Topology and its Applications 156 (2008) 138–141
Contents lists available at ScienceDirect
Topology and its Applications
www.elsevier.com/locate/topol

A note on unbounded strongly measure zero subgroups of the Baer–Specker group

Tomasz Weiss

Institute of Mathematics, Akademia Podlaska, 08-110 Siedlce, Poland

ARTICLE INFO

Article history:
Received 14 September 2007
Received in revised form 28 February 2008
Accepted 4 March 2008

MSC:
primary 54G20, 54G15, 54D20

Keywords:
Selection principles
Menger property
Hurewicz property

ABSTRACT

We show that it is consistent with ZFC that there exist:

(1) An unbounded (with respect to $\leqslant_*$) and strongly measure zero subgroup of $\mathbb{Z}^{\mathbb{N}}$, but without the Menger property.
(2) An unbounded (with respect to $\leqslant_*$) and strongly measure zero subgroup of $\mathbb{Z}^{\mathbb{N}}$ with the Menger property which does not have the Rothberger property.

This answers the last two problems which remained from a classification project of Machura and Tsaban.

1. Introduction

We denote by $\mathbb{Z}^{\mathbb{N}}$ the Baer–Specker group, that is the set of all countable sequences of integers with the group operation $+$ of the coordinatewise addition.

The topology on $\mathbb{Z}^{\mathbb{N}}$ is defined by the usual metric: for $x, y \in \mathbb{Z}^{\mathbb{N}}$, $x \neq y$,

$$d(x, y) = \frac{1}{\min\{n\colon\ x(n) \neq y(n)\} + 1}.$$

If $X \subseteq \mathbb{Z}^{\mathbb{N}}$, then by $\operatorname{diam}(X)$ we denote the diameter of X, which is defined as $\sup\{d(x, y)\colon\ x, y \in X\}$.

Throughout this paper we use standard terminology and notation. $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ is the one point compactification of the set $\mathbb{N}$, $\mathbb{N}^{\mathbb{N}\uparrow}$ denotes the set of all increasing elements of $\mathbb{N}^{\mathbb{N}}$, and we write $P(\mathbb{N})$ for the set of all subsets of $\mathbb{N}$. The quantifiers $\exists^\infty n$ and $\forall^\infty n$ stand for "there exist infinitely many n" and "for all except finitely many n", respectively. If $X \subseteq \mathbb{Z}^{\mathbb{N}}$, then $\langle X\rangle$ denotes the subgroup of $\mathbb{Z}^{\mathbb{N}}$ generated by X. For $X, Y \subseteq \mathbb{Z}^{\mathbb{N}}$, we define the algebraic sum $X + Y = \{x + y\colon\ x \in X \text{ and } y \in Y\}$. If $i \in \mathbb{Z}$ and $x \in \mathbb{Z}^{\mathbb{N}}$, then ix is the element of $\mathbb{Z}^{\mathbb{N}}$ obtained by the coordinatewise multiplication of x by i, and for $X \subseteq \mathbb{Z}^{\mathbb{N}}$, $iX = \{ix\colon\ x \in X\}$.

An $X \subseteq \mathbb{Z}^{\mathbb{N}}$ is said to be $\leqslant_*$ bounded (or dominated) if there is a function $f \in \mathbb{N}^{\mathbb{N}}$ with the property: $\forall g \in X$ $\forall^\infty_n |g(n)| \leqslant f(n)$, that is $\forall g \in X$ $|g| \leqslant_* f$. Otherwise, we call X an unbounded set with respect to $\leqslant_*$, or shortly, unbounded. By $\mathfrak{b}$ we denote the minimal cardinality of an unbounded subset of $\mathbb{N}^{\mathbb{N}}$, and $\mathfrak{d}$ is equal to the minimal cardinality of a dominating subset of $\mathbb{N}^{\mathbb{N}}$. The name $\operatorname{cov}(\mathcal{M})$ stands for the minimal cardinality of a set $X \subseteq \mathbb{N}^{\mathbb{N}}$ which satisfies: there is no $f \in \mathbb{N}^{\mathbb{N}}$ such that $\forall g \in X \exists^\infty_n f(n) = g(n)$. We shall say that $X \subseteq \mathbb{Z}^{\mathbb{N}}$ is a κ*-Lusin set* iff X has cardinality κ and for every meager set M, the intersection $X \cap M$ is of cardinality smaller than κ.

E-mail address: tomaszweiss@go2.pl.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2008.03.026

In the following definition, we restrict our attention to subsets of $\mathbb{Z}^{\mathbb{N}}$, however it is clear that an X can be a subset of a less concrete topological (or metric) space.

Definition 1. Suppose that X is a subset of $\mathbb{Z}^{\mathbb{N}}$.

(1) X is *strongly measure zero* if for every sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive real numbers, there exists a sequence $\{X_n\}_{n\in\mathbb{N}}$ of subsets of X, with $\operatorname{diam}(X_n) \leqslant \varepsilon_n$, for $n \in \mathbb{N}$, such that $X \subseteq \bigcup_{n\in\mathbb{N}} X_n$,
(2) X has the *Menger property* if for each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of open covers of X, there are finite sets $U_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\bigcup_{n\in\mathbb{N}} U_n$ is a cover of X,
(3) X has the *Rothberger property* if for each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of open covers of X, there are $U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, with $X \subseteq \bigcup_{n\in\mathbb{N}} U_n$, and finally,
(4) X has the *Hurewicz property* if for each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of open covers of X, there exist finite $U_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, so that every $x \in X$ is in all but finitely many sets of the form $\bigcup U_n$, $n \in \mathbb{N}$.

2. The main theorems

In Problem 9.14 of [4], M. Machura and B. Tsaban ask whether there could exist an unbounded strongly measure zero subgroup G of $\mathbb{Z}^{\mathbb{N}}$, which does not have the Menger property. We give a positive answer.

Theorem 2. *It is consistent with ZFC that there is a strongly measure zero subgroup $\widetilde{G}$ of $\mathbb{Z}^{\mathbb{N}}$, unbounded with respect to $\leqslant_*$, and without the Menger property.*

Proof. To get a required subgroup $\widetilde{G}$, we introduce an auxiliary property $\mathcal{P}$, and then we apply the following sequence of lemmas and a theorem of M. Scheepers. Let X be a subgroup of $\mathbb{Z}^{\mathbb{N}}$. We shall say that X has the $\mathcal{P}$-property if for every $x, y \in X$, $x + y = \mathbb{O}$ (constant zero sequence) or $\exists^\infty n\ (x + y)(n) \neq 0$. Notice that this is just the Vitali equivalence relation, and that every subgroup $X \subseteq \mathbb{Z}^{\mathbb{N}}$ with the property $\mathcal{P}$ picks at most one representative of each class.

Lemma 3. *Suppose that $X \subseteq \mathbb{Z}^{\mathbb{N}}$ is an infinite subgroup with the $\mathcal{P}$-property and of cardinality smaller than $\kappa \leqslant \mathfrak{c}$. Assume also that $Y \subseteq \mathbb{Z}^{\mathbb{N}}$ is a set of cardinality κ. Then there is $y \in Y$ such that $\langle X \cup \{y\}\rangle$ has the $\mathcal{P}$-property.*

Proof. It suffices to notice that $\langle X \cup \{y\}\rangle = \bigcup_{i\in\mathbb{Z}}(X + i\{y\})$. □

Lemma 4. *Suppose that $\operatorname{cov}(\mathcal{M}) = \mathfrak{c}$. Then there is a set $\widetilde{L} \subseteq \mathbb{Z}^{\mathbb{N}}$ such that $\langle\widetilde{L}\rangle$ is strongly measure zero, unbounded, and satisfies the $\mathcal{P}$-property.*

Proof. By $\operatorname{cov}(\mathcal{M}) = \mathfrak{c}$, there is L, a $\mathfrak{c}$-Lusin set in $\mathbb{Z}^{\mathbb{N}}$ such that L^n is a Rothberger set, for every $n \in \mathbb{N}$ (see [3, Theorem 2.13] or [6] for an easy proof). Clearly, L is unbounded, and $\langle L\rangle$ is strongly measure zero as the Rothberger property is closed under taking continuous images and countable unions. Using Lemma 3 and the fact that for every $f \in \omega^\omega$, the set $L \setminus \{g \in \mathbb{Z}^{\mathbb{N}}: \forall^\infty_n |g(n)| \leqslant f(n)\}$ has cardinality $\mathfrak{c}$, we define inductively $\widetilde{L} \subseteq L$ as required. □

Lemma 5. *Assume that $\operatorname{cov}(\mathcal{M}) = \mathfrak{c}$. Then there exists a strongly measure zero, unbounded subgroup $G \subseteq \mathbb{Z}^{\mathbb{N}}$, satisfying the $\mathcal{P}$-property, and such that $G \cap \langle\{0, 1\}^{\mathbb{N}}\rangle = \{\mathbb{O}\}$.*

Proof. Define $\Psi : \mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}^{\mathbb{N}}$ as follows:

$$\Psi(x)(n) = x(n) \cdot (2n + 1).$$

Then, Ψ is additive and uniformly continuous. Thus, $G = \Psi[\langle\widetilde{L}\rangle]$ is an unbounded and strongly measure zero subgroup of $\mathbb{Z}^{\mathbb{N}}$ with the property,

$$G \cap \left\{g \in \mathbb{Z}^{\mathbb{N}}:\ \forall^\infty n \left|g(n)\right| \leqslant n\right\} = \{\mathbb{O}\}.$$

This finishes the proof as $\langle\{0, 1\}^{\mathbb{N}}\rangle \subseteq \{g \in \mathbb{Z}^{\mathbb{N}}:\ \forall^\infty_n |g(n)| \leqslant n\}$. □

We now follow the terminology from [2]. Assume that $\mathcal{Z}$ is a subset of $\overline{\mathbb{N}}^{\mathbb{N}}$ that consists of nondecreasing functions f which satisfy the following condition: for every n, if $f(n) < \infty$, then $f(n) < f(n + 1)$. If q is a finite increasing sequence of natural numbers, then we define $q' \in \mathcal{Z}$ as follows: $q'(k) = q(k)$ if $k < \operatorname{length}(q)$ and $q'(k) = \infty$, otherwise.

Let $\overline{Q}$ denote the set of all q' which satisfy the above condition. To each $f \in \mathcal{Z}$, we assign the set $\tau(f) = \{k \in \mathbb{N}: k \in \operatorname{range}(f)\}$. Clearly, $\tau\colon\ \mathcal{Z} \to P(\mathbb{N})$ is a bijective function, and if we identify every $x \in P(\mathbb{N})$ with its characteristic function, then $\tau : \mathcal{Z} \to \{0, 1\}^{\mathbb{N}}$ is a homeomorphism [7]. For the rest of this paper we shall denote the set $\tau[\mathbb{N}^{\mathbb{N}\uparrow}]$ by $[\mathbb{N}]^\omega$.

Assume now that in addition to $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$, we have that $\mathfrak{b} = \mathfrak{d}$. Hence there exists a scale $B = \{f_\alpha\colon \alpha < \mathfrak{c}\} \subseteq \mathbb{N}^{\mathbb{N}\uparrow}$, that is a set well ordered by $\leqslant_*$, and such that for every $g \in \mathbb{N}^{\mathbb{N}}$, there exists f_α with $g \leqslant_* f_\alpha$. It is easy to see that we may suppose without loss of generality that for every $\alpha, \beta < \mathfrak{c}$, $\alpha < \beta$, the intersection $\mathrm{range}(f_\alpha) \cap \mathrm{range}(f_\beta)$ is finite.

Let us put $H = \overline{Q} \cup B$. By Theorem 10 of [2], H is strongly measure zero, and H^n is a Hurewicz set for every $n \in \mathbb{N}$.

Theorem 6. *(See Scheepers [5].) If X is a strongly measure zero set with the Hurewicz property, and Y is a strongly measure zero set, then the product $X \times Y$ has strongly measure zero.*

Proof. See Theorem 10 in [5]. □

Corollary 7. *The subgroup $\langle \tau[H] \rangle \subseteq \mathbb{Z}^{\mathbb{N}}$ is a strongly measure zero set with the Hurewicz property.*

Proof. Use the fact that

$$\bigl\langle \tau[H] \bigr\rangle = \bigcup_{n\in\mathbb{N}} \bigl\{ i_1\tau[H] + \cdots + i_n\tau[H]\colon\ i_1, \ldots, i_n \in \{-1, 1\} \bigr\},$$

and repeatedly apply Theorem 6. □

Lemma 8. *The subgroup $\langle \tau[B] \rangle \subseteq \mathbb{Z}^{\mathbb{N}}$ is not a Menger set.*

Proof. Suppose that e denotes the continuous function $e : [\mathbb{N}]^\omega \to \mathbb{N}^{\mathbb{N}\uparrow}$ of assigning an increasing enumeration to an infinite subset of $\mathbb{N}$. Then e maps $\tau[B]$ onto a dominating subset of $\mathbb{N}^{\mathbb{N}}$, hence $\tau[B]$ is not a Menger set. Let $B' = \{x \in [\mathbb{N}]^\omega\colon\ x$ is of the form $x_1 + \cdots + x_k$, for some $x_1, \ldots, x_k \in \tau[B]$ and $k \in \mathbb{N}\}$. Clearly, B' is not a Menger set either. On the other hand, by pairwise almost disjointness of elements of $\tau[B]$, we have that $\langle \tau[B] \rangle \cap \{0, 1\}^{\mathbb{N}} = B'$ which finishes the proof. □

Let $G_1 = \langle \tau[B] \rangle \subseteq \langle \tau[H] \rangle$. To complete the proof of Theorem 2, we apply Scheepers' theorem again, and we obtain that $G + \langle \tau[H] \rangle$ is a strongly measure zero subgroup. Thus, a smaller set $\widetilde{G} = G + G_1$ is a strongly measure zero subgroup as well. Let us assume that $x \in G$, $x \neq \mathbb{O}$, $y \in G_1$. We have that,

$$\exists^\infty n \bigl| x(n) + y(n) \bigr| > n.$$

Consequently,

$$\widetilde{G} \cap \bigl\{ g \in \mathbb{Z}^{\mathbb{N}}\colon\ \forall^\infty n \bigl| g(n) \bigr| \leqslant n \bigr\} = G_1.$$

Hence the intersection of $\widetilde{G}$ with an F_σ set is equal to G_1. This implies that $\widetilde{G}$ is an unbounded and strongly measure zero subgroup of $\mathbb{Z}^{\mathbb{N}}$ without the Menger property. □

In [4, Problem 9.15], the authors ask whether there could exist an unbounded, Menger, strongly measure zero subgroup of $\mathbb{Z}^{\mathbb{N}}$, which does not have the Rothberger property. The next theorem gives an affirmative answer to this question.

Theorem 9. *It is consistent with ZFC that there is $\widetilde{G}$, an unbounded (with respect to $\leqslant_*$) and strongly measure zero subgroup of $\mathbb{Z}^{\mathbb{N}}$ that has the Menger property, and does not have the Rothberger property.*

Proof. We follow the same scenario as in the proof of Theorem 2 above.

Lemma 10. *There is M, a model of ZFC theory, satisfying $\aleph_1 = \mathfrak{b} = \mathrm{cov}(\mathcal{M}) < \mathfrak{d} = \aleph_2$ in which there exists an unbounded, strongly measure zero subgroup $G \subseteq \mathbb{Z}^{\mathbb{N}}$ such that $G \cap \{g\colon\ \forall_n^\infty |g(n)| \leqslant n\} = \{\mathbb{O}\}$.*

Proof. Let V be a model of ZFC which satisfies $\aleph_1 < \mathfrak{b} = \aleph_2 = \mathfrak{c}$. Suppose that $P_{\aleph_1}$ is an $\aleph_1$-iteration of the random forcing with finite supports. If F is a V-generic filter on $P_{\aleph_1}$, then we have that $\aleph_1 = \mathfrak{b} = \mathrm{cov}(\mathcal{M}) < \mathfrak{d} = \aleph_2 = \mathfrak{c}$ holds in $M = V[F]$ (see [1, p. 382]). Assume that L is a set of $\aleph_1$ generic Cohen reals added by F. Then L is an $\aleph_1$-Lusin set in $\mathbb{Z}^{\mathbb{N}}$ such that L^n has the Rothberger property, for every $n \in \mathbb{N}$ (this was pointed out by B. Tsaban). Thus, using Lemma 5 from above, we can construct an unbounded, strongly measure zero subgroup $G \subseteq \mathbb{Z}^{\mathbb{N}}$ which satisfies

$$G \cap \bigl\{ g\colon\ \forall_n^\infty \bigl| g(n) \bigr| \leqslant n \bigr\} = \{\mathbb{O}\}. \qquad \square$$

Next we combine methods from Lemma 9.12 in [4] and from Theorem 2 above. Let $\eta : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be any bijection that satisfies $\eta(m, n) \geqslant m, n$, for every $m, n \in \mathbb{N}$. We may assume for example that η is a function defined by $\eta(m, n) = \frac{1}{2}(m^2 + 2mn + n^2 + 3m + n)$. Let us put $\Psi(f, g)(n) = \eta((f, g)(n))$. Clearly, $\Psi : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is a homeomorphism, and $\Phi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ defined by $\Phi(h) = \pi_1 \Psi^{-1}(h)$, where π_1 denotes the projection onto the first coordinate, is a continuous function.

Lemma 11. *(See Machura and Tsaban [4].) Assume that* $\aleph_1 = \mathfrak{b} = \mathrm{cov}(\mathcal{M})$. *Then there exists a subgroup* D *of the Cantor group* $\{0,1\}^{\mathbb{N}}$ *which has cardinality* $\aleph_1$, *and satisfies the following two conditions.*

(1) *D is generated by $\tau[B]$, where B is an unbounded and well ordered by $\leqslant_*$ subset of $\mathbb{N}^{\mathbb{N}\uparrow}$ of cardinality $\aleph_1$,*
(2) *$D \subseteq [\mathbb{N}]^{\omega} \cup \{\mathbb{O}\}$ and $\Phi \circ e[D \setminus \{\mathbb{O}\}]$ is a family of size $\aleph_1$ that witnesses* $\mathrm{cov}(\mathcal{M}) = \aleph_1$.

Proof. See Lemma 9.12 in [4]. □

Lemma 12. *The subgroup $\langle \tau[B]\rangle \subseteq \mathbb{Z}^{\mathbb{N}}$ does not have the Rothberger property.*

Proof. Define as in Lemma 8.1 from [4] the continuous group homomorphism $\overline{\Phi}: \mathbb{Z}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ by $\overline{\Phi}(f)(n) = f(n) \bmod 2$. Then,

$$\overline{\Phi}\big[\big\langle \tau[B]\big\rangle\big] = \overline{\Phi}\big[\langle D\rangle\big] = D.$$

By Lemma 12, we know that D does not have the Rothberger property. Thus, $\langle \tau[B]\rangle$ is not a Rothberger set either. □

Let $G_1 = \langle \tau[B]\rangle$ and $H = \overline{Q} \cup B$. Since $\langle \tau[H]\rangle$ is a strongly measure zero subgroup with the Hurewicz property, we obtain that $\widetilde{G} = G + G_1$ is an unbounded and strongly measure zero subgroup of $\mathbb{Z}^{\mathbb{N}}$ without the Rothberger property. Clearly, $\widetilde{G}$ is of cardinality $\aleph_1 < \mathfrak{d}$, hence it is a Menger subgroup as was required. □

Remark 13. Notice that Theorems 2 and 10 prove that the settings (2.b) and (3.c) of Fig. 1 in [4] are consistent with ZFC.

Acknowledgements

We thank Roman Pol and Boaz Tsaban for making useful remarks on this paper.

References

[1] T. Bartoszyński, H. Judah, On the Structure of the Real Line, A.K. Peters, Massachussets, 1995.
[2] T. Bartoszyński, B. Tsaban, Hereditary topological diagonalizations and the Menger–Hurewicz conjectures, Proc. Amer. Math. Soc. 134 (2) (2006) 605–615.
[3] W. Just, A.W. Miller, M. Scheepers, P.J. Szeptycki, The combinatorics of open covers II, Topology Appl. 73 (1996) 241–266.
[4] M. Machura, B. Tsaban, The combinatorics of the Baer–Specker group, Israel J. Math., in press.
[5] M. Scheepers, Finite powers of strong measure zero sets, J. Symbolic Logic 64 (3) (1999) 1295–1306.
[6] B. Tsaban, *o*-Bounded groups and other topological groups with strong combinatorial properties, Proc. Amer. Math. Soc. 134 (2006) 881–891.
[7] B. Tsaban, L. Zdomskyy, Scales, fields, and a problem of Hurewicz, J. Eur. Math. Soc. 10 (2008) 837–866.