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FUNCTORS OF THE CATEGORY OF COMBINATORIAL GEOMETRIES AND STRONG MAPS

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The motivation of this study was to prove the following conjecture by Rota: Given a pregeometry $G(r, s)$ and an integer $k \geq 1$, the geometrization of the function kr is a functor of the category of finite pregeometries and strong maps into itself. In addition to a proof of this fact, properties of other classes of functors based on expansions and geometrizations are presented in this paper.

1. Introduction

As in the case with most mathematical structures, an important question in the theory of combinatorial geometries is to develop constructions for obtaining new geometries from old ones. Several geometric constructions are well-known, e.g. deletion, contraction, truncation, direct sums, etc. ... and are generally extensions to combinatorial geometries of existing operations on projective geometries, graphs or lattices. In this paper we introduce a new class of constructions based on the ideas of expansion and geometrization. More precisely, we will study several classes of functors of the category $\mathcal{S}$ of combinatorial pregeometries and strong maps into itself. The motivation of this study was the following conjecture of Rota: Given a pregeometry $G(r, s)$ and an integer $k \geq 1$, the geometrization of the function kr is a functor of $\mathcal{S}$ into itself. After a brief review of the basic concepts of the theory of combinatorial geometries in Section 2, the conjecture is proved in Section 3 as a consequence of a general study concerning a broader class of functors. In Section 4, a generalization is considered which yields a method for constructing new classes of functors.

2. Basic concepts

A *combinatorial pregeometry* $G(S)$, or simply a pregeometry G, is a set S together with a closure relation $A \to \bar{A}^G$ (or $\bar{A}$ if no ambiguity) for all A, $A \subset S$, which satisfies the following two axioms:

Exchange axiom: if $a, b \in S$, $A \subset S$, and $a \in \overline{A \cup b} - \bar{A}$ then $b \in \overline{A \cup a}$.

Finite basis property: if $A \subset S$, there is a finite subset $A_0 \subset A$ such that $\bar{A}_0 = \bar{A}$.

A pregeometry is a *geometry* if $\emptyset$ and all single-element subsets are closed. The *flats* of $G(S)$ are the closed subsets of S. The set of all flats of $G(S)$ ordered by inclusion is a *geometric lattice*, i.e. a semi-modular point lattice with finite rank.

A subset $A \subset S$ is *independent* if for no $a \in A$, $A \subset \overline{A - a}$. If A is not independent, then A is *dependent*. If $B \subset A \subset S$ and $A \subset \bar{B}$, we say that B *spans* A.

A *basis* of A, for $A \subset S$, is an independent subset of A which spans A. All bases of A have the same cardinality, $r(A)$, the *rank* of A. If A is finite, the *nullity* of A is $n(A) = |A| - r(A)$. The flats of $G(S)$ of rank $1, 2, r(G) - 1$, $r(G) - 2$ are called point, line, copoint, coline respectively. $r(G)$ is the *rank of G*.

A *circuit* is a minimal dependent set. A *cyclic flat* is a flat which is a union of circuits.

The pregeometry $G^*(S)$ *dual* to $G(S)$ is the pregeometry on S whose bases are $S - B$ where B is a basis of G. If r is the rank-function of G, the rank-function r^* of $G^*(S)$ is $\forall A \subset S$, $r^*(A) = |A| - r(G) + r(S - A)$. The function $n : \forall A \subset S \rightarrow n(A) = |A| - r(A)$ is the *nullity function*. $r^*(A) = n(S - A) - n(A)$.

$A \subset S$, the *subgeometry* of G defined on A, $G - A$, is the pregeometry on A whose closure relation is: $U \subset A \rightarrow \bar{U} \cap A$. The *contraction* of G by A, G/A, is the pregeometry on $S - A$, with closure: $U \subset S - A \rightarrow \overline{U \cup A} - A$. A point $x \in A$ is an *isthmus* of G if $r(G - x) = r(G) - 1$. x is a *loop* of G if $r(G/s) = r(G)$. x is a *relative isthmus* of a set $A \subset S$ if $r(A - x) = r(A) - 1$.

Given two geometric lattices L_1 and L_2, a *strong map* from L_1 to L_2 is a function $\sigma : L_1 \rightarrow L_2$ which is supremum-preserving and cover-preserving. A strong map σ between two pregeometries G(S) and $H(T)$ is a strong map between the corresponding geometric lattices of flats of $G(S)$ and $H(T)$. With the expedient of adjoining a point 0 to each point set S and T, σ determines a function $\bar{\sigma}$ from the point set $S \cup 0$ to the point set $T \cup 0$, with $\bar{\sigma}(0) = 0$. $\bar{\sigma}$ is said to *extend* to the strong map σ. A function $\bar{\sigma}$ from $S \cup 0$ to $T \cup 0$ such that $\bar{\sigma}(0) = 0$ extends to a strong map $G(S)$ to $H(T)$ if and only if the inverse image of any flat of $H(T)$ is a flat of $G(S)$. When the identity extends to a strong map from $G(S)$ to $H(S)$, H is called a *quotient* of G and G is a *lift*. The category whose objects are pregeometries and morphisms are strong maps is called $\mathcal{S}$.

S being a finite set, a real-valued function f defined on the power set of S is *semi-modular* if and only if:

$$\forall A, \quad B \subset S, \quad f(A) + f(B) \geq f(A \cap B) + f(A \cup B).$$

Given an integer-valued. semi-modular, non decreasing function f on S, the pregeometry defined by the following family of independent sets

$$\{I : I \subset S, \forall I' \subset I, I' \neq \emptyset, |I'| \leq f(I')\}$$

is called the *geometrization* of f and denoted $G(f, S)$. The rank-function of $G(f, S)$ is r:

$$\forall T \subset S,\quad T \neq \emptyset,\quad r(T) = \inf_{A \subset T}\{f(A) + |T - A|\}.$$

Given an integer-valued, semi-modular, non decreasing, non negative function f on S, the *free expansion* (or simply *expansion* in this paper) of f is the pregeometry $E(f)$ defined (1) on the set $X = \dot{\bigcup}_{a \in S} X_a$ where $|X_a| = f(a)$ (*notation*: $\forall A \subset S$, $X_A = \bigcup_{a \in A} X_a$), (2) by the following family of independent sets:

$$\{I : I \subset X, \forall A \subset S, |I \cap X_A| \leq f(A)\}.$$

The rank function of $E(f)$ is

$$\rho : \forall T \subset X, \rho(T) = \inf_{A \subset S}\{f(A) + |T - X_A|\}.$$

In particular $\forall A \subset S$, $\rho(X_A) = f(A)$. s is the *natural surjection* of X onto S: $\forall T \subset X$, $s(T) = \{a : a \in S, X_a \cap T \neq \emptyset\}$. The fundamental relationship [8] between $G(f, S)$ and $E(f)$ is that $G(f, S)$ is the subgeometry of $E(f)$ defined on a set $Y \subset X$ such that $\forall a \in S$, $|Y \cap X_a| = 1$.

Notation:
S is a finite set,
f is a semi-modular, integer-valued, non decreasing, non negative function, defined on S,
$G(f, S)$ is the geometrization of f,
$G(r, S)$ is the pregeometry on S with rank-function r,
$E(f)$ is the expansion of f, defined on the set $X = \dot{\bigcup}_{a \in S} X_a$,
$\forall A \subset S$, $X_A = \bigcup_{a \in A} X_a$,
ρ is the rank-function of $E(f)$, i.e. $E(f) = G(\rho, X)$,
$\forall T \subset X$, $s(T) = \{a : a \in S, X_a \cap T \neq \emptyset\}$,
$\bar{A}^G$ is the closure of A in the pregeometry G (or simply $\bar{A}$ if there is no ambiguity).
$\mathcal{S}$ is the category of pregeometries and strong maps.

We will use the simplified set notation: $A \cup e$, $A - e$, etc. for $A \cup \{e\}$, $A - \{e\}$, etc. ...

3. A class of functors of $\mathcal{S}$ into itself

Given a pregeometry $G(r, S)$ and an integer $k \geq 1$, we define $E_k(G)$ to be the expansion of the function kr. The motivation in this section is first to study some properties of E_k, in particular to characterize the flats of $E_k(G)$, and finally to show that E_k defines a functor of $\mathcal{S}$ into itself.

The following general properties of the expansions of a semi-modular function will be needed.

Lemma 3.1. *If K is a circuit of $E(f)$, then $\rho(K) = f(s(K))$.*

Proof. K is dependent in $E(f)$: $\exists A \subset S$ such that $|K \cap X_A| > f(A)$. On the other hand, $\forall x \in K$, $K - x$ is independent and $|(K - x) \cap X_A| \leq f(A)$. Thus necessarily, $K \subset X_A$ and $|X \cap X_A| = f(A) + 1$. Now consider $s(K)$: $K \subset X_{s(K)} \cdot s(K)$ being the smallest subset T of S such that $K \subset X_T$, we have $s(K) \subset A$. If $|K \cap X_{s(K)}| = |K| \leq f(s(K))$ we would have $|K| \leq f(A)$ which is a contradiction. Thus $|K \cap X_{s(K)}| > f(s(K))$ and by repeating the above argument used with A, we can show that $|K| = |K \cap X_{s(K)}| = f(s(K)) + 1$.

Lemma 3.2. *If K is a circuit of $E(f)$, then for any $x \in K$, we have $X_{s(x)} \subset \bar{K}$.*

Proof. Excluding the trivial case when $K = \{x\}$ is a loop, suppose that $X_{s(x)} = \{x_1, x_2, x_3, \ldots, x_m\}$ $(f(s(x)) = m \geq 1)$ and that $\{x_1, x_2, \ldots, x_i\} \subset K$, $1 \leq i < m$; we want to show that $x_j \in \bar{K}$, for $j = i + 1, \ldots, m$.

Consider $y \in K$, $y \notin X_{s(x)}$ and x_j, $i + 1 \leq j \leq m$.
$K - y$ is independent in $E(f)$ whereas $(K - y) \cup x_j$ is dependent because

$$|[(K - y) \cup x_j] \cap X_{s(K)}| = |K \cap X_{s(K)}| > f(s(K)).$$

Thus $x_j \in \overline{K - y} = K$.

3.1. *Circuits and cyclic flats of $E_k(G)$*

From the general properties of the expansion of a semi-modular function, $E_k(G)$ is defined on a set $X = \dot{\bigcup}_{a \in S} X_a$ where $|X_a| = kr(a)$, with the following family of independent set $\{I : I \subset X, \forall A \subset S, |I \cap X_A| \leq kr(A)\}$ and the rank function

$$r_k : \forall T \subset X, r_k(T) = \inf_{A \subset S} \{kr(A) + |T - X_A|\}.$$

The following results are easy to prove: a set X_A is independent if and only if A is independent, and X_A is closed if and only if A is closed.

In general, the circuits and flats of an expansion $E(f)$ have no neat characterization but in the case of $E_k(G)$ it turns out that we have the following:

Theorem 3.3. *A subset F of X is a cyclic flat of $E_k(G)$ if and only if $s(F)$ is a cyclic flat of G and $F = X_{s(F)}$.*

In order to prove Theorem 3.3 we need the following intermediary results:

Proposition 3.4. *If C is a circuit of G, any subset $K \subset X_C$ such that $|K| = kr(C) + 1$ is a circuit of $E_k(G)$.*

Proof. As $E_k(G)$ is the expansion of kr, we have $r_k(X_C) = kr(C)$ and any subset $K \subset X_C$, $|K| = kr(C) + 1$ is dependent. To prove Proposition 3.4, we have to show that any subset $T \subset X_C$, $|T| = kr(C)$ is independent in $E_k(G)$. Let $T \subset X_C$ and $|T| = kr(C)$. $\forall A \subset S$, $|T \cap X_A| = |T \cap X_{A \cap C}|$. If $C \cap A \neq C$, then $C \cap A$ is independent in G and

$$|T \cap X_{A\cap C}| \leq |X_{A\cap C}| \leq k\,|A \cap C| = kr(A \cap C) \leq kr(A).$$

If $C \cap A = C$ then

$$|T \cap X_{A\cap C}| = |T \cap X_C| = |T| = kr(C) = kr(A \cap C) \leq kr(A).$$

In all cases we have $|T \cap X_A| \leq kr(A)$ and thus T is independent in $E_k(G)$.

Lemma 3.5. *If K is a circuit of $E_k(G)$, $\forall x \in K$, $s(x)$ is not a relative isthmus of $s(K)$.*

As a consequence of Lemma 3.5, we have:

Proposition 3.6. *The natural surjection extends to a strong map from $E_k(G)$ to G.*

Proposition 3.7. *If K is a circuit of $E_k(G)$, then*

$$\bar{K} = X_{\overline{s(K)}}.$$

Proof. By Lemma 3.2 we know that $X_{s(K)} \subset \bar{K}$.

Let $a \in S$, $a \in \overline{s(K)}$: there is a circuit C in G such that $C - a \subset s(K)$ and $a \in C$. By Proposition 3.4, $\forall T \subset X_C$, $|T| = kr(C) + 1$, T is a circuit of $E_k(G)$; then for any element $a_1 \in X_a$, $L = X_{C-a} \cup a_1$ is a circuit of $E_k(G)$,

$$X_{C-a} \subset X_{s(K)} \Longrightarrow \bar{X}_{C-a} \subset \overline{X_{s(K)}} \subset \bar{K}.$$

By Lemma 3.2,

$$X_a \subset \bar{L} = \bar{X}_{C-a} \Longrightarrow X_a \subset \bar{K}.$$

Thus $\forall a \in \overline{s(K)}$, $X_a \subset \bar{K} \Longrightarrow X_{\overline{s(K)}} \subset \bar{K}$.

Conversely, let $x \in \bar{K}$: there is a circuit L of $E_k(G)$ such that $L - x \subset K$, $x \in L$.

By Lemma 3.5, $s(x)$ is not an isthmus of $s(L)$: there exists a circuit C containing $s(x)$ and $C \subset s(L)$, but

$$L - x \subset K \Longrightarrow s(L) - s(x) \subset s(K) - s(x)$$

and

$$\bar{C} = \overline{C - s(x)} \Longrightarrow s(x) \in \overline{C - s(x)} = \bar{C} \subset \overline{s(L)} \subset \overline{s(K)} \Longrightarrow x \in X_{s(x)} \subset X_{\overline{s(K)}}.$$

Finally $X_{\overline{s(K)}} = \bar{K}$.

A description of all cyclic flats of $E_k(G)$ is at hand. Let F be a cyclic flat of $E_k(G)$: F is the union of closures of circuits contained in F, say $F = \bigcup_{i=1}^{m} \bar{K}_i$ where each K_i is a circuit of $E_k(G)$:

$$F = \bigcup_{i=1}^{m} \bar{K}_i = \bigcup_{i=1}^{m} X_{\overline{s(K_i)}} = X_{\bigcup_{i=1}^{m} \overline{s(K_i)}} = X_{s(F)}$$

$X_{s(F)}$ is a flat in E_k $\Longrightarrow$ $s(F)$ is a flat in G. $\forall x \in S$, $x \in \bigcup_{i=1}^{m} s(K_i)$, by Lemma 3.5, x is not an isthmus of $\bigcup_{i=1}^{m} s(K_i)$ and thus $s(F)$ does not contain any isthmus. $s(F)$ is a cyclic flat of G.

Conversely, let H be a cyclic flat of G, say $H = \bigcup_{i=1}^{p} \bar{C}_i$ where C_i is a circuit of G, $i = 1, \ldots, p$. $X_{\bar{C}_i}$ is a cyclic flat in $E_k(G)$ because, by Proposition 3.4, there is a circuit K_i of $E_k(G)$ such that $s(K_i) = C_i$ and then

$$\bar{K}_i = X_{\overline{s(K_i)}} = X_{\bar{C}_i}.$$

H being a flat in G, X_H is a flat in $E_k(G)$, as

$$X_H = X_{\bigcup_{i=1}^{p} \bar{C}_i^G} = \bigcup_{i=1}^{p} X_{\bar{C}_i^G} = \bigcup_{i=1}^{p} \bar{K}_i^{E_k},$$

X_H is a cyclic flat of E_k. Theorem 3.3 is thus proved.

3.2. *Properties of the functor E_k*

The transformation E_k defined above induces an action on morphisms of $\mathcal{S}$ as follows. Given a strong map σ from $G_1(r_1, S_1)$ to $G_2(r_2, S_2)$, with the classical convention of adding a point 0 to S_1 and S_2, the map σ induces a function of the set $S_1 \cup 0$ to $S_2 \cup 0$ (by definition $\sigma(0) = 0$ and $0 \in \bar{\emptyset}$). $E_k(G_1)$ is a geometry on a set X_1, $E_k(G_2)$ is defined on X_2 and we consider the following function σ_k from $X_1 \cup 0$ to $X_2 \cup 0$:

(i) for any $a \in S_1$, X_a has either no element if $r_i(a) = 0$ or k elements if $r_i(A) = 1$, in the latter case we can write $X_a = \{a_1, a_2, \ldots, a_k\}$;

(ii) for any element a of S_1, if $\sigma(a)$ is 0 we set $\sigma_k(a_i) = 0$, $\forall a_i \in X_a$ and if $\sigma(a) = \alpha \in S_2$, we set $\sigma_k(a_i) = \alpha_i$, $\forall a_i \in X_a$, where $X_\alpha = \{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k\}$.

We will say that σ induces the function σ_k from $X_1 \cup 0$ to $X_2 \cup 0$. For E_k to be a functor, among other requirements, we need to prove that the function σ_k extends to a strong map from $E_k(G_1)$ to $E_k(G_2)$, i.e. that we have the following:

Theorem 3.8. *σ being a strong map from $G_1(r_1, S_1)$ to $G_2(r_2, S_2)$, the point map σ_k induced by σ extends to a strong map $E_k(\sigma)$ from $E_k(G_1)$ to $E_k(G_2)$.*

We will first consider 2 special cases: first when σ is an embedding and then when σ is a single point contraction.

Proposition 3.9. *If $G_1(r_1 S_1)$ is a subgeometry of $G_2(r_2, S_2)$, $E_k(G_1)$ is a subgeometry of $E_k(G_2)$.*

Proposition 3.10. *If $G_2(r_2, S_2)$ is the contraction of $G_1(r_1, S_1)$ by a point $e \in S_1$, then $E_k(G_2)$ is the contraction of $E_k(G_1)$ by the set X_e.*

Proof. We will suppose that e is not a loop of G.

As $S_2 = S_1 - e$, the geometry $E_k(G_2)$ is defined on the set $X_2 = X_1 - X_e$: let σ_k be the map induced by the contraction $G_1 \to G_1/e = G_2$ from $X_1 \cup 0$ to $X_2 \cup 0$: $\sigma_k(X_e) = 0$.

Let F be a flat of $E_k(G_2)$. We want to show that $\sigma_k^{-1}(F) = F \cup X_e$ is a flat of $E_k(G_1)$. If it is not the case, there exist an $x \in X$ and a circuit K of $E_k(G_1)$, $K - x \subset F \cup X_e$ with $x \in K$, $x \notin F \cup X_e$.

$x \in K \implies s(x)$ is not a relative isthmus in $s(K)$ (Lemma 3.5). In the contraction by e, $s(x)$ does not become a relative isthmus of $s(K) - e$ in G_1/e so there is a circuit L of G_1/e such that $s(x) \in L$, $L \subset s(K) - e \subset s(F)$. $L \subset s(F)$ and F being a flat of $E_k(G_2)$ we have by Lemma 3.2 and Theorem 3.3,

$$x \in X_{s(x)} \subset \bar{X}_L^{E_k(G_2)} \subset F$$

which is a contradiction.

Thus $F \cup X_e$ is a flat of $E_k(G_1)$ and consequently F is a flat of $E_k(G_1)/X_e$: $E_k(G_2)$ is thus a quotient of $E_k(G_1)/X_e$. Furthermore, it is clear that $E_k(G_2)$ and $E_k(G_1)/X_e$ are geometries with the same rank so finally, $E_k(G_2) = E_k(G_1)/X_e$.

In order to prove Theorem 3.8, we will invoke the fundamental result of Higgs [7] stating that any strong map can be decomposed into a surjection followed by an injection: if σ is a strong map from G_1 to G_2, then $\sigma = i \circ s$ where

$$G_1 \xrightarrow{s} Q = \sigma(G_1) \xrightarrow{i} G_2, \qquad G_1 \xrightarrow{\sigma} G_2$$

s is a surjection of G_1 into a quotient Q of G_1 and i is an embedding of $\sigma(G_1)$ (isomorphic to Q) into G_2. ($\sigma(G_1)$ is a sybgeometry of G_2.)

Furthermore, the surjection $G_1 \overset{s}{\dashrightarrow} Q$ can be decomposed into a sequence of elementary quotients:

$$G_1 \xrightarrow{s_1} Q_1 \xrightarrow{s_2} Q_2 \longrightarrow \cdots \xrightarrow{s_k} Q_k = Q.$$

Any elementary quotient is decomposed further as an embedding into a single-element extension followed by the contraction by that element.

Any strong map is thus decomposed into the products of single-element contractions and extensions. We know that Theorem 3.8 holds for those elementary strong maps. It is easy to see that Theorem 3.8 still holds when we compose elementary strong maps (this is a direct consequence of rule for composing strong maps) and thus Theorem 3.8 holds in general.

Let us consider the map F of $\mathscr{S}$ into itself defined as follows:

(i) for any pregeometry $G(r, S)$, $F(G) = E_k(G)$;

(ii) for any morphism $\sigma \in \mathrm{Hom}(G, G')$, $F(\sigma) = E_k(\sigma) \in \mathrm{Hom}[E_k(G), E_k(G')]$.

Theorem 3.11. *F is a faithful functor of $\mathscr{S}$ into itself.*

Proof. F is a functor because

(a) if i_G is the identity of $G(r, S)$, clearly

$$F(i_G) = E_k(i_G) = i_{E_k}(G),$$

(b) if a morphism σ is the composition $\sigma = \sigma_1 \circ \sigma_2$ then

$$F(\sigma) = F(\sigma_1 \circ \sigma_2) = E_k(\sigma_1 \circ \sigma_2) = E_k(\sigma_1) \circ E_k(\sigma_2) = F(\sigma_1) \circ F(\sigma_2).$$

F is faithful because

$$F(\sigma) = F(\sigma') \Leftrightarrow E_k(\sigma) = E_k(\sigma') \Leftrightarrow \sigma_k = \sigma'_k \Leftrightarrow \sigma = \sigma'.$$

In the following we will investigate some further properties of E_k. A remarkable result is the following:

Theorem 3.12. *If $G^*(r^*, S)$ is the dual of a given pregeometry $G(r, S)$ which has no isthmuses or loops, then $E_k(G^*)$ is the dual of $E_k(G)$.*

Proof. We want to show that the following diagram is commutative:

$$\begin{array}{ccc} G & \overset{*}{\longleftrightarrow} & G^* \\ {\scriptstyle s}\uparrow & & \uparrow{\scriptstyle s^*} \\ E_k(G) & \overset{*}{\longleftrightarrow} & E_k(G^*) \end{array}$$

First we note that $E_k(G)$ and $E_k(G^*)$ are defined on the same set X, $|X| = k|S|$ (because $\forall a \in S$, $r(a) = r^*(a) = 1$).

The system of bases of $E_k(G)$ is

$$\mathscr{B} = \{B : B \subset X, |B| = kr(S), \forall A \subset S, |B \cap X_A| \leq kr(A)\}$$

and the family of bases of $E_k(G^*)$ is

$$\mathscr{B}^* = \{B^* : B^* \subset X, |B^*| = kr^*(S), \forall A \subset S, |B \cap X_A| \leq kr^*(A)\}.$$

Call $\mathscr{B}' = \{B' : B' = X - B, B \in \mathscr{B}\}$. We want to show that $\mathscr{B}' = \mathscr{B}^*$.

$$\forall B' \in \mathscr{B}', \forall A \subset S,$$

let $A' = S - A$. We have

$$|B' \cap X_A| = |(X - B) \cap (X - X_{A'})| = |X - (X_{A'} \cup B)|$$
$$= |X| - |X_{A'} \cup B| = |X| - |X_{A'}| - |B| + |X_{A'} \cup B|$$

but $B \in \mathscr{B} \implies |X_{A'} \cup B| \leq kr(A')$, so

$$|B' \cap X_A| \leq k|S| - k|A'| - kr(S) + kr(A')$$
$$\leq kn(S) - kn(A') \quad (n = \text{nullity function})$$
$$\leq kn(S) - kn(S - A) = kr^*(A)$$

thus $B' \in \mathscr{B}^* \implies \mathscr{B}' \subset \mathscr{B}^*$.

If we now call $\mathscr{B}^{*\prime} = \{B^{*\prime}: B^{*\prime} = X - B^*, B^* \in \mathscr{B}^*\}$ because of reciprocity of G and G^*, we would have

$$\mathscr{B}^{*\prime} \subset \mathscr{B} \Longrightarrow \mathscr{B}^* \subset \mathscr{B}',$$

finally

$$\mathscr{B}' = \mathscr{B}^*.$$

We give without proofs some other interesting properties of E_k:

Theorem 3.13. *TG being the truncation of a given pregeometry $G(r, S)$, $E_k(TG)$ is the k-truncation of $E_k(G)$ (i.e., obtained by k successive truncations of $E_k(G)$).*

Corollary 3.14. *If G is an erection of G', $E_k(G)$ is an erection of $E_k(G')$.*

Theorem 3.15. *If $G = G_1 \oplus G_2$ then $E_k(G) = E_k(G_1) \oplus E_k(G_2)$.*

3.3. *Properties of the functor G_k*

Given a pregeometry $G(r, S)$, for any integer $k \geq 1$, we consider the geometrization of kr, $G(kr, S)$, which we will write $G_k(G)$.

$G_k(G)$ is a subgeometry of $E_k(G)$ and expectedly, we will be able to derive some properties of G_k which are similar to those of E_k, and prove the initial conjecture. As a consequence of Proposition 3.6 we have

Proposition 3.16. *G is a quotient of $G_k(G)$.*

Theorem 3.17. *Let σ be a strong map from $G_1(r_1, S_1)$ to $G_2(r_2, S_2)$, then σ induces a function from $S_1 \cup 0$ to $S_2 \cup 0$ which extends to a strong map $G_k(\sigma)$ from $G_k(G_1)$ to $G_k(G_2)$.*

As for Theorem 3.8, the proof of Theorem 3.17 reduces to showing that it holds in the cases when σ is an embedding and when σ is a single-element contraction.

Proposition 3.18. *Let $G_1(r_1, S_1)$ be a subgeometry of $G_2(r_2, S_2)$, then $G_k(G_1)$ is a subgeometry of $G_k(G_2)$.*

Proposition 3.19. *Let $G_2(r_2, S_2)$ be the contraction of $G_1(r_1, S_1)$ by a point $e \in S$: the function induced by the contraction from $S_1 \cup 0$ to $S_2 \cup 0$ extends to a strong map from $G_k(G_1)$ to $G_k(G_2)$.*

Proof. To prove the proposition directly is not easy. We will use the fact that $G_k(G)$ is a subgeometry of $E_k(G)$ by the means of the following lemma:

Lemma 3.20. *Given a pregeometry $H(S)$ and a subgeometry $H(T)$ of $H(S)$, $T \subset S$, let $H(S)/A$ be the contraction of $H(S)$ by the set $A \subset S$ and $H'(T')$ be the* *subgeometry of $H(S)/A$ defined on the set $T' = T \cap (S - A) = T - A$. Then the contraction $H(S) \to H(S)/A$ induces a function from $T \cup 0$ to $T' \cup 0$ which extends to a strong map from $H(T)$ to $H'(T')$.*

Proof. We have to show that given any flat F of $H'(T')$, the set $F \cup (A \cap T)$ is a flat of $H(T)$.

As $H'(T')$ is a subgeometry of $H(S)/A$, there is a flat F' of $H(S)/A$ such that $F \subset F'$ and $F' \cap T' = F$. Then $F' \cup A$ is a flat in $H(S)$ and $(F' \cup A) \cap T$ is a flat of $H(T)$. The proof is completed by noting that:

$$(F' \cup A) \cap T = (F' \cap T) \cup (A \cap T) = F \cup (A \cap T).$$

We can now prove Proposition 3.19: let us consider $E_k(G_1)$ and $E_k(G_2)$: by Proposition 3.10, $E_k(G_2)$ is the contraction of $E_k(G_1)$ by the set X_e. $G_k(G_1)$ is the subgeometry of $E_k(G_1)$ defined on the set $Y_1 \subset X_1$ ($\forall a \in S$, $|Y_1 \cap X_a| = 1$), and $G_k(G_2)$ is the subgeometry of $E_k(G_2)$ defined on $Y_2 = Y_1 - Y_e$. By the above lemma, the function of $Y_1 \cup 0$ to $Y_2 \cup 0$ induced by the contraction by X_e, extends to a strong map of $G_k(G)$ to $G_k(G')$.

Theorem 3.17 is thus proved: to any strong map σ from $G_1(r_1)$ to $G_2(r_2)$ is associated a strong map $G_k(\sigma)$ from $G_k(G_1)$ to $G_k(G_2)$. The following diagram is commutative in $\mathscr{S}$:

$$\begin{array}{ccc} G_1(r_1) & \xrightarrow{\sigma} & G_2(r_2) \\ {\scriptstyle id}\uparrow & & \uparrow{\scriptstyle id} \\ G_k(G_1) & \xrightarrow{G_k(\sigma)} & G_k(G_2) \end{array}$$

id is the strong map induced by the identity on the ground set.

As for E_k, the functor F' defined by:

$$\forall \text{ pregeometry } G(r, S),\ F'(G) = G_k(G),$$

$$\forall \sigma \in \operatorname{Hom}(G, G'), F'(\sigma) = G_k(\sigma)$$

is faithful.

$G_k(G)$ being a subgeometry of $E_k(G)$, the following results hold:

(i) if $G = G_1 \oplus G_2$ then $G_k(G) = G_k(G_1) \oplus G_k(G_2)$;

(ii) if $G'(r', S')$ is the truncation of $G(r, S)$, then $G_k(G')$ is obtained from $G_k(G)$ by a sequence of truncations.

If we consider duality, with evident notations, we have the following commutative diagram:

$$\begin{array}{ccccc} G_k(G) & \xrightarrow{id} & G(r) & \xrightarrow{id} & [G_k(G^*)] \\ \updownarrow * & & \updownarrow * & & \updownarrow * \\ G_k(G)^* & \longleftarrow & G^*(r^*) & \longleftarrow & G_k(G^*) \end{array}$$

The whole picture is clear if we consider $E_k(G)$ and its dual $[E_k(G)]^* = E_k(G^*)$: $[G_k(G)]^*$ and $G_k(G^*)$ can be considered as defined on a same set, $[G_k(G)]^*$ being a contraction of $E_k(G^*)$ whereas $G_k(G^*)$ is a subgeometry of $E_k(G^*)$.

So far, most of the results we have derived for $G_k(G)$ are consequences of properties of $E_k(G)$. The following result is particular to G_k.

Theorem 3.21. *For any pregeometry $G(r, S)$, and any two integers $k \geq k' \geq 1$, $G_{k'}(G)$ is a quotient of $G_k(G)$.*

Proof. We need the two following lemmas. We call r_k and $r_{k'}$ the rank-functions of $G_k(G)$ and $G_{k'}(G)$ and we set $k = k' + d$, $d \geq 0$.

Lemma 3.22. *If K is a circuit of $G_k(G)$ then $r_{k'}(K) = k'r(K)$.*

Proof. K being a circuit of $G_k(G)$, $|K| = kr(K) + 1$ and $r_k(K) = |K| - 1 = kr(K)$. Another way to write $r_k(K)$ is

$$r_k(K) = \inf_{A \subset K} \{kr(A) + |K - A|\}.$$

Consider K in $G_{k'}(G)$:

$$r_{k'}(K) = \inf_{A \subset K} \{k'r(A) + |K - A|\} = \inf_{A \subset K} \{kr(A) - dr(A) + |K - A|\}$$

$$\geq \inf_{A \subset K} \{kr(A) + |K - A|\} - dr(K) = r_k(K) - dr(K)$$

$$= kr(K) - dr(K) = k'r(K)$$

thus $r_{k'}(K) \geq k'r(K)$.

On the other hand, letting $A = K$ in the formula, we have $r_{k'}(K) \leq k'r(K)$ so finally $r_{k'}(K) = k'r(K)$.

Lemma 3.23. *K being a circuit of $G_k(G)$, K has no relative isthmus in $G_{k'}(G)$.*

Proof. If it is not the case, there is a point $x \in K$ such that

$$r_{k'}(K) = r_{k'}(K - x) + 1$$

so

$$r_{k'}(K - x) = \inf_{A \subset K - x} \{k'r(A) + |(K - x) - A|\}.$$

Suppose the inf is attained for a set $B \subset K - x$:

$$k'r(B) + |(K - x) - B| = r_{k'}(K - x) = r_{k'}(K) - 1 = k'r(K) - 1$$

then by adding $dr(B)$ to both sides of the equality

$$k'r(K)+dr(B)-1=kr(B)+|(K-x)-B|$$

so

$$kr(B)+|(K-x)-B|<k'r(K)+dr(B)\leq k'r(K)+dr(K)=kr(K)$$

which is a contradiction as

$$kr(K)=r_k(K)=r_k(K-x)=\inf_{A\subset K-x}\{kr(A)+|(K-x)-A|\}.$$

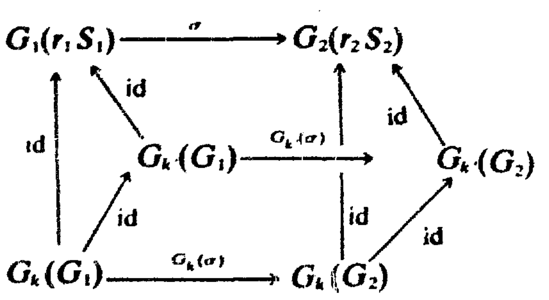
The proof of Theorem 3.21 is now straightforward. Let F be a flat of $G_{k'}(G)$: we have to show that F is a flat of $G_k(G)$; if it is not the case, there is a circuit K of $G_k(G)$ and a point $x\in K$ such that $x\notin F$ and $K-x\subset F$. But now,

$$r_k(K)=r_{k'}(K-x)\implies x\in\overline{K-x}^{G_k(G)}\subset F$$

which is a contradiction. Thus F is a flat of $G_k(G)$.

Corollary 3.24. *For any loopless pregeometry $G(r,S)$ and two integers $k\leq k'\leq 1$, $G_k(G)\equiv G_{k'}(G)$ if and only if $G_{k'}(G)$ is a Boolean algebra.*

Theorem 3.25. *Let σ be a strong map between $G_1(r_1,S_1)$ and $G_2(r_2,S_2)$, then for any two integers $k\leq k'\leq 1$, the following diagram is commutative in $\mathcal{S}_s$:*



The quotient map $G_k(G)\rightarrow G_{k'}(G)$ is a natural transformation between the functors associated with G_k and $G_{k'}$.

Proof. The proof is immediate by checking the composition of the respective induced functions.

4. A generalization

In the case of $k=2$, given a pregeometry $G(r,S)$, $E_2(G)$ is the expansion of the function $r+r$: a generalization which is interesting to investigate is to consider the expansion of $r+r'$ where r' is the rank-function of some pregeometry $G'(r',S)$ (defined on the same ground set as G) associated to G. We will use the notation

$E(G+G')$ to denote the expansion of $r+r'$. More precisely, we will prove the following:

Theorem 4.1. *If α is a functor of $\mathcal{S}$ into itself, the image of a given pregeometry $G(r,S)$ being $\alpha(G)=G'(r',S)$ the transformation associating to $G(r,S)$ the expansion $E(G+G')$ of the function $r+r'$ is a functor of $\mathcal{S}$ into itself.*

Theorem 4.1 will be proved as a consequence of the following 3 results.

Proposition 4.2. *Given 2 pregeometries $G_1(r_1,S)$ and $G_2(r_2,S)$, if H_1 and H_2 are the respective subgeometries defined on the same set $S-A$, $E(H_1+H_2)$ is a subgeometry of $E(G_1+G_2)$.*

Proposition 4.3. *Given 2 pregeometries $G_1(r_1,S)$ and $G_2(r_2,S)$, e being an isthmus of G_2, if $G_1'(r_1',S-e)$ is the contraction G_1/e of G_1 by e and $G_2'(r_2',S-e)$ is the subgeometry of G_2 on $S-e$, then $E(G_1'+G_2')$ is the contraction of $E(G_1+G_2)$ by the set X_e.*

Proof. Let $E_1=E(G_1+G_2)$, $E_1'=E_1/X_e$ and $E_2=E(G_1'+G_2')$. We will prove that E_1' is equal to E_2 by showing that E_1' and E_2 are defined by the same family of independent sets. We will exclude the trivial case when e is a loop of G_1. If X is the ground set of E_1, both E_1' and E_2 are defined on $X-X_e$. Consider an independent set $I\subseteq X-X_e$ of E_1' and suppose that it is dependent in E_2.

$\exists A\subset S-e$ such that

$$|I\cap X_A|>r_1'(A)+r_2'(A)\geqslant r_1(A)-1+r_2(A).$$

On the other hand I independent in E_1' is also independent in E_1 and

$$|I\cap X_A|\leqslant r_1(A)+r_2(A).$$

Combining the 2 inequalities, we get $|I\cap X_A|=r_1(A)+r_2(A)$ and also $r_1'(A)=r_1(A)-1$, i.e. $e\in\bar{A}^{G_1}$ or $r_1(A\cup e)=r_1(A)$.

Let ρ_1 and ρ_1' be the rank functions of E_1 and E_1' respectively:

$$\forall T\subset X-X_e,\quad \rho_1'(T)=\rho_1(T\cup X_e)-\rho_1(X_e)=\rho_1(T\cup X_e)-2.$$

We have

$$\rho_1(X_A)=r_1(A)+r_2(A)=|I\cap X_A|=\rho_1(I\cap X_A)$$

and

$$\begin{aligned}\rho_1'(I\cap X_A)&=\rho_1[(I\cap X_A)\cup X_e]-2=\rho_1(X_A\cup X_e)-2\\&=\rho_1(X_{A\cup e})-2=r_1(A\cup e)+r_2(A\cup e)-2\\&=r_1(A)+(r_2(A)+1)-2=|I\cap X_A|-1,\end{aligned}$$

which is a contradiction because I is independent in E_1' and we must have $\rho_1'(I\cap X_A)=|I\cap X_A|$. Thus I is also independent in E_2.

Conversely suppose that a set $I \subset X - X_e$ is independent in E_2 and dependent in E'_1.

I is then independent in E_1 because

$$\forall A \subset S, \quad |I \cap X_A| = |I \cap X_{A-e}| \leq r'_1(A-e) + r'_2(A-e)$$
$$\leq r_1(A) + r_2(A).$$

So for I to be dependent in E'_1 we must have

$$X_e \cap \bar{I}^{E_1} \neq \emptyset$$

or: $\exists K$, circuit of E_1, $K - X_e \subset I$, $X_e \cap K \neq \emptyset$. As $X_e = \{e_1, e_2\}$, w.l.o.g. suppose that $e_1 \in K$. Then

$$|(I \cup e_1) \cap X_{s(K)}| \geq |K| - 1 = r_1(s(K)) + r_2(s(K)) \quad \text{by Lemma 3.1}$$

and also

$$|(I \cup e_1) \cap X_{s(K)}| = |I \cap X_{s(K)}| + 1 = |I \cap X_{s(K)-e}| + 1$$
$$\leq r'_1(s(K)-e) + r'_2(s(K)-e) + 1$$
$$\text{(because } I \text{ is independent in } E_2)$$
$$= (r_1(s(K)) - 1) + (r_2(s(K)) - 1) + 1$$
$$= r_1(s(K)) + r_2(s(K)) - 1,$$

which is a contradiction and thus I is also independent in E'_1.

Propositions 4.2 and 4.3 are special cases of the following situation: 2 strong maps are given $\sigma_1 : G_1(r_1, S) \to G'_1(r'_1, S')$, $\sigma_2 : G_2(r_2, S) \to G'_2(r'_2, S')$ and they induce a point map u from $X \cup 0$ to $X' \cup 0$ where X and X' are the ground sets of $E(G_1 + G_2)$ and $E(G'_1 + G'_2)$ respectively.

For a point a of S, X_a may have 0, 1 or 2 elements as $|X_a| = r_1(a) + r_2(a)$. By convention we will write $X_a = \{a_1, a_2\}$ with $a_i \neq 0$ (i.e. a_i is a proper point of X) if and only if $r_i(a) = 1$. The same notation being used for X', u is then defined as follows: $\forall a \in S$, if $\sigma_i(a) = \alpha$ then $u(a_i) = \alpha_i$.

Propositions 4.2 and 4.3 say that in the special cases considered, u extends to a strong map from $E(G_1 + G_2)$ to $E(G'_1 + G'_2)$. More generally, we have

Proposition 4.4. *Any 2 strong maps* $\sigma_1 : G_1(r_1, S) \to G'_1(r'_1, S)$ *and* $\sigma_2 : G_2(r_2, S) \to G'_2(r'_2, S')$ *induce a strong map from* $E(G_1 + G_2)$ *to* $E(G'_1 + G'_2)$.

Proof. Using Higgs' decomposition, we can write:

$$\sigma_1 = i_1 \circ g_m \circ g_{m-1} \circ \cdots \ g_2 \circ g_1$$

where $g = g_m \circ \cdots \circ g_1$ is the product of elementary quotients bringing G_1 onto a quotient $Q = \sigma_1(G_1)$ and i_1 is an embedding of $\sigma_1(G_1)$ into G'_1:

$$G_1 \xrightarrow{g_1} Q_1 \xrightarrow{g_2} Q_2 \xrightarrow{g_3} \cdots \to Q_{m-1} \xrightarrow{g_m} Q = \sigma_1(G_1) \xrightarrow{i_1} G'_1.$$

We claim that the following diagram is commutative:

$$
\begin{array}{ccc}
G_1, G_2 & \xrightarrow{g_1, \text{id}} & g_1(G_1), G_2 \\
\downarrow & & \downarrow \\
E(G_1 + G_2) & \xrightarrow{u_1} & E(g_1(G_1) = G_2)
\end{array}
$$

where u_1 is the map induced by g_1 and the identity of G_2.

g_1, being an elementary quotient, is the composition of a single-element extension, say by a point e, giving the pregeometry $\hat{G}_1$ defined on $S \cup e$, followed by the contraction by e giving $\hat{G}_1/e = Q_1$. Let $\hat{G}_2$ be the single-element extension of G_2 with e being an isthmus; the following diagram is commutative as a consequence of Propositions 4.2 and 4.3:

$$
\begin{array}{ccccc}
G_1, G_2 & \xrightarrow{\text{embedding}} & \hat{G}_1, \hat{G}_2 & \xrightarrow{/e, /e} & Q_1, G_2 \\
\downarrow & & \downarrow & & \downarrow \\
E(G_1 + G_2) & \xrightarrow{\text{embedding}} & E(\hat{G}_1 + \hat{G}_2) & \xrightarrow{/X_e} & E(Q_1 + G_2)
\end{array}
$$

proving our claim.

By repeating the argument, we can show that the following diagram is also commutative

$$
\begin{array}{ccccccccc}
G_1, G_2 & \xrightarrow{g_1, \text{id}} & Q_1, G_2 & \xrightarrow{g_2, \text{id}} & Q_2, G_2 & \longrightarrow \cdots \longrightarrow & & \xrightarrow{g_m, \text{id}} & Q, G_2 \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
E(G_1 + G_2) & \xrightarrow{u_1} & E(Q_1 + G_2) & \xrightarrow{u_2} & E(Q_2 + G_2) & \longrightarrow \cdots \longrightarrow & & \xrightarrow{u_m} & E(Q + G_2)
\end{array}
$$

which is equivalent to the diagram:

$$
\begin{array}{ccc}
G_1, G_2 & \xrightarrow{g, \text{id}} & Q, G_2 \\
\downarrow & & \downarrow \\
E(G_1 + G_2) & \xrightarrow{u} & E(Q + G_2)
\end{array}
$$

Similarly, we have $\sigma_2 = i_2 \circ p$ where $p(G_2)$ is a quotient R of G_2 and i_2 is an embedding of $R \simeq \sigma_2(G_2)$ into G_2'. The following diagram is commutative:

$$
\begin{array}{ccc}
Q, G_2 & \xrightarrow{\text{id}, p} & Q, R \\
\downarrow & & \downarrow \\
E(Q + G_2) & \longrightarrow & E(Q + R)
\end{array}
$$

Finally, Proposition 4.4 is proved by showing that the following diagram is commutative

$$\begin{array}{ccc} Q, R & \xrightarrow{i_1, i_2} & G'_1, G'_2 \\ \downarrow & & \downarrow \\ E(Q+R) & \xrightarrow{\text{embedding}} & E(G'_1 + G'_2) \end{array}$$

which is a consequence of Proposition 4.2.

As a consequence of Proposition 4.4 given a functor α of $\mathcal{S}$ into itself, the following transformation T:

(a) $\forall$ pregeometry $G(r, S)$ whose image by α is denoted $\alpha(G) = G'(r', S)$, $T(G) = E(G + G')$,

(b) $\forall \sigma \in \mathrm{Hom}(G_1, G_2)$, $T(\sigma)$ is the element of $\mathrm{Hom}(T(G_1), T(G_2))$ induced by σ and $\alpha(\sigma)$,

is a functor of $\mathcal{S}$ into itself. Theorem 4.1 is thus proved.

As examples of functors that one can define using Theorem 4.1, one may take for α any known functor of $\mathcal{S}$ into itself: truncation, dual functor, G_k defined in Section 3, or any of the functors obtained by using Theorem 4.5 below.

Considering geometrizations as subgeometries of expansions and using the same arguments as in Section 3, one obtains:

Theorem 4.5. *If α is a functor of $\mathcal{S}$ into itself, the image of a given pregeometry $G(r, S)$ being $\alpha(G) = G'(r', S)$, the transformation associating to $G(r, S)$ the geometrization of the function $r + r'$ is a functor of $\mathcal{S}$ into itself.*

By alternating or repetitive applications of Theorems 4.1 and 4.5 one can get large sets of new functors of $\mathcal{S}$ into itself.

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