On dimensional properties of topological products

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A B S T R A C T

For any \( n = 2, 3, \ldots \), there exist a metrizable compactum \( \Phi_n \) and a compactum \( Y_n \) such that \( \dim \Phi_n = \dim Y_n = n \) and \( \dim(\Phi_n \times Y_n) = n + 1 < 2n = \operatorname{ind}(\Phi_n \times Y_n) \).

For any Tychonoff space \( X \) and any metrizable space \( Y \), we have:

1. \( \operatorname{ind} X \times Y \leq \operatorname{ind} X + \dim Y \) and
2. \( \operatorname{ind} X \times Y \leq \dim X + \operatorname{ind} Y \) if, additionally, \( Y \) is strongly metrizable (in particular, strongly paracompact, or separable, or compact).

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Below a space is a topological regular \( T_1 \)-space, a compactum is a Hausdorff compact space, a map is a continuous mapping and “nbd” is used instead of “neighborhood”.

1. Introduction

It is well known that \( \dim X \times Y \leq \dim X + \dim Y \) for any compacta \( X, Y \) and there exist even metrizable compacta \( X \) and \( Y \) with \( \dim X \times Y < \dim X + \dim Y \). Hence for compacta \( X \) and \( Y \) with \( \dim X = \operatorname{ind} X \) and \( \dim Y = \operatorname{ind} Y \), two ways are possible for providing of the inequality \( \dim X \times Y < \operatorname{ind} X \times Y \) (recall that \( \dim Z \leq \operatorname{ind} Z \) for any compactum \( Z \)):

(a) to obtain the inequality \( \dim X + \dim Y < \operatorname{ind} X \times Y \);
(b) to obtain the inequalities \( \dim X \times Y < \dim X + \dim Y \leq \operatorname{ind} X \times Y \).

Both ways have been realized.

Way (a). In 1972, V.V. Filippov [5] constructed compacta \( X \) and \( Y \) with \( \dim X = \operatorname{ind} X = \operatorname{Ind} X = 1, \dim Y = \operatorname{ind} Y = \operatorname{Ind} Y = 2 \) and \( \operatorname{ind} X \times Y = 4 > 3 = \dim X \times Y \). In 1999, D.V. Malykhin [9] obtained a similar result even with a linearly ordered \( X \).

Note that way (a) is not possible if one of factors \( X, Y \) is metrizable because \( \dim X \times Y \leq \dim X + \operatorname{ind} Y \) in this case (see Corollary 3 in this paper or [4], 2.4.1(d)).

Way (b). In 1973, B.A. Pasynkov [8] presented the first pair of compacta \( X \) and \( Y \) realizing way (b). More general and strong result was obtained in [1]. Yet more strong result is obtained in [2]:

for any \( n = 2, 3, \ldots \) there exist a metrizable compactum \( \chi_n \) and a first countable compactum \( X_n \) such that \( \dim X_n = \dim X_n = \operatorname{ind} X_n = n \) and \( \dim \chi_n \times X_n = \dim(X_n)^2 = 2n - 1 < 2n = \operatorname{ind} \chi_n \times X_n \leq \operatorname{ind}(X_n)^2 \).

It is established for the mentioned examples of products that the difference between dimensions \( \dim \) and \( \operatorname{ind} \) of these products either is equal to 1 or is not less than 1 (but without exact evaluation of the difference). It is possible to obtain the following more strong result.
**Main Theorem (Theorem 6).** For any \( n = 2, 3, \ldots \), there exist a metrizable compactum \( \Phi_n \) and a compactum \( Y_n \) such that \( \dim \Phi_n = \dim Y_n = \text{ind} Y_n = n \) and \( \dim(\Phi_n \times Y_n) = n + 1 < 2n = \text{ind}(\Phi_n \times Y_n) \).

Note that, \( \dim Z^2 \geq 2 \dim Z - 1 \) for any compactum \( Z \). Hence \( X \) and \( Y \) in Main Theorem are not homeomorphic. Note also that, for any non-0-dimensional compacta \( X \) and \( Y \), we have that \( \dim Y + 1 \leq \dim X \times Y \) and \( \text{ind} X \times Y \leq \text{ind} X + \text{ind} Y \) if \( X \) or \( Y \) is metrizable (see Addition). Hence, in Main Theorem, the maximal possible difference between dimensions \( \dim \) and \( \text{ind} \) of the product \( \Phi \times Y \) of an \( n \)-dimensional metrizable compactum \( \Phi \) and a compactum \( Y \) with \( \dim Y = \text{ind} Y = n \) has been reached.

**Question 1.** Is it possible to construct compacta \( X \) and \( Y \) with coinciding dimensions \( \dim \) and \( \text{ind} \) and with \( \dim X \times Y < \dim X + \dim Y = \text{ind} X + \text{ind} Y < \text{ind} X \times Y \)?

**Question 2.** Is Main Theorem true in the class of 1-st countable or perfectly normal compacta?

### 2. Some constructions

#### 2.1. Embeddings in openly ind-homogeneous spaces

For a space \( X \) and its closed subset \( X_0 \), define the space \( A(X, X_0) \) in the following way. Let the set \( A = A(X, X_0) \) be the disjoint union of \( X \) and copies \( X_s \) of \( X \) with \( x \in X \setminus X_0 \). Define \( r = r(X, X_0) : A \to X \) setting \( r(t) = t \) if \( t \in X \), and \( r(t) = x \) if \( t \in X_s \) with \( x \in X \setminus X_0 \). The family of all open subsets of all copies \( X_s \) of the space \( X \) with \( x \in X \setminus X_0 \), and all sets \( U(O, F) = O \cup r^{-1}(O \setminus F) \), where \( O \) is open in \( X \) and \( F \) is a finite subset of \( X \), is a base of a topology \( \tau = \tau_{A(X, X_0)} \) on \( A \). Below \( A = A(X, X_0) \) denotes the space which is the set \( A = A(X, X_0) \) with the topology \( \tau = \tau_{A(X, X_0)} \). Evidently, all \( X_s, x \in X \setminus X_0 \), are homeomorphic to \( X \) and are open-closed in \( A \); \( A \) as a subspace of \( A \), coincides with the space \( X \); \( r \) is a retraction of \( A \) onto \( X \); \( A \) is a regular \( T_1 \) -space; \( w(A) = \max(|X|, w(X)) \) (in particular, \( w(A) = |X| \) if \( X \) is an infinite compactum); \( A \) is compact if \( X \) is compact.

Let us prove that

\[
\dim A = \dim X.
\]

Let \( O \) be a functionally open subset of \( X \) and, for any \( x \in O \), let \( Ox \) be an nbd of \( x \) in \( X \) contained in \( O \) then \( V = \bigcup_{x \in O} r^{-1}(O \setminus \{x\}) \) is functionally open in \( A \). Indeed, if the function \( \varphi : X \to [0, 1] \) is such that \( O = \varphi^{-1}(0, 1] \), then the function \( \psi : A \to [0, 1] \) such that \( \psi(A \setminus V) = [0] \) and \( \psi|_V = \varphi \circ r|_V \) is continuous (because the set \( \{x \in O : r^{-1}x \setminus \{x\} \cap (A \setminus V) \} \) is discrete in \( O \)).

Now take a finite functionally open cover \( \mu \) of \( A \). Then \( \{U \cap X : U \in \mu\} \) is a finite functionally open cover of \( X \) and so there exists its finite functionally open refinement \( v = \{O_i : \text{order } \leq \dim X\} \). Fix \( U(i) \in \mu \) such that \( i \subseteq U(i), i = 1, \ldots, s \). For any \( x \in O_i \), choose its nbd \( O_i x \) in \( X \) so that \( V_i = \bigcup_{x \in O_i} r^{-1}(O_i x \setminus \{x\}) \). As it was noted, the sets \( V_i \) are functionally open in \( A \) and, evidently, the order of the family \( \eta = \{V_i : i = 1, \ldots, s\} \) is not greater than \( \dim X \).

The set \( B = A \setminus \bigcup \eta \) is closed-open in \( A \) and is the discrete union of copies of \( X \). Hence there exists a finite functionally open refinement \( \zeta \) of \( \{U \cap B : U \in \mu\} \) of order \( \leq \dim X \). Then \( \eta \cup \zeta \) is a finite functionally open refinement of \( \mu \) of order \( \leq \dim X \). Hence \( \dim A \leq \dim X \). Since \( X \) is a retract of \( A \), we have that \( A \leq \dim X \).

Let us prove that

\[
\text{ind} A = \text{ind} X.
\]

Obviously, \( \text{ind} A \leq \text{ind} X \) for any \( t \in X \), and \( x \in X \setminus X_0 \). Let \( t \in X \) and \( Ot \) be an nbd of \( t \) in \( A \). Take an nbd \( U \) of \( t \) in \( X \) such that \( V = U \cup r^{-1}(U \setminus \{t\}) \) is a nbd and \( \text{ind} \text{bd}_X U \leq \text{ind} X - 1 \). Then \( \text{bd}_X V = \text{bd}_X U \) and so \( \text{ind} \text{bd}_X U \leq \text{ind} X - 1 \).

From this place, we shall consider the spaces \( A(X, X_0) \) only when \( X \) has no isolated points.

It follows from this that:

1. \( A = A(X, X_0) \) has no isolated points and
2. if \( X_0 \) is nowhere dense in \( X \) then \( X \) is nowhere dense in \( A \) and an open subset \( O \) of \( A \) meeting \( X \) contains a copy of \( X \) (and so \( \text{ind} O = \text{ind} X = \text{ind} A \)).
Thus we can consider the following sequence of spaces and maps:  
\[ A^0(X) = X, \ A^1(X) = A(A^0(X), \emptyset), \ A^{n+1} = A(A^n(X), A^{n-1}(X)), r_1 = r(A^0(X), \emptyset), r_{n+1} = r(A^n(X), A^{n-1}(X)), n = 1, 2, \ldots \]

Note that \( r_{n+1} \) is a retraction of \( A^{n+1} \) onto \( A^n \). It is not difficult to prove (by induction) that for any \( n = 0, 1, \ldots \), any \( t \in A^n(X) \) and any its nbhd \( O_t \) in \( A^{n+1}(X) \), there exists an nbhd \( U \) of \( t \) in \( A^n(X) \) such that

\[ U \cup r_{n+1}^{-1}(U \setminus \{t\}) \subset O_t, \]

an open subset \( O \) of \( A^{n+1}(X) \) contains a copy of \( X \) and \( \text{ind} A^{n+1}(X) = \text{ind} O = \text{ind} X \) if \( O \cap A^n(X) \neq \emptyset \). \( \square \)

\[ \text{dim} A^n(X) = \text{dim} X, \]

\[ w(A^n(X)) \leq \max(|X|, w(X)) \quad \text{and} \quad w(A^n(X)) \leq |X| \quad \text{if} \ X \text{ is a compactum}, \]

\[ r_{n+1}^{-1}A^n(X) = A^n(X). \]

Let \( A^\infty(X) \) be the limit of the inverse sequence \( \Sigma(A(X)) = \{A^n(X), r_n; n = 0, 1, \ldots \} \) and \( \pi_n \) be the projection of the limit \( A^\infty(X) \) to \( A^n(X) \), \( n = 0, 1, \ldots \) Evidently, \( X = A^0(X) \subset A^1(X) \subset \cdots \). \( A^n(X) \setminus A^{n-1}(X) \) is dense in \( A^n(X) \), \( r_n \) is identical on \( A^{n-1}(X) \), \( n = 1, 2, \ldots \). So we may suppose that \( A^\infty(X) = \bigcup \{A^n(X): n = 0, 1, \ldots \} \) is a subset of \( A^\infty(X) \) and \( \pi_n \) is a retraction of \( A^\infty(X) \) onto \( A^n(X) \), \( n = 0, 1, \ldots \). It is also evident that \( A^\infty(X) \) is dense in \( A^\infty(X) \).

It follows from (4) and (5) that

\[ w(A^\infty(X)) \leq \max(|X|, w(X)) \quad \text{and} \quad w(A^\infty(X)) \leq |X| \quad \text{if} \ X \text{ is a compactum}, \]

\[ \pi_{n+1}^{-1}A^n(X) = A^n(X), \quad n = 0, 1, \ldots \]

Prove that \( \text{ind} A^\infty(X) \leq \text{ind} X \). Take \( x \in A^\infty(X) \).

First, let \( x \in A_n(X) = A(X, X_0, n) \). It follows from this that \( \pi_{n+1} X \in A^{n+1}(X) \setminus A^n(X) \) for all \( n \). If \( O_x \) is an nbhd of \( x \) then there exist \( n \) and an nbhd \( O \) of \( x_n = \pi_n x \) such that \( U = \pi_{n+1}^{-1} O \subset O_x \). By the construction of \( A^{n+1}(X) = A(A^n(X), A^{n-1}(X)) \), the set \( A^n(X) \setminus A^{n-1}(X) \) is open-closed in \( A^{n+1}(X) \) and \( r_{n+1} A^n(X) \setminus A^{n-1}(X) \) is an open-closed nbhd of \( x_n \) contained in \( U \) and so in \( O_x \). Thus \( \text{ind} A^\infty(X) = 0 \) for any \( x \in A_n(X) \).

Now let \( x \in A^\infty(X) \) and \( O \) be its nbhd. Then there exist \( n \) and an nbhd \( V \) of \( \pi_n x \) in \( A^n(X) \) such that \( \pi_n x = x \) and \( \pi_{n+1}^{-1} V \subset O_x \). By (1), we can take an nbhd \( U_x \) of \( x \) in \( A^n(X) \) with \( U_x = U \setminus r_{n+1}^{-1} U \setminus \{x\} \subset \pi_{n+1}^{-1} V \), \( \text{bd} A^{n+1}(X) \setminus A^n(X) \subset \pi_{n+1}^{-1} V \), \( \text{bd} A^{n+1}(X) \setminus A^n(X) \subset \pi_{n+1}^{-1} V \), and \( \text{ind} A^{n+1}(X) \setminus A^n(X) \subset \text{ind} X - 1 \). It follows from this that \( F = U_x \cup \text{bd} A^n(X) \setminus A^n(X) \subset \text{ind} X - 1 \). Hence \( \text{ind} A^\infty(X) = 0 \).

We shall say that a space \( X \) is \( \text{openly ind-homogeneous} \), if \( \text{ind} O = \text{ind} X \) for any open subset \( O \neq \emptyset \) of \( X \).

We have proved the following.

\[ \text{Theorem 1. Any space} \ X \ (\text{without isolated points}) \ (\text{is a retract of the space}) \ A^\infty(X), \ \text{ind} A^\infty(X) = \text{ind} X, \ \text{every nonempty open in} \ A^\infty(X) \ \text{set contains a copy of} \ X \ (\text{and so} \ A^\infty(X) \ \text{is openly ind-homogeneous}), \ \text{w}(A^\infty(X)) \leq \max(|X|, w(X)). \text{If} \ X \text{ is compact, then} \ A^\infty(X) \text{ is also compact, dim} A^\infty(X) = \text{dim} X \text{ and } w(A^\infty(X)) \leq |X|. \]

Generalize our construction a little. Additionally, take a subset \( S \) of \( X \) \( \setminus X_0 \). Then the subspace \( A(S) = A(X, X_0, S) \) is a closed subset of \( A(X, X_0) \) (see (*)) is a closed subset of \( A(X, X_0) \). Evidently, \( \text{ind} A(S) = \text{ind} X \) and if \( X \) is compact then \( A(S) \) is also compact and \( \text{dim} A(S) = \text{dim} X \).

If \( S \) is dense in \( X \) then every nonempty open set in \( S \) meeting \( X \) contains a copy of \( X \). If \( X \) is compact metrizable and \( S \) is countable then \( A(S) \) is also compact metrizable (because it has a countable network). Let \( r(X, X_0, S) \) be the restriction of \( r(X, X_0, S) \) to \( A(S) \) and \( S(x, X_0, S) \) the union of copies \( S_x \) of \( S \) in all \( x \), \( x \in S \). For a space \( X \) and \( S \subset X \), put

\[ A^0(X, S) = X, \ S_0 = S, \ A^1(X, S) = A(A^0(X, S), \emptyset, S_0), \ S_1 = S(A^0(X, S), \emptyset, S_0), \ S_2 = S(A^1(X, S), \emptyset, S_0) \text{ and } A^{n+1}(X, S) = A(A^n(X, S), A^{n-1}(X, S), S_n), \ S_{n+1} = S(A^n(X, S), A^{n-1}(X, S), S_n), \ s_1 = r(A^0(X, S), \emptyset, S_0), \ s_{n+1} = r(A^n(X, S), A^{n-1}(X, S), S_n), \ n = 1, 2, \ldots \]

and let \( A^\infty(X, S) \) be the limit of the inverse sequence \( \Sigma(A(X, S)) = \{A^n(X, S), S_n; n = 0, 1, \ldots \} \) and \( \psi_n \) be the projection of the limit \( A^\infty(X, S) \) to \( A^n(X, S), n = 0, 1, \ldots \).
Evidently, $A^\infty(X, S)$ is a closed subset of $A^\infty(X)$ and so: $\text{ind } A^\infty(X, S) \leq \text{ind } A^\infty(X) = \text{ind } X$; $\dim A^\infty(X, S) \leq \dim A^\infty(X) = \dim X$ if $X$ is compact: $w(A^\infty(X, S), S) \leq w(A^\infty(X), S)$; $\gamma_n$ is a retraction of $A^\infty(X, S)$ onto $A^n(X, S), n = 0, 1, \ldots$, and so $\text{ind } A^\infty(X, S) \geq \text{ind } X, \dim A^\infty(X, S) \geq \dim X$. Hence, $\text{ind } A^\infty(X, S) = \text{ind } X; \dim A^\infty(X, S) = \dim X$ if $X$ is compact. As above, if $S$ is dense in $X$, then any open subset $O \neq \emptyset$ of $A^n(X, S)$ meeting $A^{n-1}(X, S), n = 1, 2, \ldots$, and $A^\infty(X, S)$ contain a copy of $X$. Evidently, if $X$ is compact metrizable and $S$ is countable then all $A^n(X, S)$ and so $A^\infty(X, S)$ are compact metrizable.

Let $X$ and $Y$ be metrizable compacta and $S$ be countable. Evidently, $Z = A^\infty(X, S) \times Y$ is the limit of the inverse sequence $\{Z_n = A^n(X, S) \times Y, S_n \times \text{id}_Y : n = 0, 1, \ldots\}$. It is also evident that $A^\infty(X, S)$ is the countable union of copies of $X$ and, by induction, the same is true for all $A^n(X, S)$. Hence every $Z_n$ is the countable union of copies of the product $X \times Y$ and so $\dim Z_n = \dim X \times Y$ for $n = 1, 2, \ldots$. It follows from this that $\dim Z \leq \dim X \times Y$. But since $X \subset A^\infty(X, S)$, we have that $\dim Z \geq \dim X \times Y$ and so $\dim Z = \dim X \times Y$.

Hence we have the following.

**Theorem 2.** For any compact metrizable space $X$ (without isolated points), there exists a compact metrizable space $A^\infty(X, S)$ such that $X$ is a retract of it, $A^\infty(X, S) = X$, every nonempty open set in $A^\infty(X, S)$ contains a copy of $X$ (and so $A^\infty(X, S)$ is openly ind-homogeneous) and, for every compact metrizable space $Y$, we have $\dim A^\infty(X, S) \times Y = \dim X \times Y$.

Evidently, every nonempty open set in $A^\infty(X, S)$ contains a copy of $A^\infty(X, S)$.

### 2.2. Open tailings

**Definition 1.** ([1], Definition 2.1, and [2], Addition to Section 5) For a space $\Phi$, a space $X$ is called an open tailing (OT, for short) of $\Phi$ if $\Phi \subset X$ and either $X = \emptyset$ or for $\Phi \neq \emptyset$,

1. there exists a retraction $r : X \to \Phi$;
2. there exist a regular cardinal number $\tau \geq \max(\omega_0, w(\Phi)^+)$ and for any $x \in \Phi$, a system of closed sets $\Phi_x$ in $X$, $x \in X$, such that $|\Phi_x| \geq \tau$, and for any $x \in X$, the restriction $r_x = r|_{\Phi_x}$ is open and $\text{ind } r_x^{-1} \tau > 1$;
3. for any $x \in \Phi$ and any system $\nu$ of nbds of $\Phi$ in $X$ of cardinality $|\nu| < \tau$, we have $|\{|x \in \Phi : \Phi_x \cap \nu| \geq \tau\}| > \tau$.

Since $\tau$ is regular, the following condition (3′) is sufficient for (3);

3′ for any $x \in \Phi$ and any nbhd $O$ of $\Phi$ in $X$, the set $A_{xO} = \{x \in \Phi : \Phi_x \subseteq O\}$ has cardinality $|A_{xO}| < \tau$.

**Remark 1.** ([1], Remark 2.2, and [2], Addition to Section 5) It follows from (3) (and (3′)) that, for any system $\nu$ of nbds of $\Phi$ in $X$ of cardinality $|\nu| < \tau$, the spaces $\bigcap \nu$ and (consequently) $\overline{\bigcap \nu}$ are OTs of $\Phi$.

Below we shall use the notations of Definition 1. Put $A = \bigcup \{A_x : x \in \Phi\}$.

Recall some necessary assertions (see [1], Preliminaries, and [2], Addition to Section 5).

**Lemma 1.** Let $F$ be either

(a) a one-point subset of $\Phi$ for an arbitrary $\Phi$ or
(b) a closed subset of $\Phi$ for a normal $\Phi$ and let $U$ be an nbhd of $F$ in $X$. Then there exist nbhds $V$ of $F$ in $\Phi$ and $O$ of $\Phi$ in $X$ such that $F \subset O \cap r^{-1}V \subset U$.

**Corollary 1.** Let $F$ be closed in $\Phi$ and $\Phi$ be normal. Then $r^{-1}F$ is an OT of $F$.

**Lemma 2.** Let $X$ be an OT of $\Phi$, $U$ be open in $X$ and $O = U \cap \Phi \neq \emptyset$, $\overline{\text{cl } O} \neq \emptyset$. Then either $\Phi \cap \text{bd } U$ contains an open and nonempty subset of $\Phi$ or $\text{bd } O \subset \text{cl } O \cap \text{cl } (\Phi \setminus \text{cl } U)$ and there exists a system $\nu$ of nbds of $\Phi$ in $X$ such that $|\nu| \leq w(\Phi) < \tau$ and

$$r^{-1}\text{bd } \text{cl } O \cap \bigcup_{\Phi} \bigg\{\Phi_a : \alpha \in A, \Phi_a \subset \bigcap \nu \bigg\} \subset \text{bd } U \cap \text{bd } (X \setminus \text{cl } U) \quad \text{and}$$

$$r_x : r^{-1}\text{bd } \text{cl } O \cap \Phi_x \rightarrow \text{bd } \text{cl } O \quad \text{is open for all } \alpha \in A.$$

**Theorem 3.** Let $X$ be an OT of a normal space $\Phi$ and $\text{ind } O \geq n$ for any open set $O \neq \emptyset$ in $\Phi$. Then $\text{ind } X \geq n + 1, n = 0, 1, \ldots$.

**Proof.** The case $n = 0$ is evident. Let our statement be true for all $n < m$, $m > 0$, and let $n = m$.

Take $x \in \Phi$ and its nbhd $O_x$ in $\Phi$ such that $\text{ind } \text{bd } \text{cl } O \geq m - 1$ for any nbhd $O$ of $x$ in $\Phi$ with $\overline{\text{cl } O} \subset O_x$. Let an open set $U_x$ in $X$ be such that $U_x \cap \Phi = O_x$. Take an nbhd $U$ of $x$ in $X$ with $\overline{\text{cl } U} \subset U_x$. Then $\overline{\text{cl } U} \subset O_x$ for $O = U \cap \Phi$ and so $\text{ind } \text{bd } \text{cl } O \geq m - 1$.

If $F = \Phi \cap \text{bd } U$ contains a nonempty open subset $V$ of $\Phi$ then $\text{ind } \text{bd } U \geq \text{ind } V \geq m$. 
Suppose that $F$ does not contain any nonempty open subset of $\Phi$. By Lemma 2 and Corollary 1, there exists a system $v$ of nbds of $\Phi$ in $X$ such that $T = (r^{-1} \mathcal{B}_c O) \cap (\bigcap v) \cap \mathcal{B}_c X \setminus \mathcal{B}(X \cup O)$. Hence, by inductive hypothesis, $\text{ind} \mathcal{B}_c X \geq 1 \text{ind} \mathcal{B}_c Y$. Thus $\text{ind} \mathcal{B}_c X \geq m + 1$ and so $\text{ind} \mathcal{B}_c X \geq m + 1$. □

We shall also need the following assertion (see Propositions 2.12 and 2.13 from [1] and Addition to Section 5 from [2]).

**Proposition 1.** Let $\Phi$ be compact and $X$ be an OT of $\Phi$. Then (a) $X^2$ is an OT of $\Phi^2$ and (b) for any compact $\Psi$ with $w(\Psi) \leq w(\Phi)$, the product $X \times \Psi$ is an OT of $\Phi \times \Psi$.

2.3. Special open tailings

Recall some constructions and results from Section 3 of [1].

Fix a space $\Phi \neq \emptyset$, a compactum $K$ with $\text{ind} K > 0$, an onto map $c : C \to K$ of a zero-dimensional compactum $C$ and let $\tau$ be a regular cardinal number $\geq \max(\omega, |\Phi|, w(\Phi)^{\omega})$, $\omega_\tau$ be the first ordinal number of cardinality $\tau$, $T(\omega_\tau)$ be the space of all ordinals $\leq \omega_\tau$ and $C_\alpha$ be a copy of $C$, $\alpha < \omega_\tau$ (we suppose that $C_\alpha \cap C_\beta = \emptyset$ if $\alpha \neq \beta$). Put

$$T(\omega_\tau, C) = T(\omega_\tau) \cup \bigcup (C_\alpha : \alpha < \omega_\tau).$$

Define a topology on $T(\omega_\tau, C)$ a base of which consists of all open subsets of spaces $C_\alpha, \alpha < \omega_\tau$, and of all sets

$$O_{\rho \gamma} = \{ \alpha \geq \omega_\tau : \beta < \alpha < \gamma \} \bigcup \bigcup (C_\alpha : \beta < \alpha < \gamma \}, \quad \beta < \gamma \leq \omega_\tau.$$

(This $T(\omega_\tau, C)$ is the result of “placing $C_\alpha$ between $\alpha$ and $\alpha + 1$” for $\alpha < \omega_\tau$.) Evidently, $T(\omega_\tau, C)$ is a compactum and the space $T(\omega_\tau)$ may be identified with the subspace $T(\omega_\tau)$ of $T(\omega_\tau, C)$. Choose disjoint sets $A_\alpha \subseteq T(\omega_\tau) \setminus \{ \omega_\tau \}$ for all $\alpha \in \Phi$ so that $|A_\alpha| = \tau$.

Define an equivalence relation $E$ on $T(\omega_\tau, C) \times \Phi$ supposing $(t, x)E(t', x')$ if $x = x'$ and either $t = t'$ or there exists $\alpha \in A_\alpha$ such that $t, t' \in C_\alpha$ and $c(t) = c(t')$. Put $T(\Phi, K, C, \tau) = (T(\omega_\tau, C) \times \Phi)/E$. Let $q$ be the canonical quotient map of $T(\omega_\tau, C) \times \Phi$ onto $T(\Phi, K, C, \tau)$. Evidently $q$ is perfect and so $T$ is regular and $T_1$. Also it is evident that there exists a continuous (even perfect) and open mapping $r : T \to \Phi$ such that $pr = r \circ q$ where $pr$ is the projection of the product $T(\omega_\tau, C) \times \Phi$ onto the factor $\Phi$. It is easily seen that the restriction of $q$ to $T(\omega_\tau, C) \times \Phi$ is a topological embedding and the restrictions of $pr$ and $r \circ q$ to $T(\omega_\tau, C) \times \Phi$ coincide. That is why we shall identify: $T(\omega_\tau, C) \times \Phi$ and $q(T(\omega_\tau, C) \times \Phi)$, the restrictions $pr$ and $r$ to $T(\omega_\tau, C) \times \Phi$ (by means of $q$).

Let us identify points $x \in \Phi$ and $(\omega_\tau, x) \in \{ \omega_\tau \} \times \Phi \subseteq T$. Then $\Phi$ will be identified with $\{ \omega_\tau \} \times \Phi$ and $r$ will become a retraction of $T$ onto $\Phi$.

Evidently, $T$ is an OT of $\Phi$ if $\text{ind} K \geq 1$.

**Theorem 4.** ([1], Theorem 3.4) If either (a) $C$ is extremally disconnected and $c$ is irreducible or (b) $K$ and $C$ are metrizable compacta then

$$\text{ind} T \leq \max(\text{ind} \Phi + 1, \text{ind} K). \quad (\ast)$$

If, additionally, $\Phi$ is normal and openly $\text{ind}$-dimensionally homogeneous then

$$\text{ind} T \geq \text{ind} \Phi + 1 \quad (\ast\ast)$$

and if $\Phi$ is paracompact then $T$ is also paracompact and

$$\text{dim} T = \max(\text{dim} \Phi, \text{dim} K). \quad (\ast\ast\ast)$$

**Proof.** In the case (a), $(\ast)$ is proved in Theorem 3.4 from [1]. In the case (b), the proof of it is similar (but Lemma 3.2 from [1] is used instead of Lemma 3.1). The inequality $(\ast\ast)$ follows from Theorem 3 and $(\ast\ast\ast)$ may be proved as Theorem 3.6 from [1]. □

**Corollary 2.** If $\text{ind} K = 1$, $\Phi$ is normal and openly $\text{ind}$-dimensionally homogeneous and either (a) $C$ is extremally disconnected and $c$ is irreducible or (b) $K$ and $C$ are metrizable compacta then

$$\text{ind} T = \text{ind} \Phi + 1.$$

If, additionally, $\Phi$ is paracompact and $\text{dim} K \leq \text{dim} \Phi$ then $T$ is also paracompact and

$$\text{dim} T = \text{dim} \Phi.$$

**Theorem 5.** For all numbers $n = 1, 2, \ldots$, there exist openly $\text{ind}$-homogeneous compacta $I_n$ such that $I_1 = I = [0, 1]$; $\text{ind} I_n = n$; all components of $I_n$ are either points or copies of $I$ (and so $\text{dim} I_n = 1$); for $n > 1$, any nonempty open subset of $I_n$ contains a copy of $I' = T(I_{n-1}, I, c, r_n)$, where $r_n > w(I_{n-1})$ and $c$ is a continuous mapping of the Cantor set $C$ onto $I$ with one-point or two-point inverse images of points.
Proof. Let $c$ be a map of the Cantor set $C$ onto $I$ with one-point or two-point inverse images of points (it may be supposed that $c$ is monotone).

Let the necessary compacta $I_m$ have been constructed for all $m < n, n > 1$.

Put (see Section 2.1) $I'_n = T(I_{n-1}, I, c, \tau_n > w(I_{n-1}))$ and $I_n = A^n(I_n)$. By Corollary 2 and Theorem 1, $\dim I_n = \dim I'_n = 1$, $\text{ind } I_n = \text{ind } I'_n = n$ and any nonempty open subset $O$ of $I_n$ contains a copy of $I'_n$. It follows from this that $I_n$ is openly $\text{ind}$-homogeneous. By their constructions and by the inductive supposition, all components of $I'_n$ and $I_n$ are either points or copies of $I$. \hfill \Box

3. Dimension of products

Let $L(l)$ be the class of all metrizable compacta $\Phi$ such that $\dim \Phi = l$ and every nonempty open subset $O$ of $\Phi$ contains a compactum $X = X(O)$ with $\dim X = l, l = 1, 2, \ldots$. Evidently, if $\Phi \in L(l)$, then $\text{cl } O \in L(l)$ for every nonempty open subset $O$ of $\Phi$ and every nonempty open subset $O$ of $\Phi$ contains a compactum from $L(l)$.

Proposition 2. For $I_n$ from Theorem 5, any $\Phi \in L(l)$ and $P_{ml} = I_m \times \Phi$, we have $\dim P_{ml} = 1 + l$, $\text{ind } P_{ml} = m + l$ and $P_{ml}$ is openly $\text{ind}$-homogeneous, $m, l = 1, 2, \ldots$.

Proof. It is well known that $\dim Y \times Z = \dim Y + 1$ for compacta $Y$ and $Z$ if $\dim Z = 1$. By this and since all components of $P_{ml}$ are contained (topologically) in $I \times \Phi$ (by Theorem 5), we have that $\dim P_{ml} = 1 + l, m, l = 1, 2, \ldots$.

Fix $l$.

First, let $m = 1$. Take nonempty open sets $U$ and $V$ in $I_1$ and $\Phi \in L(l)$, respectively. Then $U$ contains a copy $l'$ of $I_1$, $V$ contains a compactum $\Psi \in L(l)$ and $l' \times \Psi \subset U \times V$. Since $I_1$ and $\Phi$ are metrizable, $\text{ind } I_1 \times \Phi = \dim I_1 \times \Phi = 1 + l = m + l$ and $l' \times \Psi = 1 + l = m + l$. Hence $I_1 \times \Phi$ is openly $\text{ind}$-homogeneous.

Suppose that our assertion is true for all $m < n, n > 1$, and let $m = n$. Take $\Phi \in L(l)$. Since $\Phi$ is metrizable, $\text{ind } I_n \times \Phi \leq \text{ind } I_n + \text{ind } \Phi = n + l$ (see Corollary 3 below or [4], 2.4.4).

Now take nonempty open sets $U$ and $V$ in $I_n$ and $\Phi$, respectively. By the previous theorem, $U$ contains a copy $l''$ of $I_n$, $V$ contains a compactum $\Psi \in L(l)$ and $l'' \times \Psi \subset U \times V$. By Proposition 1, $l'' \times \Psi$ is an OT of $P = I_{n-1} \times \Psi$. By the inductive hypothesis, $\text{ind } P = (n - 1) + l$ and $P$ is openly $\text{ind}$-homogeneous. By Theorem 3, $\text{ind } l'' \times \Psi \geq n + l$. Since $\text{ind } l'' \times \Phi \geq \text{ind } l'' \times \Psi$, we have that $\text{ind } l'' \times \Psi = \text{ind } l'' \times \Phi = n + l$. \hfill \Box

Let $\Psi(m)$ and $\Psi'(m)$ be metrizable compacta such that $\dim \Psi(m) = \dim \Psi'(m) = m$ and $\dim \Psi(m) \times \Psi'(m) = m + 1, m = 2, 3, \ldots$ (for example, it is possible to take compacta $FQ_{p,m}$ and $FQ_{q,m}$ with $p \neq q$ from [3]). By Theorem 2, take a compact metrizable space $\Phi(m)$ such that $\dim \Phi(m) = m, \dim \Psi(m) \times \Phi(m) = m + 1$ and every nonempty open subset of $\Phi(m)$ contains a copy of $\Psi'(m)$ (hence $\Phi(m) \in L(m)$).

For the discrete union $J_m$ of $I_m$ (from Theorem 5) and $\Psi(m)$, we have $\dim J_m = \text{ind } J_m = m$. Take $J_m \times \Phi(m)$. Then $\dim J_m \times \Phi(m) = \text{ind } J_m \times \Phi(m) \times \Phi(m) = m + 1$. It follows from Proposition 2 that $\dim I_m \times \Phi(m) = m + 1, \text{ind } I_m \times \Phi(m) = 2m$. Hence $\dim J_m \times \Phi(m) = m + 1, \text{ind } J_m \times \Phi(m) = 2m, m = 1, 2, \ldots$. We have proved the following.

Theorem 6. For any $m = 2, 3, \ldots$, there exist a compactum $J_m$ and a metrizable compactum $\Phi(m)$ such that

$$\dim J_m = \text{ind } J_m = \dim \Phi(m) = (\text{ind } \Phi(m)) = m$$

and

$$\dim(J_m \times \Phi(m)) = m + 1 < 2m = \text{ind}(J_m \times \Phi(m)).$$

4. Addition

The following assertion was proved by me more than 30 years ago (as a reaction to Filippov’s inequality $\text{ind } X \times I^n \leq \text{ind } X + n$ for any Tychonoff space $X$ and the $n$-cube $I^n$ [6]) but it was not published.

Theorem 7. For any Tychonoff space $X$ and any metrizable space $Y$, we have

$$\text{ind } X \times Y \leq \text{ind } X + \text{dim } Y.$$
every uniformly 0-dimensional map is decomposing. Katětov proved [7] that every metric space $X$ with $\dim X \leq n$ has an uniformly 0-dimensional map to $n$-cube $I^n$, $n = 0, 1, 2, \ldots$.

**Proof of Theorem 7.** Let a space $X$ be Tychonoff and a space $Y$ be metrizable with $\dim Y = n$. Take a metric on $Y$ generating the topology of $Y$. Then there exists an uniformly 0-dimensional map $f$ of $Y$ to $n$-cube $I^n$. By [6], $\text{ind } X \times I^n \leq \text{ind } X + n$. Evidently, the map $id_X \times f$ is decomposing. Hence $\text{ind } X \times Y \leq \text{ind } X \times I^n \leq \text{ind } X + n = \text{ind } X + \dim Y$. \[\square\]

Recall that (see [10,11]):

Tychonoff space $X$ is called **strongly metrizable** if there exists a base in $X$ that is a countable union of star-finite covers:

$$\dim X = \text{ind } X$$

for any strongly metrizable $X$.

**Corollary 3.** For any Tychonoff space $X$ and any strongly metrizable (in particular, strongly paracompact metrizable, or separable metrizable, or metrizable compact) $Y$ we have

$$\text{ind } X \times Y \leq \text{ind } X + \text{ind } Y.$$ 

**Remark 2.** Theorem 7 is also true for arbitrary Tychonoff (even for regular $T_1$-) space $Y$ having a decomposing map onto a metrizable space and with $\dim Y \leq n$, $n = 1, 2, \ldots$. (It may be proved by means of the factorization theorem for maps to metrizable spaces (see [4], Theorem 4.2.5 and Problem 4.2.F).)

**Acknowledgement**

I am grateful for referee’s helpful remarks.

**References**


