying the hypotheses of Theorem* 2.1. *Then,*

$$y_{\varepsilon}(t) \rightharpoonup y(t) \quad \text{in } X, \text{ as } \varepsilon \to 0, \text{ uniformly on } [0, T], \tag{2.18}$$

where y_{ε} is the mild solution to

$$\begin{aligned}&\frac{dy_{\varepsilon}}{dt}(t) + A^{\varepsilon}y_{\varepsilon}(t) = f(t), \quad t \in (0, T), \\ &y_{\varepsilon}(0) = x.\end{aligned} \tag{2.19}$$

Proof. If (2.1) is fulfilled, the study of the convergence of the solution to (2.19) to the solution to (2.17) reduces to the convergence of

$$y_\varepsilon(t) = e^{-tA^\varepsilon}x + \int_0^t e^{-(t-s)A^\varepsilon} f(s)\,ds.$$

We have

$$\left|\left(y_\varepsilon(t) - y(t), \varphi\right)\right| \leqslant \left|\left(e^{-tA^\varepsilon}x - e^{-tA}x, \varphi\right)\right| + \int_0^t \left|\left(e^{-(t-s)A^\varepsilon} f(s) - e^{-(t-s)A} f(s), \varphi\right)\right| ds.$$

Since $e^{-sA^\varepsilon} f(s) - e^{-sA} f(s) \rightharpoonup 0$ for any s, by Theorem 2.1 and

$$\left|\left(e^{-(t-s)A^\varepsilon} f(s) - e^{-(t-s)A} f(s), \varphi\right)\right| \leqslant 2e^{\lambda_0^{-1}t}\left\| f(s)\right\| \|\varphi\| \quad \text{a.e. } s \in (0,T),$$

we have by the Lebesgue dominated convergence theorem that the integral converges to zero and so

$$y_\varepsilon(t) \rightharpoonup y(t) \quad \text{as } \varepsilon \to 0 \text{ for any } t \in [0,T].$$

This ends the proof. □

Remark 2.3. We remark that by the inspection of the proof, it is sufficient to formulate condition (2.1) for a sequence $(\lambda_j) \subset (0,\infty)$ with $\lambda_j \to 0$ instead of the interval $(0,\lambda_0)$.

3. Example: a singular perturbed equation

Let $\Omega = (0,L) \times \Omega_1$, where Ω_1 is an open bounded subset of $\mathbb{R}^{N-1}$ with the boundary $\partial\Omega_1$ smooth enough. Let us denote $x = (x_1, x_2, \ldots, x_N)$ and $x' = (x_2, \ldots, x_N)$. We consider the problem

$$\frac{\partial y}{\partial t} + \frac{\partial y}{\partial x_1} - \sum_{i=2}^N \frac{\partial}{\partial x_i}\left(a_i(x)\frac{\partial y}{\partial x_i}\right) = f \quad \text{in } Q = (0,T) \times \Omega, \tag{3.1}$$

$$y(0,x) = y_0(x) \quad \text{in } \Omega, \tag{3.2}$$

$$y\left(t,\left(0,x'\right)\right) = \left(By(t)\right)\left(x'\right) \quad \text{on } \Sigma_0 = (0,T) \times \Omega_1, \tag{3.3}$$

$$y(t,x) = 0 \quad \text{on } \Sigma_1 = (0,T) \times (0,L) \times \partial\Omega_1. \tag{3.4}$$

Here $B \in \mathcal{L}(L^2(\Omega), L^2(\Omega_1))$, and let $B_M = \|B\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega_1))}$. We assume that $f \in L^1(0,T;L^2(\Omega))$ and

$$a_i \in L^\infty(\Omega), \qquad a_i \geqslant a_0 > 0 \quad \text{a.e. in } \Omega,\ i = 2,\ldots,N.$$

Such a problem arises for instance in population dynamics with age-structure and diffusion in space, where x_1 stands for the age and Ω_1 is the space habitat. In this case (3.3) reads

$$y\left(t,\left(0,x'\right)\right) = \left(By(t)\right)\left(x'\right) = \int_0^L \beta\left(x_1,x'\right) y\left(t,\left(x_1,x'\right)\right) dx_1, \tag{3.5}$$

with β a known function which describes the demographic process (see [1,2]). Further we shall use this form of B, assuming that

$$\beta \in C^2(\overline{\Omega}), \qquad \beta\left(x_1,x'\right) \in [0,\beta_M], \quad \forall\left(x_1,x'\right) \in \Omega.$$

We intend to show the existence of the solution to (3.1)–(3.4) in a constructive manner by applying the result previously given.

To this end we consider the perturbed problem for $\varepsilon > 0$,

$$\frac{\partial y_\varepsilon}{\partial t} + \frac{\partial y_\varepsilon}{\partial x_1} - \varepsilon\frac{\partial^2 y_\varepsilon}{\partial x_1^2} - \sum_{i=2}^N \frac{\partial}{\partial x_i}\left(a_i(x)\frac{\partial y_\varepsilon}{\partial x_i}\right) = f \quad \text{in } Q = (0,T) \times \Omega, \tag{3.6}$$

$$y_\varepsilon(0,x) = y_0(x) \quad \text{in } \Omega, \tag{3.7}$$

$$\left(y_\varepsilon - \varepsilon\frac{\partial y_\varepsilon}{\partial x_1}\right)\left(t,\left(0,x'\right)\right) = \left(By_\varepsilon(t)\right)\left(x'\right) \quad \text{on } \Sigma_0 = (0,T) \times \Omega_1, \tag{3.8}$$

$$y_\varepsilon(t,x)=0 \quad \text{on } \Sigma_1=(0,T)\times(0,L)\times\partial\Omega_1, \tag{3.9}$$

$$\frac{\partial y_\varepsilon}{\partial x_1}\big(t,\big(L,x'\big)\big)=0 \quad \text{on } \Sigma_0=(0,T)\times\Omega_1, \tag{3.10}$$

obtained by completing the operator acting in (3.1) up to a parabolic operator and assigning appropriate boundary conditions. We shall prove that this perturbed problem approximates in a weak sense (3.1)–(3.4). Moreover, if the aim is the numerical computation of the solution to (3.1)–(3.4), this can be fulfilled by computing the more stable solution to (3.6)–(3.10).

We denote $X=L^2(\Omega)$, by V the Sobolev space

$$H_0^1(\Omega_1)=\left\{u\in L^2(\Omega_1);\ \frac{\partial u}{\partial x_i}\in L^2(\Omega_1),\ \forall i=2,\dots,N,\ u|_{\partial\Omega_1}=0\right\}$$

and by $V'=H^{-1}(\Omega_1)$ the dual of V. The notation $u|_{\partial\Omega_1}$ means the trace of u on $\partial\Omega_1$. We also denote $W=L^2(0,L;V)$, $W'=L^2(0,L;V')$ and define the duality between W' and W by

$$\langle v,\psi\rangle_{W',W}=\int_0^L\big\langle v(x_1),\psi(x_1)\big\rangle_{V',V}\,dx_1, \quad \text{for any } v\in W',\ \psi\in W.$$

We write (3.1)–(3.4) in the abstract form

$$\begin{aligned}&\frac{dy}{dt}(t)+Ay(t)=f(t) \quad \text{a.e. } t\in(0,T),\\ &y(0)=y_0\end{aligned} \tag{3.11}$$

where the operator A is defined below in two steps. First we introduce

$$A_0:D(A_0)\subset W\to W', \quad D(A_0)=\left\{v\in W;\ \frac{\partial v}{\partial x_1}\in W',\ v\big(0,x'\big)=(Bv)\big(x'\big)\right\},$$

$$\langle A_0v,\psi\rangle_{W',W}=\left\langle\frac{\partial v}{\partial x_1},\psi\right\rangle_{W',W}+\sum_{i=2}^N\int_\Omega a_i\frac{\partial v}{\partial x_i}\frac{\partial\psi}{\partial x_i}\,dx, \quad \text{for any } v\in W,\ \psi\in W.$$

Then we define

$$A:D(A)\subset L^2(\Omega)\to L^2(\Omega), \quad D(A)=\big\{v\in D(A_0);\ A_0v\in L^2(\Omega)\big\},$$

$$Av=A_0v=\frac{\partial v}{\partial x_1}-\sum_{i=2}^N\frac{\partial}{\partial x_i}\left(a_i(x)\frac{\partial v}{\partial x_i}\right) \quad \text{for any } v\in D(A).$$

The singular perturbed problem is written as an abstract Cauchy problem

$$\begin{aligned}&\frac{dy_\varepsilon}{dt}(t)+A^\varepsilon y_\varepsilon(t)=f(t) \quad \text{a.e. } t\in(0,T),\\ &y_\varepsilon(0)=y_0\end{aligned} \tag{3.12}$$

by introducing

$$A^\varepsilon:D\big(A^\varepsilon\big)\subset L^2(\Omega)\to L^2(\Omega),$$

$$D\big(A^\varepsilon\big)=\left\{\begin{array}{l}v\in H^2(\Omega);\ v(x_1,x')=0 \text{ on } (0,L)\times\partial\Omega_1,\\ (v-\varepsilon\frac{\partial v}{\partial x_1})(0,x')=(Bv)(x') \text{ in } \Omega_1,\ \frac{\partial v}{\partial x_1}(L,x')=0 \text{ in } \Omega_1\end{array}\right\},$$

$$A^\varepsilon v=\frac{\partial v}{\partial x_1}-\varepsilon\frac{\partial^2 v}{\partial x_1^2}-\sum_{i=2}^N\frac{\partial}{\partial x_i}\left(a_i(x)\frac{\partial v}{\partial x_i}\right) \quad \text{for any } v\in D\big(A^\varepsilon\big).$$

In the sequel we shall show that Corollary 2.2 applies.

First we show that A^ε is quasi m-accretive on $L^2(\Omega)$. This means that $I+\lambda A^\varepsilon$ is positive semidefinite for $0<\lambda<\lambda_0$ and that the equation $(I+\lambda A^\varepsilon)v=g$ has a solution $v\in D(A)$ for any $g\in L^2(\Omega)$.

For simplicity we denote the norm and scalar product in $L^2(\Omega)$ without subscript. Let us compute

$$\begin{aligned}
\left(A^\varepsilon v, v\right) &= \int_0^L \int_{\Omega_1} \frac{\partial}{\partial x_1}\left(v - \varepsilon \frac{\partial v}{\partial x_1}\right) v \, dx_1 \, dx' + \sum_{i=2}^N \int_\Omega a_i \left(\frac{\partial v}{\partial x_i}\right)^2 dx \\
&= \int_{\Omega_1} \left(v - \varepsilon \frac{\partial v}{\partial x_1}\right) v|_0^L \, dx' - \int_\Omega \left(v - \varepsilon \frac{\partial v}{\partial x_1}\right) \frac{\partial v}{\partial x_1} dx + \sum_{i=2}^N \int_\Omega a_i \left(\frac{\partial v}{\partial x_i}\right)^2 dx \\
&= \left\| v(L) \right\|^2_{L^2(\Omega_1)} - \int_{\Omega_1} (Bv) v(0) \, dx' - \frac{1}{2} \left\| v(L) \right\|^2_{L^2(\Omega_1)} + \frac{1}{2} \left\| v(0) \right\|^2_{L^2(\Omega_1)} \\
&\quad + \sum_{i=2}^N \int_\Omega a_i \left(\frac{\partial v}{\partial x_i}\right)^2 dx + \varepsilon \left\| \frac{\partial v}{\partial x_1} \right\|^2 \\
&\geqslant \frac{1}{2} \left\| v(L) \right\|^2_{L^2(\Omega_1)} - \frac{1}{2} B_M^2 \|v\|^2 + \varepsilon \left\| \frac{\partial v}{\partial x_1} \right\|^2 + a_0 \sum_{i=2}^N \left\| \frac{\partial v}{\partial x_i} \right\|^2
\end{aligned}$$

with $B_M = \beta_M \sqrt{L}$, which shows that $(I + \lambda A^\varepsilon v, v) \geqslant 0$ if $0 < \lambda \leqslant \lambda_0 = \frac{2}{B_M^2}$, λ_0 being independent of ε.

For the quasi m-accretivity we consider the elliptic problem

$$\frac{\partial v_\varepsilon}{\partial x_1} - \varepsilon \frac{\partial^2 v_\varepsilon}{\partial x_1^2} - \sum_{i=2}^N \frac{\partial}{\partial x_i}\left(a_i(x) \frac{\partial v_\varepsilon}{\partial x_i}\right) + \frac{v_\varepsilon}{\lambda} = \frac{g}{\lambda} \quad \text{in } \Omega, \tag{3.13}$$

$$\left(v_\varepsilon - \varepsilon \frac{\partial v_\varepsilon}{\partial x_1}\right)\left(0, x'\right) = B v_\varepsilon(x), \quad x' \in \Omega_1, \tag{3.14}$$

$$v_\varepsilon(x) = 0 \quad \text{on } (0, L) \times \partial\Omega_1, \tag{3.15}$$

$$\frac{\partial v_\varepsilon}{\partial x_1}\left(L, x'\right) = 0, \quad x' \in \Omega_1, \tag{3.16}$$

with $g \in L^2(\Omega)$. Applying a fixed point theorem we shall show that this problem has a unique solution $v_\varepsilon \in H^2(\Omega)$. To this end we fix $\omega \in H^1(\Omega)$ and replace the boundary condition at $x_1 = 0$ by

$$\left(v_\varepsilon - \varepsilon \frac{\partial v_\varepsilon}{\partial x_1}\right)\left(0, x'\right) = (B\omega)\left(x'\right), \quad x' \in \Omega_1. \tag{3.17}$$

Problem (3.13), (3.15)–(3.17) has a unique solution ensured by the general theory of elliptic equations (see [4]). Thus, if $\omega \in L^2(\Omega)$ it follows that $v_\varepsilon \in H^1(\Omega)$ and if $\omega \in H^1(\Omega)$ we get $v_\varepsilon \in H^2(\Omega)$.

Then we can define a mapping $\Phi : L^2(\Omega) \to L^2(\Omega)$, $\Phi\omega = v_\varepsilon$ (the solution to (3.13), (3.15)–(3.17)) which is a contraction for $\lambda \in (0, \lambda_0)$. This follows by a few computations leading to the estimate

$$\|v_\varepsilon - \overline{v_\varepsilon}\|^2 \leqslant \frac{\lambda}{2} B_M^2 \|\omega - \overline{\omega}\|^2.$$

Since $\frac{\lambda}{2} B_M^2 < 1$ for $\lambda < \lambda_0$ we deduce that Φ has a fixed point which turns out to be the unique solution to (3.13)–(3.16).

Next we have to show that

$$\left(I + \lambda A^\varepsilon\right)^{-1} g \rightharpoonup (I + \lambda A)^{-1} g \quad \text{as } \varepsilon \to 0, \text{ for any } g \in L^2(\Omega). \tag{3.18}$$

By multiplying (3.13) by v_ε and integrating over Ω we get the estimate

$$\left(1 - \frac{\lambda}{\lambda_0}\right) \|v_\varepsilon\|^2 + \lambda \left(\frac{1}{2} \left\| v_\varepsilon(L) \right\|^2_{L^2(\Omega_1)} + \varepsilon \left\| \frac{\partial v_\varepsilon}{\partial x_1} \right\|^2 + a_0 \sum_{i=2}^N \left\| \frac{\partial v_\varepsilon}{\partial x_i} \right\|^2\right) \leqslant \|g\| \|v_\varepsilon\|, \tag{3.19}$$

whence

$$\|v_\varepsilon\| \leqslant \frac{1}{1 - \frac{\lambda}{\lambda_0}} \|g\|. \tag{3.20}$$

We deduce that

$$v_\varepsilon \rightharpoonup v \quad \text{in } L^2(\Omega) \text{ as } \varepsilon \to 0,$$
$$\frac{\partial v_\varepsilon}{\partial x_i} \rightharpoonup \frac{\partial v}{\partial x_i} \quad \text{in } L^2(\Omega) \text{ as } \varepsilon \to 0,\ i = 2, \dots, N.$$

Moreover, still by (3.19) there exists $\xi \in L^2(\Omega)$ such that

$$\sqrt{\varepsilon}\frac{\partial v_\varepsilon}{\partial x_1} \rightharpoonup \xi \quad \text{in } L^2(\Omega) \text{ as } \varepsilon \to 0.$$

Combining these results we can write that

$$v_\varepsilon \rightharpoonup v \quad \text{in } L^2(0, L; V) \text{ as } \varepsilon \to 0,$$
$$\varepsilon\frac{\partial v_\varepsilon}{\partial x_1} \rightharpoonup 0 \quad \text{in } L^2(\Omega) \text{ as } \varepsilon \to 0,$$
$$v_\varepsilon - \varepsilon\frac{\partial v_\varepsilon}{\partial x_1} \rightharpoonup v \quad \text{in } L^2(\Omega) \text{ as } \varepsilon \to 0.$$

On the other hand by (3.13) we compute

$$\left\| \frac{\partial}{\partial x_1}\left(v_\varepsilon - \varepsilon\frac{\partial v_\varepsilon}{\partial x_1}\right)\right\|_{L^2(0,L;V')} = \sup_{\substack{\psi \in L^2(0,L;V) \\ \|\psi\|_{L^2(0,L;V)} \leqslant 1}} \int_\Omega \left(\sum_{i=2}^N a_i(x)\frac{\partial v_\varepsilon}{\partial x_i}\frac{\partial \psi}{\partial x_i} - \frac{v_\varepsilon}{\lambda}\psi + \frac{g}{\lambda}\psi\right) dx$$

which is bounded independently of ε. Therefore

$$\frac{\partial}{\partial x_1}\left(v_\varepsilon - \varepsilon\frac{\partial v_\varepsilon}{\partial x_1}\right) \rightharpoonup \frac{\partial v}{\partial x_1} \quad \text{in } L^2\big(0, L; V'\big) \text{ as } \varepsilon \to 0.$$

Writing now that

$$\left(v_\varepsilon - \varepsilon\frac{\partial v_\varepsilon}{\partial x_1}\right)\big(x_1, x'\big) - \left(v_\varepsilon - \varepsilon\frac{\partial v_\varepsilon}{\partial x_1}\right)\big(0, x'\big) = \int_0^{x_1} \frac{\partial}{\partial x_1}\left(v_\varepsilon - \varepsilon\frac{\partial v_\varepsilon}{\partial x_1}\right)\big(x_1, x'\big)\, dx_1$$

we deduce that $\{(v_\varepsilon - \varepsilon\frac{\partial v_\varepsilon}{\partial x_1})(0)\}_{\varepsilon>0}$ is bounded in V' and so there exists $l \in V'$ such that

$$\left(v_\varepsilon - \varepsilon\frac{\partial v_\varepsilon}{\partial x_1}\right)(0) \rightharpoonup l \quad \text{in } V' \text{ as } \varepsilon \to 0.$$

By passing to the weak limit in the previous equality we get

$$v\big(x_1, x'\big) - l = \int_0^{x_1} \frac{\partial v}{\partial x_1}\big(x_1, x'\big)\, dx_1,$$

whence we see that $l = v(0, x')$.

On the other hand by passing to the limit in (3.14) we get that

$$v\big(0, x'\big) = \int_0^L \beta\big(x_1, x'\big)v\big(x_1, x'\big)\, dx_1.$$

In conclusion, putting together all properties of the limit v we get that $v \in D(A)$.

Passing now to the limit in the weak form of (3.13)–(3.16), i.e.,

$$\lambda\int_0^L\int_{\Omega_1} \frac{\partial}{\partial x_1}\left(v_\varepsilon - \varepsilon\frac{\partial v_\varepsilon}{\partial x_1}\right)\phi\, dx_1\, dx' + \int_\Omega\left(\lambda\sum_{i=2}^N a_i\frac{\partial v_\varepsilon}{\partial x_i}\frac{\partial \phi}{\partial x_i} + v_\varepsilon\phi\right) dx = \int_\Omega g\phi\, dx \quad \text{for any } \phi \in L^2(0, L; V),$$

we obtain on the basis of the previous convergences that

$$\lambda\int_0^L\left\langle\frac{\partial v}{\partial x_1}(x_1), \phi(x_1)\right\rangle_{V',V} dx_1 + \int_\Omega\left(\lambda\sum_{i=2}^N a_i\frac{\partial v}{\partial x_i}\frac{\partial \phi}{\partial x_i} + v\phi\right) dx = \int_\Omega g\phi\, dx$$

which is the weak form of the problem $(I+\lambda A)^{-1}g=v$, equivalently to

$$\lambda\frac{\partial v}{\partial x_1}-\lambda\sum_{i=2}^{N}\frac{\partial}{\partial x_i}\left(a_i(x)\frac{\partial v}{\partial x_i}\right)+v=g \quad \text{in } \Omega, \tag{3.21}$$

$$v\big(0,x'\big)=(Bv)\big(x'\big),\quad x'\in\Omega_1, \tag{3.22}$$

$$v(x)=0 \quad \text{on } (0,L)\times\partial\Omega_1. \tag{3.23}$$

We have proved in fact (3.18).

Analyzing the previous proof we remark that due to the lack of compactness in $L^2(\Omega)$ the family of operators $\{A^\varepsilon\}_{\varepsilon>0}$ is not strongly convergent in resolvents and so the classical Trotter–Kato theorem is not applicable. Here, the result given in Section 2 plays its role.

Relation (3.18) also implies that A is quasi m-accretive. Indeed, by (3.20), (3.18) and the lower semicontinuity property of the norm we have that

$$\big\|(I+\lambda A)^{-1}g\big\|\leqslant\liminf_{\varepsilon\to 0}\big\|\big(I+\lambda A^\varepsilon\big)^{-1}g\big\|\leqslant\frac{1}{1-\frac{\lambda}{\lambda_0}}\|g\|$$

which shows that A is quasi m-accretive.

Next we show that for $k=2,3,\dots$ we have

$$\big(I+\lambda A^\varepsilon\big)^{-k}g\rightharpoonup(I+\lambda A)^{-k}g \quad \text{as } \varepsilon\to 0,\ \text{for any } g\in L^2(\Omega). \tag{3.24}$$

Again we denote

$$\big(I+\lambda A^\varepsilon\big)^{-1}g=v_\varepsilon,\qquad (I+\lambda A)^{-1}g=v$$

and we have to prove first that

$$\big(I+\lambda A^\varepsilon\big)^{-2}g=\big(I+\lambda A^\varepsilon\big)^{-1}v_\varepsilon\rightharpoonup(I+\lambda A)^{-1}v=(I+\lambda A)^{-2}g \quad \text{as } \varepsilon\to 0, \tag{3.25}$$

knowing by (3.18) that $v_\varepsilon\rightharpoonup v$ in $L^2(\Omega)$. We have to study the solution to the equation

$$z_\varepsilon+\lambda A^\varepsilon z_\varepsilon=v_\varepsilon.$$

We proceed exactly in the same way as for the problem (3.13)–(3.16) and take into account that the function g in (3.13) is replaced here by v_ε which tends weakly to v in $L^2(\Omega)$. We deduce that

$$z_\varepsilon\rightharpoonup z \quad \text{in } L^2(0,L;V),\ \text{as } \varepsilon\to 0,$$
$$\frac{\partial z_\varepsilon}{\partial x_1}\rightharpoonup\frac{\partial z}{\partial x_1} \quad \text{in } L^2\big(0,L;V'\big),\ \text{as } \varepsilon\to 0,$$

where z is the solution to the equation $z+\lambda Az=v$. This means that (3.25) is verified. For the next powers $k=3,4,\dots$ we proceed by induction.

Finally we have to check (2.2). Let $x\in X$. Then there exists a sequence $x_n\in C_0^\infty((0,L)\times\Omega_1)$ such that

$$x_n\to x \quad \text{as } n\to\infty.$$

Let $z_n\in C^\infty([0,L])$ have the properties

$$z_n(0)=1,\qquad z_n(L)=0,\qquad z_n'(0)=0,\qquad z_n'(L)=0$$

and

$$z_n\to 0 \quad \text{as } n\to\infty.$$

Arguing as in [5] we consider the sequence

$$u_n\big(x_1,x'\big)=k_n\big(x'\big)z_n(x_1)+x_n\big(x_1,x'\big) \tag{3.26}$$

where k_n is computed such that u_n satisfies the boundary condition at $x_1=0$, i.e., $u_n(0,x')=\int_0^L\beta(x_1,x')u_n(x_1,x')\,dx_1$.

We get

$$k_n\big(x'\big)=\frac{\int_0^L\beta(x_1,x')x_n(x_1,x')\,dx_1}{1-\int_0^L\beta(x_1,x')z_n(x_1)\,dx_1}$$

and we remark that for n large the denominator does not vanish since $z_n\to 0$.

Also we note that $u_n \in D(A^\varepsilon)$ for all $\varepsilon > 0$. Moreover, we easily check that

$$\sup_{\varepsilon \in (0,1]} \left\| A^\varepsilon u_n \right\| < \infty \quad \text{for each } n \geqslant 1,$$

i.e., $u_n \in \mathcal{X}$ defined in (2.2) for all $n \geqslant 1$. Since by (3.26)

$$u_n \to x \quad \text{in } X, \text{ as } n \to \infty$$

we deduce that $\overline{\mathcal{X}} = X$.

Hence the hypothesis of Corollary 2.2 being verified we conclude that $y_\varepsilon(t)$ the solution to (3.6)–(3.10) converges to $y(t)$ the solution to (3.1)–(3.4),

$$y_\varepsilon(t) \rightharpoonup y(t) \quad \text{in } L^2(\Omega) \text{ uniformly on } [0, T].$$

In this way the sequence of solutions $y_\varepsilon(t)$ serves as an approximating solution to $y(t)$.

The technique can be used if for instance one has to study the asymptotic behavior of the solution to the problem (3.6)–(3.10) depending on the small parameter ε, when $\varepsilon \to 0$. Thus it can be proved that the solution behaves at limit as the solution to (3.1)–(3.4).

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