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Critical Points of Convex Perturbations of Quadratic Functionals

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Abstract—A relationship between PS-condition and convexity for functionals exhibiting resonance type behavior at infinity is established. This leads to new existence and multiplicity results for critical points.

Keywords—Critical point, Palais-Smale condition, Convex functionals, Resonance at infinity, Hamiltonian equation.

1. INTRODUCTION

We consider here the problem of existence of critical points for the function

$$f(x) = \frac{1}{2}(Lx \mid x) + H(x). \tag{1}$$

In case of a convex and subquadratic H a typical proof is based on Clarke's famous dual action principle [1] as say in [2–5]. It seems, however, natural to ask whether the classical minimax variational methods (the theory of Ljusternik-Schnirelman, the mountain pass of Ambrosetti-Rabinowitz, saddle-point, or linking theorems [6], etc.) can be applied in this case. In the paper, we give a positive answer to the question.

Namely, we show in Theorem 1 that convexity of H implies a compactness property of f, a *weighted PS-condition*, similar to that introduced in 1978 by Cerami [7] which provides a sufficient amount of compactness needed to establish existence of critical points using minimax methods of calculus of variations (modified here for nondifferentiable functionals in the spirit of [8–10]). It is really surprising that the implication remained unnoticed for a long time.

An immediate advantage of our approach is a possibility to weaken the convexity requirements in general and more specialized theorems as will be shown in Theorems 5–7, the last two concerned with periodic solutions of Hamiltonian systems. We believe, however, that the approach can be applied beyond the specific context of these three theorems.

In fact, even in the convex case we get some new information, by weakening requirements on H and, especially, by being able to prove multiplicity results for solutions in case of even

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subquadratic H (Theorem 3). Note that in this case too, the convexity requirement can be weakened in a similar way.

Another element of our approach is that we allow L to be an unbounded closed operator (cf. e.g., [11]). Because of that the deformation lemma of Section 3 is new even for the case of a smooth H. This generalization seems rather technical at first glance but it allows us to further weaken the Palais-Smale condition which is essential in the proof of Theorems 6 and 7.

We finally mention a new abstract multiplicity result of Theorem 4.

2. PRELIMINARIES

We adopt the following hypotheses throughout the paper.

(A_1) X is a separable Hilbert space; L is a closed self-adjoint linear operator in X with dense domain $\operatorname{dom} L$;

(A_2) $\sigma(L)$, the spectrum of L, is purely discrete and every eigenvalue has a finite multiplicity;

(A_3) there is a $\mu > 0$ such that $[-\mu, 0) \cap \sigma(L) = \emptyset$ and

$$\limsup_{|x|\to\infty} |x|^{-2}|H(x)| < \frac{\mu}{2};$$

(A_4) $H(x) \to \infty$ if $|x| \to \infty$, $x \in \operatorname{Ker} L$;

(A_5) H satisfies the Lipschitz condition on every ball.

We denote by $\partial H(x)$ Clarke's generalized gradient of H of x and say that x is a critical point of f if $x \in \operatorname{dom} L$ and

$$0 \in Lx + \partial H(x). \tag{2}$$

Finally, we shall denote

$$\rho(x) = \operatorname{dist}(0, Lx + \partial H(x)).$$

This means that critical points are characterized by the relation $\rho(x) = 0$.

3. WEIGHTED PALAIS-SMALE CONDITION

The following is a weakened version of the PS-condition we shall use (cf. [3,7,12]).

DEFINITION 1. *A sequence $\{x_n\}$ is a weighted Palais-Smale sequence (at level $c \in \mathbb{R}$) if $f(x_n) \to c$ and $\rho(x_n)(1 + \|x_n\|) \to 0$ as $n \to \infty$. We say that f satisfies the weighted PS-condition at level c if any weighted PS-sequence at that level contains a convergent subsequence.*

THEOREM 1. *We assume that (A_1)–(A_4) hold and H is convex. Then f satisfies the weighted PS-condition at every level. Moreover, the limit of any convergent weighted PS-sequence is a critical point of f.*

By (A_2), the theorem will be proved if we show that every weighted PS-sequence is bounded. (The last statement then follows from the facts that L is closed and the set-valued mapping $x \to \partial H(x)$ is norm-to-weak upper semicontinuous.) The latter is an immediate corollary of the following basic observation.

PROPOSITION 1. *Under the assumptions of the theorem for any $a \le b$ and any $\delta > 0$, there is a $K \ge 0$ such that $\|x\| \le K$ for any x satisfying $a \le f(x) \le b$ and $\rho(x)(1 + \|x\|) \le \delta$.*

4. DEFORMATION TECHNIQUES

The following is a brief description of the deformation technique which is in the heart of proofs of the main theorems stated in the next section. We refer to [8–10] for the (almost equivalent) definitions of some basic concepts of "nonsmooth critical point theory" which play the central role in proofs but are not explicitly mentioned in this paper.

Denote by E_n the subspace of X spanned by all eigenvectors of L corresponding to eigenvalues λ with $|\lambda| \leq n$. Clearly, each E_n is L-invariant, $\dim E_n < \infty$, $E_n \subset E_{n+1} \subset \cdots$ and the union $\cup E_n$ is dense in E. Recall that

$$H^0(x,h) = \limsup_{u \to x, t \to 0} t^{-1}(H(u+th) - H(u))$$

is Clarke's directional derivative of H at x along h and $\partial H(x) = \{y;\ (y \mid h) \leq H^0(x,h),\ \forall\, h \in E\}$.

LEMMA 1. *Assume* (A_1), (A_2), *and* (A_5). *Let G be an open subset of X. Suppose that $\varepsilon > 0$, $r > 0$ are such that $\rho(x) > \varepsilon$ for all $x \in \bar{G} \bigcap B_r$, $x \in$ dom L. Then there is an $\bar{n} \in \mathbb{N}$ such that whenever $n \geq \bar{n}$, $\|x\| \leq r$, $x \in E_n \bigcap G$, we can find an $h \in E_n$, $\|h\| = 1$ such that*

$$(Lx \mid h) + H^0(x;h) \leq -\varepsilon. \tag{3}$$

Moreover, let $n(\varepsilon, r)$ be the minimal of such $\bar{n}$. Then the function $(\varepsilon, r) \mapsto n(\varepsilon, r)$ is upper semicontinuous on the set of pairs (ε, r) for which (3) holds.

With the help of this lemma and using similar technique as in the proof of the Basic deformation lemma of [10], we can establish the following deformation result.

DEFORMATION LEMMA. *Let $G \subset X$ be an open set. Then under the assumptions* (A_1), (A_2), *and* (A_5) *the following alternative holds for any $\delta > 0$:*

- *either there is an $x \in \bar{G} \bigcap \operatorname{dom} L$ such that $\rho(x)(1 + \|x\|) < \delta$,*
- *or for any positive $\delta' < \delta$ there is a continuous deformation $\Phi : [0,1] \times X \to X$ such that*
  - (a) *$\Phi(\lambda, x) \in$ dom L if $x \in$ dom L. Moreover, for any $r > 0$ there is an $n(r) \in \mathbb{N}$ such that $\Phi(\lambda, x) \in E_n$, whenever $n \geq n(r)$, $x \in E_n$, and $\|x\| \leq r$;*
  - (b) *$f(x) - f(\Phi(\lambda, x)) \geq \delta'(1 + \|x\|)^{-1}\|x - \Phi(\lambda, x)\|$, $\forall\, \lambda \in [0,1]$, $\forall\, x \in \operatorname{dom} L$;*
  - (c) *$\Phi(\lambda, x) = x$ if and only if $\lambda = 0$ or $x \notin G$;*

We state below three existence theorems for critical points. The proof of the first of them in which the existence of at least one critical points is stated does not actually require the deformation lemma, but the multiplicity results of Theorems 3 and 4 need already the full power of the lemma.

5. EXISTENCE THEOREMS: THE GENERAL CASE

THEOREM 2. *Under the assumptions of Theorem 1 ((*A_1*)–(*A_4*) and convexity of H), f has at least one critical point.*

If dim $X < \infty$, the proof of the theorem is the standard application of the minimax techniques of Ambrosetti-Rabinowitz, e.g., [6,13] (or rather its nonsmooth counterparts—see [8,9]) jointly with Proposition 1. In the general case, we use this fact in combination with a Galerkin-type approximation procedure.

6. EXISTENCE THEOREMS: THE CASE OF AN EVEN H

Ljusternik-Schnirelman theory based on Theorem 1 and a "symmetric" version of the Deformation lemma allow us to obtain estimates for the number of solutions in the presence of symmetry, in particular, for even functionals. So, we assume in this section that H is even: $H(x) = H(-x)$.

Let E_n be as defined in the beginning of Section 4: the invariant subspace of L spanned by all eigenspaces corresponding to eigenvalues λ with $|\lambda| \leq n$. For any linear self-adjoint operator A, we denote by $N(A)$ the dimension of the total negative space in the spectral decomposition for A and set $\overline{N}(A) = N(A) + \dim(\operatorname{Ker} A)$. Further, let P_n be the projection of H on E_n and $A_n = P_n A|_{E_n}$, the restriction of $P_n A$ to E_n.

THEOREM 3. *In addition to the assumptions of Theorem 1, we assume that H is even, C^1-smooth in a neighborhood of zero and, moreover, in this neighborhood $\nabla H(x) = Bx + o(x)$, where B is a bounded and self-adjoint linear operator in X. Set $p = \lim_{n\to\infty}(N(L_n) - \overline{N}(L_n + B_n)) > 0$ and assume that $p > 0$. Then f has at least p nontrivial distinct pairs of symmetric critical points.*

Theorem 3 is an easy consequence of a more general result stated below (if we apply it to the function $f(0) - f(x)$ instead of $f(x)$).

Let gen D denote the genus (in the standard sense, e.g., see [6,14]) of an even subset $D \subset X\backslash\{0\}$.

DEFINITION 2. *For a nonempty even set $D \subset X \cap \operatorname{dom} L$ we define the relative genus of D by*

$$\operatorname{genr} D = \limsup_{n\to\infty}[\operatorname{gen}(D \cap E_n) - \dim(E_n \cap X^-)].$$

(Observe that an alternative concept of a *limit relative category* was introduced in [15]. The interrelationship between the two, though, is not clear.)

Consider now function $f(x)$ defined by (1) with $f(0) = 0$. For any set $D \subset X \cap \operatorname{dom} L$ set

$$\tilde{f}(D) = \sup_{x\in D} f(x).$$

Denote by Γ_k^r the collection of closed bounded even subsets of the *relative* genus not smaller than k. Observe that for every integer $k \geq 0$ the collection $\Gamma_k^r \neq \emptyset$. Namely, if $S_1 = \{x : \|x\| + \|Lx\| = 1\}$ and $V \subset (X^0 + X^+)$ is a finite-dimensional subspace, then $\operatorname{genr}(S_1 \cap (V + X^-)) = \dim V$.

DEFINITION 3. *The integer $k \geq 0$ is the genus at infinity (respectively, genus at zero) of f if this is the minimal integer such $\tilde{f}$ is bounded below (respectively, nonnegative) on Γ_{k+1}^r. We shall denote the geni of f at infinity and zero by* gen f *and* Gen f, *respectively, and set* gen $f = \infty$ *(respectively,* Gen $f = \infty$*) if $\tilde{f}$ is not bounded from below (respectively, nonnegative) on Γ_k^r for any k.*

Observe that although our function f may be discontinuous (and even not everywhere defined on X), the deformation lemma allows us to apply these definitions (as well as with standard concepts of geni) exactly as if the function were continuous.

THEOREM 4. *Suppose H is even and (A_1), (A_2), and (A_5) are satisfied. Suppose also that the weighted PS-condition is satisfied at any level $c < 0$. Then inclusion (2) has at least $p =$ (Gen f − gen f) distinct pairs of nontrivial solutions at the levels $c_i = \inf_{\{D\subset\Gamma_{k+i}^r\}} \tilde{f}(D)$, $i = 1,\ldots,p$, $k =$ gen f.*

7. WEAKENING OF THE CONVEXITY REQUIREMENT

We state below two results in which convexity of H is replaced by a weaker condition. In both cases, we use the fact that under (A_2) and (A_5), f satisfies the following "*bounded PS-condition*": if $\{x_n\}$ and $\{f_n\}$ are bounded sequences and $\rho(x_n) \to 0$, then $\{x_n\}$ is precompact. Now the possibility to weaken the convexity requirements becomes almost obvious: $H(x)$ must be close enough to a convex function as $\|x\| \to \infty$ to make sure that an unbounded weighted PS-sequence cannot appear.

THEOREM 5. *Assume that (A_1)–(A_5) hold. Then the conclusions of Theorems 1 and 2 remain valid if H is "convex at infinity" in the following sense: $H(x) = \tilde{H}(x) + \varphi(x)$, where $\tilde{H}$ is convex continuous, and φ is bounded and locally Lipschitz with* Lip $\varphi(x) =\leq C/(1 + \|x\|)$ (Lip $\varphi(x)$ *being the Lipshitz constant of φ at x).*

THEOREM 6. *(See [3,5].) Consider on the segment $[0,T]$, $T > 0$ the Hamiltonian inclusion*

$$-J\dot{x} \in \partial H(t,x) \tag{4}$$

under the assumptions:

(A_6) *$H(\cdot, x)$ is summable on $[0,t]$ for every x, $H(t,\cdot)$ is locally Lipschitz in x for almost any t and $H(t,x) \to \infty$ as $|x| \to \infty$ uniformly in t;*

(A_7) *there is a positive $\alpha < \pi/T$ and a summable nonnegative function $r(t)$ such that $|H(t,x)| \leq \alpha\|x\|^2 + r(t)$;*

(A_8) *$H(t,x) = \tilde{H}(t,x) + \varphi(t,x)$, where $\tilde{H}$, as a function of x, is convex continuous with the Lipschitz constant satisfying $\operatorname{Lip}\varphi(t,x) \leq C/(1+\|x\|)$ uniformly in t.*

Then there exists a solution of (4) satisfying the periodic boundary condition $x(0) = x(T)$.

Theorem 6 does not follow from Theorem 5 and, moreover, the function $f(x(\cdot)) = \int_0^T [(1/2)(J\dot{x} \mid x) + H(t,x)]\,dt$ under the assumptions may not satisfy a suitable $(\mathrm{PS})_c$-condition on $H^{1/2}$ or $W^{1,2}$. The key element of the proof is the demonstration that the weighted (PS)-condition is satisfied for $f(\cdot)$ in the L_2-metric.

Observe finally that Theorems 5 and 6 have their counterparts in multiplicity critical point results for even functionals (in the spirit of Theorems 3 and 4). For example, combining Theorems 4 and 6, we get the following.

THEOREM 7. *We assume (A_6)–(A_8) and, in addition, that $H(t\cdot)$ is even. Suppose there are symmetric operators B_- and B_+ in $\mathbb{R}^{2n}$ such that uniformly in t,*

$$(B_-x \mid x) + o\left(\|x\|^2\right) \leq 2H(t,x) \leq (B_+x \mid x) + o\left(\|x\|^2\right)$$

in a neighborhood of zero. Consider the numbers

$$\theta_+ = \sum_{k=-\infty}^{\infty} \left(N\left(ikJ + \left(\frac{T}{2\pi}\right) B_+ \right) - \bar{N}(ikJ) \right),$$

$$\theta_- = \sum_{k=-\infty}^{\infty} \left(N(ikJ) - \bar{N}\left(ikJ + \left(\frac{T}{2\pi}\right) B_- \right) \right),$$

and suppose that $p = \max\{\theta_+, \theta_-\} > 0$. Then (4) has at least p distinct pairs of symmetric nonzero T-periodic solutions.

Observe that for $k \neq 0$, we have $N(ikJ) = \bar{N}(ikJ) = n$.

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