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# Optimal Control of the Irrigation Problem: Characterization of the Solution

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**Abstract**

This article studies the optimal solution of an irrigation problem. It consists in optimizing the planning of water used so by the water amount in the soil (trajectory) fulfils the cultivation water requirements. We characterize the optimal solution by applying the necessary conditions of optimality in the form of the Maximum Principle. We also compare the results obtained analytically and numerically.

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*Keywords:* Optimal control; Maximum Principle; Irrigation.

## 1. Introduction

We propose to study an optimal control problem in the context of water resource management. Given an agricultural field of potatoes with unit area located in the region of Lisbon, Portugal, we intend to minimize the amount of water used in that field such that the crop is kept in a good preservation state.

In [5], the authors proposed a mathematical model for this problem. There, we also obtain a numerical solution for the so-called "initial plan" problem considering different weather scenarios. Such scenarios are simulated by multiplying by a certain "precipitation factor" the known rainfall monthly averages. The model is later improved in [7]: an extra term taking account the rainfall of the previous month is added (this rainfall model was statistically proven to be significant). A comparison between this new model and the solution knowing a priori the rainfall is shown. In order to consider uncertainties of the weather, the initial model is replaned in [6] where *hard* state constraints are replaced by *soft* state constraints. For the numerical resolution, the authors have used the solver IPOPT and KNITRO. In [9] a new problem is considered. An yearly planning problem was taken into account such that the objective is now to design a water resevoir to fulfill the water requirements of a crop. Once again, a precipitation factor is considered.

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If this factor is smaller than 1, the reservoir will have a greater amount of water than it is needed, ensuring that in a drought year the water needs are fulfilled.

Here, we intend to apply the necessary conditions of optimality in the form of the Maximum Principle in order to characterize the solutions to the irrigation problem. We also compare the results obtained analytically and numerically.

## 2. MODEL

Our problem consists in optimizing the planning of the water used in the irrigation of farm fields by means of the optimal control, where the trajectory is the water in the soil and the control is the flow (amount per unit time) of water introduced in the soil via its irrigation system. The formulation is given by:

$$\text{OCP:} \quad \begin{aligned} & \min_u \ \tfrac{1}{2}\int_0^T u^2(t)dt \\ & \text{subject to:} \\ & \dot{x}(t) = f(t, x(t), u(t)) \ \text{a.e.}\ t \in [0, T] \\ & x(t) \geq x_{\min} \qquad \forall\, t \in [0, T] \\ & u(t) \geq 0 \qquad \text{a.e.}\ t \in [0, T] \\ & x(0) = x_0, \end{aligned} \tag{1}$$

where $x$ is the trajectory, $u$ is the control, $f$ is the hydrological balance function, $x_{\min}$ is the hydrological need of the crop (according to [4]), $x_0$ is an initial state and $T$ is a given time.

We assume that the initial state strictly satisfies the hydrological needs $x_0 \geq x_{\min}$. In the dynamic equation that represents the hydrological balance, $f$ is defined by $f(t, x(t), u(t)) = u(t) + g(t) - \beta x(t)$, where $g(t)$ is the rainfall minus the evapotranspiration and $\beta$ is the percentage of losses of water due to the runoff and deep infiltration. A detailed description of this models is given in [7].

## 3. CONDITIONS OF OPTIMALITY

To characterize the solution, we are interested in writing the optimality conditions in the normal form, that is guaranteeing that the multiplier associate to the objective function is not zero (see [1] and [3] for discussion of normal forms of the MP for optimal control problems with state constraints).

Let $(t, x) \to h(t, x)$ be the inequality state constraints function (in our case $h(x) = x_{\min} - x$).

Following Rampazzo and Vinter [3], the Maximum Principle can be written with $\lambda = 1$, if there exists a continuous feedback $u = \eta(t, \xi)$ such that

$$\frac{dh(t, \xi(t))}{dt} = h_t(t, \xi) + h_x(t, \xi) \cdot f(t, \xi, \eta(t, \xi)) < -\gamma' \tag{2}$$

for some positive $\gamma'$, whenever $(t, \xi)$ is close to the graph of $\bar{x}(\cdot)$ and $\xi$ is near to the state constraint boundary. So, there exists a control (the flow of water introduced in the soil via its irrigation system) that pulls the state variable away from the state constraint boundary (this guarantees that the crop survives).

In our problem from (2), we may write

$$\frac{dh(\xi(t))}{dt} = \nabla_x h(\xi(t)) \cdot f(t, \xi, \eta(t, \xi)) = -(\eta(t, \xi) + \triangle(t, \xi)) \leq -\gamma', \tag{3}$$

where $\triangle(t, \xi) = g(t) - \beta\xi$. For $\xi$ in a neighbourhood of $\bar{x}$, we can choose $\eta$ sufficiently large. So that satisfies equation (3).

We define the Hamilton function $H(t, x, p, u) = p(t)f(t, x, u) - \frac{1}{2}\lambda u^2$ where $p(t)$ and $\lambda$ represent the Lagrange multipliers. Since the inward pointing condition (3) is satisfied, the Maximum Principle (with $\lambda = 1$ in the Hamilton function) writes:

$$-\dot{p}(t) = H_x(t, \bar{x}(t), q(t), \bar{u}(t));$$

$$H(t, \bar{x}(t), p(t), \bar{u}(t)) = max_{v\in[0,\infty[}H(t, \bar{x}(t), p(t), v)\text{a.e.};$$

$$supp\{\mu\} \subset \{t \in [0, T] : h(\bar{x}(t)) = 0\};$$

$$q(T) = 0;$$

where $q(t)$ is defined as

$$q(t) = \begin{cases} p(t) - \int_{[0,t)} \mu(ds), & t \in [0, T) \\ p(T) - \int_{[0,T]} \mu(ds), & t = T. \end{cases}$$

In our problem the Hamilton function is given by:

$$H(t, x, p, u) = p(t)\,(u + g(t) - \beta x) - \frac{1}{2}u^2. \tag{4}$$

Therefore from the Maximum Principle, we obtain:

$$\dot{p}(t) = \beta q(t)$$

$$q(t)(\bar{u}(t) - u(t)) - \tfrac{1}{2}(\bar{u}^2(t) - u^2(t)) \geq 0$$

$$\text{supp}\{\mu\} \subset \{t \in [0, T] : \bar{x}(t) = x_{\min}\}$$

$$q(T) = 0$$

## 4. CHARACTERIZATION OF THE SOLUTION

We start by studying (OCP) when the trajectory never touches the boundary. In this case, our problem works as if there was no state constraints and therefore we may obtain the optimal solution using the MP.

If the state constraint never touches the boundary (i.e. $x \neq x_{\min}$), we have $\mu \equiv 0$ and consequently $p = q$. So, the MP is written as:

$$\begin{aligned} &\dot{p}(t) = \beta p(t), \\ &p(t)(\bar{u}(t) - u(t)) - \tfrac{1}{2}(\bar{u}^2(t) - u^2(t)) \geq 0, \\ &p(T) = 0. \end{aligned} \tag{5}$$

Since $p(T) = 0$ and $\dot{p}(t) = \beta p(t)$, by Gronwall's inequality, we conclude that $p(t) = 0$ and $\bar{u} \leq u$, $\forall u \in \mathbb{R}_0^+$. Thus, we conclude that $\bar{u} = 0$. This means that if the state constraint is never active, then it is not necessary to supply water to the crop.

Appealing to regularity results on [8], we can deduce the following: If at the endpoints, the trajectory does not touch the boundary, then $\mu$ is an absolutely continuous function and $u$ is a continuous function on $[0, T]$.

Now, we characterize the optimal solution for (OCP) in a more general case, by applying the necessary conditions of optimality. For that, we apply the Hamiltonian Condition of the Maximum Principle for $\bar{u} = 0$ and $\bar{u} > 0$.

If $\bar{u} = 0$, we have: for all $u(t) \geq 0$,

$$q(t)u(t) - \frac{1}{2}u^2(t) \leq 0 \Leftrightarrow u(t)(q(t) - \frac{1}{2}u(t)) \leq 0 \Leftrightarrow u(t) \geq 2q(t).$$

So, we may say that $q(t) \leq 0$.

On the other hand, if $\bar{u} > 0$, then (4) the fact that $H_u(t, \bar{x}, q, \bar{u}) = 0$ implies $q(t) - \bar{u}(t) = 0$. So $\bar{u}(t) = q(t)$.

Observe that $H_{uu}(t, \bar{x}, p, \bar{u}) = -1 \neq 0$. So, if $\bar{u} > 0$, then $\bar{u}$ is the unique solution, as we would expect from the convexity of the problem.

So, we conclude that if $\bar{u} = 0$ then $q(t) \leq 0$ and if $\bar{u} > 0$ then $\bar{u} = q(t)$, that is

$$\bar{u} = \max\{q(t), 0\}. \tag{6}$$

As we said, the regularity conditions allow to conclude that $\mu$ is an absolutely continuous function. This means that there exists an integrable function $\psi(t)$ such that $\int_0^t \psi(s)ds = \int_{[0,T]} \mu(ds)$. Therefore, $q$ is absolutely continuous function on $[0, T]$ and $\dot{q}(t) = \dot{p}(t) - \psi(t)$.

By the MP, we know that $\dot{p}(t) = \beta q(t)$, so

$$\psi(t) = \beta q(t) - \dot{q}(t). \tag{7}$$

On the other hand, two cases can occur: either the trajectory is on the boundary of the state constraint or it is not.

Suppose that on the interval $[t_0, t_1]$, the trajectory is on the boundary of the state constraint. In this case, for $t \in [t_0, t_1]$, we have $\bar{x}(t) = x_{\min}$ and consequently

$$\dot{\bar{x}}(t) = 0 \Leftrightarrow \bar{u}(t) + \triangle(t, x_{min}) = 0 \Leftrightarrow \bar{u}(t) = -\triangle(t, x_{min}), \quad \forall t \in ]t_0, t_1[. \tag{8}$$

We may assume throughout that $g(t) < \beta x_{\min}$. From (8) and (6), we conclude that

$$q(t) = \bar{u}(t) = -\triangle(t, x_{min}). \tag{9}$$

Therefore, equation (7) can be written as: $\psi(t) = -\beta\triangle(t, x_{min}) + \dot{g}(t)$.

In the interval that the state constraint is not active, we have $\psi(t) = 0$.

This problem was solved numerically in [9]. We confirm that the numerical solution satisfies the necessary conditions of optimality and we compare the results obtained analytically and numerically.

## 5. Numerical results

We consider now the numerical problem of optimizing the planning of the water used in the irrigation of farm fields. The discrete formulation of this problem is as follows:

$$\begin{array}{ll}
\min \dfrac{1}{2}\displaystyle\sum_{i=1}^{N-1} u_i^2 & \\
\text{such that:} & \\
x_{i+1} = x_i + hg(t_i, x_i, u_i),\ a.e.\ i = 1, \dots,\ N-1, & \\
x_1 = x_0 & \\
x_i \geq x_{\min}, & i = 1, \dots,\ N, \\
u_i \geq 0, & a.e.\ i = 1, \dots,\ N-1,
\end{array}$$

where $x = (x_1, \dots, x_N)$ is the trajectory, $u = (u_1, \dots, u_{N-1})$ is the control, $g$ is the hydrological balance function, $x_{\min}$ is the hydrological need of the crop, $x_0$ is an initial state, $h$ is the time step discretization, and $N = 12/h$. In the dynamic equation, that represents the hydrological balance, $f$ is defined by

$$g(t_i, x_i, u_i) = u_i + \text{rainfall}(t_i) - \text{evapotranspiration}(t_i) - \text{losses}(x_i), \tag{10}$$

where the evapotranspiration is the evaporation of the soil and the transpiration of the crop and the losses are the losses of water due to the runoff and deep infiltration. The Rainfall model [7] is based on a linear combination of average monthly rainfall from the last 10 years and the amount of rainfall in the previous month. The evapotranspiration model is a crop coefficient (in our case potatoes) multiplied by the reference value of evapotranspiration in Lisbon, given by Pennman-Monteith methodology, see [2]. The losses are 15% of the water in the soil.

The state constraint ($x_i \geq x_{\min}$) is based on the fact that the plant needs a minimum amount of water to survive. A detail description of these models is in [7].

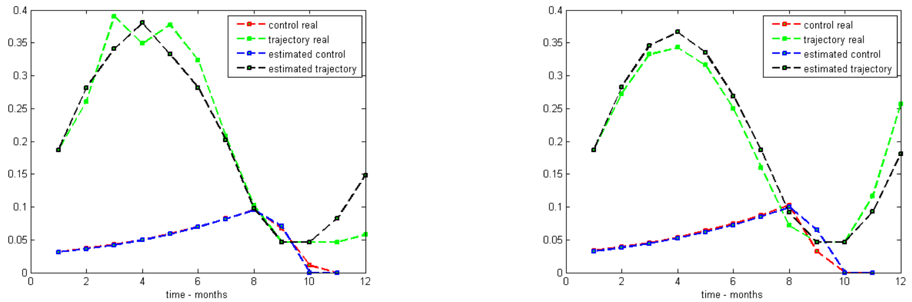
Our study considers field of potatoes ($x_{\min} = 0.56/12h$) in the region of Lisbon (defining the evapotranspiration table) with unit area.

To obtain the numerical solution for this optimal control problem we have transcribed the problem into a nonlinear programming problem.

To implement this optimization problem, we have used *fmincon* function of MatLab with the algorithm "active set", by default. Although it is a local search method, the convexity of the problem allows us to conclude that the solution obtained is the global solution.

In Fig. 1, we show the results that compare the solutions obtained from our model with solutions obtained having a prior knowledge of the rainfall in the years 2008 and 2010. These results suggest that there is a good approximation between the prediction by the model and the real data.

**YEAR 2008** **YEAR 2010**




Estimated Needs 0.5334 Estimated Needs 0.5491
Real Needs 0.5451 Real Needs 0.5318

Fig. 1. Comparison of our model against solutions obtained having a prior knowledge of the rainfall (left-2008, right-2010).

In order to validate the numerical results, in Fig. 2 we plot the multipliers obtained from our code for the year 2010 (we note that the green line is the hydrological need of the crop).

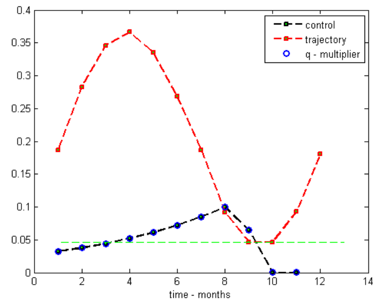



Fig. 2. Multipliers obtained from our code for the year 2010.

We can observe that $\bar{u} = \max\{q(t), 0\}$, as expected from section 4. From here, we can say that although the analytical solution was not completely obtained explicitly, the numerical solution fulfils the characterization given by the necessary optimality conditions.

## 6. Conclusions

We studied the optimal solution to the irrigation problem described as the optimizing the planning of water used so that the water amount in the soil (trajectory) fulfils the cultivation water requirements of a crop of potatoes.

We prove that if the state constraint is never active then it is not necessary to supply water to the crop.

We also characterize the solution applying the necessary conditions of optimality in the form of the Maximum Principle and we conclude that the optimal flow is either zero or adjoint multiplier follows the $\bar{u} = max\{q(t), 0\}$.

Finally, we compare the results obtained analytically and numerically and conclude that the numerical solution fulfils the necessary optimality conditions.

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