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Minimum Numbers of Circuits in Affine Sets

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Motivated by a question due to J. Eckhoff, we look for the minimum of the number of circuits contained in a subset of s points in a d-dimensional affine space, with fixed s and d.

In matroid theory (see e.g. [4]), a circuit is any minimal dependent set; for instance, a circuit in an affine space $AG(d, K)$ of dimension d over the skew-field K consists of $n+2$ points in general position in an n-dimensional subspace. In his stimulating survey paper on Radon's theorem, J. Eckhoff sets a question [2, Problem (2.8)] about primitive Radon partitions which is easily seen to be equivalent to the following extremal problem (see [1]): *to minimize the number of circuits contained in a subset of s points of a d-dimensional real affine space.* More generally, we want to determine the exact values of the following numbers, where $c(X)$ denotes as in [4] the number of circuits contained in X:

$$c(s, d, K) = \min\{c(X) | X \subset AG(d, K) \text{ and } |X| = s\},$$

$$c(s, d) = \min\{c(M) | M \text{ matroid}, rk(M) = d+1, |M| = s\}$$

(of course one could also introduce corresponding numbers for projective or vector spaces). Notice that the similar *maximum* possible numbers of circuits are obtained for X in general position or when M is a uniform matroid [4, Chapter 16].

The precise value of $c(s, d)$ will be given below. For $c(s, d, K)$ we prove partial results and formulate a quite general conjecture. The main proofs rely on the Gale transform technique or its abstract setting, the Whitney duality (see [3] and its references, especially [4, Chapter 2] for duality).

PROPOSITION 1. *Assuming $|K| \geq 3$, one has:*

$$c(s, d, K) = \begin{cases} 0 & \text{if } s \leq d+1, \\ s-d-1 & \text{if } d+1 \leq s \leq \frac{3}{2}(d+1), \\ s-d & \text{if } s = \frac{3}{2}d+2, \\ s-d+1 & \text{if } s = \frac{3}{2}d+\frac{5}{2}. \end{cases}$$

Moreover, $c(s, d, K) > s-d+2$ if $s > \frac{3}{2}d+\frac{5}{2}$.

PROOF. The independent sets are the only sets having no circuit; they have at most $d+1$ points. Assuming now $s \geq d+1$, we first remark that $c(s, d, K)$ is obtained for at least one generating set (replacing one point of a non-generating set by a point not dependent on the set does not increase the number of circuits). We then take a finite generating family $X = (p_i)$ of $AG(d, K)$ and consider its Gale transform $\bar{X} = (\bar{p}_i)$ which is a family of vectors, with the same indices, having zero sum and generating a vector space $\bar{V}$ of dimension $s-d-1$ over the same field. A subfamily Y in X is a circuit iff the corresponding complementary family $\bar{X} \setminus \bar{Y}$ is induced on $\bar{X}$ by a unique hyperplane. So we want to minimize the number $h(\bar{X})$ of hyperplanes generated by vectors of $\bar{X}$. By taking in $\bar{X}$ a basis of $\bar{V}$, we see $h(\bar{X}) \geq s-d-1$, with equality iff $\bar{X}$ is contained in the union of the rays generated by basis vectors. But in this case we have to put at least

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three non-zero vectors of $\bar{X}$ on each of these rays (otherwise all vectors but two would be in a hyperplane, which means that there are coincident points in X; now we want to have s distinct points). The construction of $\bar{X}$ is thus always possible (eventually with coincident vectors), if we have $3s \geq s-d-1$, that is $s \leq \frac{3}{2}(d+1)$.

For the following values of s as indicated in the statement, we know $c(s, d) > s-d-1$. If $s = \frac{3}{2}d+2$, one constructs $\bar{X}$ with $h(\bar{X}) = s-d$ by taking five vectors in a plane with two pairs of proportional vectors, and three vectors on each of $s-d-3$ rays independent together with the plane (notice $3(s-d-3) = s-5$).

If $s = \frac{3}{2}d + \frac{5}{2}$, then $\bar{X}$ can be taken with four vectors in a plane, no two of them proportional, and $s-4$ vectors on $s-d-3$ rays again independent with the plane. Thus, $h(\bar{X}) = s-d+1$ and one has to show that no $\bar{X}$ with $h(\bar{X}) = s-d$ exists. If B is a basis contained in such a $\bar{X}$, take p in $\bar{X}$ depending on k basis vectors, with $k>1$, k minimal. So $k \geq 2$ and $h(\bar{X}) \geq (s-d+1)+\frac{1}{2}k(k-1)$. Hence $h(\bar{X}) = s-d$ iff all vectors of $\bar{X}$ are on basis rays and one supplementary ray in a basis plane. But to ensure that the points in X are distinct, this needs five vectors in the plane and thus more than $\frac{3}{2}d + \frac{5}{2}$ vectors.

Finally, the last assertion is proved by the same technique: one uses the existence of at least one supplementary ray not in any basis plane or at least two supplementary rays in basis planes.

REMARK 1. It is to be noticed that for $|K| \geq 3$ and $s \leq \frac{3}{2}(d+1)+1$, the value of $c(s, d, K)$ is independent of K. This is not true for all s, since $c(7, 2, K)$ is equal to 14 or 17, the characteristic of K being 2 or not (with the exceptional values 0 and 17 for $|K| = 2$, $|K| = 4$ respectively).

REMARK 2. The proof gives also the types of the sets having the minimum number of circuits. For $d+1 \leq s \leq \frac{3}{2}(d+1)$, one takes $s-d-1$ skew affine subspaces, each one provided with a maximum circuit, and eventually adds points together skew with these subspaces; here subspaces are said to be skew if the projective hull of any one does not meet the projective hull of the union of the remaining ones. One can also describe the minimum sets if $s = \frac{3}{2}d+2$ or $s = \frac{3}{2}d + \frac{5}{2}$.

REMARK 3. Another easy way of proving $c(s, d, K) = s-d-1$ is by first showing $c(s+1, d, K) \geq c(s, d, K)+1$ and then giving examples with at most $s-d-1$ circuits. But this does not seem to lead to a complete classification of all the minimum sets.

PROPOSITION 2. *For matroids, one has* $c(s, d) = s-d-1$.

PROOF. Given a matroid M of rank $d+1$ on s points, we consider its dual matroid M^* of rank $s-d-1$ on the same s points. As a circuit in M is the complement of a hyperplane in M^*, we try to minimize the number $h(M^*)$ of hyperplanes in M^*. If B is a basis of M^*, the closures of B minus one of its elements give us $s-d-1$ hyperplanes, and it is easy to see that there are no more hyperplanes in M^* iff M^* is the union of the closures of the elements of B (that is, the simple matroid determined by M^* is free [4, pp. 10, 54]).

REMARK 4. The matroids of rank $d+1$ on s points minimizing the number of circuits are those having disjoint circuits (possibly loops or pairs of parallel elements).

We now turn to the plane affine case with base field K. The circuits consist in either three collinear points or four points in general position; we call them respectively 3-circuits and 4-circuits.

PROPOSITION 3. *If char $K \neq 2$ and card $K \geqslant s-2$, one has for $s \geqslant 4$ and $s \neq 6$:*

$$c(s, 2, K) = 1 + \binom{s-2}{3} + \binom{s-3}{2} = \tfrac{1}{6}(s^3 - 6s^2 + 5s + 18)$$

and moreover $c(6, 2, K) = 7$ which is one less than the value given by the above formula. The minimum sets for $s \geqslant 8$ are of a unique type, consisting in $s-2$ collinear points together with two more points collinear with one of those. For $s \leqslant 7$, the minimum sets are those pictured in [1].

PROOF. The number of circuits in a plane set X only depends on the numbers n_i of lines that contain exactly i points of X; more precisely,

$$c(X) = \binom{s}{4} - \sum_{i \geqslant 3} n_i \left[\binom{i}{3}(s-i-1) + \binom{i}{4} \right].$$

Using a list of possible configurations, one easily establishes the thesis for $s \leqslant 9$. Assume now $s \geqslant 10$ and proceed by induction. If X has exactly i points on some line, with $4 \leqslant i \leqslant s-3$, it contains $\binom{i}{3}$ 3-circuits on this line, at least

$$\binom{s-i-2}{3} + \binom{s-i-3}{2}$$

circuits outside the line (by the induction) and at least

$$\left(1 + \binom{i-1}{2}\right)\binom{s-i}{2}$$

composite circuits (since two exterior points either form a 4-circuit with any two points of the line, or form a 3-circuit with one of them and a 4-circuit with any two of the remaining ones). We subtract from this total number of circuits the number appearing in the thesis, thus obtaining for a fixed i an excess polynomial $P = P(s)$. Developing the binomial coefficients, one sees that $P(s)$ is of degree 2 if $i \neq 4$. From the values $P(i+1) = i-1$, $P(i+2) = 1$, $P(i+3) = \frac{1}{2}(i-5)(i-2)$, it is clear that $P(s) > 0$ for the cases we consider, so a minimal set X cannot have exactly i points on a line with $4 < i \leqslant s-3$. To exclude now $i = 4$, we remark that any three non-collinear exterior points form a circuit with at least one point on the line; thus we add at least $\frac{1}{6}(s-4)(s-5)(s-8)$ circuits and form a new excess polynomial which is easily seen to be positive for $s \geqslant 10$. We then show that a set X with at most three points on a line cannot be minimal. Since

$$c(X) = \binom{s}{4} - n_3(s-4)$$

and $n_3 \leqslant \frac{1}{6}s(s-1)$ we see that $c(X)$ exceeds the number in the thesis (recall $s \geqslant 10$). Thus we conclude that our minimum set X must have $s-2$ points on a line, and the thesis easily follows.

We cannot prove a general result for all s and d, but we have some reasons to think that the following statement is true.

CONJECTURE. Assume $|K| \geqslant s$ and char $K \neq 2$. If $s > \frac{3}{2}d + \frac{5}{2}$ and $d > 2$, then any set X with $c(X)$ minimum is of the following type.

(i) For d odd, take $\frac{1}{2}(d+1)$ skew lines, and write $s = \frac{1}{2}(d+1)n + m$ where $n, m \in \mathbb{N}$ and $m < \frac{1}{2}(d+1)$; then X consists of $n+1$ points on each of m of these lines and n points on each of the remaining ones. Thus

$$c(s, d, K) = \tfrac{1}{2}(d+1)\binom{n}{3} + m\binom{n}{2}.$$

(ii) For d even, take $\frac{1}{2}(d-2)$ lines skew together with a plane, and define t to be the largest integer such that, after writing $t = \frac{1}{2}(d-2)n + m$ where n, $m \in \mathbb{N}$ and $m < \frac{1}{2}(d-2)$, one has

$$t + \tfrac{1}{2}\sqrt{(9 + 4n(n-1))} \leq s - \tfrac{5}{2};$$

then X consists of $n+1$ points on each of $m+1$ of the lines, n points on the remaining ones and $s-t-1$ points forming a minimum set in the plane. Thus

$$c(s, d, K) = \tfrac{1}{2}(d-2)\binom{n}{3} + (m+1)\binom{n}{2} + \binom{s-t-3}{3} + \binom{s-t-4}{2} + 1.$$

However, this number must be lowered by one unit when $s-t-1=6$, that is $2d+2 \leq s \leq \frac{5}{2}d+1$. Also, for some exceptional values of s and d, a second type of set X is obtained by moving a point from the plane to a line (more precisely, this can be done when the former X has seven points in the plane and exactly five points on some line).

This statement is easily derived from the following conjectured lemma: if $d > 2$, any set X with the minimum number of circuits can be split into two subsets lying in skew subspaces. In fact, this implies that X lies on skew lines (and a plane if d is even). One checks that X must then be partitioned as equally as possible on these lines.

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