Approximation orders of formal Laurent series by $\beta$-expansions

Shuai Ling Wang

School of Mathematics and Statistics, Wuhan University, Wuhan, Hubei 430072, PR China

Received 29 April 2008; revised 5 June 2008
Available online 10 July 2008
Communicated by Gary L. Mullen

Abstract

We prove that almost all (with respect to Haar measure) formal Laurent series are approximated with the linear order $-(\deg \beta)n$ by their $\beta$-expansions convergents. Hausdorff dimensions of sets of Laurent series which are approximated by all other orders, are determined. In contrast, the corresponding theory of real case has not been established.

© 2008 Elsevier Inc. All rights reserved.

Keywords: $\beta$-Expansion; Approximation; Laurent series; Finite field; Hausdorff dimension

1. Introduction

Let $\mathbb{F}$ be a finite field of $q$ elements. Denote by $\mathbb{F}[X]$ the ring of polynomials with coefficients in $\mathbb{F}$ and $\mathbb{F}(X)$ the field of fractions. Let $\mathbb{F}((X^{-1}))$ be the field of formal Laurent series, i.e.,

$$\mathbb{F}((X^{-1})) = \left\{ \sum_{n=n_0}^{+\infty} x_n X^{-n} : x_n \in \mathbb{F} \text{ and } n_0 \in \mathbb{Z} \right\}.$$  

For $x = \sum_{n=n_0}^{+\infty} x_n X^{-n} \in \mathbb{F}((X^{-1}))$, $\deg(x) = -\inf\{n \in \mathbb{Z} : x_n \neq 0\}$ is called the degree of $x$, with the convention, $\deg(0) = -\infty$.

The norm on $\mathbb{F}((X^{-1}))$ is defined as $\| \cdot \| = q^{\deg(\cdot)}$. For any $x, y \in \mathbb{F}((X^{-1}))$, the following is known:

$E-mail$ $address$: wangshuailing@yahoo.com.cn.

1071-5797/$ – see front matter © 2008 Elsevier Inc. All rights reserved.
doi:10.1016/j.ffa.2008.06.003
(1) \( \|x\| \geq 0; \) moreover, \( \|x\| = 0 \) if and only if \( x = 0; \)
(2) \( \|xy\| = \|x\| \cdot \|y\|; \)
(3) for \( \alpha, \beta \in \mathbb{F}, \|\alpha x + \beta y\| \leq \max(\|x\|, \|y\|); \)
(4) for \( \alpha, \beta \in \mathbb{F}, \alpha \neq 0, \beta \neq 0, \) if \( \|x\| \neq \|y\|, \)
then
\[
\|\alpha x + \beta y\| = \max(\|x\|, \|y\|). \tag{1.1}
\]

That is to say, \( \| \cdot \| \) is a non-Archimedean norm on the field \( \mathbb{F}((X^{-1})) \). It is known that \( \mathbb{F}((X^{-1})) \) is a complete metric space under the metric \( d \) defined by 
\[
d(x, y) = \|x - y\|.
\]

**Remark.** By the non-Archimedean property of the norm \( \| \cdot \| \), we know the following facts:

(i) Every point in a ball may be regarded as the center of this ball.
(ii) If two balls intersect, the one with larger radius must contain the other.

Let \( I = \{x \in \mathbb{F}((X^{-1})): \|x\| < 1\} \), which is isomorphic to \( \prod_{n \geq 1} \mathbb{F} \). The set \( I \) is an Abel compact group. As a result, there exists a unique normalized Haar measure \( \mu \) on \( I \) given by
\[
\mu(B(a, q^{-r})) = q^{-r},
\]
where \( B(a, q^{-r}) = \{x \in \mathbb{F}((X^{-1})): \|x - a\| < q^{-r}\} \) is the ball with the center \( a \in I \) and the radius \( q^{-r} \) \((r \in \mathbb{N})\). Note that \( \mu(I) = 1 \) and \( (I, \mathcal{B}(I), \mu) \) is a probability space, where \( \mathcal{B}(I) \) is Borel field on \( I \). Every \( x \in \mathbb{F}((X^{-1})) \) has a unique (Artin) decomposition (see [1]) as \( x = [x] + \{x\} \), where the integral part \([x]\) belongs to \( \mathbb{F}[X]\) and the fractional part \(\{x\}\) belongs to \( I \).

Let us give some descriptions about the \( \beta \)-expansions of formal Laurent series introduced by K. Scheicher [13], M. Hbaib and M. Mkaouar [8] independently.

Let \( \beta \in \mathbb{F}((X^{-1})) \) with \( \|\beta\| > 1 \). The \( \beta \)-transformation \( T_\beta \) on \( I \) is defined as
\[
T_\beta x = \beta x - [\beta x],
\]
where \([x]\) denotes the integer part of \( x \), that is, the polynomial part of \( x \). Then every \( x \in I \) can be written as
\[
x = \frac{\varepsilon_1(x)}{\beta} + \frac{\varepsilon_2(x)}{\beta^2} + \cdots + \frac{\varepsilon_n(x)}{\beta^n} + \cdots \tag{1.2}
\]
where \( \varepsilon_1(x) = [\beta x] \) and \( \varepsilon_n(x) = \varepsilon_1(T_\beta^{n-1} x) \) for all \( n \geq 2 \). We call the form (1.2) the \( \beta \)-expansion of \( x \) in base \( \beta \), for simplicity, denoted by
\[
(\varepsilon_1(x), \varepsilon_2(x), \ldots, \varepsilon_n(x), \ldots).
\]
Moreover, \( \varepsilon_1(x), \varepsilon_2(x), \ldots, \varepsilon_n(x), \ldots \) are called the \( \beta \)-digits of \( x \).

Since \( \varepsilon_1(x) = [\beta x], \varepsilon_n(x) = [\beta T_\beta^{n-1} x], \) and \( \|T_\beta^{n-1} x\| < 1 \), then \( \|\varepsilon_n(x)\| < \|\beta\| \) (i.e., \( \deg(\varepsilon_n(x)) < \deg(\beta) \)) for all \( x \in I \) and \( n \geq 1 \). Put
\[
P = \{\varepsilon \in \mathbb{F}[X]: \|\varepsilon\| < \|\beta\|\}.
\]
K. Scheicher [13] also pointed that, for any given sequence \( \{\varepsilon_n\}_{n \geq 1} \) with \( \varepsilon_n \in P \), there exists a unique \( x \in I \) such that \( \varepsilon_n(x) = \varepsilon_n \) for all \( n \geq 1 \). Note that \( \#P = \|\beta\| \), that is, the number of all possible digits is \( \|\beta\| \). Some metric properties are studied by B. Li, J. Wu, and J. Xu [10]. They obtained that \( T_\beta \) is invariant and ergodic with respect to the Haar measure \( \mu \), the \( \beta \)-digits functions \( \varepsilon_n(\cdot) \) are independent and identically distributed as a random variables sequence. They also obtained some limit theorems for \( \beta \)-digits and Hausdorff dimensions of some exceptional sets (for details, see [10]).

Such an expansion is analogue to the \( \beta \)-expansion of real numbers, which was introduced by A. Rényi [12], where he proved that there is a \( T_\beta \)-invariant measure \( \nu \) which is equivalent to the Lebesgue measure \( m \), i.e., there exists a constant \( C > 0 \) such that \( \frac{1}{C} \nu(E) \leq m(E) \leq C \nu(E) \) for every Lebesgue measurable set \( E \). Later, A.O. Gel’fond [7] and W. Parry [11] obtained independently this unique normalized \( T_\beta \)-invariant measure. The arithmetic properties of \( \beta \)-expansions were also studied extensively. For example, many researchers tried to describe the numbers which have the eventually periodic expansions (see [3,14]) and the numbers with the finite expansions (see [2,6,9]). For the formal Laurent series case, the authors ([8] and [13]) independently gave a complete characterization of the sets of series with eventually periodic and finite expansions. However, in the real case, one can only get a direction and the other is also the conjecture (for details, see [14]). For the relationship between the real and the Laurent series case of \( \beta \)-expansion, see [8,10,13].

**Definition 1.1.** For any given block \( (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \) with \( \varepsilon_i \in P \) \( (1 \leq i \leq n) \),

\[
J(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) = \{ x \in I : \varepsilon_1(x) = \varepsilon_1, \varepsilon_2(x) = \varepsilon_2, \ldots, \varepsilon_n(x) = \varepsilon_n \}
\]

is called an \( n \)th cylinder of the \( \beta \)-expansion.

From [10], we know the measures of \( n \)th cylinders are very regular. In fact, we have the following.

**Proposition 1.2.** (See [10].) For any cylinder \( J(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \), we have

\[
J(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) = B\left(\frac{\varepsilon_1}{\beta} + \frac{\varepsilon_2}{\beta^2} + \cdots + \frac{\varepsilon_n}{\beta^n}, \frac{1}{\|\beta\|^n}\right).
\]

(1.3)

As a consequence, \( \mu(I(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)) = \|\beta\|^{-n} \).

From (1.3), we know every cylinder is a ball. Conversely, we have

**Proposition 1.3.** Let \( B(x, r) \subset I \) be a ball. Then there exists \( n_0 \in \mathbb{N} \) such that

\[
J(\varepsilon_1(x), \varepsilon_2(x), \ldots, \varepsilon_{n_0+1}(x)) \subset B(x, r) \subset J(\varepsilon_1(x), \varepsilon_2(x), \ldots, \varepsilon_{n_0}(x)).
\]

(1.4)

For any \( x \in I \), we denote the partial sums of the form (1.2) by

\[
\omega_n(x) = \frac{\varepsilon_1(x)}{\beta} + \frac{\varepsilon_2(x)}{\beta^2} + \cdots + \frac{\varepsilon_n(x)}{\beta^n}
\]

and call them the convergents of the \( \beta \)-expansion of \( x \). In this paper, we consider the convergence speeds of \( \beta \)-expansions, i.e., the speeds of convergence of \( \omega_n(x) \). We have the following.
**Theorem 1.4.** For \( \mu \)-almost all \( x \in I \),

\[
\lim_{n \to \infty} \frac{1}{n} \log_q \| x - \omega_n(x) \| = -\deg \beta.
\]

Roughly speaking, Theorem 1.4 means that for \( \mu \)-almost all \( x \in I \), we have \( \| x - \omega_n(x) \| \approx q^{-(\deg \beta)n} \). We say \( x \) is approximated by its convergents \( \omega_n(x) \) with the linear order \( -\deg \beta n \).

We would like to know how many Laurent series can be approximated with other orders \( -\phi(n) \) with \( \phi \) being a nonnegative increasing function on \( \mathbb{N} \) and \( \phi(n) \to \infty \) as \( n \to \infty \), that is to say, we want to know the size of the following set:

\[
\left\{ x \in I : \lim_{n \to \infty} \frac{1}{\phi(n)} \log_q \| x - \omega_n(x) \| = -1 \right\}.
\]

(1.5)

In [5], A.H. Fan and J. Wu considered the approximation, with polynomial and exponential orders, of Laurent series by Oppenheim rational functions.

The set (1.5) needs that the limsup and liminf of the quantity \( \frac{1}{\phi(n)} \log_q \| x - \omega_n(x) \| \) are both equal to \(-1\). However, we have the following.

**Proposition 1.5.** Let \( \phi \) be a nonnegative increasing function on \( \mathbb{N} \) and \( \phi(n) \to \infty \) as \( n \to \infty \). If \( \eta := \lim \inf_{n \to \infty} \frac{\phi(n)}{n} > \deg \beta \), then the set

\[
\left\{ x \in I : \limsup_{n \to \infty} \frac{1}{\phi(n)} \log_q \| x - \omega_n(x) \| = -1 \right\}
\]

is countable at most.

Note that for \( \mu \)-almost all \( x \in I \),

\[
\liminf_{n \to \infty} \frac{1}{n} \log_q \| x - \omega_n(x) \| = -\deg \beta.
\]

Therefore, we replace the limit with liminf in the set (1.5). We have the following.

**Theorem 1.6.** Let \( \phi \) be a nonnegative increasing function on \( \mathbb{N} \) and \( \phi(n) \to \infty \) as \( n \to \infty \). Define \( \eta = \lim \inf_{n \to \infty} \frac{\phi(n)}{n} \)

Define \( A_\phi \) as

\[
A_\phi = \left\{ x \in I : \liminf_{n \to \infty} \frac{1}{\phi(n)} \log_q \| x - \omega_n(x) \| = -1 \right\}.
\]

We denote by \( \dim_{\text{Haus}} \) the Hausdorff dimension.

(i) If \( 0 \leq \eta < \deg \beta \), then \( \dim_{\text{Haus}} A_\phi = 0 \).
(ii) If \( \deg \beta \leq \eta < +\infty \), then

\[
\dim_{\text{Haus}} A_\phi = \frac{\deg \beta}{\eta}.
\]

(iii) If \( \eta = +\infty \), then \( \dim_{\text{Haus}} A_\phi = 0 \).
Take $\phi(n) = n\alpha$, Theorem 1.6 gives the dimensions of the following level sets.

**Corollary 1.7.** For $\alpha > \deg \beta$, we have

$$\dim_H \left\{ x \in I : \liminf_{n \to \infty} \frac{1}{n} \log_q \| x - \omega_n(x) \| = -\alpha \right\} = \frac{\deg \beta}{\alpha}.$$  

From Corollary 1.7, we know that the above level sets have a rich multifractal structure.

Finally we consider a limsup set of Laurent series which is approximated by some orders. We have

**Theorem 1.8.** Let $\phi(n)$ be a nonnegative function. Suppose $\liminf_{n \to \infty} \frac{\phi(n)}{n} = \xi$. Then

$$\dim_H \left\{ x \in I : \| x - \omega_n(x) \| \leq \| \beta \|^{-\phi(n)} \text{ infinitely often} \right\} = \frac{1}{\xi}.$$  

### 2. Convergence speeds of $\beta$-expansions and metric properties of $L_n$

Let $x \in I$ and $n \in \mathbb{N}$, put

$$L_n(x) = \sup \{ k \geq 0 : \varepsilon_{n+j}(x) = 0 \text{ for all } j \text{ with } 1 \leq j \leq k \}, \quad (2.1)$$

that is, the maximal length of all strings of 0’s behind the $\beta$-digit $\varepsilon_n(x)$. $L_n(x)$ is a key quantity of the proofs. In order to study the approximation theory of $\beta$-expansions, we just need to consider the quantity $L_n(x)$ by the following property.

**Proposition 2.1.** For any $x \in I$,

$$\log_q \| x - \omega_n(x) \| = \deg(\varepsilon_{n+L_n(x)+1}(x)) - (n + L_n(x) + 1) \deg \beta. \quad (2.2)$$

**Proof.** For any $n \geq 1$, since $\varepsilon_{n+L_n(x)+1}(x) \neq 0$, we have

$$\frac{\| \varepsilon_{n+L_n(x)+k}(x) \|}{\beta^{n+L_n(x)+k}} < \frac{\| \beta \|}{\| \beta \|^{n+L_n(x)+k}} \leq \frac{1}{\| \beta \|^{n+L_n(x)+1}} \leq \frac{\| \varepsilon_{n+L_n(x)+1}(x) \|}{\beta^{n+L_n(x)+1}} \quad (2.3)$$

for all $k > 1$. By the definition of $\omega_n(x)$ and (1.2), we have

$$\| x - \omega_n(x) \| = \| \varepsilon_{n+1}(x) \| + \frac{\varepsilon_{n+2}(x)}{\beta^{n+2}} + \cdots + \frac{\varepsilon_{n+L_n(x)+1}(x)}{\beta^{n+L_n(x)+1}} + \cdots$$

by (1.1) and (2.3).

$$= \frac{\varepsilon_{n+L_n(x)+1}(x)}{\beta^{n+L_n(x)+1}} + \cdots \quad (by \ (2.1))$$

(by (1.1))
Therefore
\[
\log_q \| x - \omega_n(x) \| = \log_q q^{\deg(\varepsilon_n + L_n(x) + 1(x)) - (n + L_n(x) + 1) \deg \beta} \\
= \deg(\varepsilon_n + L_n(x) + 1(x)) - (n + L_n(x) + 1) \deg \beta.
\]

From Proposition 2.1, we know the set \( A_\phi \) in Theorem 1.6 is just
\[
\left\{ x \in I : \limsup_{n \to \infty} \frac{n + L_n(x)}{\phi(n) / \deg \beta} = 1 \right\}.
\]
(2.4)

So we just need to consider the dimensions of the set (2.4).

In the following, we prove Proposition 1.5 by using Proposition 2.1.

**Proof of Proposition 1.5.** Let \( x \in I \) whose \( \beta \)-digits are not ultimately zero, that is, there is a subsequence of digits \( \{\varepsilon_{n_k}(x)\} \) with \( \varepsilon_{n_k}(x) \neq 0 \) for all \( k \). Then \( L_{n_k} - 1(x) = 0 \) by the definition of \( L_n \). So
\[
\liminf_{n \to \infty} \frac{n + L_n(x)}{\phi(n)} \deg \beta \leq \liminf_{n \to \infty} \frac{n_k - 1}{\phi(n_k - 1)} \deg \beta \leq \frac{1}{n} \deg \beta < 1.
\]

Then
\[
\limsup_{n \to \infty} \frac{1}{\phi(n)} \log_q \| x - \omega_n(x) \| > -1
\]
by Proposition 2.1. Thus we get the desired result, since the set of Laurent series whose \( \beta \)-digits are ultimately zero, is countable. \( \square \)

The following describes the probabilistic property of the sequence \( \{L_n(\cdot)\}_{n \geq 1} \).

**Proposition 2.2.** The random variables sequence \( \{L_n(\cdot)\}_{n \geq 1} \) is identically distributed. Moreover, for any nonnegative \( N \) and for all \( n \geq 1 \), we have
\[
\mu\left( \left\{ x \in I : L_n(x) = N \right\} \right) = \frac{\| \beta \| - 1}{\| \beta \| N + 1}.
\]

**Remark.** The random variables sequence \( \{L_n(\cdot)\}_{n \geq 1} \) is not independent. In fact, for example, \( \mu(\{ x \in I : L_1(x) = 1, L_2(x) = 1 \}) = 0 \), however,
\[
\mu(\{ x \in I : L_1(x) = 1 \}) \mu(\{ x \in I : L_2(x) = 1 \}) = \frac{(\| \beta \| - 1)^2}{\| \beta \|^4}.
\]

**Proof.** Since
\[
\{ x \in I : L_n(x) = N \} = \bigcup_{\varepsilon_1, \ldots, \varepsilon_n \in P} \bigcup_{\varepsilon_{n+N+1} \in P \setminus \{0\}} J_{n+N+1},
\]
where \( J_{n+N+1} = J(\varepsilon_1, \ldots, \varepsilon_n, 0, \ldots, 0, \varepsilon_{n+N+1}) \), then

\[
\mu\left( \left\{ x \in I : L_n(x) = N \right\} \right) = \sum_{\varepsilon_1, \ldots, \varepsilon_n \in P} \sum_{\varepsilon_{n+1} \in P \setminus \{0\}} \mu(J_{n+N+1}) = \|\beta\|^n (\|\beta\| - 1) \|\beta\|^{-(n+N+1)} = \|\beta\| - 1 \|\beta\|^{N+1},
\]

where the second equality is because \( \mu(J_{n+N+1}) = \|\beta\|^{-(n+N+1)} \) and \( \#P = \|\beta\| \).

The following is a metric theorem on \( L_n(x) \). We turn to study the \( \limsup \) of \( L_n(x) \) instead of its limit since \( \liminf_{n \to \infty} L_n(x) = 0 \).

**Proposition 2.3.** For \( \mu \)-almost all \( x \in I \), we have

\[
\limsup_{n \to \infty} \frac{L_n(x)}{\log_{\|\beta\|} n} = 1.
\]

**Proof.** Step 1. For \( \mu \)-almost all \( x \in I \), we have \( \limsup_{n \to \infty} \frac{L_n(x)}{\log_{\|\beta\|} n} \leq 1 \).

In fact, for any \( \varepsilon > 0 \), let

\[
A_n(\varepsilon) = \left\{ x \in I : L_n(x) > (1 + \varepsilon) \log_{\|\beta\|} n \right\}
\]

and \( m = \left\lfloor (1 + \varepsilon) \log_{\|\beta\|} n \right\rfloor \), i.e., the greatest integer less or equal to \( (1 + \varepsilon) \log_{\|\beta\|} n \). Then

\[
\mu(A_n(\varepsilon)) = \sum_{k=m+1}^{\infty} \mu(\left\{ x \in I : L_n(x) = k \right\}) = \frac{1}{\|\beta\|^{m+1}}
\]

by Proposition 2.2. Thus

\[
\sum_{n=1}^{\infty} \mu(A_n(\varepsilon)) = \sum_{n=1}^{\infty} \frac{1}{\|\beta\|^{m+1}} \leq \sum_{n=1}^{\infty} \frac{1}{\|\beta\|^{(1+\varepsilon) \log_{\|\beta\|} n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} < +\infty.
\]

Using the Borel–Cantelli lemma, we obtain

\[
\mu(\left\{ x \in I : L_n(x) > (1 + \varepsilon) \log_{\|\beta\|} n \text{ infinite often (i.o.)} \right\}) = 0.
\]

Thus

\[
\mu\left( \left\{ x \in I : \limsup_{n \to \infty} \frac{L_n(x)}{\log_{\|\beta\|} n} \leq 1 \right\} \right) = 1. \tag{2.5}
\]

Step 2. Let \( (\varepsilon_1(x), \varepsilon_2(x), \ldots, \varepsilon_n(x), \ldots) \) be the \( \beta \)-expansion of \( x \in I \).
\[ R_{m,n}(x) = \max\{k \geq 0: \varepsilon_{i+1}(x) = \cdots = \varepsilon_{i+k}(x) = 0 \text{ for some } m \leq i \leq n - k\}, \]

that is, the maximal length of strings of 0’s between \( \varepsilon_{m+1}(x) \) and \( \varepsilon_{n}(x) \). Denote \( r_n(x) := R_{0,n}(x) \). We claim that

\[ \mu\left( \left\{ x \in I: \liminf_{n \to \infty} \frac{r_n(x)}{\log_{\|\beta\|} n} \geq 1 \right\} \right) = 1. \quad (2.6) \]

In fact, for any \( \varepsilon > 0 \), we denote \( t_n = \left(1 - \varepsilon\right) \log_{\|\beta\|} n \) and \( k_n = \lfloor n/t_n \rfloor \), the random variables \( R_{0,t_n}(\cdot), R_{t_n,2t_n}(\cdot), \ldots, R_{(k-1)t_n,kt_n}(\cdot) \) are independent and identically distributed, and

\[ \mu\left( \left\{ x \in I: R_{(i-1)t_n,kt_n}(x) = t_n \right\} \right) = \frac{1}{\|\beta\|^{t_n}} \quad \text{for all } 1 \leq i \leq k_n. \]

Since \( R_{(i-1)t_n,kt_n}(x) \leq t_n \) for all \( 1 \leq i \leq k_n \), then

\[ \mu\left( \left\{ x \in I: R_{(i-1)t_n,kt_n}(x) < t_n \right\} \right) = 1 - \frac{1}{\|\beta\|^{t_n}}. \]

So we have

\[ \mu\left( \left\{ x \in I: R_{0,n}(x) < t_n, \ldots, R_{(k-1)t_n,kt_n}(x) < t_n \right\} \right) = \left(1 - \frac{1}{\|\beta\|^{t_n}}\right)^{k_n}. \]

For

\[ \left\{ x \in I: r_n(x) < t_n \right\} \subset \left\{ x \in I: R_{0,n}(x) < t_n, \ldots, R_{(k-1)t_n,kt_n}(x) < t_n \right\}, \]

we obtain

\[ \sum_{n=1}^{\infty} \mu\left( \left\{ x \in I: r_n(x) < t_n \right\} \right) \leq \sum_{n=1}^{\infty} \left(1 - \frac{1}{\|\beta\|^{t_n}}\right)^{k_n} < +\infty. \]

By the Borel–Cantelli lemma, we have for any \( \varepsilon > 0 \),

\[ \mu\left( \left\{ x \in I: r_n(x) < \left[(1 - \varepsilon) \log_{\|\beta\|} n \right] \text{ i.o.} \right\} \right) = 0. \]

Thus we obtain (2.6).

**Step 3.** For all \( x \in I \), we have

\[ \limsup_{n \to \infty} \frac{L_n(x)}{\log_{\|\beta\|} n} \geq \liminf_{n \to \infty} \frac{r_n(x)}{\log_{\|\beta\|} n}. \quad (2.7) \]

In fact, for any \( n \in \mathbb{N} \) and \( x \in I \), by the definitions of \( L_n(x) \) and \( r_n(x) \), we know \( \max_{1 \leq k \leq n} L_k(x) = r_n + L_n(x) \). We choose an integer \( 1 \leq k_n \leq n \) such that \( L_{k_n}(x) = \max_{1 \leq k \leq n} L_k(x) \). Therefore
Thus we obtain (2.7).

Combining (2.6) and (2.7), we get

\[
\mu \left( \left\{ x \in I : \limsup_{n \to \infty} \frac{L_n(x)}{\log_{\| \beta \|} n} \geq 1 \right\} \right) = 1. \tag{2.8}
\]

We obtain the desired by (2.5) and (2.8). \(\square\)

We end this section with the proofs of Theorem 1.4 and Proposition 1.3.

**Proof of Theorem 1.4.** From Proposition 2.3, we know

\[
\mu \left( \left\{ x \in I : \lim_{n \to \infty} \frac{L_n(x)}{n} = 0 \right\} \right) = 1. \tag{2.9}
\]

By Proposition 2.1 and (2.9), we obtain for \(\mu\)-almost all \(x \in I\),

\[
\lim_{n \to \infty} \frac{1}{n} \log_q \| x - \omega_n(x) \| = -\left( 1 + \lim_{n \to \infty} \frac{L_n(x)}{n} \right) \deg \beta = -\deg \beta. \tag{2.10}
\]

**Proof of Proposition 1.3.** There exists \(n_1 \in \mathbb{N}\) such that \(q^{-(n_1 + 1)} < r \leq q^{-n_1}\). So

\[
B(x, r) = B(x, q^{-n_1}) \tag{2.10}
\]

because \(d\) is a discrete metric. We choose \(n_0 \in \mathbb{N}\) such that

\[
n_0 \deg \beta \leq n_1 < (n_0 + 1) \deg \beta. \tag{2.11}
\]

From (2.11), we know \(q^{-n_1} < q^{-n_0 \deg \beta}\), so

\[
B(x, q^{-n_1}) \subset B(x, q^{-n_0 \deg \beta}) = J(\varepsilon_1(x), \ldots, \varepsilon_{n_0}(x)), \tag{2.12}
\]

where the equality is because of Proposition 1.2.

Since \(x \in B(x, q^{-n_1}) \cap B(x, q^{-(n_0 + 1) \deg \beta})\) and \(q^{-n_1} > q^{-(n_0 + 1) \deg \beta}\), then

\[
J(\varepsilon_1(x), \ldots, \varepsilon_{n_0 + 1}(x)) = B(x, q^{-(n_0 + 1) \deg \beta}) \subset B(x, q^{-n_1}) \tag{2.13}
\]

by the non-Archimedean property of \(\| \cdot \|\). Thus we obtain (1.4) by the equality (2.10), (2.12), and (2.13). \(\square\)
3. Dimensional propositions of $L_n$

The definition of Hausdorff measure on $I$ is the same as that on $\mathbb{R}^n$ (see [4]). Given $s > 0$ and a subset $E$ of $I$, the s-Hausdorff measure is given by

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \left\{ \inf \sum_j (\text{diam}(D_j))^s \right\},$$

where the infimum is taken over all covers of $E$ by disks $D_j$ of diameter (in the sense of the metric $d$) at most $\delta$ and diam denotes the diameter of a set. The Hausdorff dimension of $E$ is defined by

$$\dim_H(E) = \inf \{ s \geq 0 : \mathcal{H}^s(E) = 0 \}.$$

We first state the mass distribution principle (see [4, Proposition 4.2]) that will be used later.

**Lemma 3.1.** Let $E \subset I$ be a Borel set and $\mu$ be a probability measure with $\mu(E) > 0$. If there exist the constants $C > 0$ and $\delta > 0$ such that

$$\mu(D) \leq C (\text{diam}(D))^s$$

(3.1)

for all disks $D$ with $\text{diam}(D) \leq \delta$, then

$$\dim_H(E) \geq s.$$

Now we describe the maximal length of all strings of 0’s (behind $\varepsilon_n(x)$) by the Hausdorff dimension. In this section, we always assume $\phi$ be a nonnegative function on $\mathbb{N}$. Denote

$$E_\phi = \{ x \in I : L_n(x) \geq \phi(n) \text{ infinitely often} \}$$

and

$$F_\phi = \left\{ x \in I : \limsup_{n \to \infty} \frac{L_n(x)}{\phi(n)} = 1 \right\}.$$

Note that $F_\phi \subset E_{\phi-\varepsilon}$ for any $\varepsilon > 0$. We will study the Hausdorff dimensions of sets $E_\phi$ and $F_\phi$.

Firstly we obtain an upper bound of the Hausdorff dimension of the set $E_\phi$.

**Lemma 3.2.**

$$\dim_H(E_\phi) \leq \frac{1}{1 + \liminf_{n \to \infty} \frac{\phi(n)}{n}}.$$  (3.2)
Proof. We know

\[ E_\phi \subset \bigcap_{N=1}^{\infty} \bigcup_{n \geq N} \{ x \in I : L_n(x) \geq \phi(n) \} \]

and

\[ \{ x \in I : L_n(x) \geq \phi(n) \} \subset \bigcup_{k=[\phi(n)]}^{\infty} \bigcup_{\varepsilon_1, \ldots, \varepsilon_n \in P, \varepsilon_{n+k+1} \in \{0\}} J_{n+k+1}, \]

where \( J_{n+k+1} = J(\varepsilon_1, \ldots, \varepsilon_n, 0, \ldots, 0, \varepsilon_{n+k+1}) \). Denote

\[ \mathcal{C}_N = \{ J_{n+k+1} : n \geq N, k \geq [\phi(n)], \varepsilon_1, \ldots, \varepsilon_n \in P, \varepsilon_{n+k+1} \in P \setminus \{0\} \}. \]

Note that \( \mathcal{C}_N = \bigcup_{n=N}^{\infty} \bigcup_{k=[\phi(n)]}^{\infty} \bigcup_{\varepsilon_1, \ldots, \varepsilon_n \in P} \bigcup_{\varepsilon_{n+k+1} \in \{0\}} J_{n+k+1} \).

Therefore, for any \( N \geq 1 \), \( \mathcal{C}_N \) is a cover of the set \( E_\phi \).

Put \( s = 1/(1 + \lim \inf_{n \to \infty} \phi(n)/n) \). For any \( \varepsilon > 0 \), by the definition of the Hausdorff measure, we have

\[ \mathcal{H}^{s+\varepsilon}(E_\phi) \leq \sum_{J_{n+k+1} \in \mathcal{C}_N} (\text{diam}(J_{n+k+1}))^{s+\varepsilon} \]

\[ = \sum_{n=N}^{\infty} \sum_{k=[\phi(n)]}^{\infty} \sum_{\varepsilon_1, \ldots, \varepsilon_n \in P} \sum_{\varepsilon_{n+k+1} \in P \setminus \{0\}} (2 \| \beta \|^{-(n+k+1)})^{s+\varepsilon} \]

\[ = (2 \| \beta \|^{s+\varepsilon} (\| \beta \| - 1) - 1) \sum_{n=N}^{\infty} \| \beta \|^{nt_n}, \]

where \( t_n = 1 - (s + \varepsilon)(1 + \frac{1}{n} + \frac{[\phi(n)]}{n}) \).

By the definition of \( s \), we know \( \limsup_{n \to \infty} t_n = -\varepsilon (1 + \liminf_{n \to \infty} \frac{\phi(n)}{n}) < -\varepsilon \), then the series \( \sum_{n=N}^{\infty} \| \beta \|^{nt_n} \) is convergent, i.e., \( \mathcal{H}^{s+\varepsilon}(E_\phi) < +\infty \). Therefore

\[ \dim_H(E_\phi) \leq \frac{1}{1 + \lim \inf_{n \to \infty} \frac{\phi(n)}{n}} + \varepsilon. \]

Letting \( \varepsilon \to 0^+ \), we obtain (3.2). \( \Box \)

The following lemma gives a lower bound of the Hausdorff dimension of the set \( F_\phi \) by constructing a Cantor-like subset.
Lemma 3.3. Suppose \( \phi \) be increasing and \( \phi(n) \to \infty \) as \( n \to \infty \). Then we have
\[
\dim_H(F_\phi) \geq \frac{1}{1 + \lim \inf_{n \to \infty} \frac{\phi(n)}{n}}.
\]

Proof. Let \( \varepsilon > 0 \) be an arbitrary real number. Let \( \{n_k\}_{k \geq 1} \) be a sequence with
\[
\lim_{k \to \infty} \frac{\phi(n_k)}{n_k} = \lim \inf_{n \to \infty} \frac{\phi(n)}{n}.
\]
(3.3)

We choose a subsequence \( \{n_{k_i}\}_{i \geq 1} \) of \( \{n_k\}_{k \geq 1} \) (for simplicity, we still denote by \( \{n_k\}_{k \geq 1} \)) such that
\[
\frac{1}{\lceil \phi(n_1) \rceil} < \frac{\varepsilon}{4}, \quad n_1 > 1, \quad n_{k+1} > n_k + \lceil \phi(n_k) \rceil, \quad \text{and}
\]
\[
\frac{\varepsilon}{2}(n_k + \lceil \phi(n_k) \rceil) > \sum_{i=1}^{k-1} \left( \frac{n_{i+1} - (n_i + \lceil \phi(n_i) \rceil)}{\phi(n_i)} + 2 + \lceil \phi(n_i) \rceil \right).
\]
(3.4)

The condition \( 1/\lceil \phi(n_1) \rceil < \varepsilon/4 \) assures that the sequence satisfying (3.4) can be chosen. Denote the two subsets of integers
\[
I_1 = \{ n_k + j: k \geq 1, \ 1 \leq j \leq \lceil \phi(n_k) \rceil \}
\]
and
\[
I_2 = \{ n_k, n_k + \lceil \phi(n_k) \rceil + 1, n_k + j\lceil \phi(n_k) \rceil: k \geq 1, \ 1 < j \leq \frac{n_{k+1} - n_k}{\phi(n_k)} \}.
\]

We denote by \( \mathcal{D}_n \) the set of \( n \)-cylinders \( J(\varepsilon_1, \ldots, \varepsilon_n) \) satisfying that \( \varepsilon_k = 0 \) if \( k \in I_1, \varepsilon_k \in P \setminus \{0\} \) if \( k \in I_2, \) and \( \varepsilon_k \in P \) if \( k \not\in I_1 \cup I_2 \) for all \( 1 \leq k \leq n \). Let \( \mathcal{D} = \bigcup_{n=1}^{\infty} \mathcal{D}_n \). Put
\[
F = \bigcap_{n=1}^{\infty} \bigcup_{J(\varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{D}_n} J(\varepsilon_1, \ldots, \varepsilon_n).
\]

From the construction of \( F \) and the increasing property of \( \phi \), we know
\[
F \subset F_\phi = \left\{ x \in I: \lim_{n \to \infty} \frac{L_n(x)}{\phi(n)} = 1 \right\}.
\]

In the following we get a lower bound of \( F \) by the mass distribution principle. We need to give a mass distribution on \( F \), i.e., a measure \( \nu \) which is supported on \( F \).

We give the definition of \( \nu \) on cylinders firstly. Since \( n_1 > 1 \), then \( 1 \not\in I_1 \cup I_2 \). Let \( \nu(J(\varepsilon_1)) = \| \beta \|^{-1} \) for any \( \varepsilon_1 \in P \). Suppose \( \nu(J(\varepsilon_1, \ldots, \varepsilon_{n-1})) \) is well defined, we define \( \nu(J(\varepsilon_1, \ldots, \varepsilon_n)) \) \( (J(\varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{D}) \) as follows:
\[
v(J(\varepsilon_1, \ldots, \varepsilon_n)) = \begin{cases} 
\nu(J(\varepsilon_1, \ldots, \varepsilon_{n-1})) & \text{if } n \in I_1, \\
(\|\beta\| - 1)^{-1} \nu(J(\varepsilon_1, \ldots, \varepsilon_{n-1})) & \text{if } n \in I_2, \\
\|\beta\|^{-1} \nu(J(\varepsilon_1, \ldots, \varepsilon_{n-1})) & \text{otherwise};
\end{cases}
\]

and \(v(J(\varepsilon_1, \ldots, \varepsilon_n)) = 0\) if \((\varepsilon_1, \ldots, \varepsilon_n) \notin \mathcal{D}\). The measure \(v\) is defined well on all cylinders because we can verify that

\[
\sum_{\varepsilon_{n+1} \in P} v(J(\varepsilon_1, \ldots, \varepsilon_{n+1})) = v(J(\varepsilon_1, \ldots, \varepsilon_n)) \quad \text{and} \quad \sum_{\varepsilon_1, \ldots, \varepsilon_n \in P} v(J(\varepsilon_1, \ldots, \varepsilon_n)) = 1.
\]

By Kolmogorov's extension theorem, the measure \(v\) is defined on the measurable space \((I, \mathcal{B}(I))\).

In the following we prove that the measure \(v\) satisfies the condition (3.1). That is, for any \(\varepsilon > 0\), there exist two constants \(C > 0\) and \(\delta > 0\) such that

\[
v(B(x, r)) \leq Cr^{s-\varepsilon}
\]

for any ball \(B(x, r)\) with \(r < \delta\), where \(s = 1/(1 + \liminf_{n \to \infty} \phi(n)/n)\).

We show (3.5) is true for all cylinders at first. There are two cases:

Case I: \((\varepsilon_1, \ldots, \varepsilon_n) \notin \mathcal{D}\). We know \(v(J(\varepsilon_1, \ldots, \varepsilon_n)) = 0\), which implies the inequality (3.5).

Case II: \((\varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{D}\).

(i). \(n_k < n \leq n_k + [\phi(n_k)]\) for some \(k \in \mathbb{N}\). By the construction of the measure \(v\), we have

\[
v(J(\varepsilon_1, \ldots, \varepsilon_n)) \leq \|\beta\|^{-n}(n_k - \Sigma_{i=1}^{k-1} [\phi(n_i)] - [\phi(n_k)])
\]

Since \(\text{diam}(J(\varepsilon_1, \ldots, \varepsilon_n)) = 2\|\beta\|^{-n}\), then

\[
\frac{v(J(\varepsilon_1, \ldots, \varepsilon_n))}{(\text{diam}(J(\varepsilon_1, \ldots, \varepsilon_n)))^{s-\varepsilon}} \leq 2^{s-\varepsilon} \|\beta\|^n(s-\varepsilon) - n_k + \frac{\phi(n_k)}{2}
\]

where \(t_k = (1 + \frac{\phi(n_k)}{n_k})s - 1 - (1 + \frac{\phi(n_k)}{n_k})\frac{\varepsilon}{2}\), the second inequality is because \(n \leq n_k + [\phi(n_k)]\). By the definition of \(s\) and \(t_k\), we have \(\limsup_{k \to \infty} t_k \leq -\frac{s}{2}\) for (3.3). So, there exists a constant \(C_1\) such that \(\|\beta\|^{n_k t_k} \leq C_1\). Let \(C = 2^{s-\varepsilon}C_1\), we obtain that (3.5) holds for the measure \(v\) and such a \(C\).

(ii). \(n_k + [\phi(n_k)] < n \leq n_{k+1}\) for some \(k \in \mathbb{N}\). We have

\[
v(J(\varepsilon_1, \ldots, \varepsilon_n)) \leq \|\beta\|^{-n+\Sigma_{i=1}^{k-1} [\phi(n_i)] - [(\Sigma_{i=1}^{k-1} n_i + [\phi(n_i)])] + 2\|\beta\|^{n-n_k + [\phi(n_k)] + 1} + [\phi(n_k)]}\]

Let \(\mathcal{D} = \{J(\varepsilon_1, \ldots, \varepsilon_n) : n_k < n \leq n_k + [\phi(n_k)]\}\).
\[ \leq \| \beta \|^n + \frac{x}{2} (n_k + [\phi(n_k)]) + \frac{1}{2} n + [\phi(n_k)] \]
\[ \leq \| \beta \|^n (1 - \frac{1}{2} \xi) + [\phi(n_k)] , \]
where the third equality is because \( \frac{1}{\phi(n_k)} \) \( \leq \frac{1}{\phi(n_1)} < \frac{1}{\xi} \) \( \frac{n_k + [\phi(n_k)]}{\phi(n_k)} \geq 1 \), and (3.4), the last inequality is because \( n_k + [\phi(n_k)] \leq n \). Thus
\[ \nu(J(\varepsilon_1, \ldots, \varepsilon_n)) \] (diam \( J(\varepsilon_1, \ldots, \varepsilon_n) \)) \( s - \xi \) \( \leq \frac{2}{s^r} \| \beta \|^{s - \xi} r^{s - \xi} . \]

Thus (3.5) is true for any ball.

Therefore, applying the mass distribution principle to the set \( F \) and the measure \( \nu \), we obtain
\[ \dim_H (F) \geq \frac{1}{1 + \liminf_{n \to \infty} \frac{\phi(n)}{n}} - \varepsilon . \]

Letting \( \varepsilon \to 0^+ \), we have
\[ \dim_H (F) \geq \frac{1}{1 + \liminf_{n \to \infty} \frac{\phi(n)}{n}} . \]

Thus we obtain the result since \( F \subset F_\phi \). \( \square \)

Combining Lemmas 3.2 and 3.3, we can obtain
Proposition 3.4. Suppose \( \phi \) be increasing and \( \phi(n) \to \infty \) as \( n \to \infty \). Denote \( \liminf_{n \to \infty} \frac{\phi(n)}{n} = \gamma \). Then we have

\[
\dim_H \left\{ x \in I : \limsup_{n \to \infty} \frac{L_n(x)}{\phi(n)} = 1 \right\} = \frac{1}{1 + \gamma}.
\]

During the construction of the Cantor-like set \( F \) in Lemma 3.3, we replace \( \phi(n) \) by \( \phi(n) - n \), we can obtain the following lemma.

Lemma 3.5. Suppose \( \phi \) be increasing and \( \phi(n) \to \infty \) as \( n \to \infty \). If \( \liminf_{n \to \infty} \frac{\phi(n)}{n} \geq 1 \), then we have

\[
\dim_H \left\{ x \in I : \limsup_{n \to \infty} \frac{n + L_n(x)}{\phi(n)} = 1 \right\} \geq \frac{1}{\liminf_{n \to \infty} \frac{\phi(n)}{n}}.
\]

By Lemmas 3.2 and 3.5, we obtain

Proposition 3.6. Suppose \( \phi \) be increasing and \( \phi(n) \to \infty \) as \( n \to \infty \). If \( \alpha = \liminf_{n \to \infty} \frac{\phi(n)}{n} \geq 1 \), then we have

\[
\dim_H \left\{ x \in I : \limsup_{n \to \infty} \frac{n + L_n(x)}{\phi(n)} = 1 \right\} = \frac{1}{\alpha}.
\]

Similar with Lemma 3.3, we can construct a Cantor-like subset \( E \) of \( E_\phi \). We choose a sequence \( \{n_k\}_{k \geq 1} \) such that \( \lim_{k \to \infty} \frac{\phi(n_k)}{n_k} = \liminf_{n \to \infty} \frac{\phi(n)}{n} \) and \( n_{k+1} > n_k + [\phi(n_k)] \). Denote by \( \varepsilon_n \) the set of \( n \)-cylinders \( J(\varepsilon_1, \ldots, \varepsilon_n) \) satisfying \( \varepsilon_k \in P \setminus \{0\} \) if \( k = n_i + [\phi(n_i)] \) for some \( i \in \mathbb{N} \); otherwise, \( \varepsilon_k \in P \). Put

\[
E = \bigcap_{n=1}^{\infty} \bigcup_{J(\varepsilon_1, \ldots, \varepsilon_n) \in \varepsilon_n} J(\varepsilon_1, \ldots, \varepsilon_n).
\]

Then \( E \subset E_\phi \). We can defined a measure supported on \( E \) like the measure \( \nu \) on \( F \). Then it can be shown that this measure satisfies the mass distributed principle by similar skill in Lemma 3.3. Thus we obtain the following. Note that the assumptions of increasing and tending to \( \infty \) can be relaxed because these conditions just assure that \( F \) is a subset of \( F_\phi \).

Lemma 3.7.

\[
\dim_H (E_\phi) \geq \frac{1}{1 + \liminf_{n \to \infty} \frac{\phi(n)}{n}}.
\]

Combining Lemmas 3.2 and 3.7, we obtain

Proposition 3.8. Denote \( \liminf_{n \to \infty} \frac{\phi(n)}{n} = \tau \), then

\[
\dim_H \left\{ x \in I : L_n(x) \geq \phi(n) \ i.o. \right\} = \frac{1}{1 + \tau}.
\]
4. Proofs of Theorems 1.6 and 1.8

In this section, we prove Theorems 1.6 and 1.8.

**Proof of Theorem 1.6.** (i) Note that if $\eta = 0$, we can obtain that

$$\liminf_{n \to \infty} \frac{1}{\phi(n)} \log_q \|x - \omega_n(x)\| = -\infty$$

for all $x \in I$, by Proposition 2.1. If $0 < \eta < \deg \beta$, then $\liminf_{n \to \infty} \phi(n) = +\infty$. Let $\{n_k\}$ be a sequence such that $\eta = \lim_{k \to \infty} \phi(n_k)/n_k$. By (2.2), note that $L_n(x)$ is nonnegative, we have

$$\liminf_{k \to \infty} \frac{1}{\phi(n_k)} \log_q \|x - \omega_{n_k}(x)\| = -\left(\limsup_{k \to \infty} \frac{n_k + L_{n_k}(x)}{\phi(n_k)}\right) \deg \beta$$

$$\leq -\frac{1}{\eta} \deg \beta < -1.$$

So the set $A_\phi$ is empty, thus the dimension is 0.

(ii) Let $\phi(n)/\deg \beta$ be the function of Proposition 3.6. We obtain the result from (2.4).

(iii) For any $0 < \varepsilon < 1/\deg \beta$, by Proposition 2.1, note that $\eta = +\infty$, we have

$$A_\phi = \left\{x \in I: \limsup_{n \to \infty} \frac{L_n(x)}{\phi(n)} = \frac{1}{\deg \beta}\right\}$$

$$\subset \left\{x \in I: L_n(x) \geq \left(\frac{1}{\deg \beta} - \varepsilon\right) \phi(n) \text{ i.o.}\right\}.$$

Applying Lemma 3.2 to the above limsup set, we obtain $\dim_H(A_\phi) = 0$ since

$$\liminf_{n \to \infty} \frac{(1/\deg \beta - \varepsilon) \phi(n)}{n} = +\infty. \quad \square$$

**Proof of Theorem 1.8.** From Proposition 2.1, we obtain

$$\left\{x \in I: \|x - \omega_n(x)\| \leq \|\beta\|^{-\phi(n)} \text{ i.o.}\right\} = \left\{x \in I: L_n(x) \geq \phi(n) - n \text{ i.o.}\right\}. \quad (4.1)$$

Applying Proposition 3.8 to the right set of (4.1), we have

$$\dim_H \left\{x \in I: \|x - \omega_n(x)\| \leq \|\beta\|^{-\phi(n)} \text{ i.o.}\right\} = \frac{1}{\xi}$$

since $\liminf_{n \to \infty} \frac{\phi(n) - n}{n} = \xi - 1. \quad \square$

References


