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Fuzzy Random Variables*

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In this paper we define the concepts of fuzzy random variable and the expectation of a fuzzy random variable. The new definition of expectation generalizes the integral of a set-valued function. We derive some properties of these new concepts. By considering a suitable generalization of the Hausdorff metric, we derive the Lebesgue-dominated convergence type theorem.

1. INTRODUCTION

In practice we are often faced with random experiments whose outcomes are not numbers (or vectors in $\mathbb{R}^n$) but are expressed in inexact linguistic terms. As an example, consider a group of individuals chosen at random who are questioned about the weather on a particular city on a particular winter day. Some possible answers would be "cold," "more or less cold," "very cold," "extremely cold," and so on. A natural question which arises with reference to this example is: What is the average opinion about the weather in that particular city?

A possible way of handling situations like this is by using the concepts of fuzzy sets and fuzzy functions [24] found useful in many applications, notably in pattern recognition, clustering, information retrieval, and systems analysis (cf. [16]).

Motivated by examples of the type given above and related problems (especially concerning group opinions), we introduce fuzzy random variables and their expectations, and we investigate some of their properties.

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Fuzzy random variables (or fuzzy variables) generalize random variables and random vectors; they also generalize random sets [15]. The expected value of a fuzzy variable is a natural generalization of the integral of a set-valued function [2].

Kwakernaak [14] introduced the notion of a fuzzy random variable as a function $F\colon \Omega \to \bar{\mathscr{F}}(\mathbb{R})$ (subject to certain measurability conditions), where $(\Omega, \mathscr{A}, P)$ is a probability space, and $\bar{\mathscr{F}}(\mathbb{R})$ denotes all piecewise continuous functions $u\colon \mathbb{R} \to [0, 1]$.

Féron [8] defined a fuzzy random set as a measurable function $F\colon \Omega \to \mathscr{F}(\mathscr{X})$, where $\mathscr{X}$ is a topological space, $\mathscr{F}(\mathscr{X}) = \{u\colon \mathscr{X} \to [0, 1]\}$, and $\{x \in \mathscr{X}\colon F(\omega)(x) \geqslant \alpha\}$ are closed subsets of $\mathscr{X}$ for each $0 \leqslant \alpha \leqslant 1$, $\omega \in \Omega$.

Relationships between fuzzy sets and random sets were studied by Fortet and Kambouzia [9] and by Goodman [10].

However, in the work of the authors mentioned above, no attempt is made to define the expected value of a fuzzy variable and to study its properties. Our definition of the expected value is new and it provides a natural generalization of the set valued function (that is, random set) setting.

In Section 2, we briefly state some results related to the integral calculus for set-valued functions. These results will be frequently referred to in the subsequent sections.

In Section 3, we introduce the notion of a fuzzy random variable slightly different than that of Kwakernaak [14]. We define it as a function (subject to certain measurability requirements) $X\colon \Omega \to \mathscr{F}_0(\mathbb{R}^n)$, where $(\Omega, \mathscr{A}, P)$ is a probability space, and $\mathscr{F}_0(\mathbb{R}^n)$ denotes all functions (fuzzy subsets of $\mathbb{R}^n$) $u\colon \mathbb{R}^n \to [0, 1]$ such that $\{x \in \mathbb{R}^n\colon u(x) \geqslant \alpha\}$ is nonempty and compact for each $0 < \alpha \leqslant 1$. In this setting we define the expected value $E(X)$ of a fuzzy variable X.

In Section 4 we study some properties of this expected value. To this end, we first define a metric in $\mathscr{F}_0(\mathbb{R}^n)$ which generalizes the Hausdorff metric in the space of compact subsets of $\mathbb{R}^n$. We show that, under certain conditions $E(X)$ is a fuzzy convex set. The main result of this section is a Lebesgue-dominated convergence type theorem. This generalizes the corresponding results of Aumann [2] and Debreu [6]. Some of the results of this section can possibly be extended to Banach space valued fuzzy variables, that is, functions $X\colon \Omega \to \mathscr{F}_1(\mathscr{Y})$, where $(\Omega, \mathscr{A}, P)$ is a probability space, $\mathscr{Y}$ is a Banach space, and $\mathscr{F}_1(\mathscr{Y})$ denotes all functions (fuzzy sets) $u\colon \mathscr{Y} \to [0, 1]$ whose levels $\{y \in \mathscr{Y}\colon u(y) \geqslant \alpha\}$, $\alpha \neq 0$ are closed, bounded, and convex subsets of $\mathscr{Y}$. However, this generalization will not be carried out in this paper.

Finally, in Section 5 we state some problems of further research interest.

2. Integral Calculus for Set-Valued Functions

The concept of an integral of a set-valued function was first introduced by Kudō [12] in connection with the theory of experiments in statistics. Later, using Kudō [12] and Richter [23], this concept of an integral was extended by Aumann [2] who proved some important convergence properties. Debreu [6] defined another concept of an integral of a set-valued function in a more general context, studied its properties, and showed that under suitable hypotheses, it coincided with the integral of Aumann [2].

Applications of these integrals are found in economics [3], control theory [11], and probability theory [1, 5, 20].

We define two different types of convergence for sequences of sets. Let A and B be two nonempty bounded subsets of $\mathbb{R}^n$. The distance between A and B is defined by the *Hausdorff metric*,

$$d_H(A, B) = \max[\sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\|], \tag{2.1}$$

where $\|\cdot\|$ denotes the usual euclidean norm in $\mathbb{R}^n$.

We denote the *Hausdorff semimetric* by $\rho(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|$. It is clear that

$$\rho(A, B) = 0 \Leftrightarrow A \subset \bar{B} \tag{2.2}$$

and

$$\rho(A, C) \leqslant \rho(A, B) + \rho(B, C), \tag{2.3}$$

where A, B, C are nonempty bounded subsets of $\mathbb{R}^n$, and $\bar{B}$ denotes the closure of B.

Also $d_H(A, B) = \max[\rho(A, B), \rho(B, A)]$ and

$$d_H(A, B) = 0 \Leftrightarrow \bar{A} = \bar{B}. \tag{2.4}$$

If $Q(\mathbb{R}^n)$ denotes the set of all nonempty, compact subsets of $\mathbb{R}^n$, it is clear that $(Q(\mathbb{R}^n), d_H)$ becomes a metric space. The following theorem gives a more precise result.

Theorem 2.1. *The metric space $(Q(\mathbb{R}^n), d_H)$ is complete and separable.*

Another type of convergence for a sequence of sets was defined by Kuratowski [13].

We say that a sequence of sets $\{C_k\}_k$, $C_k \subset \mathbb{R}^n$, *converges* to a set $C \subset \mathbb{R}^n$, denoted by $C = \lim_k C_k$, if

$$C = \liminf C_k = \limsup C_k, \tag{2.5}$$

where

$$\liminf C_k = \{x \in \mathbb{R}^n : x = \lim_{k \to \infty} x_k, x_k \in C_k\}, \tag{2.6}$$

$$\limsup C_k = \bigcap_{k=1}^{\infty} \left(\overline{\bigcup_{m=k}^{\infty} C_m} \right). \tag{2.7}$$

We mention that for sequences of closed sets, convergence in the Hausdorff metric implies convergence in the sense of Kuratowski. On $Q(\mathbb{R}^n)$, both types of convergence are equivalent provided the limit set is nonempty, that is, the sequence is bounded (cf. [15]).

Now, let $(\Omega, \mathscr{A}, P)$ be a probability space where the probability measure P is assumed to be *nonatomic.*

A *set-valued function* is a function $F: \Omega \to \mathscr{P}(\mathbb{R}^n)$ such that $F(\omega) \neq \varnothing$ for every $\omega \in \Omega$. By $L^1(P)$ (or by $L^1(P, \mathbb{R}^n)$) we denote the space of P-integrable functions $f: \Omega \to \mathbb{R}^n$. We denote by $S(F)$ the set of all $L^1(P)$ selections of F, that is,

$$S(F) = \{f \in L^1(P) : f(\omega) \in F(\omega) \text{ a.e.}\} \tag{2.8}$$

The *Aumann integral* of F is defined by

$$(A) \int F = \left\{ \int_\Omega f \, dP : f \in S(F) \right\} \tag{2.9}$$

For easy reference, we state the following results due to Richter [23], Aumann [2], and Debreu [6].

THEOREM 2.2. *If $F: \Omega \to \mathscr{P}(\mathbb{R}^n)$ is a set-valued function, then $(A) \int F$ is a convex subset of $\mathbb{R}^n$.*

Observe, however, that $(A) \int F$ may be empty in general.

A function $F: \Omega \to \mathscr{P}(\mathbb{R}^n)$ is called *measurable* if its graph $\{(\omega, x): x \in F(\omega)\}$ belongs to $\mathscr{A} \times \mathscr{B}$ (where $\mathscr{B}$ denotes the Borel subsets of $\mathbb{R}^n$). A function $F: \Omega \to \mathscr{P}(\mathbb{R}^n)$ is called *integrably bounded* if there exists a function $h: \Omega \to \mathbb{R}$, $h \in L^1(P, \mathbb{R})$ such that $\|x\| \leqslant h(\omega)$ for all x, ω with $x \in F(\omega)$.

THEOREM 2.3 [2]. *If $F: \Omega \to \mathscr{P}(\mathbb{R}^n)$ is measurable and integrably bounded, then $(A) \int F \neq \varnothing$.*

Another result about the structure of $(A) \int F$ is given by

THEOREM 2.4 [2]. *If $F: \Omega \to \mathscr{P}(\mathbb{R}^n)$, $F(\omega)$ is closed for every $\omega \in \Omega$, and if F is integrably bounded, then $(A) \int F$ is a compact subset of $\mathbb{R}^n$.*

The following theorem is a generalization of the Lebesgue dominated convergence theorem.

THEOREM 2.5 [2]. *If* $F_k\colon \Omega \to \mathscr{P}(\mathbb{R}^n)$ *are measurable and if there exists* $h \in L^1(P, \mathbb{R})$ *such that* $\sup_{k \geqslant 1} \|f_k(\omega)\| \leqslant h(\omega)$ *for every* $f_k \in S(F_k)$, *and if* $F_k(\omega) \to F(\omega)$ *(in the sense of Kuratowski), then* $(A)\int F_k \to (A)\int F$.

In Debreu [6], a concept of an integral is defined for more general set-valued functions $F\colon \Omega \to K(\mathscr{X})$, where $\mathscr{X}$ is a Banach space and $K(\mathscr{X})$ denotes all nonempty compact convex subsets of $\mathscr{X}$. The key result used in defining the Debreu integral is an embedding theorem for $K(\mathscr{X})$ due to Rådström (1952).

Without going through the details of this construction here, we may mention that the Aumann integral can also be defined in this more general setting and it is possible to prove the equivalence of the Aumann and Debreu integrals without assuming that $\mathscr{X}$ is reflexive (see Byrne [4]).

Remark. It is important to observe that Theorem 2.5 can be stated in a different form by replacing convergence in the sense of Kuratowski by convergence in the Hausdorff metric. The statement of the theorem remains unchanged provided we assume that all functions take values in $Q(\mathbb{R}^n)$. We shall use this version of Theorem 2.5 in Section 4, where we shall generalize the Lebesgue dominated convergence theorem to fuzzy variables.

3. FUZZY VARIABLES AND THEIR EXPECTATIONS

Let $(\Omega, \mathscr{A}, P)$ be a probability space where P is a probability measure. Let $\mathscr{F}_0(\mathbb{R}^n)$ denote the set of fuzzy subsets $u\colon \mathbb{R}^n \to [0, 1]$ with the following properties:

(a) $\{x \in \mathbb{R}^n\colon u(x) \geqslant \alpha\}$ is compact for each $\alpha > 0$, (3.1)

(b) $\{x \in \mathbb{R}^n\colon u(x) = 1\} \neq \varnothing$. (3.2)

DEFINITION 3.1. A *fuzzy random variable* (or *fuzzy variable*) is a function $X\colon \Omega \to \mathscr{F}_0(\mathbb{R}^n)$ such that

$$\{(\omega, x)\colon x \in X_\alpha(\omega)\} \in \mathscr{A} \times \mathscr{B} \tag{3.3}$$

for every $\alpha \in [0, 1]$, where $X_\alpha\colon \Omega \to \mathscr{P}(\mathbb{R}^n)$ is defined by

$$X_\alpha(\omega) = \{x \in \mathbb{R}^n\colon X(\omega)(x) \geqslant \alpha\}. \tag{3.4}$$

A fuzzy variable X is called *integrably bounded* if X_α is integrably bounded for all $\alpha \in (0, 1]$, i.e., if for any $\alpha \in (0, 1]$, there exists $h_\alpha \in L^1(\Omega)$ such that $\|x\| \leqslant h_\alpha(\omega)$ for each x, ω with $x \in X_\alpha(\omega)$.

Here $L^1(\Omega)$ denotes all functions $h: \Omega \to \mathbb{R}$ which are integrable with respect to the probability measure P.

Motivated by examples similar to the one given in the introduction, we define the expected value $E(X)$ of a fuzzy variable $X: \Omega \to \mathscr{F}_0(\mathbb{R}^n)$ in such a way that the following conditions are satisfied:

$$E(X) \in \mathscr{F}_0(\mathbb{R}^n), \tag{3.5}$$

$$\{x \in \mathbb{R}^n: (E(X))(x) \geqslant \alpha\} = \int X_\alpha \text{ for each } \alpha \in [0, 1]. \tag{3.6}$$

The next theorem shows that under certain assumptions, there is a unique fuzzy set satisfying these requirements. The proof is based on the following lemma.

LEMMA 3.1. *Let M be a set and let $\{M_\alpha: \alpha \in [0, 1]\}$ be a family of subsets of M such that*

(i) $M_0 = M$,

(ii) $\alpha \leqslant \beta$ *implies* $M_\beta \subseteq M_\alpha$,

(iii) $\alpha_1 \leqslant \alpha_2 \leqslant \cdots$, $\lim_{n \to \infty} \alpha_n = \alpha$ *implies* $M_\alpha = \bigcap_{n=1}^\infty M_{\alpha_n}$.

Then, the function $\phi: M \to [0, 1]$ defined by $\phi(x) = \sup\{\alpha \in [0, 1]: x \in M_\alpha\}$ has the property that $\{x \in M: \phi(x) \geqslant \alpha\} = M_\alpha$ for every $\alpha \in [0, 1]$.

Proof. See [16].

THEOREM 3.1. *If $X: \Omega \to \mathscr{F}_0(\mathbb{R}^n)$ is an integrably bounded fuzzy variable, there exists a unique fuzzy set $v \in \mathscr{F}_0(\mathbb{R}^n)$ such that*

$$\{x \in \mathbb{R}^n: v(x) \geqslant \alpha\} = \int X_\alpha \textit{ for every } \alpha \in [0, 1] \tag{3.7}$$

Proof. Let $M_\alpha = \int X_\alpha$, $\alpha \in [0, 1]$. Since X_α is measurable and integrably bounded, it follows from Theorem 2.3 that $M_\alpha \neq \varnothing$. Since $X_\alpha(\omega) = \{x: X(\omega)(x) \geqslant \alpha\}$ are closed subsets of $\mathbb{R}^n$ for all $\omega \in \Omega$, it follows from Theorem 2.4 that $M_\alpha = \int X_\alpha$ is compact.

Consider now the family $\{M_\alpha: \alpha \in [0, 1]\}$ of subsets of $\mathbb{R}^n$. Note that $X_0(\omega) = \{x: X(\omega)(x) \geqslant 0\} = \mathbb{R}^n$, for all $\omega \in \Omega$. Thus $\int X_0 = \mathbb{R}^n$. If $\alpha \leqslant \beta$, then clearly $X_\alpha(\omega) \supseteq X_\beta(\omega)$ for $\omega \in \Omega$. Thus $M_\alpha \supseteq M_\beta$.

We now apply Lemma 3.1. To do so we have to check that $\alpha_1 \leqslant \alpha_2 \leqslant \cdots$, $\lim_{n \to \infty} \alpha_n = \alpha$, $\alpha \neq 0$ implies $M_\alpha = \bigcap_{n=1}^\infty M_{\alpha_n}$. Observe that $\bigcap_{n=1}^\infty X_{\alpha_n}(\omega) =$

$X_{\alpha 0}(\omega)$ for all $\omega \in \Omega$. Since $X_{\alpha_n}(\omega)$ are compact, $n \in \mathbb{N}$, it follows by a simple argument that

$$X_{\alpha_n}(\omega) \to X_{\alpha 0}(\omega), \tag{3.8}$$

where converge is in the Hausdorff metric.

Now $X_{\alpha_1}(\omega) \supseteq X_{\alpha_2}(\omega) \supseteq \cdots$, and since X_{α_1} is integrably bounded there exists $h \in L^1(\Omega)$ such that $\|f(\omega)\| \leqslant h(\omega)$ for every $f \in S(X_{\alpha_1})$. It follows that $\|g(\omega)\| \leqslant h(\omega)$ for every $g \in S(X_{\alpha_n})$, $n \in \mathbb{N}$. Thus $\{X_{\alpha_n}, n \geqslant 1\}$ is bounded by the same integrable function, and since the X_{α_n} are also measurable, it follows from Theorem 2.5 that $\int X_{\alpha_n} \to \int X_{\alpha 0}$ (in the Hausdorff metric). Observe that $\{\int X_{\alpha_n}, n \geqslant 1\}$ is a decreasing sequence of compact sets and so it must converge to its intersection. Thus we obtain

$$\bigcap_{n=1}^{\infty} \int X_{\alpha_n} = \int X_{\alpha 0}. \tag{3.9}$$

By Lemma 3.1, the fuzzy set defined by $v(x) = \sup\{\alpha \in [0, 1]\colon x \in M_\alpha\}$ satisfies

$$\{x \in \mathbb{R}^n\colon v(x) \geqslant \alpha\} = M_\alpha = \int X_\alpha, \qquad \alpha \in [0, 1]. \tag{3.10}$$

The uniqueness of v is obvious, since if two fuzzy sets v and w satisfy (3.12), then $\{x\colon v(x) \geqslant \alpha\} = \{x\colon w(x) \geqslant \alpha\}$ for every α and this implies $v = w$.

Finally, $v \in \mathscr{F}_0(\mathbb{R}^n)$ since $\{x\colon v(x) \geqslant \alpha\} = M_\alpha$ is a compact set and $\{x\colon v(x) = 1\} = M_1 = \int X_1 \neq \varnothing$. This completes the proof.

We use the above theorem to define the expected value of a fuzzy random variable $X\colon \Omega \to \mathscr{F}_0(\mathbb{R}^n)$ which is integrably bounded.

DEFINITION 3.2. The *expected value* of X, denoted by $E(X)$, is the fuzzy set $v \in \mathscr{F}_0(\mathbb{R}^n)$ such that $\{x \in \mathbb{R}^n\colon v(x) \geqslant \alpha\} = \int X_\alpha$ for every $\alpha \in [0, 1]$.

Note that the existence and uniqueness of v are established in Theorem 3.1.

Thus

$$(E(X))(x) = \sup\left\{\alpha \in [0, 1]\colon x \in \int X_\alpha\right\} \tag{3.11}$$

and its level sets are given by

$$\{x\colon (E(X))(x) \geqslant \alpha\} = \int X_\alpha, \qquad \alpha \in [0, 1] \tag{3.12}$$

4. Properties of the Expectation

Our aim is to extend the Lebesgue dominated convergence theorem to fuzzy random variables. To this end, we first define a metric in $\mathscr{F}_0(\mathbb{R}^n)$ which generalizes the Hausdorff metric.

Let $u, v \in \mathscr{F}_0(\mathbb{R}^n)$, and set

$$d(u, v) = \sup_{\alpha > 0} d_H(L_\alpha(u), L_\alpha(v)), \tag{4.1}$$

where d_H is the Hausdorff metric, and we denote by $L_\alpha(u) = \{x: u(x) \geqslant \alpha\}$, $L_\alpha(v) = \{x: v(x) \geqslant \alpha\}$.

PROPOSITION 4.1. *$(\mathscr{F}_0(\mathbb{R}^n), d)$ is a metric space.*

Proof. (i) $d(u, v) = 0$ implies $d_H(L_\alpha(u), L_\alpha(v)) = 0$ for each $\alpha > 0$, and this implies $L_\alpha(u) = L_\alpha(v)$, $\alpha > 0$, which implies $u = v$. (Recall that $L_\alpha(u)$ and $L_\alpha(v)$ are compact).

(ii) Obviously $d(u, v) = d(v, u)$.

(iii) The triangle inequality $d(u, v) \leqslant d(u, w) + d(w, v)$ follows from the corresponding inequality for d_H.

Note that if A and B are compact subsets of $\mathbb{R}^n$, and χ_A and χ_B are their respective characteristic functions, then $d(\chi_A, \chi_B) = d_H(A, B)$.

The following result generalizes Theorem 2.1.

THEOREM 4.1. *The metric space $(\mathscr{F}_0(\mathbb{R}^n), d)$ is complete.*

The proof of this theorem is given in the Appendix.

Theorem 2.2 can easily be extended to fuzzy variables by using the concept of fuzzy convexity [24].

A fuzzy set $u: \mathbb{R}^n \to [0, 1]$ is called a *fuzzy convex set*, if

$$u(\lambda x + (1 - \lambda) y) \geqslant \min[u(x), u(y)] \quad \text{for ever} \quad x, y \in \mathbb{R}^n,\ \lambda \in [0, 1]. \tag{4.2}$$

THEOREM 4.2. *If the probability measure P is nonatomic, and if $X: \Omega \to \mathscr{F}_0(\mathbb{R}^n)$ is integrably bounded fuzzy variable, then $E(X)$ is a fuzzy convex set.*

Proof. Let $v = E(X)$. Since $\{v \geqslant \alpha\} = \int X_\alpha$, it follows from Theorem 2.2 that $\{v \geqslant \alpha\}$ is a convex set, $\alpha \in [0, 1]$. The proof now follows by noting that (4.2) is equivalent to the convexity of $\{u \geqslant \alpha\}$ for every $\alpha \in [0, 1]$.

The following theorem extends the Lebesgue dominated convergence Theorem 2.5 (see the remark at the end of Section 2). We assume that the probability measure P is nonatomic.

THEOREM 4.3. *Let $\{X_k, k \geq 1\}$ and X be integrably bounded fuzzy variables such that $X_k(\omega) \to^d X(\omega)$ for almost every $\omega \in \Omega$. Suppose there exists an $h \in L^1(\Omega)$ such that $\sup_{x \in X_{k,\alpha}(\omega)} \|x\| \leq h(\omega)$ for all $k \geq 1$ and $\alpha > 0$, where $X_{k,\alpha}(\omega) = L_\alpha(X_k(\omega)) = \{X_k(\omega) \geq \alpha\}$. Then $E(X_k) \to^d E(X)$.*

Proof. To prove this theorem, we shall use a technique similar to the one in Debreu [7, pp. 366; 367]). Assume first that X_k, X are such that $L_\alpha(X_k(\omega))$ and $L_\alpha(X(\omega))$ are convex subsets of $\mathbb{R}^n$ for every $\alpha \geq 0$.

Then for every $\alpha > 0$,

$$\begin{aligned} d_H(L_\alpha(E(X_k)), L_\alpha(E(X))) &= d_H\left(\int L_\alpha(X_k), \int L_\alpha(X)\right) \\ &\leq \int d_H(L_\alpha(X_k), L_\alpha(X)) \leq \int d(X_k(\omega), X(\omega))\, dP(\omega) \end{aligned} \tag{4.3}$$

implying $d(E(X_k), E(X)) \leq \int d(X_k(\omega), X(\omega))\, dP(\omega)$. Now $d(X_k(\omega), X(\omega)) \to 0$, a.e. and also, $d(X_k(\omega), X(\omega)) \leq d(X_k(\omega), 0) + d(0, X(\omega))$, where 0 denotes the set $\{0\}$. It follows that

$$\begin{aligned} d(X_k(\omega), 0) &= \sup_{\alpha > 0} d_H(L_\alpha(X_k(\omega)), 0) \\ &= \sup_{\alpha > 0} \sup_{x \in X_{k,\alpha}(\omega)} \|x\| \leq h(\omega), h \in L^1(\Omega). \end{aligned}$$

Using now the classical Lebesgue dominated convergence theorem, we obtain $d(E(X_k), E(X)) \to 0$.

Assume now that X_k, X are as in the statement of the theorem. For any subset A of $\mathbb{R}^n$, denote by $\operatorname{Co} A$ its convex hull. For a random set $F: \Omega \to \mathscr{P}(\mathbb{R}^n)$, denote by $(\operatorname{Co} F)(\omega) = \operatorname{Co}(F(\omega))$, $\omega \in \Omega$. Note that, since P is nonatomic, $\int F = \int \operatorname{Co} F$ (provided the integral exists).

Also, if F and G are random sets (not necessarily convex valued), we have $d_H(\int F, \int G) = d_H(\int \operatorname{Co} F, \int \operatorname{Co} G) \leq \int d_H(\operatorname{Co} F, \operatorname{Co} G) \leq \int d_H(F, G)$, the last inequality following from Price [17]. Now a simple argument similar to the one above (see (4.3)) concludes the proof.

5. COMPUTATION OF $E(X)$

In this section, we provide some examples to compute the expected value of a fuzzy random variable.

EXAMPLE 1. Toss a fair coin. Denote the outcomes "Tail" by T and "Head" by H. Suppose a player loses *approximately* \$10 if the outcome is T, and wins an amount *much larger* than \$100 but *not much larger* than \$1000 if the outcome is H. The fuzzy r.v. then is $X: \{T, H\} \to \mathscr{F}_0(\mathbb{R})$, where $X(T) =$ "approximately -10," and $X(H) =$ "much larger than 100 but not much larger than 1000." For (intuitively plausible) technical reasons, let us write $X(T) = u$ and $X(H) = v$, where $u, v: \mathbb{R} \to [0, 1]$ are given by, say, $u(x) = [1 - (x + 10)^2/4]^+$ and $v(x) = [1 - (x - 630)^2/380^2]^+$, where $f^+ = \max(f, 0)$. Since u and v are continuous with compact support, it is easy to show by using (3.13) that $E(X)(x) = \sup_{y+z=2x} \min\{[1 - (y + 10)^2/4]^+, [1 - (z - 630)^2/380^2]^+\}$.

In particular, the support of $E(X)$ is included in the interval $[119, 501]$.

EXAMPLE 2 (Extension of Example 1). Let $X: \Omega \to \mathscr{F}_0(\mathbb{R})$ be a fuzzy variable such that $P(X = u_i) = p_i$, $i = 1, ..., r$, where $u_i: \mathbb{R} \to [0, 1]$ are continuous with compact support. Then

$$E(X) = \sum_{i=1}^{r} p_i u_i \tag{5.1}$$

The sum of two fuzzy sets is defined as $(u + v)(x) = \sup_{y+z=x} \min[u(y), v(z)]$, $x \in \mathbb{R}$, and the product of a scalar and a fuzzy set is defined as

$$(\lambda u)(x) = \begin{cases} u(\lambda^{-1}x) & \text{if } \lambda \neq 0 \\ 0 & \text{if } \lambda = 0,\ x \neq 0 \\ \sup_{y \in \mathbb{R}} u(y) & \text{if } \lambda = 0,\ x = 0. \end{cases}$$

(These definitions generalize the corresponding operations with sets).

EXAMPLE 3. Consider now a fuzzy r.v. of *discrete* type, i.e., $X: \Omega \to \mathscr{F}_0(\mathbb{R})$ such that $P(X = u_i) = p_i$, $i = 1, 2, ...$, where $u_i: \mathbb{R} \to [0, 1]$ are continuous and have compact support. The infinite sum of fuzzy sets is defined by $(\sum_{j=1}^{\infty} v_j)(x) = \sup \inf_{j \geqslant 1} [v_j(y_j)]$, where the supremum is taken over all sequence $\{y_1, y_2, ...\}$ such that $x = \sum_{j=1}^{\infty} y_j$. Then by using Theorem 3.1 and the fact that $v(A) = \int_A F\, dP$ is a set-valued measure for every random set F, (see [7]), it is easy to check (as a generalization of (5.1)) that

$$E(X) = \sum_{i=1}^{\infty} p_i u_i. \tag{5.2}$$

EXAMPLE 4 (Nondiscrete case). Although the theory developed in Sections 3 and 4 applies to fuzzy r.v.'s of general type, the computation of the expected value becomes somewhat complicated in the case of nondiscrete r.v.

Let X denote the diameter of a hole made by a drillpress and, because of the errors of measurement, let the values of X be fuzzy sets rather than numbers. Let $X: \Omega \to \mathscr{F}_0(\mathbb{R})$ such that $X(\omega)$ is continuous and has compact support, $\omega \in \Omega$. Then, to compute $E(X)$ we use Example 2 along with Theorem 4.3.

If $s = \sum_{i=1}^{r} u_i \chi_{A_i}$ is a simple function, $u_i \in \mathscr{F}_0(\mathbb{R})$, $A_i \in \mathscr{A}$, then $E(s) = \sum_{i=1}^{r} u_i P(A_i)$. If X is an integrably bounded fuzzy r.v., then $E(X) = \lim_{k \to \infty} E(s_k)$, where s_k is a simple function and $s_k \to X$, (all these limits are in the metric d defined by (4.1)).

Remark. In the above examples, we have made a somewhat restrictive assumption that the fuzzy r.v.'s take on values which have compact support. If this assumption is dropped, then the addition of fuzzy sets defined in Example 2 (and the corresponding formula (5.1)) should be replaced by a different operation defined as follows: $(u+v)(x) = \sup\{\alpha \in [0, 1]: x \in L_\alpha(u) + L_\alpha(v)\}$, where $u, v \in \mathscr{F}_0(\mathbb{R})$, $L_\alpha(u) = \{x \in \mathbb{R}: u(x) \geqslant \alpha\}$, $L_\alpha(v) = \{x \in \mathbb{R}: v(x) \geqslant \alpha\}$. For details see Puri and Ralescu [18, 21].

6. Concluding Remarks

Motivated by the study of asymptotic or limiting opinion (see the example in the Introduction), it is desirable to explore different limit theorems for sequences of independent fuzzy random variables. Of particularly great interests would be the theorems which generalize the classical law of large numbers and the central limit theorem. (See Artstein and Vitale [1], Cressie [5], and Puri and Ralescu [20] for generalizations of some of these theorems to random sets.)

7. Appendix

Proof of Theorem 4.1. Let $\{u_n, n \geqslant 1\}$ be a Cauchy sequence in $\mathscr{F}_0(\mathbb{R}^n)$. Consider a fixed $\alpha > 0$. Then $\{L_\alpha(u_n), n \geqslant 1\}$ is a Cauchy sequence in $(Q(\mathbb{R}^n), d_H)$ where $Q(\mathbb{R}^n)$ denotes all nonempty compact subsets of $\mathbb{R}^n$.

Since $(Q(\mathbb{R}^n), d_H)$ is complete (Theorem 2.1), it follows that

$$L_\alpha(u_n) \xrightarrow{d_H} M_\alpha \in Q(\mathbb{R}^n).$$

Actually, from the Definition (4.1) of d and from the continuity of d_H, it is easy to see that $L_\alpha(u_n) \to^{d_H} M_\alpha$, uniformly in $\alpha \in [0, 1]$.

Consider now the family $\{M_\alpha : \alpha \in [0, 1]\}$, where $M_0 = \mathbb{R}^n$. Take $\alpha \leqslant \beta$ and denote by ρ the Hausdorff semimetric in $Q(\mathbb{R}^n)$ (see Sect. 2). We then have

$$\rho(M_\beta, M_\alpha) \leqslant \rho(M_\beta, L_\beta(u_n)) + \rho(L_\beta(u_n), L_\alpha(u_n)) + \rho(L_\alpha(u_n), M_\alpha).$$

Since $L_\beta(u_n) \subseteq L_\alpha(u_n)$, it follows that $\rho(L_\beta(u_n), L_\alpha(u_n)) = 0$. Thus $\rho(M_\beta, M_\alpha) \leqslant \rho(M_\beta, L_\beta(u_n)) + \rho(L_\alpha(u_n), M_\alpha) < \varepsilon$ if n is large enough. Hence $\rho(M_\beta, M_\alpha) = 0$, and since M_β, M_α are closed, we have $M_\beta \subseteq M_\alpha$.

Now take $\alpha > 0$, $\alpha_n \uparrow$, and $\lim_{n\to\infty} \alpha_n = \alpha$. We have to show that $M_\alpha = \bigcap_{n=1}^\infty M_{\alpha_n}$.

It is clear that

$$M_\alpha \subseteq \bigcup_{n=1}^{\infty} M_{\alpha_n}. \tag{7.1}$$

Using again the Hausdorff semimetric, we get

$$\rho\left(\bigcap_{n=1}^{\infty} M_{\alpha_n}, M_\alpha\right) \leqslant \rho\left(\bigcap_{n=1}^{\infty} M_{\alpha_n}, \bigcap_{n=1}^{\infty} (L_{\alpha_n}(u_j)\right) + \rho\left(\bigcap_{n=1}^{\infty} L_{\alpha_n}(u_j), L_\alpha(u_j)\right)$$
$$+ \rho(L_\alpha(u_j), M_\alpha) \qquad \text{for fixed } j.$$

But $\rho(\bigcap_{n=1}^\infty L_{\alpha_n}(u_j), L_\alpha(u_j)) = 0$. Consequently, for every $\varepsilon > 0$, there exists j_ε such that

$$\rho\left(\bigcap_{n=1}^{\infty} M_{\alpha_n}, M_\alpha\right) \leqslant \varepsilon + \rho\left(\bigcap_{n=1}^{\infty} M_{\alpha_n}, \bigcap_{n=1}^{\infty} L_{\alpha_n}(u_j)\right)$$

for $j \geqslant j_\varepsilon$, since $L_\alpha(u_j) \to M_\alpha$.

Now

$$\rho\left(\bigcap_{n=1}^{\infty} M_{\alpha_n}, \bigcap_{n=1}^{\infty} L_{\alpha_n}(u_j)\right) \leqslant \rho\left(\bigcap_{n=1}^{\infty} M_{\alpha_n}, M_{\alpha_p}\right) + \rho(M_{\alpha_p}, L_{\alpha_p}(u_j))$$
$$+ \rho\left(L_{\alpha_p}(u_j), \bigcap_{n=1}^{\infty} L_{\alpha_n}(u_j)\right) \qquad \text{for any } p \geqslant 1.$$

Since $\bigcap_{n=1}^\infty M_{\alpha_n} \subseteq M_{\alpha_p}$, we obtain

$$\rho\left(\bigcap_{n=1}^{\infty} M_{\alpha_n}, \bigcap_{n=1}^{\infty} L_{\alpha_n}(u_j)\right) \leqslant \rho(M_{\alpha_p}, L_{\alpha_p}(u_j)) + \rho\left(L_{\alpha_p}(u_j), \bigcap_{n=1}^{\infty} L_{\alpha_n}(u_j)\right).$$

Now $\rho(M_{\alpha_p}, L_{\alpha_p}(u_j)) < \varepsilon$ for $j \geqslant j_0$. Note that (since the convergence $L_\alpha(u_j) \to M_\alpha$ is uniform in α) j_0 does not depend on p. Since $\{L_{\alpha_p}(u_j), p \geqslant 1\}$ decreases to $\bigcap_{n=1}^\infty L_{\alpha_n}(u_j)$, if follows that $\rho(L_{\alpha_{p_0}}(u_j), \bigcap_{n=1}^\infty L_{\alpha_n}(u_j)) < \varepsilon$ for some p_0 (depending on j). Thus $\rho(\bigcap_{n=1}^\infty M_{\alpha_n}, \bigcap_{n=1}^\infty L_{\alpha_n}(u_j)) < 2\varepsilon$, if j is large.

Finally, by taking j large enough, we obtain

$$\rho\left(\bigcap_{n=1}^{\infty} M_{\alpha_n}, M_\alpha\right) \leqslant 3\varepsilon, \qquad \text{i.e.,} \bigcap_{n=1}^{\infty} M_{\alpha_n} \subseteq M_\alpha. \tag{7.2}$$

Equations (7.1) and (7.2) yield $\bigcap_{n=1}^\infty M_{\alpha_n} = M_\alpha$. Thus Lemma 3.1 is applicable and there exists $u \in \mathscr{F}_0(\mathbb{R}^n)$ with $L_\alpha(u) = M_\alpha$ for every $\alpha \in [0, 1]$. It follows that $L_\alpha(u_n) \to^{d_H} L_\alpha(u)$. It remains to show that $u_n \to u$ in $(\mathscr{F}_0(\mathbb{R}^n), d)$.

Let $\varepsilon > 0$. Then, since $\{u_n\}$ is Cauchy, there exists n_ε such that $n, m > n_\varepsilon$ implies $d(u_n, u_m) < \varepsilon$.

Let $n(> n_\varepsilon)$ be fixed. Then

$$\begin{aligned} d_H(L_\alpha(u_n), L_\alpha(u)) &= \lim_{m \to \infty} d_H(L_\alpha(u_n), L_\alpha(u_m)) \\ &\leqslant \overline{\lim_{m \to \infty}} \sup_{\alpha > 0} d_H(L_\alpha(u_n), L_\alpha(u_m)) \\ &= \overline{\lim_{m \to \infty}} d(u_n, u_m) < \varepsilon. \end{aligned}$$

Thus $\sup_{\alpha > 0} d_H(L_\alpha(u_n), L_\alpha(u)) \leqslant \varepsilon$, i.e., $d(u_n, u) \leqslant \varepsilon$ for $n > n_\varepsilon$, implying that $u_n \to u$ in the metric d. The proof terminates.

References

1. Z. Artstein and R. A. Vitale, A strong law of large numbers for random compact sets, *Ann. Probab.* (1975), 879–882.
2. R. J. Aumann, Integrals of set-valued functions, *J. Math. Anal. Appl.* **12** (1965), 1–12.
3. R. J. Aumann and M. Perles, A variational problem arising in economic. *J. Math. Anal. Appl.* **11** (1965), 488–503.
4. C. L. Byrne, Remarks on the set-valued integrals of Debreu and Aumann, *J. Math. Anal. Appl.* **62** (1978), 243–246.
5. N. Cressie, A central limit theorem for random sets, *Z. Wahrsch. Verw. Gebiete* **49** (1979), 37–47.
6. G. Debreu, Integration of correspondences, *in* "Proc. Fifth Berkeley Symp. Math. Statist. Probability," pp. 351–372, Univ. of California Press, Berkeley, 1966.
7. G. Debreu and D. Schmeidler, The Radon–Nikodým derivative of a correspondence, *in* "Proc. Sixth Berkeley Symp. Math. Statist. Probability," pp. 41–56, Univ. of California Press, Berkeley, 1970.
8. R. Féron, Ensembles aléatoires flous. *C. R. Acad. Sci. Paris Ser. A* **282** (1976), 903–906.

9. R. Fortet and M. Kambouzia, Ensembles aléatoires et ensembles flous, *Publ. Économétriques* **9** (1976), 1–23.
10. I. R. Goodman, Fuzzy sets as equivalence classes of random sets, *in* "Recent Developments in Fuzzy Set and Possibility Theory" (R. Yager, Ed.), Pergamon, Elmsford, N.Y., 1980.
11. H. Hermes, Calculus of set-valued functions and control, *J. Math. Mech.* **18** (1968), 47–59.
12. H. Kudō, Dependent experiments and sufficient statistics, *Natur. Sci. Rep. Ochanomizu Univ.* **4** (1953), 151–163.
13. C. Kuratowski, "Topologie I," Monografie Matematyczne, Warsaw, 1948.
14. H. Kwakernaak, Fuzzy random variables: Definition and theorems, *Inform. Sci.* **15** (1978), 1–29.
15. G. Matheron, "Random Sets and Integral Geometry," Wiley, New York, 1975.
16. C. V. Negoita and D. Ralescu, "Applications of Fuzzy Sets to Systems Analysis," Wiley, New York, 1975.
17. G. B. Price, The theory of integration, *Trans. Amer. Math. Soc.* **47** (1940), 1–50.
18. M. L. Puri and D. Ralescu, Différentielle d'une fonction floue. *C. R. Acad. Sci. Paris Sér. I.* **293** (1981), 237–239.
19. M. L. Puri and D. Ralescu, Integration on fuzzy sets, *Advan. Appl. Math.* **3** (1982), 430–434.
20. M. L. Puri and D. Ralescu, Strong law of large numbers for Banach space valued random sets, *Ann. Probab.* **11** (1983), 222–224.
21. M. L. Puri and D. Ralescu, Differentials of fuzzy functions, *J. Math. Anal. Appl.* **91** (1983), 552–558.
22. H. Rådström, An embedding theorem for spaces of convex sets, *Proc. Amer. Math. Soc.* **3** (1952), 165–169.
23. H. Richter, Verallgemeinerung eines in der Statistik benötigten Satzes der Masstheorie, *Math. Ann.* **150** (1963), 85–90.
24. L. A. Zadeh, Fuzzy sets, *Inform. Control* **8** (1965), 338–353.