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Factorizing τ-spectra of mappings

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ABSTRACT

In this paper by a spectrum of mappings we mean a morphism of spectra of spaces. However, using the notion of a mapping of mappings, we give the definition of a spectrum of mappings similar to that of a spectrum of spaces. In this case, the formulations of the given results are also similar to the formulations of the corresponding results concerning the spectra of spaces.

For the spectra of mappings we define the notion of a τ-spectrum of mappings factorizing in a special sense and prove a version of the Spectral Theorem for such spectra. Furthermore, to a given indexed collection **F** of mapping we associate a τ-spectrum factorizing in the above special sense whose mappings are Containing Mappings for **F** constructed in Iliadis (2005) [4]. These associated τ-spectra and the corresponding version of the Spectral Theorem imply that for a given indexed collection **F** of mappings any so-called "natural" τ-spectrum for **F** factorizing in the special sense contains a cofinal and τ-closed subspectrum whose mappings are Containing Mapping for **F**. Thus, Containing Mappigs for **F** appear here without any concrete construction. The associated τ-spectra are used also in order to define and characterize the so-called second-type saturated classes of mappings (which are "saturated" by universal elements).

1. Introduction

In this introduction we recall in suitable for our consideration form some notions and results which will be used in the next sections. First we recall the notion of an inverse spectrum of spaces and some related notions. In particular, we recall the notion of a τ-spectrum factorizing with respect to a surjective subset of the limit space (that is a subset with the property that the restriction on this set of any limit projection is onto) and the corresponding version of the Spectral Theorem. As in [4], the notion of a factorizing (in the above special sense) spectrum and the Spectral Theorem are given here (Proposition 1.3) in the realm of T_0-spaces of weight $\leqslant \tau$. In the realm of completely regular spaces of weight $\leqslant \tau$ for the definition of a factorizing spectrum it is considered real functions on the limit space, that is mappings of the limit space into the real line. Here, we consider mappings into an arbitrary T_0-space. Of course, these definitions coincide in the realm of completely regular spaces. In the end of the paper we make a remark concerning the case, where the realm of spaces is the class of regular or the class of completely regular spaces of weight $\leqslant \tau$. (The notion of a factorizing τ-spectrum and the corresponding Spectral Theorem originally appear in [5]. For some (historical) comments and further bibliography see [1,2].)

Furthermore, for an indexed collection **S** of mutually disjoin spaces we recall the construction of Containing Spaces for **S** (see [4]) and the construction of the associated to **S** a τ-spectrum $\bar{D}[\mathbf{S}]$ (Proposition 1.4) whose spaces are Containing Spaces for **S**. This spectrum is factorizing with respect to a surjective subset S of the limit space (and, therefore, the limit projections of $\bar{D}[\mathbf{S}]$ are onto) such that S is homeomorphic to the free union of elements of **S**. The interesting case is that, where the cardinality of this collection is large than the weights of its elements, that is large than τ. The Spectral Theorem

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makes important the concrete factorizing τ-spectra indicating to the existence of a relation between the limit space of such a spectrum and the spaces of this spectrum.

We recall also the following result (Corollary 1.5), which follows by the Spectral Theorem and Proposition 1.4: if a τ-spectrum $\overline{D}$ on a corresponding directed set is factorizing with respect to a surjective subset S of the limit space such that S is homeomorphic to the free union of elements of an indexed collection **S** of spaces, then there exists a cofinal and τ-closed subspectrum of $\overline{D}$ whose spaces are Containing Spaces for **S**. Thus, Containing Spaces appear here without any concrete construction.

Finally, using the associated τ-spectra $\overline{\mathrm{D}}[\mathbf{S}]$ we formulate and prove a characterization of the so-called second-type saturated classes of spaces. This result is not indicated in [4].

All notions and results mentioned in this introduction will be extended in the next sections to spectra of mappings.

Agreement. The realm of spaces is the class of T_0-space of weight less than or equal to a fixed infinite cardinal τ. In particular, the spaces of the considered spectra are such spaces. However, the weight of the limit space of a spectrum may be large than τ. All mappings of spaces are assumed to be continuous.

Spectra of spaces. A system

$$\overline{D} \equiv \left\{X(\lambda), \varpi_{\lambda}^{\lambda'}, \Lambda\right\}, \tag{1.1}$$

where:

(a) Λ is a directed set whose order is denoted by $\prec$,
(b) $X(\lambda)$, $\lambda \in \Lambda$, is a space (a *space* of $\overline{D}$), and
(c) $\varpi_{\lambda}^{\lambda'}$ is a mapping of $X(\lambda')$ into $X(\lambda)$ (a *projection* of $\overline{D}$ or the *projection* of $X(\lambda')$ into $X(\lambda)$) such that $\varpi_{\lambda}^{\lambda'} \circ \varpi_{\lambda'}^{\lambda''} = \varpi_{\lambda}^{\lambda''}$ whenever $\lambda \prec \lambda' \prec \lambda''$,

is said to be an (*inverse*) *spectrum of spaces*.

By $\mathbb{P}(\overline{D})$ we denote the class consisting of all spaces which are homeomorphic to spaces of $\overline{D}$.

Let (1.1) be a spectrum. We denote by $\lim(\overline{D})$ the set of all mappings

$$p : \Lambda \to \bigcup\left\{X(\lambda):\ \lambda \in \Lambda\right\}$$

such that $p(\lambda) \in X(\lambda)$ and $\varpi_{\lambda}^{\lambda'}(p(\lambda')) = p(\lambda)$ whenever $\lambda \prec \lambda'$. By ϖ_{λ} we denote the *limit mapping* of $\lim(\overline{D})$ into X_{λ} defined by relation $\varpi_{\lambda}(p) = p(\lambda)$. The *limit space* of $\overline{D}$ is the set $\lim(\overline{D})$ with the topology for which the sets of the form $\varpi_{\lambda}^{-1}(U)$, where $\lambda \in \Lambda$ and U is an open subset of $X(\lambda)$ compose a subbasis.

A subset S of $\lim(\overline{D})$ is said to be *surjective* if $\varpi_{\lambda}(S) = X_{\lambda}$ for every $\lambda \in \Lambda$.

By a *mapping of a space* X *into the spectrum* $\overline{D}$ we mean an indexed set

$$\overline{f} \equiv \{f_{\lambda}:\ \lambda \in \Lambda\} \tag{1.2}$$

where f_{λ} is a mapping of X into X_{λ} such that $f_{\lambda} = \varpi_{\lambda}^{\lambda'} \circ f_{\lambda'}$ whenever $\lambda \prec \lambda'$. Suppose that (1.2) is a mapping of X into $\overline{D}$. Then, this set defines a mapping of X into $\lim(\overline{D})$, denoted by $\lim(\overline{f})$, as follows: if x is a point of X, then $\lim(\overline{f})(x)$ is the point p of $\lim(\overline{D})$ for which $p(\lambda) = f_{\lambda}(x)$, $\lambda \in \Lambda$. If all mappings f_{λ}, $\lambda \in \Lambda$, are homeomorphisms into (onto), then $\overline{f}$ is also a homeomorphism into (onto). The mapping $\lim(\overline{f})$ is called the *limit mapping induced by* $\overline{f}$.

Let Λ' be a directed subset of Λ (not necessarily cofinal) and $\lambda_0 \in \Lambda$ such that $\lambda \prec \lambda_0$ for every $\lambda \in \Lambda'$. We denote by $\overline{D}|_{\Lambda'}$ the *subspectrum*

$$\left\{X(\lambda), \varpi_{\lambda}^{\lambda'}, \Lambda'\right\}$$

of $\overline{D}$. The indexed set

$$\left\{\varpi_{\lambda}^{\lambda_0}:\ \lambda \in \Lambda\right\}$$

is a mapping of $X(\lambda_0)$ into the spectrum $\overline{D}|_{\Lambda'}$. By $\varpi_{\Lambda'}^{\lambda_0}$ we denote the limit mapping of $X(\lambda_0)$ into $\lim(\overline{D}|_{\Lambda'}$ induced by $\{\varpi_{\lambda}^{\lambda_0}:\ \lambda \in \Lambda\}$. This limit mapping will be called the *natural mapping* of $X(\lambda_0)$ into $\lim(\overline{D}|_{\Lambda'})$.

Now, we suppose that Λ' is a cofinal subset of Λ. In this case, each point p of $\lim(\overline{D})$ corresponds to the point p' of $\lim(\overline{D}|_{\Lambda'})$ defined by relation $p'(\lambda) = p(\lambda)$, $\lambda \in \Lambda'$. This correspondence is actually a homeomorphism of $\lim(\overline{D})$ onto $\lim(\overline{D}|_{\Lambda'})$ which is called the *natural homeomorphism*. In what follows, we shall identify the point p with the point p' and, therefore, the space $\lim(\overline{D})$ will be identified with the space $\lim(\overline{D}|_{\Lambda'})$. In the case, where Λ' is a cofinal (and τ-closed) subset of Λ, $\overline{D}|_{\Lambda'}$ will be called a *cofinal* (*and* τ*-closed*) *subspectrum* of $\overline{D}$.

Let

$$\overline{D} \equiv \{X(\lambda), \varpi_{\lambda}^{\lambda'}, \Lambda\} \quad \text{and} \quad \overline{R} \equiv \{Y(\lambda), \pi_{\lambda}^{\lambda'}, \Lambda\}$$

be two spectra of spaces. By a *morphism* of the spectrum $\overline{D}$ into the spectrum $\overline{R}$ we mean an indexed set

$$\overline{f} \equiv \{f_\lambda \colon \lambda \in \Lambda\} \tag{1.3}$$

where f_λ is a mapping of X_λ into Y_λ such that

$$f_\lambda \circ \varpi_{\lambda}^{\lambda'} = \pi_{\lambda}^{\lambda'} \circ f_{\lambda'}$$

whenever $\lambda \prec \lambda'$. In the case, where all mappings f_λ are homeomorphisms into (onto), $\overline{f}$ is called an *isomorphism into* (*onto*).

Suppose that (1.3) is a morphism of $\overline{D}$ into $\overline{R}$. This morphism defines a mapping of $\lim(\overline{D})$ into $\lim(\overline{R})$, denoted by $\lim(\overline{f})$, as follows: if $p \in \lim(\overline{D})$, then $\lim(\overline{f})(p) = q$ where q is the point of $\lim(\overline{R})$ defined by relation $q(\lambda) = f_\lambda(p(\lambda))$, $\lambda \in \Lambda$. The mapping $\lim(\overline{f})$ is called the *limit mapping induced by the morphism* $\overline{f}$. In the case, where $\overline{f}$ is an isomorphism into (onto) the limit mapping induced by $\overline{f}$ is a homeomorphism into (onto).

τ-Complete sets. Let Λ be a directed set whose order is denoted by $\prec$. The lower upper bound of a subset Λ_0 of Λ is denoted (when it exists) by $\sup(\Lambda_0)$. A subset Λ_0 of Λ is said to be a *chain* if every two elements of Λ_0 are comparable. A subset Λ' of Λ is said to be τ*-closed* in Λ if for every chain $\Lambda_0 \subset \Lambda'$ with $|\Lambda_0| \leqslant \tau$ we have $\sup(\Lambda_0) \in \Lambda'$ whenever $\sup(\Lambda_0)$ exists. The directed set Λ is said to be τ*-complete* if for every chain Λ_0 with $|\Lambda_0| \leqslant \tau$ there exists the lower upper bound of Λ_0.

The following two lemmas give the main properties of τ-complete sets which are used below. (See [5].)

Lemma 1.1. *The intersection of not more than τ many cofinal τ-closed subsets of a τ-complete set is also cofinal and τ-closed.*

Lemma 1.2. *Let Λ be a τ-complete set and R a relation on Λ (that is, $R \subset \Lambda \times \Lambda$) satisfying the following conditions:*

(1) *Existence: For every $\lambda \in \Lambda$ there exists $\lambda' \in \Lambda$ such that $(\lambda, \lambda') \in R$.*
(2) *Majorantness: if $(\lambda, \lambda') \in \Lambda$ and $\lambda' \prec \lambda'' \in \Lambda$, then $(\lambda, \lambda'') \in R$.*
(3) *τ-Closeness: if λ_1 is a fixed element of Λ and Λ' is a chain with $|\Lambda'| \leqslant \tau$ such that $(\lambda, \lambda_1) \in R$ for every $\lambda \in \Lambda'$, then $(\lambda', \lambda_1) \in R$, where $\lambda' = \sup(\Lambda')$.*

Then, the set $\Lambda^R \equiv \{\lambda \in \Lambda \colon (\lambda, \lambda) \in R\}$ (that is, the set of all R-reflexive elements of Λ) is a cofinal and τ-closed subset of Λ.

In the class of all τ-complete sets we define an equivalence relation, denoted by $\sim^\tau$, as follows: two τ-complete sets Λ_0 and Λ_1 are $\sim^\tau$*-equivalent* if they have isomorphic cofinal and τ-closed subsets. Lemma 1.1 provides that the defined relation is really an equivalence relation. If Λ is a τ-complete set, then by $\tilde{\Lambda}$ we shall denote the class of all τ-complete sets, which are $\sim^\tau$-equivalent to Λ.

Factorizing τ-spectra of spaces. ([1,4]) Suppose that (1.1) is a spectrum of spaces and $\mathbb{P}$ is a class of spaces. The spectrum $\overline{D}$ is said to be $\mathbb{P}$*-factorizing with respect to a surjective subset* S of $\lim(\overline{D})$ if for every mapping f of S into an element Y of $\mathbb{P}$ there exist an element λ of Λ and a mapping f_λ of $X(\lambda)$ into Y such that $f_\lambda \circ \varpi_\lambda = f$. It is easy to see that if $\overline{D}$ is $\mathbb{P}$-factorizing, then any cofinal subspectrum is also $\mathbb{P}$-factorizing. In the case, where $\mathbb{P}$ is the class of all spaces, then instead of "$\mathbb{P}$-factorizing" we shall write "factorizing".

The spectrum $\overline{D}$ is said to be τ*-continuous* if for every chain Λ' of Λ with $|\Lambda'| \leqslant \tau$ and $\lambda = \sup(\Lambda')$ the natural mapping $\varpi_{\Lambda'}^{\lambda}$ is an embedding of $X(\lambda)$ into $\lim(\overline{D}|_{\Lambda'})$. In what follows, whenever $\overline{D}$ is considered to be τ-continuous, then we shall consider the space $X(\lambda)$ as a subspace of $\lim(\overline{D}|_{\Lambda'})$ identifying the point x of $X(\lambda)$ with the point $\varpi_{\Lambda'}^{\lambda}(x)$ of the space $\lim(\overline{D}|_{\Lambda'})$. A τ-continuous spectrum $\overline{D}$ is said to be a τ*-spectrum* if the set Λ is τ-complete. (We recall that according to our Agreement the weights of all spaces of a τ-spectrum are $\leqslant \tau$.)

Proposition 1.3 *(The Spectral Theorem). Let $\overline{D}_0$ and $\overline{D}_1$ be two τ-spectra of spaces on the same directed set such that $\overline{D}_0$ is $\mathbb{P}(\overline{D}_1)$-factorizing with respect to a surjective subset S_0 of $\lim(\overline{D}_0)$. Then, each mapping f of S_0 into $\lim(\overline{D}_1)$ is extended to a mapping $\overline{f}$ of $\lim(\overline{D}_0)$ into $\lim(\overline{D}_1)$ such that $\overline{f}$ is induced by a morphism of cofinal and τ-closed subspectra. If moreover $\overline{D}_1$ is $\mathbb{P}(\overline{D}_0)$-factorizing with respect to a surjective subset S_1 of $\lim(\overline{D}_1)$ and f is a homeomorphism of S_0 onto S_1, then f is induced by an isomorphism onto of cofinal and τ-closed subspectra.*

Containing Spaces. ([3,4]) Let $\mathbf{S}$ be an indexed collection of spaces. We recall that a *co-mark* of $\mathbf{S}$ is an indexed set

$$\mathbf{M} \equiv \{\{U_\delta^X:\ \delta \in \tau\}:\ X \in \mathbf{S}\}$$

where $\{U_\delta^X:\ \delta \in \tau\}$ is an indexed base for the open subsets of $X \in \mathbf{S}$. For every $\delta \in \tau$ the indexed set $\{U_\delta^X:\ X \in \mathbf{S}\}$ is said to be a *component* of $\mathbf{M}$ or, more precisely, the δ-component of $\mathbf{M}$.

A family

$$\mathrm{R} \equiv \{\sim^s:\ s \in \mathcal{F}\}$$

of equivalence relations on $\mathbf{S}$, where $\mathcal{F}$ is the set of all finite subsets of τ, is said to be *admissible* if the following conditions are satisfied: (a) for every $s \in \mathcal{F}$ the number of equivalence classes of $\sim^s$ is finite, (b) $\sim^s \subset \sim^t$ if $t \subset s$ and (c) $\sim^\emptyset = \mathbf{S} \times \mathbf{S}$. The equivalence relation $\sim^s$ is called the *s-relation* of R and it is denoted also by $\sim_{\mathrm{R}}^s$.

Let $\mathbf{M}$ be a co-mark of $\mathbf{S}$. By $\mathrm{R}_{\mathbf{M}}$ we denote the admissible family of equivalence relations on $\mathbf{S}$ such that two element X and Y of $\mathbf{S}$ are $\sim_{\mathrm{R}_{\mathbf{M}}}^s$-equivalent if and only if there exists an isomorphism i of the algebra of subsets of X generated by the set $\{U_\delta^X:\ \delta \in s\}$ onto the algebra of subsets of Y generated by the set $\{U_\delta^Y:\ \delta \in s\}$ such that $i(U_\delta^X) = U_\delta^Y$, $\delta \in s$.

Let R_0 and R_1 be two admissible families of equivalence relations on $\mathbf{S}$. We say that R_1 is a *final refinement* of R_0 if for every $s \in \mathcal{F}$ there exists $t \in \mathcal{F}$ such that the t-relation of R_1 is contained in the s-relation of R_0.

An admissible family of equivalence relations on $\mathbf{S}$ is said to be $\mathbf{M}$-*admissible* if R is a final refinement of $\mathrm{R}_{\mathbf{M}}$.

Let $\mathbf{M}$ be a co-mark of $\mathbf{S}$ and R an $\mathbf{M}$-admissible family of equivalence relations on $\mathbf{S}$. On the set of all pairs (x, X) where $x \in X \in \mathbf{S}$, we define an equivalence relation, denoted by $\sim_{\mathrm{R}}^{\mathbf{M}}$, as follows: we say that two such pairs (x, X) and (y, Y) are $\sim_{\mathrm{R}}^{\mathbf{M}}$-*equivalent* if (a) the point x belongs to U_δ^X for some $\delta \in \tau$ if and only if the point y belongs to U_δ^Y for the same δ and (b) X and Y are $\sim_{\mathrm{R}}^s$-equivalent for every $s \in \mathcal{F}$.

Denote by $\mathrm{T} \equiv \mathrm{T}(\mathbf{M}, \mathrm{R})$ the set of all equivalence classes of the relation $\sim_{\mathrm{R}}^{\mathbf{M}}$. Let $\mathbf{H}$ be an equivalence class of some equivalence relation of R and $\delta \in \tau$. By $U_\delta^{\mathrm{T}}(\mathbf{H})$ we denote the subset of T consisting of all elements $\mathbf{a}$ containing a pair (x, X) for which $x \in U_\delta^X$ and $X \in \mathbf{H}$. On the set T we consider the topology for which the sets of the form $U_\delta^{\mathrm{T}}(\mathbf{H})$ compose a subbase for the open subsets. The space T is called a *Containing Space* for $\mathbf{S}$.

For every $X \in \mathbf{S}$ we denote by i_{T}^X the mapping of X into T such that for every $x \in X$, $i_{\mathrm{T}}^X(x) = \mathbf{a}$, where $\mathbf{a}$ is the point of T containing the pair (x, X). The mapping i_{T}^X is a homeomorphism of X into T.

Associated factorizing τ-spectra. ([4]) Let $\mathbf{S}$ be an indexed collection of spaces. Denote by $\mathrm{P}(\mathbf{S})$ the set of all pairs $(\mathbf{M}, \mathrm{R})$, where $\mathbf{M}$ is a co-mark of $\mathbf{S}$ and R is an $\mathbf{M}$-admissible family of equivalence relations on $\mathbf{S}$. On the set $\mathrm{P}(\mathbf{S})$ we define a preorder, denoted by $\prec_{af}^{cm}$, as follows: for two elements $(\mathbf{M}_0, \mathrm{R}_0)$ and $(\mathbf{M}_1, \mathrm{R}_1)$ of $\mathrm{P}(\mathbf{S})$ we say that $(\mathbf{M}_0, \mathrm{R}_0) \prec_{af}^{cm} (\mathbf{M}_1, \mathrm{R}_1)$ if (a) each component of $\mathbf{M}_0$ is also a component of $\mathbf{M}_1$, and (b) the family R_1 is a final refinement of R_0.

On $\mathrm{P}(\mathbf{S})$ we define also an equivalence relation, denoted by $\sim^{\mathbf{S}}$, as follows: for two elements $(\mathbf{M}_0, \mathrm{R}_0)$ and $(\mathbf{M}_1, \mathrm{R}_1)$ of $\mathrm{P}(\mathbf{S})$ we set $(\mathbf{M}_0, \mathrm{R}_0) \sim^{\mathbf{S}} (\mathbf{M}_1, \mathrm{R}_1)$ if $(\mathbf{M}_0, \mathrm{R}_0) \prec_{af}^{cm} (\mathbf{M}_1, \mathrm{R}_1)$ and $(\mathbf{M}_1, \mathrm{R}_1) \prec_{af}^{cm} (\mathbf{M}_0, \mathrm{R}_0)$.

Denote by $\mathrm{C}(\mathbf{S})$ the set of all equivalence classes of the relation $\sim^{\mathbf{S}}$. On the set $\mathrm{C}(\mathbf{S})$ we define an order, denoted by $\prec \equiv \prec^{\mathbf{S}}$, as follows: for two elements c_0 and c_1 of $\mathrm{C}(\mathbf{S})$ we write $c_0 \prec c_1$ if $(\mathbf{M}_0, \mathrm{R}_0) \prec_{af}^{cm} (\mathbf{M}_0, \mathrm{R}_1)$ where $(\mathbf{M}_0, \mathrm{R}_0) \in c_0$, and $(\mathbf{M}_1, \mathrm{R}_1) \in c_1$.

For every $c \in \mathrm{C}(\mathbf{S})$ we denote by $\mathrm{T}(c)$ the Containing Space $\mathrm{T}(\mathbf{M}, \mathrm{R})$, where $(\mathbf{M}, \mathrm{R}) \in c$. The space $\mathrm{T}(c)$ is independent of elements of c.

For every $c, c' \in \mathrm{C}(\mathbf{S})$ with $c \prec c'$ we define a mapping $p_c^{c'}$ of $\mathrm{T}(c')$ onto $\mathrm{T}(c)$ as follows: if $\mathbf{a} \in \mathrm{T}(c')$ and $(x, X) \in \mathbf{a}$, then we set $p_c^{c'}(\mathbf{a}) = \mathbf{b}$ where $\mathbf{b}$ is the point of $\mathrm{T}(c)$ containing the pair (x, X). The mapping $p_c^{c'}$ is called the *natural mapping* of $\mathrm{T}(c')$ onto $\mathrm{T}(c)$.

Let $\mathbf{S}$ be an indexed collection of mutually disjoin spaces. By a *free union* of elements of $\mathbf{S}$ we mean the set X which is the union of all elements of $\mathbf{S}$ equipped with the topology defined as follows: a subset U of X is open in X if and only if the intersection of U with any element of $\mathbf{S}$ is open in this element.

Proposition 1.4. *([4]) For every indexed collection $\mathbf{S}$ of mutually disjoin spaces the system*

$$\overline{\mathrm{D}}[\mathbf{S}] \equiv \{\mathrm{T}(c), p_c^{c'}, \mathrm{C}(\mathbf{S})\} \tag{1.4}$$

is a τ-spectrum (the associated to $\mathbf{S}$ τ-spectrum) whose the limit space contains a surjective subest S which is homeomorphic to the free union of elements of $\mathbf{S}$ such that $\overline{\mathrm{D}}[\mathbf{S}]$ is factorizing with respect to S

Definition. Let $\mathbf{S}$ be an indexed collection of mutually disjoin spaces. Any τ-spectrum

$$\overline{D} \equiv \{X_c, \varpi_c^{c'}, \mathrm{C}\}$$

with $\mathrm{C} \in \widetilde{\mathrm{C}}(\mathbf{S})$ whose the limit space contains a surjective subset S which is homeomorphic to the free union of elements of $\mathbf{S}$, is said to be a *natural τ-spectrum for* $\mathbf{S}$. The subset S is said to be a *kernel* of $\lim(\overline{D})$. If moreover $\overline{D}$ is factorizing (respectively, $\mathbb{P}$-factorizing, where $\mathbb{P}$ is a class of spaces) with respect to S, then $\overline{D}$ is said to be *kernel factorizing* (respectively, *kernel $\mathbb{P}$-factorizing*) natural τ-spectrum.

Corollary 1.5. *([4]) Let* **S** *be an indexed collection of mutually disjoin spaces. Then, each kernel factorizing natural τ-spectrum for* **S** *contains a cofinal and τ-closed subspectrum whose spaces are Containing Spaces for* **S**.

Definition. A class $\mathbb{P}$ of spaces is said to be a *second-type saturated class* if for every indexed collection **S** of mutually disjoin elements of $\mathbb{P}$ the associated τ-spectrum $\overline{\mathrm{D}}[\mathbf{S}]$ contains a cofinal and τ-closed subspectrum whose spaces belong to $\mathbb{P}$.

Proposition 1.6. *A class $\mathbb{P}$ of spaces is second-type saturated if and only if for every indexed collection* **S** *of mutually disjoin elements of $\mathbb{P}$ each kernel $\mathbb{P}$-factorizing natural τ-spectrum for* **S** *contains a cofinal and τ-closed subspectrum whose spaces belong to $\mathbb{P}$ or, equivalently, if and only if for every indexed collection* **S** *of mutually disjoin elements of $\mathbb{P}$ there exists a kernel factorizing natural τ-spectrum for* **S** *containing a cofinal and τ-closed subspectrum whose spaces belong to $\mathbb{P}$.*

Proof. Let $\mathbb{P}$ be a second-type saturated class of spaces and **S** an indexed collection of mutually disjoin elements of $\mathbb{P}$. Then, $\overline{\mathrm{D}}[\mathbf{S}]$ contains a cofinal and τ-closed subspectrum

$$\overline{\mathrm{D}}_0[\mathbf{S}] \equiv \left\{\mathrm{T}(c), p_c^{c'}, \mathrm{C}_0\right\}$$

such that $\mathrm{T}(c) \in \mathbb{P}$ for every $c \in \mathrm{C}_0$. Let now

$$\overline{D} \equiv \left\{X_c, \varpi_c^{c'}, \mathrm{C}\right\}$$

be a kernel $\mathbb{P}$-factorizing natural τ-spectrum for **S**. Then, $\mathrm{C} \in \widetilde{\mathrm{C}}(\mathbf{S})$ which means that there exist isomorphic cofinal and τ-closed subsets C' and C'_0 of C and $\widetilde{\mathrm{C}}(\mathbf{S})$, respectively. Without loss of generality we can suppose that $\mathrm{C}' = \mathrm{C}'_0$. By Lemma 1.1 the set $\mathrm{C}'' = \mathrm{C}'_0 \cap \mathrm{C}_0$ is simultaneously cofinal and τ-closed subset of C and $\mathrm{C}(\mathbf{S})$. Consider the cofinal and τ-closed subspectra

$$\overline{D}' \equiv \left\{X_c, \varpi_c^{c'}, \mathrm{C}''\right\} \quad \text{and} \quad \overline{\mathrm{D}}'_0[\mathbf{S}] \equiv \left\{\mathrm{T}(c), p_c^{c'}, \mathrm{C}''\right\}$$

of $\overline{D}$ and $\overline{\mathrm{D}}_0[\mathbf{S}]$, respectively. By the Spectral Theorem there exist isomorphic cofinal and τ-closed subspectra

$$\overline{D}'' \equiv \left\{X_c, \varpi_c^{c'}, \mathrm{C}''_0\right\} \quad \text{and} \quad \overline{\mathrm{D}}''_0[\mathbf{S}] \equiv \left\{\mathrm{T}(c), p_c^{c'}, \mathrm{C}''_0\right\}$$

of $\overline{D}'$ and $\overline{D}'_0[\mathbf{S}]$, respectively. The spectrum $\overline{D}''$ can be also considered as a cofinal and τ-closed subspectrum of $\overline{D}$. Thus, since X_c is homeomorphic to $\mathrm{T}(c) \in \mathbb{P}$ for every $c \in \mathrm{C}''_0$, X_c is also belongs to $\mathbb{P}$ for every $c \in \mathrm{C}''_0$.

To complete the proof of the proposition it suffices to show that if $\overline{D}_0$ is a kernel factorizing natural τ-spectrum for an indexed collection **S** of mutually disjoin elements of $\mathbb{P}$, then $\mathbb{P}$ is second-type saturated. This sentence can be proved as the above replacing $\overline{\mathrm{D}}[\mathbf{S}]$ by $\overline{D}_0$ and $\overline{D}$ by $\overline{\mathrm{D}}[\mathbf{S}]$. □

2. Spectra of mappings and the Spectral Theorem

In this section we define the notion of an (inverse) spectrum of mappings, the notion of an $\mathbb{F}$-factorizing τ-spectrum of mappings, where $\mathbb{F}$ is a class of mappings, and prove the Spectral Theorem for such spectra. Actually, a spectrum of mappings is a morphism of a spectrum of spaces into another spectrum of spaces. However, it is convenient to consider another definition. For this purpose we give the notions of a mapping of mappings and the composition of such mappings. Using these notions, the spectrum of mappings is defined exactly as the spectrum of spaces replacing the word "space" by the word "mapping". In this case the Spectral Theorem for mappings has the same formulation as the Spectral Theorem for spaces (Proposition 1.3).

Mappings of mappings. For every mapping f we denote by D_f the domain of f and by R_f the range of f, that is if f is a mapping of a set X into a set Y, then $D_f = X$ and $R_f = Y$. A mapping g is said to be a *restriction* of a mapping f if $D_g \subset D_f$, $R_g \subset R_f$, and $g(x) = f(x)$ for every $x \in D_g$. In this case, the mapping f is said to be an *extension* of g. In particular, if D is a subset of D_f, then by $f|_D$ we denote the restriction g of f such that $D_g = D$ and $R_g = R_f$.

Let g and f be two mappings. A pair (ϖ, π), where ϖ is a mapping of D_g into (onto) D_f and π is a mapping of R_g into (onto) R_f such that $\pi \circ g = f \circ \varpi$, is said to be a *mapping of g into* (*onto*) *f*. In the case, where ϖ and π are homeomorphisms into (onto), the pair (ϖ, π) is said to be a *homeomorphism into* (*onto*). A homeomorphism into is called also an *embedding*. If (ϖ, π) is a homeomorphism onto, then the mappings g and f are called *homeomorphic*.

Let (ϖ, π) be a mapping of g into f and h a restriction of g. Then, the pair $(\varpi|_{D_g}, \pi|_{R_g})$ is a mapping of h into f. This mapping is called the *restriction* of (ϖ, π) on h and it is denoted by $(\varpi, \pi)|_h$. In this case, the mapping (ϖ, π) is called an extension of $(\varpi, \pi)|_h$.

Let (ϖ_0, π_0) be a mapping of g into f and (ϖ_1, π_1) a mapping of f into a mapping h. By a *composition* of mappings (ϖ_0, π_0) and (ϖ_1, π_1), denoted by $(\varpi_1, \pi_1) \circ (\varpi_0, \pi_0)$, we mean the mapping $(\varpi_1 \circ \varpi_0, \pi_1 \circ \pi_0)$ of g into h.

Spectra of mappings. A system

$$\overline{F} \equiv \{f_\lambda, (\varpi_\lambda^{\lambda'}, \pi_\lambda^{\lambda'}), \Lambda\}, \tag{2.1}$$

where:

(a) Λ is a directed set whose order is denoted by $\prec$,
(b) f_λ, $\lambda \in \Lambda$, is a mapping (a *mapping* of $\overline{F}$), and
(c) $(\varpi_\lambda^{\lambda'}, \pi_\lambda^{\lambda'})$ is a mapping of $f_{\lambda'}$ into f_λ (a *projection* of $\overline{F}$ or the *projection* of $f_{\lambda'}$ into f_λ) such that

$$(\varpi_\lambda^{\lambda'}, \pi_\lambda^{\lambda'}) \circ (\varpi_{\lambda'}^{\lambda''}, \pi_{\lambda'}^{\lambda''}) = (\varpi_\lambda^{\lambda''}, \pi_\lambda^{\lambda''})$$

whenever $\lambda \prec \lambda' \prec \lambda''$,

is said to be an (*inverse*) *spectrum of mappings on* Λ.

We note that the weights of the domains, as well as, of the ranges of the mappings of a spectrum of mappings are less than or equal to τ.

By $\mathbb{F}(\overline{F})$ we denote the class of all mappings which are homeomorphic to some mapping f_λ, $\Lambda \in \Lambda$.

It is easy to see that the system (2.1) is a spectrum of mappings if and only if the systems

$$\overline{D} \equiv \{D_{f_\lambda}, \varpi_\lambda^{\lambda'}, \Lambda\} \quad \text{and} \quad \overline{R} \equiv \{R_{f_\lambda}, \pi_\lambda^{\lambda'}, \Lambda\}$$

are spectra of spaces and the indexed set

$$\overline{f} \equiv \{f_\lambda,\ \lambda \in \Lambda\} \tag{2.2}$$

is a morphism of the spectrum $\overline{D}$ into the spectrum $\overline{R}$. The spectra $\overline{D}$ and $\overline{R}$ are called the *domain-spectrum* and the *range-spectrum* of $\overline{F}$, respectively.

Suppose that the system (2.1) is a spectrum of mappings. The limit mapping induced by the morphism (2.2) is called *the limit mapping of the spectrum* (2.1) and it is denoted by $\lim(\overline{F})$.

Let ϖ_λ be the limit projection of $\lim(\overline{D})$ into D_{f_λ} and π_λ is the limit projection of $\lim(\overline{R})$ into R_{f_λ}. By the properties of the limit mapping of a morphism of spectra of spaces we have

$$\pi_\lambda \circ \lim(\overline{F}) = f_\lambda \circ \varpi_\lambda.$$

This means that the pair $(\varpi_\lambda, \pi_\lambda)$ is a mapping of $\lim(\overline{F})$ into f_λ, which is called the *limit projection* of $\lim(\overline{F})$ into f_λ.

A restriction g of the limit mapping $\lim(\overline{F})$ is said to be *surjective* if the restriction on g of the limit projections are onto.

By a *mapping of a mapping* g *into the spectrum* $\overline{F}$ we mean an indexed set

$$\overline{(i, j)} \equiv \{(i_\lambda, j_\lambda)\colon \lambda \in \Lambda\}, \tag{2.3}$$

where (i_λ, j_λ) is a mapping of g into f_λ such that

$$(i_\lambda, j_\lambda) = (\varpi_\lambda^{\lambda'}, \pi_\lambda^{\lambda'}) \circ (i_{\lambda'}, j_{\lambda'})$$

whenever $\lambda \prec \lambda'$.

Let (2.3) be a mapping of g into $\overline{F}$. Then, the indexed set $\bar{i} \equiv \{i_\lambda\colon \lambda \in \Lambda\}$ is a mapping of D_g into the domain-spectrum $\overline{D}$ of $\overline{F}$ and the indexed set $\bar{j} \equiv \{j_\lambda\colon \lambda \in \Lambda\}$ is a mapping of R_g into the range-spectrum $\overline{R}$ of $\overline{F}$. These indexed sets define respectively the induced mapping $\lim(\bar{i})$ of D_g into $\lim(\overline{D})$ and the induced mapping $\lim(\bar{j})$ of R_g into $\lim(\overline{R})$.

Lemma 2.1. *The pair* $(\lim(\bar{i}), \lim(\bar{j}))$ *is a mapping of g into* $\lim(\overline{F})$, *which is called the limit mapping induced by the indexed set* $\overline{(i, j)}$. *Moreover, if in* (2.3) *all mappings* (i_λ, j_λ), $\lambda \in \Lambda$, *are homeomorphisms into* (*onto*), *then* $(\lim(\bar{i}), \lim(\bar{j}))$ *is also a homeomorphism into* (*onto*).

Proof. We need to prove the following relation:

$$\lim(\bar{j}) \circ g = \lim(\overline{F}) \circ \lim(\bar{i}). \tag{2.4}$$

Let $x \in D_g$. Then, $\lim(\bar{j})(g(x))$ is the point q of $\lim(\overline{R})$ for which $q(\lambda) = j_\lambda(g(x))$, $\lambda \in \Lambda$. On the other hand, $\lim(\bar{i})(x)$ is the point p of $\lim(\overline{D})$ for which $p(\lambda) = i_\lambda(x)$, $\lambda \in \Lambda$, and, by definition of $\lim(\overline{F})$, $\lim(\overline{F})(p)$ is the point r of $\lim(\overline{R})$ for which $r(\lambda) = f_\lambda(i_\lambda(x))$. Since (i_λ, j_λ) is a mapping of g into f_λ we have $j_\lambda \circ g = f_\lambda \circ i_\lambda$. Therefore, for every $x \in D_g$, $j_\lambda(g(x)) = f_\lambda(i_\lambda(x))$. This means that $q = r$ proving relation (2.4). If in (2.3) all mappings (i_λ, j_λ), $\lambda \in \Lambda$, are homeomorphisms into (onto), that is the mappings i_λ and j_λ are homeomorphisms into (onto), then $\lim(\bar{i})$ and $\lim(\bar{j})$ are homeomorphisms into (onto) and, therefore, $(\lim(\bar{i}), \lim(\bar{j}))$ is homeomorphism into (onto). $\square$

Suppose that Λ' is a directed subset of Λ (not necessarily cofinal). Then, the system

$$\overline{F}|_{\Lambda'} \equiv \{f_\lambda, (\varpi_\lambda^{\lambda'}, \pi_\lambda^{\lambda'}), \Lambda'\}$$

is a spectrum of mappings on Λ', which is called a *subspectrum* of $\overline{F}$. We note that if $\overline{D}$ and $\overline{R}$ are the domain-spectrum and the range-spectrum of $\overline{F}$, respectively, then $\overline{D}|_{\Lambda'}$ and $\overline{R}|_{\Lambda'}$ are the domain-spectrum and the range-spectrum of $\overline{F}|_{\Lambda'}$, respectively.

Let λ_0 be an element of Λ such that $\lambda \prec \lambda_0$ for every $\lambda \in \Lambda'$. Then, the indexed set

$$\{(\varpi_{\lambda}^{\lambda_0}, \pi_{\lambda}^{\lambda_0}): \lambda \in \Lambda'\} \tag{2.5}$$

is a mapping of the mapping f_{λ_0} into the spectrum $\overline{F}|_{\Lambda'}$. The limit mapping induced by (2.5) will be denoted by $(\varpi_{\Lambda'}^{\lambda_0}, \pi_{\Lambda'}^{\lambda_0})$ and will be called the *natural mapping* of f_{λ_0} into $\lim(\overline{F}|_{\Lambda'})$.

Now, we suppose that Λ' is a cofinal subset of Λ. In this case there exists a homeomorphism h_d of $\lim(\overline{D})$ onto $\lim(\overline{D}|_{\Lambda'})$ such that if $h_d(p) = p'$, then $p(\lambda) = p'(\lambda)$ for every $\lambda \in \Lambda'$. Similarly, there exists a homeomorphism h_r of $\lim(\overline{R})$ onto $\lim(\overline{R}|_{\Lambda'})$ such that if $h_r(q) = q'$, then $q(\lambda) = q'(\lambda)$ for every $\lambda \in \Lambda'$.

Lemma 2.2. *The pair* (h_d, h_r) *is a homeomorphism of* $\lim(\overline{F})$ *onto* $\lim(\overline{F}|_{\Lambda'})$.

Proof. Since h_d and h_r are homeomorphisms onto, it suffices to prove that the pair (h_d, h_r) is a mapping of $\lim(\overline{F})$ into $\lim(\overline{F}|_{\Lambda'})$, that is the following relation is true:

$$h_r \circ \lim(\overline{F}) = \lim(\overline{F}|_{\Lambda'}) \circ h_d. \tag{2.6}$$

Let $p \in \lim(\overline{D})$. Then, $\lim(\overline{F})(p)$ is the point q of $\lim(\overline{R})$ such that $q(\lambda) = f_\lambda(p(\lambda))$ and $h_r(q)$ is the point q' of $\lim(\overline{F}|_{\Lambda'})$ such that $q'(\lambda) = f_\lambda(p(\lambda))$, $\lambda \in \Lambda'$. On the other hand, $h_d(p)$ is the point p' of $\lim(\overline{D}|_{\Lambda'})$ such that $p'(\lambda) = p(\lambda)$, $\lambda \in \Lambda'$, and $\lim(\overline{F}|_{\Lambda'})(p')$ is the point q'' of $\lim(\overline{R}|_{\Lambda'})$ such that $q''(\lambda) = f_\lambda(p(\lambda))$, $\lambda \in \Lambda'$, which means that $q' = q''$ proving relation (2.6). □

In what follows we shall identified any point p of $\lim(\overline{D})$ with the point $h_d(p)$ of $\lim(\overline{D}|_{\Lambda'})$ and any point q of $\lim(\overline{R})$ with the point $h_r(q)$ of $\lim(\overline{R}|_{\Lambda'})$ and, therefore, we shall identified the mapping $\lim(\overline{F})$ with the mapping $\lim(\overline{F}|_{\Lambda'})$.

Let

$$\overline{F}_0 \equiv \{f_{0,\lambda}, (\varpi_{0,\lambda}^{\lambda'}, \pi_{0,\lambda}^{\lambda'}), \Lambda\} \quad \text{and}$$
$$\overline{F}_1 \equiv \{f_{1,\lambda}, (\varpi_{1,\lambda}^{\lambda'}, \pi_{1,\lambda}^{\lambda'}), \Lambda\}$$

be two spectra of mappings. An indexed set

$$\overline{(i,j)} \equiv \{(i_\lambda, j_\lambda): \lambda \in \Lambda\}, \tag{2.7}$$

where (i_λ, j_λ) is a mapping of $f_{0,\lambda}$ into $f_{1,\lambda}$ such that

$$(\varpi_{1,\lambda}^{\lambda'}, \pi_{1,\lambda}^{\lambda'}) \circ (i_{\lambda'}, j_{\lambda'}) = (i_\lambda, j_\lambda) \circ (\varpi_{0,\lambda}^{\lambda'}, \pi_{0,\lambda}^{\lambda'})$$

whenever $\lambda \prec \lambda'$, is called a *morphism* of $\overline{F}_0$ into $\overline{F}_1$. In the case, where all mappings (i_λ, j_λ), $\lambda \in \Lambda$, are homeomorphisms (into) onto, this morphism is called an *isomorphism into* (*onto*).

Suppose that (2.7) is a morphism. Then, $\bar{i} \equiv \{i_\lambda: \lambda \in \Lambda\}$ is a morphism of the domain-spectrum $\overline{D}_0 \equiv \{D_{f_{0,\lambda}}, \varpi_{0,\lambda}, \Lambda\}$ of $\overline{F}_0$ into the domain-spectrum $\overline{D}_1 \equiv \{D_{f_{1,\lambda}}, \varpi_{1,\lambda}, \Lambda\}$ of $\overline{F}_1$ and $\bar{j} \equiv \{j_\lambda: \lambda \in \Lambda\}$ is a morphism of the range-spectrum $\overline{R}_0 \equiv \{R_{f_{0,\lambda}}, \pi_{0,\lambda}, \Lambda\}$ of $\overline{F}_0$ into the range-spectrum $\overline{R}_1 \equiv \{R_{f_{1,\lambda}}, \pi_{1,\lambda}, \Lambda\}$ of $\overline{F}_1$. Therefore, we can consider the limit mapping $\lim(\bar{i})$ of $\lim(\overline{D}_0)$ into $\lim(\overline{D}_1)$ induced by $\bar{i}$ and the limit mapping $\lim(\bar{j})$ of $\lim(\overline{R}_0)$ into $\lim(\overline{R}_1)$ induced by $\bar{j}$.

Lemma 2.3. *The pair* $(\lim(\bar{i}), \lim(\bar{j}))$ *is a mapping of* $\lim(\overline{F}_0)$ *into* $\lim(\overline{F}_1)$, *which is called the limit mapping induced by the morphism* $\overline{(i,j)}$. *Moreover, in the case, where* $\overline{(i,j)}$ *is an isomorphism into* (*onto*), $(\lim(\bar{i}), \lim(\bar{j}))$ *is a homeomorphism into* (*onto*).

Proof. First we prove that the pair $(\lim(\bar{i}), \lim(\bar{j}))$ is a mapping of $\lim(\overline{F}_0)$ into $\lim(\overline{F}_1)$, that is the following relation is true:

$$\lim(\overline{F}_1) \circ \lim(\bar{i}) = \lim(\bar{j}) \circ \lim(\overline{F}_0). \tag{2.8}$$

Let $p \in \lim(\overline{D}_0)$. Then, $\lim(\overline{F}_0)(p)$ is the point q of $\lim(\overline{R}_0)$ such that $q(\lambda) = f_{0,\lambda}(p(\lambda))$, $\lambda \in \Lambda$. Therefore, $\lim(\bar{j})(q)$ is the point q' of $\lim(\overline{R}_1)$ such that $q'(\lambda) = j_\lambda(f_{0,\lambda}(p(\lambda)))$, $\lambda \in \Lambda$. On the other hand, $\lim(\bar{i})(p)$ is the point p' of $\lim(\overline{D}_1)$ such that $p'(\lambda) = i_\lambda(p(\lambda))$, $\lambda \in \Lambda$, and $\lim(\overline{F}_1)(p')$ is the point q'' of $\lim(\overline{R}_1)$ such that $q''(\lambda) = f_{1,\lambda}(i_\lambda(p(\lambda)))$, $\lambda \in \Lambda$. Since the pair (i_λ, j_λ) is a mapping of $f_{0,\lambda}$ into $f_{1,\lambda}$ we have $f_{1,\lambda} \circ i_\lambda = j_\lambda \circ f_{0,\lambda}$. This relation implies that $j_\lambda(f_{0,\lambda}(p(\lambda))) = f_{1,\lambda}(i_\lambda(p(\lambda)))$, that is $q'(\lambda) = q''(\lambda)$ which means that $q' = q''$ proving relation (2.8). In the case, where $\overline{(i,j)}$ is an isomorphism into (onto), that is all mappings i_λ and j_λ, $\lambda \in \Lambda$, are homeomorphisms into (onto), then $\lim(\bar{i})$ and $\lim(\bar{j})$ are also homeomorphisms into (onto), which complete the proof of the proposition. □

Factorizing τ-spectra of mappings. Suppose that (2.1) is a spectrum of mappings and $\mathbb{F}$ is a class of mappings. The spectrum (2.1) is said to be *$\mathbb{F}$-factorizing with respect to a surjective restriction g of* $\lim(\overline{F})$ if for every mapping (i, j) of g into an element h of $\mathbb{F}$ there exist an element λ of Λ and a mapping (i_λ, j_λ) of f_λ into h such that $(i, j) = (i_\lambda, j_\lambda) \circ (\varpi_\lambda, \pi_\lambda)|_g$. In the case, where $\mathbb{F}$ is the class of all mappings instead of "$\mathbb{F}$-factorizing" we shall write "factorizing".

The spectrum (2.1) is said to be *τ-continuous* if for every chain Λ' of Λ with $|\Lambda'| \leqslant \tau$ and $\lambda = \sup(\Lambda')$ the natural mapping $(\varpi^\lambda_{\Lambda'}, \pi^\lambda_{\Lambda'})$ of f_λ into $\lim(\overline{F}|_{\Lambda'})$ is an embedding. In this case, the domain-spectrum and the range-spectrum of $\overline{F}$ is also τ-continuous. Therefore, by the assumption concerning the τ-continuous spectra of spaces, f_λ is a restriction of $\lim(\overline{F}|_{\Lambda'})$.

A τ-continuous spectrum (2.1) is said to be a *τ-spectrum* if the set Λ is τ-complete. (We recall that the weights of the domains and ranges of all mappings of a τ-spectrum of mappings are $\leqslant \tau$.)

Proposition 2.4 *(The Spectral Theorem for mappings). Suppose that $\overline{F}_0$ and $\overline{F}_1$ are two τ-spectra of mappings on the same directed set such that $\overline{F}_0$ is $\mathbb{F}(\overline{F}_1)$-factorizing with respect to a surjective restriction g_0 of $\lim(\overline{F})$. Then, for each mapping (p, q) of g_0 into $\lim(\overline{F}_1)$ there exists a mapping $(\bar{p}, \bar{q})$ of $\lim(\overline{F}_0)$ into $\lim(\overline{F}_1)$ which is an extension of (p, q) such that $(\bar{p}, \bar{q})$ is induced by a morphism of cofinal and τ-closed subspectra. Moreover, if $\overline{F}_1$ is $\mathbb{F}(\overline{F}_0)$-factorizing with respect to a surjective restriction g_1 of $\lim(\overline{F}_1)$ and (p, q) is a homeomorphism of g_0 onto g_1, then $(\bar{p}, \bar{q})$ is induced by an isomorphism onto of cofinal and τ-closed subspectra (and, therefore, $(\bar{p}, \bar{q})$ is a homeomorphism of $\lim(\overline{F}_0)$ onto $\lim(\overline{f}_1)$).*

Proof. Let

$$\overline{F}_0 = \{f_{0,\lambda}, (\varpi^{\lambda'}_{0,\lambda}, \pi^{\lambda'}_{0,\lambda}), \Lambda\} \quad \text{and}$$
$$\overline{F}_1 = \{f_{1,\lambda}, (\varpi^{\lambda'}_{1,\lambda}, \pi^{\lambda'}_{0,'}), \Lambda\}.$$

We define a subset R of the set $\Lambda \times \Lambda$ as follows: an element (λ_1, λ_0) belongs to R if there exists a mapping (i, j) of f_{0,λ_0} into f_{1,λ_1} such that

$$(i, j) \circ (\varpi_{0,\lambda_0}, \pi_{0,\lambda_0})|_{g_0} = (\varpi_{1,\lambda_1}, \pi_{1,\lambda_1}) \circ (p, q).$$

We prove that the set R satisfies the conditions of Lemma 1.2. The existence condition follows immediately by the fact that the spectrum $\overline{F}_0$ is $\mathbb{F}(\overline{F}_1)$-factorizing and the majorantness condition is easy to see. We prove the τ-closeness condition. Suppose that Λ' is a chain with $|\Lambda'| \leqslant \tau$ and $\lambda' = \sup(\Lambda')$ such that for a fixed element λ_0 of Λ, $(\lambda, \lambda_0) \in R$ for every $\lambda \in \Lambda'$. We need to prove that $(\lambda', \lambda_0) \in R$. For every $\lambda \in \Lambda'$ we denote by $(i^{\lambda_0}_\lambda, j^{\lambda_0}_\lambda)$ a mapping of f_{0,λ_0} into $f_{1,\lambda}$ such that

$$\left(i^{\lambda_0}_\lambda, j^{\lambda_0}_\lambda\right) \circ (\varpi_{0,\lambda_0}, \pi_{0,\lambda_0})\big|_{g_0} = (\varpi_{1,\lambda}, \pi_{1,\lambda}) \circ (p, q). \tag{2.9}$$

Since the limit projections of the spectrum $\overline{F}_0$ are onto, relation (2.8) implies that

$$\left(i^{\lambda_0}_\lambda, j^{\lambda_0}_\lambda\right) = \left(\varpi^{\lambda_1}_{1,\lambda}, \pi^{\lambda_1}_{1,\lambda}\right) \circ \left(i^{\lambda_0}_{\lambda_1}, j^{\lambda_0}_{\lambda_1}\right)$$

whenever $\lambda \prec \lambda_1$. Therefore, the indexed set

$$\left\{\left(i^{\lambda_0}_\lambda, j^{\lambda_0}_\lambda\right) \colon \lambda \in \Lambda'\right\}$$

is a mapping of f_{0,λ_0} into the spectrum $\overline{F}_1|_{\Lambda'}$ of mappings. Setting

$$\bar{i} \equiv \left\{i^{\lambda_0}_\lambda \colon \lambda \in \Lambda'\right\} \quad \text{and} \quad \bar{j} \equiv \left\{j^{\lambda_0}_\lambda \colon \lambda \in \Lambda'\right\}$$

we have that the pair $(\lim(\bar{i}), \lim(\bar{j}))$ is a mapping of f_{0,λ_0} into $\lim(\overline{F}_1|_{\Lambda'})$ (see Lemma 2.1). We show that

$$\lim(\bar{i})(D_{f_{0,\lambda_0}}) \subset D_{f_{1,\lambda'}}.$$

(We recall that according to our assumption, $D_{f_{1,\lambda'}}$ is a subset of the domain of $\lim(\overline{F}_1|_{\Lambda'})$.) Indeed, let $x \in D_{f_{0,\lambda_0}}$. Since the restriction of the limit projections of $\overline{F}_0$ on g_0 are onto, there exists a point a of D_{g_0} such that $\varpi_{0,\lambda_0}(a) = x$. Let $y = \varpi_{1,\lambda'}(p(a))$. It is easy to verify that $\lim(\bar{i})(x) = y$ proving relation (2.9). Denote by $i^{\lambda_0}_{\lambda'}$ the restriction of $\lim(\bar{i})$ whose domain is the domain of f_{0,λ_0} and whose range is the domain of $f_{1,\lambda'}$ and by $j^{\lambda_0}_{\lambda'}$ the restriction of $\lim(\bar{j})$ whose domain is the range of f_{0,λ_0} and whose range is the range of $f_{1,\lambda'}$. Then, the pair $(i^{\lambda_0}_{\lambda'}, j^{\lambda_0}_{\lambda'})$ is a mapping of f_{0,λ_0} into $f_{1,\lambda'}$ satisfying relation (2.8) if λ is replaced by λ'. Therefore, $(\lambda', \lambda_0) \in R$ proving the τ-closeness condition. Hence, R satisfies all conditions of Lemma 1.2.

Thus the set Λ^R of all reflexive elements of R is a cofinal and τ-closed subset of Λ. For every $\lambda \in \Lambda^R$ denote by (i_λ, j_λ) the mapping of $f_{0,\lambda}$ into $f_{1,\lambda}$ such that

$$(i_\lambda, j_\lambda) \circ (\varpi_{0,\lambda}, \pi_{0,\lambda})|_{g_0} = (\varpi_{1,\lambda}, \pi_{1,\lambda}) \circ (p, q). \tag{2.10}$$

This relation implies that

$$(i_\lambda, j_\lambda) \circ \left(\varpi_{0,\lambda}^{\lambda_1}, \pi_{0,\lambda}^{\lambda_1}\right) = \left(\varpi_{1,\lambda}^{\lambda_1}, \pi_{1,\lambda}^{\lambda_1}\right) \circ (i_{\lambda_1}, j_{\lambda_1})$$

whenever $\lambda \prec \lambda_1$, which means that the indexed set

$$\left\{(i_\lambda, j_\lambda)\colon \lambda \in \Lambda^R\right\}$$

is a morphism of the spectrum $\overline{F}_0|_{\Lambda^R}$ into the spectrum $\overline{F}_1|_{\Lambda^R}$. It is not difficult top verify that the limit mapping of this morphism, denoted by $(\bar{p}, \bar{q})$, is a mapping of $\lim(\overline{F}_0|_{\Lambda^R})$ into $\lim(\overline{F}_1|_{\Lambda^R})$ and, therefore, a mapping of $\overline{F}_0$ into $\overline{F}_1$, which is an extension of the mapping (p, q).

Now, suppose that the τ-spectrum $\overline{F}_1$ is $\mathbb{P}(\overline{F}_0)$-factorizing with respect to a surjective restriction S_1 of $\lim(\overline{F}_1)$ and (p, q) is a homeomorphism onto. Then, as the above, there exists a cofinal and τ-closed subset Λ_R of Λ and for every $\lambda \in \Lambda_R$ a mapping (i^λ, j^λ) of $f_{1,\lambda}$ into $f_{0,\lambda}$ such that

$$\left(i^\lambda, j^\lambda\right) \circ (\varpi_{1,\lambda}, \pi_{1,\lambda}) = (\varpi_{0,\lambda}, \pi_{0,\lambda}) \circ \left(p^{-1}, q^{-1}\right), \tag{2.11}$$

which means that the indexed set

$$\left\{\left(i^\lambda, j^\lambda\right)\colon \lambda \in \Lambda_R\right\}$$

is a morphism of the spectra $\overline{F}_1|_{\Lambda_R}$ into $\overline{F}_0|_{\Lambda_R}$. We set $\Lambda_0 = \Lambda^R \cap \Lambda_R$. By Lemma 1.1 the set Λ_0 is a cofinal and τ-closed subset of Λ. Relations (2.10) and (2.11) imply that the mapping $i^\lambda \circ i_\lambda$, $\lambda \in \Lambda_0$, is the identical mapping of the domain of $f_{0,\lambda}$ and the mapping $i_\lambda \circ i^\lambda$ is the identing mapping of the domain of $f_{1,\lambda}$. This means that i_λ is a homeomorphism of the domain of $f_{0,\lambda}$ onto the domain $f_{1,\lambda}$. Similarly, the mapping j_λ is a homeomorphism of the range of $f_{0,\lambda}$ onto the domain of $f_{1,\lambda}$. Therefore, the pair (i_λ, j_λ) is a homeomorphism of $f_{0,\lambda}$ onto $f_{1,\lambda}$. Thus, the indexed set

$$\left\{(i_\lambda, j_\lambda)\colon \lambda \in \Lambda_0\right\}$$

is an isomorphism of the spectrum $\overline{F}_0|_{\Lambda_0}$ onto the spectrum $\overline{F}_1|_{\Lambda_0}$ and, therefore, is an isomorphism of the spectrum $\overline{F}_0$ onto the spectrum $\overline{F}_1$ which means that the limit mapping $(\bar{p}, \bar{q})$ of this isomorphism is a homeomorphism of $\lim(\overline{F}_0)$ onto $\lim(\overline{F}_1)$ proving the proposition. □

3. Concrete factorizing τ-spectra of mappings

In this section to every indexed collection **F** of mutually disjoin mappings (see below the definition) we associate a τ-spectrum, denoted by $\bar{\mathrm{F}}(\mathbf{F})$, factorizing with respect to a surjective restriction g of the limit mapping such that g is homeomorphic to the free union of elements of **F** (see below the definition). The mappings of $\bar{\mathrm{F}}(\mathbf{F})$ are Containing Mappings for **F** constructed in [4]. By the Spectral Theorem any τ-spectrum of mappings factorizing with respect to a surjective restriction g such that g is homeomorphic to the free union of elements of **F** contains a cofinal and τ-closed subspectrum whose mappings are Containing Mappings for **F**. Thus, the Containing Mappings and, therefore, the Containing Spaces which are domains and ranges of the Containing Mappings, appear here without any concrete construction of these mappings and spaces. Using the associated τ-spectra of mappings we define the so-called second-type saturated classes of mappings (which are "saturated" by universal elements) and give a characterization of these classes.

Free unions of mappings. Let $\mathbf{F} \equiv \{f_\mu\colon \mu \in M\}$ be an indexed collection of mappings. The collection **F** is called *mutually disjoin* if the domains, as well as, the ranges of all elements of this collection are mutually disjoin. The *free union* of such an indexed collection of mappings is a mapping g such that (a) the domain D_g of g is the free union of the domains of elements of **F**, (b) the range R_g of g is the free union of the ranges of elements of **F**, and (c) $g(x) = f_\mu(x)$ for every $x \in D_{f_\mu}$, $\mu \in M$.

Let $\{U^{R_f}\colon f \in \mathbf{F}\}$ be an indexed set, where U^{R_f} is a subset of R_f. The indexed set $\{f^{-1}(U^{R_f})\colon f \in \mathbf{F}\}$ is said to be the **F**-*preimage* of $\{U^{R_f}\colon f \in \mathbf{F}\}$.

Let $\{\sim^s\colon s \in \mathcal{F}\}$ be a family of equivalence relation on the indexed set $\mathbf{S}_r \equiv \{R_f\colon f \in \mathbf{F}\}$. The **F**-*preimage* of R is the family $\{\sim_d^s\colon s \in \mathcal{F}\}$ of equivalence relations on the indexed set $\mathbf{S}_d \equiv \{D_f\colon f \in \mathbf{F}\}$ defined as follows: two elements D_f and D_h are $\sim_d^s$-equivalent for some $s \in \mathcal{F}$ if and only if the elements R_f and R_h are $\sim^s$-equivalent.

The τ-complete set C(F). Let **F** be a mutually disjoin indexed collection of mappings,

$$\mathbf{S}_d \equiv \{D_f\colon f \in \mathbf{F}\}, \quad \text{and} \quad \mathbf{S_r} \equiv \{R_f\colon f \in \mathbf{F}\}$$

the mutually disjoin indexed collections of domains and ranges of the elements of **F**, respectively. Consider the set $\mathrm{C}(\mathbf{S}_d) \times \mathrm{C}(\mathbf{S}_r)$ with the *product order* denoted by $\prec^{\mathbf{F}}$, that is for two elements $c_0 \equiv (c_0^d, c_0^r)$ and $c_1 \equiv (c_1^d, c_1^r)$ we write $c_0 \prec^{\mathbf{F}} c_1$ if and only if $c_0^d \prec^{\mathbf{S}_d} c_1^d$ and $c_0^r \prec^{\mathbf{S}_r} c_1^r$. It is easy to prove that this ordered set is τ-complete.

We define a subset $\mathrm{C}(\mathbf{F})$ of $\mathrm{C}(\mathbf{S}_d) \times \mathrm{C}(\mathbf{S}_r)$ as follows: an element $c \equiv (c^d, c^r)$ belongs to $\mathrm{C}(\mathbf{F})$ if for some elements $(\mathbf{M}^d, \mathrm{R}^d) \in c^d$ and $(\mathbf{M}^r, \mathrm{R}^r) \in c^r$ the following conditions are satisfied: (a) the **F**-preimage of any component of $\mathbf{M}_r$ is a component of $\mathbf{M}_d$ and (b) the family R^d is a final refinement of the **F**-preimage of R^r. It is easy to verify that the set $\mathrm{C}(\mathbf{F})$ is well defined, that is the fact that c is an element of $\mathrm{C}(\mathbf{F})$ is independent of the elements $(\mathbf{M}^d, \mathrm{R}^d)$ and $(\mathbf{M}^r, \mathrm{R}^r)$ of c^d and c^r, respectively.

Lemma 3.1. *The set* $\mathrm{C}(\mathbf{F})$ *is a cofinal and* τ*-closed subset of* $\mathrm{C}(\mathbf{S}_d) \times \mathrm{C}(\mathbf{S}_r)$ *and, therefore, it is* τ*-complete.*

Proof. We prove that $\mathrm{C}(\mathbf{F})$ is a cofinal subset of $\mathrm{C}(\mathbf{S}_d) \times \mathrm{C}(\mathbf{S}_r)$. Let $c \equiv (c^d, c^r)$ be an element of $\mathrm{C}(\mathbf{S}_d) \times \mathrm{C}(\mathbf{S}_r)$, $(\mathbf{M}^d, \mathrm{R}^d) \in c^d$, and $(\mathbf{M}^r, \mathrm{R}^r) \in c^r$. Denote by $\mathbf{M}_0^d$ a co-mark of $\mathbf{S}_d$ which is simultaneously a co-extension of $\mathbf{M}^d$ and the **F**-preimage of $\mathbf{M}^r$. Also, denote by R_0^d an $\mathbf{M}_0^d$-admissible family of equivalence relations on $\mathbf{S}^d$ which is simultaneously a final refinement of the family R^d and the **F**-preimage of R^r. Let c_0^d be the element of $\mathrm{C}(\mathbf{S}_d)$ containing the pair $(\mathbf{M}_0^d, \mathrm{R}_0^d)$. Then, it is easy to verify that the element $c_0 \equiv (c_0^d, c^r)$ of $\mathrm{C}(\mathbf{S}_d) \times \mathrm{C}(\mathbf{S}_r)$ belongs to $\mathrm{C}(\mathbf{F})$ and $c \prec c_0$.

Now, we prove that $\mathrm{C}(\mathbf{F})$ is a τ-closed subset of $\mathrm{C}(\mathbf{S}_d) \times \mathrm{C}(\mathbf{S}_r)$. Let $C \equiv \{c_\delta \equiv (c_\delta^d, c_\delta^r)\colon\ \delta \in \tau\}$ be a chain in $\mathrm{C}(\mathbf{F})$. Denote by $c \equiv (c^d, c^r)$ the $\sup(C)$ in $\mathrm{C}(\mathbf{S}_d) \times \mathrm{C}(\mathbf{S}_r)$. It suffices to prove that $c \in \mathrm{C}(\mathbf{F})$, that is if $(\mathbf{M}^d, \mathrm{R}^d) \in c^d$ and $(\mathbf{M}^r, \mathrm{R}^r) \in c^r$ then (a) any component of $\mathbf{M}^r$ is a component of $\mathbf{M}^d$ and (b) R^d is a final refinement of the **F**-preimage of R^r.

We prove (a). Consider the chains $C^d \equiv \{c_\delta^d\colon\ \delta \in \tau\}$ and $C^r \equiv \{c_\delta^r\colon\ \delta \in \tau\}$ of the sets $\mathrm{C}(\mathbf{S}_d)$ and $\mathrm{C}(\mathbf{S}_r)$, respectively. It is clear that $c^d = \sup(C^d)$ and $c^r = \sup(C^r)$. For every $\delta \in \tau$ let $(\mathbf{M}_\delta^d, \mathrm{R}_\delta^d) \in c_\delta^d$ and $(\mathbf{M}_\delta^r, \mathrm{R}_\delta^r) \in c_\delta^r$. Suppose that $\mathbf{U}$ is a component of $\mathbf{M}^r$. Since $c^r = \sup(C^r)$ there exists $\delta \in \tau$ such that $\mathbf{U}$ is a component of $\mathbf{M}_\delta^r$. Since $(c_\delta^d, c_\delta^r) \in \mathrm{C}(\mathbf{F})$, $\mathbf{U}$ is a component of $\mathbf{M}_\delta^d$ and since $c_\delta^d \prec^{\mathbf{S}_d} c^d$, $\mathbf{U}$ is also a component of $\mathbf{M}^d$.

We prove (b). Let $s \in \mathcal{F}$. Since $c^r = \sup(C^r)$ we can suppose that for every $s \in \mathcal{F}$,

$$\sim_{\mathrm{R}^r}^s = \bigcap \bigl\{ \sim_{\mathrm{R}_\delta^r}^s\colon\ \delta \in s \bigr\}$$

(see the proof of Lemma 8.2.1 of [4]). Since for every $\delta \in \tau$, R_δ^d is a final refinement of the **F**-preimage of R_δ^r there exists $t(\delta) \in \mathcal{F}$ such that the equivalence relation $\sim_{\mathrm{R}_\delta^d}^{t(\delta)}$ is contained in the **F**-preimage of $\sim_{\mathrm{R}_\delta^r}^s$. Let $t = s \cup (\bigcup\{t(\delta)\colon\ \delta \in s\})$. Then, $\sim_{\mathrm{R}_\delta^d}^t$ is contained in $\sim_{\mathrm{R}_\delta^d}^{t(\delta)}$ and, therefore, is contained in the **F**-preimage of $\sim_{\mathrm{R}_\delta^r}^s$ for every $\delta \in s$, which means that $\sim_{\mathrm{R}_\delta^d}^t$ is contained in the **F**-preimage of $\sim_{\mathrm{R}^r}^s$. Since $c^d = \sup(C^d)$ we can suppose that

$$\sim_{\mathrm{R}^d}^t = \bigcap \bigl\{ \sim_{\mathrm{R}_\delta^d}^t\colon\ \delta \in t \bigr\}$$

and, therefore, $\sim_{\mathrm{R}^d}^t$ is contained in the **F**-preimage of $\sim_{\mathrm{R}^r}^s$, which means that R^d is a final refinement of the **F**-preimage of R^r completing the proof of the lemma. □

Let $c \equiv (c^d, c^r) \in \mathrm{C}(\mathbf{F})$. Suppose that $(\mathbf{M}^d, \mathrm{R}^d) \in c^d$ and $(\mathbf{M}^r, \mathrm{R}^r) \in c^r$. Then, we can consider the Containing Spaces $\mathrm{T}(c^d) \equiv \mathrm{T}(\mathbf{M}^d, \mathrm{R}^d)$ and $\mathrm{T}(c^r) \equiv \mathrm{T}(\mathbf{M}^r, \mathrm{R}^r)$. We define a mapping (the *Containing Mapping*) f_c^{T} of $\mathrm{T}(c^d)$ into $\mathrm{T}(c^r)$ as follows: if $\mathbf{a}$ is a point of $\mathrm{T}(c^d)$ and $(x, D_f) \in \mathbf{a}$, then we set $f_c^{\mathrm{T}}(\mathbf{a}) = \mathbf{b}$, where $\mathbf{b}$ is the point of $\mathrm{T}(c^r)$ containing the pair $(f(x), R_f)$. This mapping is well defined and continuous (see Lemma 6.1.1 of [4]).

Now, let $c \equiv (c^d, c^r)$ and $c' \equiv (c^{d,\prime}, c^{r,\prime})$ be two elements of $\mathrm{C}(\mathbf{F})$ such that $c \prec^{\mathbf{F}} c'$. Therefore, c^d and $c^{d,\prime}$ are elements of $\mathrm{C}(\mathbf{S}_d)$, c^r and $c^{r,\prime}$ are elements of $\mathrm{C}(\mathbf{S}_r)$, $c^d \prec^{\mathbf{S}_d} c^{d,\prime}$, and $c^r \prec^{\mathbf{S}_r} c^{r,\prime}$. We denote by $p_c^{c'}$ the natural mapping of $\mathrm{T}(c^d)$ into $\mathrm{T}(c^{d,\prime})$ and by $q_c^{c'}$ the natural mapping of $\mathrm{T}(c^r)$ into $\mathrm{T}(c^{r,\prime})$.

Lemma 3.2. *The pair* $(p_c^{c'}, q_c^{c'})$ *is a mapping of* $f_{c'}^{\mathrm{T}}$ *into* f_c^{T}.

Proof. We need to prove the following relation:

$$f_c^{\mathrm{T}} \circ p_c^{c'} = q_c^{c'} \circ f_{c'}^{\mathrm{T}}. \tag{3.1}$$

Let $(\mathbf{M}^d, \mathrm{R}^d) \in c^d$, $(\mathbf{M}^r, \mathrm{R}^r) \in c^r$, $(\mathbf{M}^{d,\prime}, \mathrm{R}^{d,\prime}) \in c^{d,\prime}$, and $(\mathbf{M}^{r,\prime}, \mathrm{R}^{r,\prime}) \in c^{r,\prime}$. Let $\mathbf{a}^{d,\prime} \in \mathrm{T}(c^{d,\prime})$ and $(x, D_f) \in \mathbf{a}^{d,\prime}$. Then, $p_c^{c'}(\mathbf{a}^{d,\prime})$ is the point $\mathbf{b}^d$ of $\mathrm{T}(c^d)$ containing the pair (x, D_f) and $f_c^{\mathrm{T}}(\mathbf{b}^d)$ is the point $\mathbf{b}^r$ of $\mathrm{T}(c^r)$ containing the pair $(f(x), R_f)$. On the other hand, $f_{c'}^{\mathrm{T}}(\mathbf{a}^{d,\prime})$ is the point $\mathbf{a}^{r,\prime}$ containing the pair $(f(x), R_f)$ and $q_c^{c'}(\mathbf{a}^{r,\prime})$ is the point $\mathbf{b}$ of $\mathrm{T}(c^r)$ containing the pair $(f(x), R_f)$, which means that $\mathbf{b} = \mathbf{b}^r$ proving relation (3.1). □

Proposition 3.3. *For every mutually disjoin indexed collection* **F** *of mappings the system*

$$\bar{\mathrm{F}}[\mathbf{F}] \equiv \bigl\{ f_c^{\mathrm{T}}, \bigl(p_c^{c'}, q_c^{c'}\bigr), \mathrm{C}(\mathbf{F}) \bigr\}$$

is a τ*-spectrum of mappings* (*the associated to* **F** τ*-spectrum*) *whose limit mapping have a surjective restriction* g *which is homeomorphic to the free union of elements of* **F** *such that* $\bar{\mathrm{F}}[\mathbf{F}]$ *is factorizing with respect to* g.

Proof. It is easy to verify that $(p_c^{c'}, q_c^{c'}) \circ (p_{c'}^{c''}, q_{c'}^{c''}) = (p_c^{c''}, q_c^{c''})$ whenever $c \prec c' \prec c''$. Therefore, Lemmas 3.1 and 3.2 imply that $\bar{\mathrm{F}}[\mathbf{F}]$ is a spectrum. The domain-spectrum and the range-spectrum of $\bar{\mathrm{F}}[\mathbf{S}]$ are subspectra of $\bar{\mathrm{D}}[\mathbf{S}_d]$ and $\bar{\mathrm{D}}[\mathbf{S}_r]$, respectively, and therefore they are τ-continuous. This fact implies that $\bar{\mathrm{F}}[\mathbf{F}]$ is also τ-continuous. Since by Lemma 3.1 $\mathrm{C}(\mathbf{F})$ is τ-complete we have that $\bar{\mathrm{F}}[\mathbf{F}]$ is a τ-spectrum.

We prove that $\lim(\bar{\mathrm{F}}[\mathbf{F}])$ have a restriction g which is homeomorphic to the free union of elements of $\mathbf{F}$. Let f be an element of $\mathbf{F}$. There exists an element $c^d \equiv (\mathbf{M}^d, \mathrm{R}^d)$ of $\mathrm{C}(\mathbf{S}_d)$ such that the singleton $\{D_f\}$ is an equivalence class of an element of R^d (see the proof of Lemma 8.2.8 of [4]). Similarly, there exists an element $c^r \equiv (\mathbf{M}^r, \mathrm{R}^r)$ of $\mathrm{C}(\mathbf{S}_r)$ such that the singleton $\{R_f\}$ is an equivalence class of an element of R^r. By Lemma 3.1 there exists an element $c \in \mathrm{C}(\mathbf{F})$ such that $(c^d, c^r) \prec c$. It is easy to verify that if $c \prec c' \equiv ((\mathbf{M}^{d,\prime}, \mathrm{R}^{d,\prime}), (\mathbf{M}^{r,\prime}, \mathrm{R}^{r,\prime})) \in \mathrm{C}(\mathbf{F})$, then $\{D_f\}$ is an equivalence class for some element of $\mathrm{R}^{d,\prime}$ and $\{R_f\}$ is an equivalence class for some element of $\mathrm{R}^{r,\prime}$. This fact implies that $\{D_f\}$ is an open subset of the limit space of the domain-spectrum of $\bar{\mathrm{F}}[\mathbf{F}]$ and $\{R_f\}$ is an open subset of the limit space of the range-spectrum of $\bar{\mathrm{F}}[\mathbf{F}]$. By the construction of the Containing Mappings the restriction of $\lim(\bar{\mathrm{F}}[\mathbf{F}])$ on $\{D_f\}$ coincides with the mapping f.

Thus, the subset $D \equiv \bigcup\{D_f\colon f \in \mathbf{F}\}$ of the limit space of the domain-spectrum of $\bar{\mathrm{F}}[\mathbf{F}]$ is the free union of the domains of elements of $\mathbf{F}$ and the subset $R \equiv \bigcup\{R_f\colon f \in \mathbf{F}\}$ of the limit space of the range-spectrum of $\bar{\mathrm{F}}[\mathbf{F}]$ is the free union of ranges of elements of $\mathbf{F}$. This means that the restriction g of $\lim(\bar{\mathrm{F}}[\mathbf{F}])$ on D is homeomorphic to the free union of elements of $\mathbf{F}$. Since the domains and ranges of all mappings f_c^{T}, $c \in \mathrm{C}(\mathbf{F})$, are Containing Spaces for $\mathbf{S}_d$ and $\mathbf{S}_r$, respectively, the restriction g is surjective.

Now we prove that $\bar{\mathrm{F}}[\mathbf{F}]$ is factorizing with respect to the surjective restriction g. Let (p, q) be a mapping of g onto a mapping h. Denote by

$$\mathrm{M}_g^d \equiv \{U_\delta^{D_h}\colon \delta \in \tau\}$$

an indexed base for the open subsets of D_g and by

$$\mathrm{M}_g^r \equiv \{U_\delta^{R_h}\colon \delta \in \tau\}$$

an indexed base for the open subsets of R_g such that for every $\delta \in \tau$ there exists $\varepsilon \in \tau$ for which $U_\varepsilon^{D_g} = U_\delta^{R_g}$. Let $c^d \equiv (\mathbf{M}^d, \mathrm{R}^d)$ be an element of $\mathrm{C}(\mathbf{S}_d)$ such that for every $\delta \in \tau$ the indexed set

$$\{(p|_{D_f})^{-1}(U_\delta^{D_h})\colon D_f \in \mathbf{S}_d\}$$

is a component of $\mathbf{M}^d$. Also, let $c^r \equiv (\mathbf{M}^r, \mathrm{R}^r)$ be an element of $\mathrm{C}(\mathbf{S}_r)$ such that for every $\delta \in \tau$ the indexed set

$$\{(q|_{R_f})^{-1}(U_\delta^{R_h})\colon R_f \in \mathbf{S}_r\}$$

is a component of $\mathbf{M}^r$. Without loss of generality we can suppose that $c \equiv (c^d, c^r) \in \mathrm{C}(\mathbf{F})$.

By Lemma 6.2.1 of [4] there exists a mapping $f_{c^d}^{\mathrm{T}}$ of $\mathrm{T}(c^d) \equiv \mathrm{T}(\mathbf{M}^d, \mathrm{R}^d)$ into D_h such that $f_{c^d}^{\mathrm{T}}(\mathbf{a}) = p|_{D_f}(x)$ where $(x, D_f) \in \mathbf{a} \in \mathrm{T}(c^d)$. Similarly, there exists a mapping $f_{c^r}^{\mathrm{T}}$ of $\mathrm{T}(c^r) \equiv \mathrm{T}(\mathbf{M}^r, \mathrm{R}^r)$ into R_h such that $f_{c^r}^{\mathrm{T}}(\mathbf{b}) = q|_{R_f}(y)$ where $(y, R_f) \in \mathbf{b} \in \mathrm{T}(c^r)$. On the other hand, by the definition of the mapping f_c^{T} the conditions $f \in \mathbf{F}$, $f(x) = y$, $(x, D_f) \in \mathbf{a}$, and $(f(x), R_f) \in \mathbf{b}$ imply that $f_c^{\mathrm{T}}(\mathbf{a}) = \mathbf{b}$. The last relation implies that the pair $(f_{c^d}^{\mathrm{T}}, f_{c^r}^{\mathrm{T}})$ is a mapping of f_c^{T} onto h. Moreover, it is easy to verify that

$$(f_{c^d}^{\mathrm{T}}, f_{c^r}^{\mathrm{T}}) \circ (p_c, q_c)|_g = (p, q),$$

where (p_c, q_c) is the limit projection of $\lim(\bar{\mathrm{F}}[\mathbf{F}])$ onto f_c^{T}, proving that the spectrum $\bar{F}[\mathbf{F}]$ is factorizing with respect to the surjective restriction g, which complete the proof of the proposition. □

Definition. Let $\mathbf{F}$ be a mutually disjoin indexed collection of mappings. Any τ-spectrum

$$\bar{F} \equiv \{f_c, (\varpi_c^{c'}, \pi_c^{c'}), \mathrm{C}\}$$

of mappings with $\mathrm{C} \in \widetilde{\mathrm{C}}(\mathbf{F})$ whose the limit mapping have a surjective restriction g which is homeomorphic to the free union of elements of $\mathbf{F}$ is said to be a *natural τ-spectrum* for $\mathbf{F}$. The restriction g is said to be *kernel* of $\lim(\bar{F})$. If moreover $\bar{F}$ is factorizing (respectively, $\mathbb{F}$-factorizing, where $\mathbb{F}$ is a class of mappings) with respect to g, then $\bar{F}$ is said to be *kernel factorizing* (respectively, *kernel $\mathbb{F}$-factorizing*) natural τ-spectrum.

The Spectral Theorem for mapping and Proposition 3.3 imply the following corollary.

Corollary 3.4. *Let* $\mathbf{F}$ *be a mutually disjoin indexed collection of mappings. Then, each kernel factorizing natural τ-spectrum for* $\mathbf{F}$ *contains a cofinal and τ-closed subspectrum whose mappings are Containing Mappings.*

Thus, Containing Mapping (and, therefore, Containing Spaces as the domains and ranges of these mappings) constructed in [4] appear here without any concrete construction.

Definition. A class $\mathbb{F}$ of mappings is said to be *second-type saturated* if for every mutually disjoin indexed collection $\mathbf{F}$ of elements of $\mathbb{F}$ the associated τ-spectrum $\bar{\mathrm{F}}(\mathbf{F})$ contains a cofinal and τ-closed subspectrum whose mappings belong to $\mathbb{F}$.

The proof of the following proposition is similar to that of Proposition 1.6.

Corollary 3.5. *A class $\mathbb{F}$ of mappings is second-type saturated if and only if for every mutually disjoin indexed collection $\mathbf{F}$ of elements of $\mathbb{F}$ each kernel $\mathbb{F}$-factorizing natural spectrum for $\mathbf{F}$ contains a cofinal and τ-closed subspectrum whose mappings belong to $\mathbb{F}$ or, equivalently, if and only if for every indexed collection $\mathbf{F}$ of mutually disjoin elements of $\mathbb{F}$ there exists a kernel factorizing natural τ-spectrum for $\mathbf{F}$ containing a cofinal and τ-closed subspectrum whose mappings belong to $\mathbb{F}$.*

Remark 3.6. 1. All notions and results of the paper are true if as realm of spaces we consider the class of all regular spaces of weight less than or equal to τ. In this case, for every indexed collection $\mathbf{S}$ of mutually disjoin regular spaces the associated to $\mathbf{S}$ τ-spectrum $\bar{\mathrm{D}}[\mathbf{S}]$ must be replaced by the spectrum

$$\bar{\mathrm{D}}_{\mathrm{reg}}[\mathbf{S}] \equiv \{\mathrm{T}(c), p_c^{c'}, \mathrm{C}_{\mathrm{reg}}(\mathbf{S})\},$$

where by $\mathrm{C}_{\mathrm{reg}}(\mathbf{S})$ we denote the subset of $\mathrm{C}(\mathbf{S})$ consisting of all elements c for which the space $\mathrm{T}(c)$ is regular. By the fact that $\bar{\mathrm{D}}[\mathbf{S}]$ is a τ-spectrum it follows that $\mathrm{C}_{\mathrm{reg}}(\mathbf{S})$ is τ-complete and the spectrum $\bar{\mathrm{D}}_{\mathrm{reg}}[\mathbf{S}]$ is τ-continuous, that is $\bar{\mathrm{D}}_{\mathrm{reg}}[\mathbf{S}]$ is a τ-spectrum. By the proof of Proposition 2.1.6 of [4] one can see that $\mathrm{C}_{\mathrm{reg}}(\mathbf{S})$ is a surjective cofinal subset of $\mathrm{C}(\mathbf{S})$, which means that $\lim(\bar{\mathrm{D}}_{\mathrm{reg}}[\mathbf{S}])$ contains a subspace S homeomorphic to the free union of elements of $\mathbf{S}$ and the spectrum $\bar{\mathrm{D}}_{\mathrm{reg}}[\mathbf{S}]$ is factorizing with respect to S.

Furthermore, for an indexed collection $\mathbf{F}$ of mutually disjoin mappings whose the indexed collection $\mathbf{S}_d$ of the domains and the indexed collection $\mathbf{S}_r$ of the ranges consist of regular spaces, the associated to $\mathbf{F}$ τ-spectrum $\bar{\mathrm{F}}[\mathbf{F}]$ must be replaced by the subspectrum

$$\bar{\mathrm{F}}_{\mathrm{reg}}[\mathbf{F}] \equiv \{f_c, (p_c^{c'}, q_c^{c'}), \mathrm{C}_{\mathrm{reg}}(\mathbf{F})\},$$

where $\mathrm{C}_{\mathrm{reg}}(\mathbf{F})$ is a subset of $\mathrm{C}(\mathbf{F})$ consisting of all elements $c \equiv (c^d, c^r)$ such that $c^d \in \mathrm{C}_{\mathrm{reg}}(\mathbf{S}_d)$ and $c^r \in \mathrm{C}_{\mathrm{reg}}(\mathbf{S}_r)$. Then, the limit mapping $\lim(\bar{\mathrm{F}}_{\mathrm{reg}}[\mathbf{F}])$ have a surjective restriction g which is homeomorphic to the free union of elements of $\mathbf{F}$ and $\bar{\mathrm{F}}_{\mathrm{reg}}[\mathbf{F}]$ is a τ-spectrum of mappings factorizing with respect to g.

Similarly, all notions and results of the paper remains true if as realm of spaces we shall consider the class of all completely regular spaces of the weight less than or equal to τ. In this case, for an indexed collection $\mathbf{S}$ of mutually disjoin completely regular spaces the τ-spectrum $\bar{\mathrm{D}}[\mathbf{S}]$ must be replaced by the τ-spectrum

$$\bar{\mathrm{D}}_{\mathrm{creg}}[\mathbf{S}] \equiv \{\mathrm{T}(c), p_c^{c'}, \mathrm{C}_{\mathrm{creg}}(\mathbf{S})\},$$

where $\mathrm{C}_{\mathrm{creg}}(\mathbf{S})$ is the subset of $\mathrm{C}(\mathbf{S})$ consisting of all elements c such that the space $\mathrm{T}(c)$ is completely regular. Also, for an indexed collection $\mathbf{F}$ of mutually disjoin mappings whose the indexed collection $\mathbf{S}_d$ of the domains and the indexed collection $\mathbf{S}_r$ of the ranges consist of completely regular spaces, the τ-spectrum $\bar{\mathrm{F}}[\mathbf{F}]$ of mappings must replaced by the τ-spectrum

$$\bar{\mathrm{F}}_{\mathrm{creg}}[\mathbf{F}] \equiv \{f_c, (p_c^{c'}, q_c^{c'}), \mathrm{C}_{\mathrm{creg}}(\mathbf{F})\},$$

where $\mathrm{C}_{\mathrm{creg}}(\mathbf{F})$ is a subset of $\mathrm{C}(\mathbf{F})$ consisting of all elements $c \equiv (c^d, c^r)$ of $\mathrm{C}(\mathbf{F})$ such that $c^d \in \mathrm{C}(\mathbf{S}_d)$ and $c^r \in \mathrm{C}(\mathbf{S}_r)$.

2. Suppose that the realm of spaces is the class of all regular spaces of the weight $\leqslant \tau$ and τ is the first infinite cardinal ω, that is the realm of spaces is the class of all separable metrizable spaces. We observe that in this case, as it is proved in [4], the following classes of separable metrizable spaces are second-type saturated:

(1) The class of all spaces,
(2) The class of all countable-dimensional spaces,
(3) The class of all strongly countable-dimensionally spaces,
(4) The class of all locally finite-dimensional spaces,
(5) The class of all spaces of dimension $\leqslant n \in \omega$, and
(6) The class of all spaces of dimension ind less than or equal to a countable ordinal.

In [4] it is proved also that if $\mathbb{D}$ and $\mathbb{R}$ are independently one of the above mentioned classes, then the following classes of mappings are saturated classes:

(7) The class of all mappings with the domain in $\mathbb{D}$ and range in $\mathbb{R}$,
(8) The class of all open mappings with the domain in $\mathbb{D}$ and range in $\mathbb{R}$,
(9) The class of all mappings f with the domain X in $\mathbb{D}$ and the range Y in $\mathbb{R}$ such that there exists a base B for the open subsets of X with the property that the set $f(\mathrm{Cl}_X(U))$ is closed in Y for every $U \in B$, and

(10) The class of all mappings f with the domain X in $\mathbb{D}$ and the range Y in $\mathbb{R}$ such that there exists a base B for the open subsets of X with the property that the set $f(X \setminus U)$ is closed for every $U \in B$.

However, it can be proved that these classes are also second-type saturated.

References

[1] A. Chigogidze, Inverse Spectra, North-Holland Mathematical Library, vol. 53, North-Holland Publishing Co., Amsterdam, 1996.
[2] V.V. Fedorchuk, A. Chigogidze, Absolyutnye Retrakty i Beskonechnomernye Mnogoobraziya, Nauka, Moscow, 1992 (in Russian).
[3] Stavros Iliadis, A construction of containing spaces, Topology and its Applications 107 (2000) 97–116.
[4] S.D. Iliadis, Universal Spaces and Mappings, North-Holland Mathematics Studies, vol. 198, Elsevier, 2005.
[5] E.V. Ščepin, Functors and uncountable powers of compacta, Uspekhi Mat. Nauk 36 (1981) 3–62 (in Russian).