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Numerical analysis of a quasi-static contact problem for a thermoviscoelastic beam

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ABSTRACT

In this paper we revisit a quasi-static contact problem of a thermoviscoelastic beam between two rigid obstacles which was recently studied in [1]. The variational problem leads to a coupled system, composed of an elliptic variational inequality for the vertical displacement and a linear variational equation for the temperature field. Then, its numerical resolution is considered, based on the finite element method to approximate the spatial variable and the implicit Euler scheme to discretize the time derivatives. Error estimates are proved from which, under adequate regularity conditions, the linear convergence is derived. Finally, some numerical simulations are presented to show the accuracy of the algorithm and the behavior of the solution.

1. Introduction

Let us denote by $\tilde{\theta}(x,t)$, $u(x,t)$ and $\tilde{\sigma}(x,t)$ the temperature, the vertical displacement and the stress of a homogeneous thermoviscoelastic beam respectively, occupying in its reference configuration the interval $I = [0, 1]$.

The beam is rigidly attached at its left end $x = 0$, and at the free end $x = 1$, the vertical displacement $u(1,t)$ is limited by the presence of two rigid obstacles at temperature zero. If there is no contact with the stops, the stress is zero; otherwise it is opposite to the displacement and there is no other possibility. Moreover, there are no moments acting on the free end and we assume that the temperature is prescribed at the boundary of I.

The above mechanical problem is then written as follows (see [1] for further details),

$$\tilde{\theta}_t(x,t) - \tilde{\theta}_{xx}(x,t) = \alpha u_{xxt}(x,t), \quad 0 < x < 1,\ t > 0, \tag{1}$$

$$\tilde{\sigma}_x(x,t) = 0, \quad 0 < x < 1,\ t > 0, \tag{2}$$

where

$$\tilde{\sigma}(x,t) = -u_{xxx}(x,t) - \zeta u_{xxxt}(x,t) - \alpha\tilde{\theta}_x(x,t), \quad 0 < x < 1,\ t > 0, \tag{3}$$

with initial conditions

$$\tilde{\theta}(x,0) = \tilde{\theta}_0(x), \qquad u(x,0) = u_0(x), \quad 0 < x < 1, \tag{4}$$

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and boundary conditions

$$u(0,t) = u_x(0,t) = 0, \quad t > 0, \tag{5}$$
$$g_1 \le u(1,t) \le g_2, \quad t > 0, \tag{6}$$
$$\tilde{\sigma}(1,t) = 0 \quad \text{if } g_1 < u(1,t) < g_2,\ t > 0, \tag{7}$$
$$\tilde{\sigma}(1,t) \ge 0 \quad \text{if } u(1,t) = g_1, \qquad \tilde{\sigma}(1,t) \le 0 \quad \text{if } u(1,t) = g_2,\ t > 0, \tag{8}$$
$$u_{xx}(1,t) + \zeta u_{xxt}(1,t) = 0, \quad t > 0, \tag{9}$$
$$\tilde{\theta}(0,t) = \theta_A, \qquad \tilde{\theta}(1,t) = 0, \quad t > 0, \tag{10}$$

where we note that Eq. (2) implies $\tilde{\sigma} = \tilde{\sigma}(t)$. Here, $\alpha > 0$ is the coefficient of thermal expansion, $\zeta > 0$ is a viscosity coefficient, the values $g_1 < 0 < g_2$ give the positions of the stops and the Dirichlet condition θ_A is assumed constant for the sake of simplicity. Moreover, in the equilibrium Eq. (2), we assumed that the density of volume forces was zero in order to simplify the presentation. In the numerical results described in Section 3 we included such a function but the results presented below can be extended in a straightforward way.

Dynamic problems for thermoelastic or thermoviscoelastic beams can be found in [2–7]. In particular, Andrews et al. proved in [2] the existence and uniqueness of a weak solution to a model for a thermoviscoelastic beam with a tip body, and they proposed a numerical approximation. A problem for a viscoelastic beam which is in frictional contact with a rigid moving surface, and which takes into account the frictional heat generation, was considered by Kuttler et al. (see [5]). In [6], Kuttler et al. established the existence of a weak solution to a model describing the frictional contact of a viscoelastic beam with a rigid rotating wheel and, in the continuation paper with Bajkowski (see [8]), the numerical resolution of this problem was provided. A boundary control problem for a thermoelastic beam was studied by Hansen and Zhang (see [4]). Arantes and Rivera provided results on the decay of solutions to a contact problem for a thermoelastic beam in [3].

The existence and uniqueness of weak solutions to the dynamic contact problem of an elastic or viscoelastic beam constrained between two stops was obtained by Kuttler and Shillor (see [9]), and some numerical simulations using finite difference methods were presented in [10] by Dumont.

The finite element method was used by Campo et al. (see [11]) to numerically approximate the solution to a dynamic frictional contact problem of a viscoelastic beam with a deformable obstacle. In the latter work, the beam was assumed to move in both horizontal and tangential directions. Error estimates were derived and some numerical experiments performed.

Here, we follow the related work of Copetti (see [1]), who considered the vertical deformations of a thermoviscoelastic beam in contact with a deformable obstacle (i.e., Signorini conditions are replaced by a penalized contact condition).

According to [1], it is convenient to transform the problem into one with homogeneous boundary conditions for the temperature. Let $\theta(x,t) = \tilde{\theta}(x,t) + \theta_A(x-1)$. Thus, the pair of functions $\{\theta, u\}$ satisfies the following system of partial differential equations, for a given final time $T > 0$,

$$\theta_t(x,t) - \theta_{xx}(x,t) = \alpha u_{xxt}(x,t), \quad 0 < x < 1,\ t \in (0,T), \tag{11}$$
$$\sigma_x(x,t) = 0, \quad 0 < x < 1,\ t \in (0,T), \tag{12}$$
$$\sigma(x,t) = -u_{xxx}(x,t) - \zeta u_{xxxt}(x,t) - \alpha\theta_x(x,t) = \tilde{\sigma}(x,t) - \alpha\theta_A, \quad 0 < x < 1,\ t \in (0,T), \tag{13}$$
$$\theta(x,0) \equiv \theta_0(x) = \tilde{\theta}_0(x) + \theta_A(x-1), \qquad u(x,0) = u_0(x), \quad 0 < x < 1, \tag{14}$$
$$u(0,t) = u_x(0,t) = 0, \quad t \in (0,T), \tag{15}$$
$$g_1 \le u(1,t) \le g_2, \quad t \in (0,T), \tag{16}$$
$$\tilde{\sigma}(1,t) = 0 \quad \text{if } g_1 < u(1,t) < g_2,\ t \in (0,T), \tag{17}$$
$$\tilde{\sigma}(1,t) \ge 0 \quad \text{if } u(1,t) = g_1, \qquad \tilde{\sigma}(1,t) \le 0 \quad \text{if } u(1,t) = g_2, \quad t \in (0,T), \tag{18}$$
$$u_{xx}(1,t) + \zeta u_{xxt}(1,t) = 0, \quad t \in (0,T), \tag{19}$$
$$\theta(0,t) = \theta(1,t) = 0, \quad t \in (0,T). \tag{20}$$

We turn now to obtain the variational formulation to the above mechanical problem. Therefore, we define the spaces:

$$H_E^1(I) = \{\chi \in H^1(I) \mid \chi(1) = 0\},$$
$$H_E^2(I) = \{\chi \in H^2(I) \mid \chi(0) = \chi_x(0) = 0\},$$

and we denote the inner product in $L^2(I)$ by $(\cdot,\cdot)$ and the associated norm by $\|\cdot\|$. The norm on the classical Sobolev space $H^m(I)$ is represented by $\|\cdot\|_m$.

The initial conditions satisfy

$$\tilde{\theta}_0 \in H_E^1(I), \qquad \tilde{\theta}_0(0) = \theta_A, \qquad u_0 \in H_E^2(I), \qquad g_1 \le u_0(1) \le g_2. \tag{21}$$

Finally, we define the admissible displacement convex set in the following form,

$$K_E(I) = \{\chi \in H_E^2(I) \mid g_1 \le \chi(1) \le g_2\}.$$

We recall the following result which states the existence of a unique weak solution to the above mechanical problem (see [1]).

Theorem 1.1. *For a given final time $T > 0$, assume that the initial conditions satisfy conditions (21). Then, there exists a unique solution to the Signorini problem (11)–(20) with the following regularity:*

$$\theta \in L^\infty(0,T;H_0^1(I)), \qquad \theta_t, \theta_{xx} \in L^2(0,T;L^2(I)),$$
$$u \in L^\infty(0,T;H_E^2(I)), \qquad u_t \in L^2(0,T;H_E^2(I)), \qquad \sigma \in L^2(0,T).$$

Furthermore, if $u_0 \in H^4(I)$ then $u_{xxx}, u_{xxxx} \in L^\infty(0,T;L^2(I))$.

Multiplying Eqs. (11)–(12) by test functions, integrating with respect to x and using the boundary conditions (14)–(20), we obtain the variational formulation to problems (11)–(20): find a temperature field $\theta : [0,T] \to H_0^1(I)$ and a displacement field $u : [0,T] \to K_E(I)$ such that, for all $w \in H_0^1(I)$ and $v \in K_E(I)$,

$$(\theta_t(t), w) + (\theta_x(t), w_x) + \alpha(u_{xt}(t), w_x) = 0, \tag{22}$$
$$(u_{xx}(t) + \zeta u_{xxt}(t), v_{xx} - u_{xx}(t)) - \alpha(\theta_x(t), v_x - u_x(t)) \geq 0. \tag{23}$$

2. Finite element approximations

In this section, the letter C represents positive constants that may depend on the problem data and the continuous solution, but it is independent of the discretization parameters, and that are not necessarily the same at each occurrence. Moreover, in order to simplify the calculations, we assume that $\theta_A = 0$ and the previous problem (11)–(20) is then equivalent to problem (1)–(10).

For $T > 0$ and N a given positive integer, define the time step $\Delta t = T/N$ and the nodes $t_n = n\Delta t$, $n = 0, 1, \ldots, N$, and consider $0 = x_0 < x_1 < \cdots < x_s = 1$ a uniform partition of I into subintervals $I_j = (x_{j-1}, x_j)$, $j = 1, \ldots, s$, of length $h = 1/s$. Then, we introduce the finite element spaces:

$$S_0^h = \{\chi \in H_0^1(I) \mid \chi \in C(\bar{I}),\ \chi|_{I_i} \text{ is a linear polynomial for } i = 1, \ldots, s\},$$
$$S_E^h = \{\chi \in H_E^2(I) \mid \chi \in C^1(\bar{I}),\ \chi|_{I_i} \text{ is a cubic polynomial for } i = 1, \ldots, s\},$$

and the discrete admissible displacement convex set given by $K_E^h = S_E^h \cap K_E(I)$; that is,

$$K_E^h = \{\chi \in K_E(I) \mid \chi \in C^1(\bar{I}),\ \chi|_{I_i} \text{ is a cubic polynomial for } i = 1, \ldots, s\}.$$

Using the classical implicit Euler scheme to approximate the time derivatives, the numerical approximation we propose is to find $\Theta^n \in S_0^h$, $U^n \in K_E^h$, for $n = 1, \ldots, N$, such that, for all $W \in S_0^h$ and $V \in K_E^h$,

$$\frac{1}{\Delta t}(\Theta^n - \Theta^{n-1}, W) + (\Theta_x^n, W_x) + \frac{\alpha}{\Delta t}(U_x^n - U_x^{n-1}, W_x) = 0, \tag{24}$$
$$(U_{xx}^n, V_{xx} - U_{xx}^n) + \frac{\zeta}{\Delta t}(U_{xx}^n - U_{xx}^{n-1}, V_{xx} - U_{xx}^n) - \alpha(\Theta_x^n, V_x - U_x^n) \geq 0, \tag{25}$$

with $\Theta^0 \in S_0^h$ and $U^0 \in K_E^h$ given approximations of the initial conditions θ_0 and u_0, respectively.

The existence of a unique discrete solution can be proved following the ideas introduced in [1] for the case of a penalized contact condition. Since the modifications are straightforward, we refer the reader there for details. In this section, our aim is to obtain error estimates on the approximate solutions.

First, we bound the error on the approximation of the temperature field. Hence, we integrate variational equation (22) between the initial time and time $t = t_n$, and for $w = W \in S_0^h$, and the discrete variational equation (24) adding it from 1 to n to obtain

$$(\theta_n - \Theta^n, W) + \left(\int_0^{t_n} \theta_x(s)\mathrm{d}s - \Delta t \sum_{j=1}^n \Theta_x^j, W_x\right) + \alpha((u_n - U^n)_x, W_x)$$
$$= (\theta_0 - \Theta^0, W) + \alpha(u_{0x} - U_x^0, W_x), \quad \forall W \in S_0^h,$$

where, for a continuous function $f(t)$, we used the notation $f_n = f(t_n)$. Thus,

$$(\theta_n - \Theta^n, W) + \left(\Delta t \sum_{j=1}^n (\theta_j - \Theta^j)_x, W_x\right) + \alpha((u_n - U^n)_x, W_x)$$
$$= (\theta_0 - \Theta^0, W) + \alpha(u_{0x} - U_x^0, W_x) - \left(\int_0^{t_n} \theta_x(s)\mathrm{d}s - \Delta t \sum_{j=1}^n \theta_{jx}, W_x\right)$$

for all $W \in S_0^h$.

Taking now $W = \Pi_{S_0^h}\theta_n - \Theta^n$, where $\Pi_{S_0^h}$ is the projection operator over S_0^h given by

$$((\Pi_{S_0^h} v - v)_x, \eta_x) = 0 \quad \forall \eta \in S_0^h,\ \forall v \in H_0^1(I),$$

keeping in mind the equations

$$\begin{aligned}\left(\Delta t \sum_{j=1}^{n} (\theta_j - \Theta^j)_x, (\Pi_{S_0^h}\theta_n - \Theta^n)_x\right) &= \left(\Delta t \sum_{j=1}^{n} (\theta_j - \Pi_{S_0^h}\theta_j)_x, (\Pi_{S_0^h}\theta_n - \Theta^n)_x\right) \\ &\quad + \left(\Delta t \sum_{j=1}^{n} (\Pi_{S_0^h}\theta_j - \Theta^j)_x, (\Pi_{S_0^h}\theta_n - \Theta^n)_x\right) \\ &= \left(\Delta t \sum_{j=1}^{n} (\Pi_{S_0^h}\theta_j - \Theta^j)_x, (\Pi_{S_0^h}\theta_n - \Theta^n)_x\right),\end{aligned}$$

$$\alpha(u_{0x} - U_x^0, W_x) = -\alpha(u_{0xx} - U_{xx}^0, W),$$

$$\left(\int_0^{t_n} \theta_x(s)\mathrm{d}s - \Delta t \sum_{j=1}^{n} \theta_{jx}, W_x\right) = -\left(\int_0^{t_n} \theta_{xx}(s)\mathrm{d}s - \Delta t \sum_{j=1}^{n} \theta_{jxx}, W\right),$$

$$(\theta_n - \Theta^n, \Pi_{S_0^h}\theta_n - \Theta^n) = (\theta_n - \Pi_{S_0^h}\theta_n, \Pi_{S_0^h}\theta_n - \Theta^n) + (\Pi_{S_0^h}\theta_n - \Theta^n, \Pi_{S_0^h}\theta_n - \Theta^n),$$

where we assumed that

$$\theta_{xx} \in L^\infty(0, T; L^2(I)), \tag{26}$$

and using several times the inequality

$$ab \le \epsilon a^2 + \frac{1}{4\epsilon} b^2, \quad a, b, \epsilon \in \mathbb{R}, \epsilon > 0, \tag{27}$$

for a parameter ϵ assumed to be small enough, and the equivalence of the $H^2(I)$-norm and the seminorm given by $|v|^2 = (v_{xx}, v_{xx})$ for all $v \in H_E^2(I)$, we find that

$$\begin{aligned}&\|\Pi_{S_0^h}\theta_n - \Theta^n\|^2 + \left(\Delta t \sum_{j=1}^{n} (\Pi_{S_0^h}\theta_j - \Theta^j)_x, (\Pi_{S_0^h}\theta_n - \Theta^n)_x\right) \\ &\le C\left(\|\theta_0 - \Theta^0\|^2 + \|u_0 - U^0\|_2^2 + \|\theta_n - \Pi_{S_0^h}\theta_n\|^2 + \left\|\int_0^{t_n} \theta_{xx}(s)\mathrm{d}s - \Delta t \sum_{j=1}^{n} \theta_{jxx}\right\|^2\right) \\ &\quad - \alpha((u_n - U^n)_x, (\Pi_{S_0^h}\theta_n - \Theta^n)_x).\end{aligned}$$

Since it follows that

$$\begin{aligned}&\left(\Delta t \sum_{j=1}^{n} (\Pi_{S_0^h}\theta_j - \Theta^j)_x, (\Pi_{S_0^h}\theta_n - \Theta^n)_x\right) \\ &= \frac{1}{2\Delta t}\left\{\|\Delta t(\Pi_{S_0^h}\theta_n - \Theta^n)_x\|^2 + \left\|\sum_{j=1}^{n} \Delta t(\Pi_{S_0^h}\theta_j - \Theta^j)_x\right\|^2 - \left\|\sum_{j=1}^{n-1} \Delta t(\Pi_{S_0^h}\theta_j - \Theta^j)_x\right\|^2\right\},\end{aligned}$$

summing over n we obtain

$$\begin{aligned}&\Delta t \sum_{j=1}^{n} \|\Pi_{S_0^h}\theta_j - \Theta^j\|^2 + \frac{1}{2}\sum_{j=1}^{n}\left\{\|\Delta t(\Pi_{S_0^h}\theta_j - \Theta^j)_x\|^2 + \left\|\sum_{j=1}^{n} \Delta t(\Pi_{S_0^h}\theta_j - \Theta^j)_x\right\|^2\right\} \\ &\le C(\|\theta_0 - \Theta^0\|^2 + \|u_0 - U^0\|_2^2) + C\Delta t \sum_{j=1}^{n} \|\Pi_{S_0^h}\theta_j - \theta_j\|^2 \\ &\quad + C\Delta t \sum_{j=1}^{n} \left\|\int_0^{t_j} \theta_{xx}\mathrm{d}s - \Delta t \sum_{l=1}^{j} \theta_{lxx}\right\|^2 - C\alpha\Delta t \sum_{j=1}^{n} ((u_j - U^j)_x, (\Pi_{S_0^h}\theta_j - \Theta^j)_x).\end{aligned}$$

Thus, taking into account that the last two terms of the left-hand side of the above inequality are positive and that

$$\|\theta_j - \Theta^j\| \le \|\Pi_{S_0^h}\theta_j - \theta_j\| + \|\Pi_{S_0^h}\theta_j - \Theta^j\|,$$

the following error estimate is obtained for the temperature field,

$$\begin{aligned}\Delta t\sum_{j=1}^n \|\theta_j - \Theta^j\|^2 \le\ & C(\|\theta_0 - \Theta^0\|^2 + \|u_0 - U^0\|_2^2)\\ &+ C\Delta t\sum_{j=1}^n \|\Pi_{S_0^h}\theta_j - \theta_j\|^2 + C\Delta t\sum_{j=1}^n \left\|\int_0^{t_j}\theta_{xx}\mathrm{d}s - \Delta t\sum_{l=1}^j \theta_{lxx}\right\|^2\\ &- C\alpha\Delta t\sum_{j=1}^n ((u_j - U^j)_x, (\Pi_{S_0^h}\theta_j - \Theta^j)_x). \end{aligned}\tag{28}$$

We turn now to prove an error estimate for the displacement field. Subtracting variational inequality (23), at time $t = t_n$ and for $v = U^n$, and variational inequality (25) we have

$$\begin{aligned}&((u_n)_{xx} - U^n_{xx}, (u_n)_{xx} - U^n_{xx}) + \zeta\left(u_{xxt}(t_n) - \frac{(U^n - U^{n-1})_{xx}}{\Delta t}, (u_n)_{xx} - U^n_{xx}\right)\\ &\quad\le \alpha((\theta_n)_x - \Theta^n_x, (u_n)_x - U^n_x) + (U^n_{xx}, V_{xx} - (u_n)_{xx}) + \frac{\zeta}{\Delta t}(U^n_{xx} - U^{n-1}_{xx}, V_{xx} - (u_n)_{xx}) - \alpha(\Theta^n_x, V_x - (u_n)_x).\end{aligned}$$

Taking into account that

$$((\theta_n - \Pi_{S_0^h}\theta_n)_x, Z) = -(\theta_n - \Pi_{S_0^h}\theta_n, Z_x) \quad \forall Z \in H^2_E(I),$$

using again the equivalence between the usual norm in $H^2_E(I)$ and the previously defined seminorm, we find that

$$\begin{aligned}&\|u_n - U^n\|_2^2 + \frac{\zeta}{\Delta t}((u_n - u_{n-1} - (U^n - U^{n-1}))_{xx}, (u_n - U^n)_{xx})\\ &\quad\le \alpha((\Pi_{S_0^h}\theta_n - \Theta^n)_x, (u_n - U^n)_x) - \alpha(\theta_n - \Pi_{S_0^h}\theta_n, (u_n - U^n)_{xx})\\ &\qquad + (U^n_{xx}, (V - u_n)_{xx}) + \frac{\zeta}{\Delta t}(U^n_{xx} - U^{n-1}_{xx}, (V - u_n)_{xx})\\ &\qquad - \alpha(\Theta^n, (V - u_n)_{xx}) + \zeta\left(\frac{(u_n - u_{n-1})_{xx}}{\Delta t} - u_{xxt}(t_n), (u_n - U^n)_{xx}\right).\end{aligned}$$

Keeping in mind that

$$\begin{aligned}&\frac{1}{\Delta t}((u_n - u_{n-1} - (U^n - U^{n-1}))_{xx}, (u_n - U^n)_{xx}) \ge \frac{1}{2\Delta t}[\|u_n - U^n\|_2^2 - \|u_{n-1} - U^{n-1}\|_2^2],\\ &(U^n_{xx}, (V - u_n)_{xx}) = ((U^n - u_n)_{xx}, (V - u_n)_{xx}) + ((u_n)_{xx}, (V - u_n)_{xx}),\\ &\frac{\zeta}{\Delta t}(U^n_{xx} - U^{n-1}_{xx}, (V - u_n)_{xx}) = \frac{\zeta}{\Delta t}((U^n - U^{n-1} - (u_n - u_{n-1}))_{xx}, (V - u_n)_{xx})\\ &\qquad + \zeta(\frac{1}{\Delta t}(u_n - u_{n-1})_{xx} - u_{xxt}(t_n), (V - u_n)_{xx}) + \zeta(u_{xxt}(t_n), (V - u_n)_{xx}),\\ &-\alpha(\Theta^n, (V - u_n)_{xx}) = -\alpha(\Theta^n - \theta_n, (V - u_n)_{xx}) - \alpha(\theta_n, (V - u_n)_{xx}),\end{aligned}$$

where we used the additional regularity

$$u_t \in L^\infty(0, T; H^2(I)),\tag{29}$$

and applying again inequality (27) several times for a parameter $\epsilon > 0$ assumed to be small enough, we get

$$\begin{aligned}&\frac{\zeta}{2\Delta t}[\|u_n - U^n\|_2^2 - \|u_{n-1} - U^{n-1}\|_2^2] + \frac{1}{2}\|u_n - U^n\|_2^2 \le C\|\theta_n - \Pi_{S_0^h}\theta_n\|^2\\ &\quad + C\left\|u_t(t_n) - \frac{u_n - u_{n-1}}{\Delta t}\right\|_2^2 + (\|(u_n)_{xx}\| + \zeta\|u_{xxt}(t_n)\| + \alpha\|\theta_n\|)\|V - u_n\|_2\\ &\quad + C\|u_n - U^n\|_2^2 + \frac{\zeta}{\Delta t}((U^n - U^{n-1} - (u_n - u_{n-1}))_{xx}, (V - u_n)_{xx})\\ &\quad + C\|V - u_n\|_2^2 + \alpha((\Pi_{S_0^h}\theta_n - \Theta^n)_x, (u_n - U^n)_x) + \frac{1}{4}\|\Theta^n - \theta_n\|^2\end{aligned}$$

for all $V \in K_E^h$. By induction the following error estimate is obtained for the displacement field,

$$\zeta\|u_n - U^n\|_2^2 + \Delta t\sum_{j=1}^n \|u_j - U^j\|_2^2 \le C\|u_0 - U^0\|_2^2 + C\Delta t\sum_{j=1}^n\left(\|\theta_j - \Pi_{S_0^h}\theta_j\|^2 + \left\|u_t(t_j) - \frac{u_j - u_{j-1}}{\Delta t}\right\|_2^2\right)$$
$$+ C\Delta t\sum_{j=1}^n(\|V_j - u_j\|_2 + \|u_j - U^j\|_2^2) + C\Delta t\sum_{j=1}^n\|V_j - u_j\|_2^2 + \alpha\Delta t\sum_{j=1}^n((\Pi_{S_0^h}\theta_j - \Theta^j)_x, (u_j - U^j)_x)$$
$$+ C\sum_{j=1}^n((U^j - U^{j-1} - (u_j - u_{j-1}))_{xx}, (V_j - u_j)_{xx}) + \frac{\Delta t}{2}\sum_{j=1}^n\|\Theta^j - \theta_j\|^2 \quad \forall\{V_j\}_{j=1}^n \subset K_E^h. \tag{30}$$

Combining now (28) and (30) we find that

$$\frac{\Delta t}{2}\sum_{j=1}^n\|\theta_j - \Theta^j\|^2 + \zeta\|u_n - U^n\|_2^2 + \Delta t\sum_{j=1}^n\|u_j - U^j\|_2^2$$
$$\le C\left\|\int_0^{t_n}\theta_{xx}(s)ds - \Delta t\sum_{j=1}^n\theta_{jxx}\right\|^2 + C\|\theta_0 - \Theta^0\|^2 + C\|u_0 - U^0\|_2^2$$
$$+ C\Delta t\sum_{j=1}^n\left(\|\theta_j - \Pi_{S_0^h}\theta_j\|^2 + \left\|u_t(t_j) - \frac{u_j - u_{j-1}}{\Delta t}\right\|_2^2 + \|u_j - U^j\|_2^2\right)$$
$$+ C\Delta t\sum_{j=1}^n[\|u_j - V_j\|_2 + \|u_j - V_j\|_2^2] + C\sum_{j=1}^n((U^j - U^{j-1} - (u_j - u_{j-1}))_{xx}, (V_j - u_j)_{xx}) \quad \forall\{V_j\}_{j=1}^n \subset K_E^h.$$

Finally, keeping in mind that (see [11])

$$\sum_{j=1}^n((U^j - U^{j-1} - (u_j - u_{j-1}))_{xx}, (V_j - u_j)_{xx}) = ((U^n - u_n)_{xx}, (V_n - u_n)_{xx}) - ((U^0 - u_0)_{xx}, (V_1 - u_1)_{xx})$$
$$+ \sum_{j=1}^{n-1}((U^j - u_j)_{xx}, (V_j - u_j)_{xx} - (V_{j+1} - u_{j+1})_{xx})$$
$$\le \epsilon\|u_n - U^n\|_2^2 + C\|V_n - u_n\|_2^2 + C\|u_0 - U^0\|_2^2 + C\|u_1 - V^1\|_2^2$$
$$+ \sum_{j=1}^{n-1}\|u_j - U^j\|_2\|u_j - V_j - (u_{j+1} - V_{j+1})\|_2,$$

where $\epsilon > 0$ is assumed to be small enough, and applying a discrete version of Gronwall's inequality (see [11] for details), we obtain the following error estimate result.

Theorem 2.1. *Let (u, θ) and $\{U^n, \Theta^n\}_{n=0}^N$ be the solutions to problems (22)–(23) and (24)–(25), respectively. If we assume the additional regularities (26) and (29) for the temperature field and the displacement field, and that $u_0 \in H^4(I)$, under assumptions (21) we have the following error estimates for all $\{V_j\}_{j=1}^n \subset K_E^h$,*

$$\max_{0\le n\le N}\|u_n - U^n\|_2^2 + \Delta t\sum_{n=1}^N(\|\theta_n - \Theta^n\|^2 + \|u_n - U^n\|_2^2)$$
$$\le C\max_{0\le n\le N}\left\|\int_0^{t_n}\theta_{xx}(s)ds - \Delta t\sum_{j=1}^n\theta_{jxx}\right\|^2 + C\|\theta_0 - \Theta^0\|^2 + C\|u_0 - U^0\|_2^2 + C\Delta t\sum_{j=1}^N\|\theta_j - \Pi_{S_0^h}\theta_j\|^2$$
$$+ C\Delta t\sum_{j=1}^N\left\|u_t(t_j) - \frac{u_j - u_{j-1}}{\Delta t}\right\|_2^2 + C\max_{1\le n\le N}\|V_n - u_n\|_2 + C\max_{1\le n\le N}\|V_n - u_n\|_2^2$$
$$+ \frac{C}{\Delta t}\sum_{j=1}^{N-1}\|u_j - V_j - (u_{j+1} - V_{j+1})\|_2^2.$$

We notice that these error estimates are the basis for the analysis of the convergence rate of the algorithm. Therefore, as an example, we assume that the discrete initial conditions are given by

$$U^0 = \Pi_{S_E^h}u_0, \qquad \Theta^0 = \Pi_{S_0^h}\theta_0,$$

where $\Pi_{S_E^h}$ is the projection operator over the finite element space S_E^h.

From the regularities stated in Theorem 1.1 we find that (see [12]),

$$\|u_0 - U^0\|_2 \le Ch\|u_0\|_{H^3(I)} \le Ch\|u\|_{L^\infty(0,T;H^3(I))},$$
$$\|\theta_0 - \Theta^0\| \le Ch\|\theta_0\|_{H^1(I)} \le Ch\|\theta\|_{L^\infty(0,T;H^1(I))},$$
$$\inf_{V_n \in K_E^h} \|u_n - V_n\|_2 \le Ch\|u\|_{L^\infty(0,T;H^3(I))}.$$

Actually, keeping in mind the extra regularity obtained there in the case where the initial condition satisfies $u_0 \in H^4(I)$, we conclude that

$$\|u_0 - U^0\|_2 \le Ch^2\|u_0\|_{H^4(I)},$$
$$\inf_{V_n \in K_E^h} \|u_n - V_n\|_2 \le Ch^2\|u\|_{L^\infty(0,T;H^4(I))}.$$

If we assume the additional regularities

$$\theta_t \in L^\infty(0,T;H^2(I)), \qquad u_{tt} \in L^\infty(0,T;H^2(I)), \qquad u \in H^1(0,T;H^3(I)), \tag{31}$$

we easily obtain

$$\max_{0\le n\le N} \left\| \int_0^{t_n} \theta_{xx}(s)\mathrm{d}s - \Delta t \sum_{j=1}^{n} \theta_{jxx} \right\|^2 \le C(\Delta t)^2 \|\theta_t\|^2_{L^\infty(0,T;H^2(I))},$$
$$\Delta t \sum_{j=1}^{N} \left\| u_t(t_j) - \frac{u_j - u_{j-1}}{\Delta t} \right\|_2^2 \le C(\Delta t)^2 \|u_{tt}\|^2_{L^\infty(0,T;H^2(I))}.$$

Finally, taking into account that (see [11])

$$\frac{1}{\Delta t} \sum_{j=1}^{N-1} \|u_j - V_j - (u_{j+1} - V_{j+1})\|_2^2 \le Ch^2\|u\|^2_{H^1(0,T;H^3(I))},$$

we have the following corollary.

Corollary 2.2. *Let the assumptions of Theorem 2.1 still hold. Under the additional regularity conditions (31) we find that there exists a positive constant $C > 0$ such that*

$$\max_{0\le n\le N} \|u_n - U^n\|_2^2 + \Delta t \sum_{n=1}^{N} (\|\theta_n - \Theta^n\|^2 + \|u_n - U^n\|_2^2) \le C(h^2 + \Delta t^2).$$

3. Numerical experiments

In this section, the numerical algorithm used to solve the discrete problem (24)–(25) is briefly described and the results of some numerical experiments are reported.

We let $\alpha = 0.017$, $\zeta = 0.1$, $g_1 = -0.02$ and $g_2 = 0.1$. Assuming that $\{\Theta^{n-1}, U^{n-1}\}$ are known, the iteration

$$\frac{1}{\Delta t}(\Theta^{n,l} - \Theta^{n-1}, W) + (\Theta_x^{n,l}, W_x) + \frac{\alpha}{\Delta t}(U_x^{n,l-1} - U_x^{n-1}, W_x) = 0,$$
$$(U_{xx}^{n,l}, V_{xx} - U_{xx}^{n,l}) + \frac{\zeta}{\Delta t}(U_{xx}^{n,l} - U_{xx}^{n-1}, V_{xx} - U_{xx}^{n,l}) - \alpha(\Theta_x^{n,l}, V_x - U_x^{n,l}) \ge 0,$$

with a given tolerance of 10^{-7} and $U^{n,0} = U^{n-1}$, was used to find $\{\Theta^n, U^n\}$. Since

$$\Theta^{n,l} = \sum_{i=1}^{s-1} c_i^{n,l}\chi_i, \qquad U^{n,l} = \sum_{i=1}^{2s} d_i^{n,l}\eta_i, \qquad V = \sum_{i=1}^{2s} v_i\eta_i,$$

where $\{\chi_i\}_{i=1}^{s-1}$ is the piecewise linear basis function for S_0^h and $\{\eta_i\}_{i=1}^{2s}$ is the piecewise Hermite cubic basis for K_E^h, at each iteration l the algebraic problems

$$(M + \Delta t K)\underline{c}^{n,l} = M\underline{c}^{n-1} + \alpha C(\underline{d}^{n-1} - \underline{d}^{n,l-1}),$$
$$[(\Delta t + \zeta)B\underline{d}^{n,l} - \underline{b}]^T(\underline{v} - \underline{d}^{n,l}) \ge 0,$$

need to be solved, where

$$M_{ij} = (\chi_i, \chi_j), \qquad K_{ij} = (\chi_{ix}, \chi_{jx}), \qquad C_{ij} = (\chi_{ix}, \eta_{jx}), \qquad B_{ij} = (\eta_{ixx}, \eta_{jxx}),$$
$$\underline{b} = \zeta B\underline{d}^{n-1} + \alpha\Delta t C^T\underline{c}^{n,l}, \qquad \{\underline{c}^{n,l}\}_i = c_i^{n,l}, \qquad \{\underline{d}^{n,l}\}_i = d_i^{n,l}.$$

The solution to the latter variational inequality was obtained by using the approach described in [13], where a Signorini contact problem for a thermoviscoelastic rod was studied.

Table 1
Computed errors when $T = 2$.

s	Δt	$\Delta t \sum_{n=1}^{N} \|\theta_n - \Theta^n\|^2$	$\max_n \|U^n - u_n\|_2^2$
10	1×10^{-3}	1.245×10^{-3}	2.006×10^{-7}
20	5×10^{-4}	7.734×10^{-5}	1.257×10^{-8}
40	2.5×10^{-4}	4.748×10^{-6}	7.720×10^{-10}
80	1.25×10^{-4}	2.858×10^{-7}	4.628×10^{-11}
160	6.25×10^{-5}	1.654×10^{-8}	2.650×10^{-12}

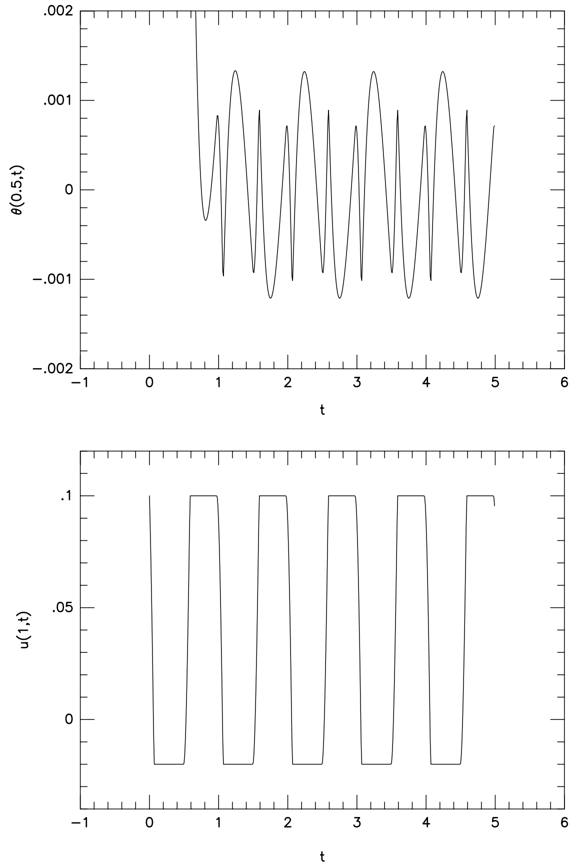

Fig. 1. The time evolution of the temperature and the displacement for the second experiment.

To examine the error estimates numerically, we run an experiment with the following exact solution:

$$\theta(x,t) = \exp(t)\sin(\pi x),$$

$$u(x,t) = \begin{cases} -3g_2\left(\dfrac{x^3}{6} - \dfrac{x^2}{2}\right)(-0.5t^2 + \sqrt{2}t), & 0 < t \le \sqrt{2}, \\ -3g_2\left(\dfrac{x^3}{6} - \dfrac{x^2}{2}\right), & \sqrt{2} < t \le 2, \end{cases}$$

$$\sigma(x,t) = \begin{cases} 0, & 0 < t \le \sqrt{2}, \\ -t, & \sqrt{2} < t \le 2, \end{cases}$$

satisfying

$$\theta_t - \theta_{xx} = -\alpha u_{xxt} + f(x,t),$$
$$\sigma_x = 0,$$

where $\sigma = -u_{xxx} - \zeta u_{xxxt} - \alpha\theta_x + G(x,t)$. The functions f, G, θ_0 and u_0 are calculated from the exact solution. The computed errors are presented in Table 1 where we observe a convergence rate of approximately 4, better than we expected from the above theoretical numerical analysis.

For completeness, we perform now a simulation with a force periodic in time $f(t) = 5\sin(2\pi t)$ in Eq. (12); that is, we solve Eq. (11) together with $\sigma_x = f(t)$. The initial conditions are $\theta_0(x) = 2\sin(\pi x)$ and $u_0(x) = 0.1x^2$. Hence, in Fig. 1 we show the evolution in time of the temperature at $x = 0.5$ and the displacement at the contact point. We note the periodic behavior and the effect of the non-penetration condition.

4. Conclusions

A quasi-static contact problem of a thermoviscoelastic beam between two rigid obstacles was numerically studied in this paper. An existence and uniqueness result of weak solutions proved in [1] was recalled. Then, fully discrete approximations were introduced by using the finite element method to approximate the spatial variable and the implicit Euler scheme to discretize the time derivatives. An a priori error estimates result was proved, Theorem 2.1, from which, under adequate additional regularity conditions, the convergence of the algorithm was derived; see Corollary 2.2.

The algorithm was implemented and two numerical examples were computed. First, a simple case was chosen in such a way as to show the numerical convergence of the algorithm. As could be seen in Table 1, the convergence of the algorithm was achieved. In fact, from these results it seems that a fourth order of convergence was obtained, improving the theoretical linear convergence, maybe due to a superconvergence property. Finally, a similar example was considered assuming a force periodic in time. As was depicted in Fig. 1, the periodic behavior and the effect of the non-penetration condition were clearly shown.

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