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Weak projections onto a braided Hopf algebra

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Abstract

We show that, under some mild conditions, a bialgebra in an abelian and coabelian braided monoidal category has a weak projection onto a formally smooth (as a coalgebra) sub-bialgebra with antipode; see Theorem 1.14. In the second part of the paper we prove that bialgebras with weak projections are cross product bialgebras; see Theorem 2.12. In the particular case when the bialgebra A is cocommutative and a certain cocycle associated to the weak projection is trivial we prove that A is a double cross product, or biproduct in Madjid's terminology. The last result is based on a universal property of double cross products which, by Theorem 2.15, works in braided monoidal categories. We also investigate the situation when the right action of the associated matched pair is trivial.

Keywords: Monoidal categories; Bialgebras in a braided category; Weak projections

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Introduction

Hopf algebras in a braided monoidal category are very important structures. Probably the first known examples are $\mathbb{Z}$-graded and $\mathbb{Z}_2$-graded bialgebras (also called superbialgebras), that already appeared in the work of Milnor–Moore and MacLane. Other examples, such as bialgebras in the category of Yetter–Drinfeld modules, arose in a natural way in the characterization as a double crossed product of (ordinary) Hopf algebras with a projection [Ra]. Some braided bialgebras have also played a central role in the theory of quantum groups.

The abundance of examples and their applications explain the increasing interest for these objects and the attempts in describing their structure. For example in [BD1,BD2,BD3,Scha] several generalized versions of the double cross product bialgebra in a braided monoidal category $\mathfrak{M}$, generically called cross product bialgebras, are constructed. All of them have the common feature that, as objects in $\mathfrak{M}$, they are the tensor product of two objects in $\mathfrak{M}$. Let A be such a cross product, and let R and B the corresponding objects such that $A \simeq R \otimes B$. Depending on the particular type of cross product, the objects R and B may have additional properties, like being algebras and/or coalgebras. These structures may also satisfy some compatibility relations. For example, we can look for those cross product bialgebras $A \simeq R \otimes B$ such that there are a bialgebra morphism $\sigma : B \to A$ and a right B-linear coalgebra map $\pi : A \to B$ satisfying the relation $\pi\sigma = \mathrm{Id}_B$ (here A is a B-module via σ). For simplicity, we will say that A is a bialgebra (in $\mathfrak{M}$) with weak projection π on B. In the case when $\mathfrak{M}$ is the category of vector spaces, the problem of characterizing bialgebras with a weak projection was considered in [Scha].

The purpose of this is two fold. We assume that $\mathfrak{M}$ is a semisimple abelian and coabelian braided monoidal category (see Definition 1.1) and that $\sigma : B \to A$ is morphism of bialgebras in $\mathfrak{M}$. First, we want to show that there is a retraction π of σ which is a right B-linear morphism of coalgebras, provided that B is formally smooth as a coalgebra and that the B-adic coalgebra filtration on A is exhaustive (see Theorem 1.14). Secondly, assuming that a retraction π as above exists, we want to show that A is factorizable and to describe the corresponding structure R that arises in this situation. For this part of the paper we use the results of [BD3], that help us to prove that A is the cross product algebra $R \rtimes B$ where R is the 'coinvariant' subobject with respect to the right coaction of B on A defined by π (see Theorem 2.12). Several particular cases are also investigated. In Theorem 2.16, under the additional assumption that A is cocommutative and a certain cocycle is trivial, we describe the structure of A as a biproduct bialgebra of a certain matched pair (see Theorem 2.15).

We would like to note that some applications of the last mentioned result and its corollaries (see Proposition 2.19) are given in [AMS3]. As a matter of fact, our interest for the problems that we study in this paper originates in our work on the structure of cocommutative Hopf algebras with dual Chevalley property from [AMS3]. In particular, [AMS3, Theorem 6.14] and [AMS3, Theorem 6.16] are direct consequences of the main results of this article.

Let K be a field. A trivial example of semisimple braided monoidal category is the category of vector spaces over K. Typical examples of nontrivial semisimple braided monoidal categories are the category $\mathcal{M}^H$ of right comodules over a cosemisimple coquasitriangular K-Hopf algebra H and the category $\mathcal{M}_H$ of right modules over a semisimple quasitriangular K-Hopf algebra H. Note that an algebra or a coalgebra in these categories are ordinary algebras and coalgebras with an extra compatible structure. Examples of semisimple braided monoidal categories where this situation does not occur in general are the category $\mathcal{M}^H$ of right comodules over a cosemisimple coquasitriangular K-quasi-Hopf algebra H and the category $\mathcal{M}_H$ of right modules over a semisimple quasitriangular K-quasi-Hopf algebra H (see [Ka, Definition XV.2.1, p. 371]). Therefore our approach applies to these categories and can be seen as a way to reconstruct a meaningful class of braided bialgebras therein via a bosonization type procedure.

1. Hopf algebras in a braided category $\mathfrak{M}$

Definition 1.1. An *abelian monoidal category* is a monoidal category $(\mathfrak{M}, \otimes, \mathbf{1})$ such that:

(1) $\mathfrak{M}$ is an abelian category;
(2) both the functors $X \otimes (-) : \mathfrak{M} \to \mathfrak{M}$ and $(-) \otimes X : \mathfrak{M} \to \mathfrak{M}$ are additive and right exact, for every object $X \in \mathfrak{M}$.

A *coabelian monoidal category* is a monoidal category $(\mathfrak{M}, \otimes, \mathbf{1})$ such that:

(1) $\mathfrak{M}$ is an abelian category;
(2) both the functors $X \otimes (-) : \mathfrak{M} \to \mathfrak{M}$ and $(-) \otimes X : \mathfrak{M} \to \mathfrak{M}$ are additive and left exact, for every object $X \in \mathfrak{M}$.

Definitions 1.2. A *braided monoidal category* $(\mathfrak{M}, \otimes, \mathbf{1}, c)$ is a monoidal category $(\mathfrak{M}, \otimes, \mathbf{1})$ equipped with a *braiding* c, that is a natural isomorphism

$$c_{X,Y} : X \otimes Y \to Y \otimes X$$

satisfying

$$c_{X \otimes Y, Z} = (c_{X,Z} \otimes Y)(X \otimes c_{Y,Z}) \quad \text{and} \quad c_{X, Y \otimes Z} = (Y \otimes c_{X,Z})(c_{X,Y} \otimes Z).$$

For further details on these topics, we refer to [Ka, Chapter XIII].

A *bialgebra* $(B, m, u, \Delta, \varepsilon)$ in a braided monoidal category $(\mathfrak{M}, \otimes, \mathbf{1}, c)$ consists of an algebra (B, m, u) and a coalgebra (B, Δ, ε) in $\mathfrak{M}$ such that the diagrams in Fig. 1 are commutative.

1.3. For any bialgebra B in a monoidal category $(\mathfrak{M}, \otimes, \mathbf{1})$ we define the monoidal category $(\mathfrak{M}_B, \otimes, \mathbf{1})$ of right B-modules in $\mathfrak{M}$ as in [BD1]. The tensor product of two right B-modules (M, μ_M) and (N, μ_N) carries a right B-module structure defined by:

$$\mu_{M \otimes N} = (\mu_M \otimes \mu_N)(M \otimes c_{N,B} \otimes B)(M \otimes N \otimes \Delta).$$

Moreover, if $(\mathfrak{M}, \otimes, \mathbf{1})$ is an abelian monoidal category, then $(\mathfrak{M}_B, \otimes, \mathbf{1})$ is an abelian monoidal category too. Assuming that $(\mathfrak{M}, \otimes, \mathbf{1})$ is abelian and coabelian one proves that $(\mathfrak{M}_B, \otimes, \mathbf{1})$ is coabelian too.

$$\begin{tikzcd}
B\otimes B\otimes B\otimes B \arrow[rr, "B\otimes c\otimes B"] & & B\otimes B\otimes B\otimes B \arrow[d, "m\otimes m"] \\
B\otimes B \arrow[u, "\Delta\otimes\Delta"] \arrow[dr, "m"'] & & B\otimes B \\
& B \arrow[ur, "\Delta"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
B\otimes B \arrow[r, "\varepsilon\otimes\varepsilon"] \arrow[d, "m"'] & \mathbf{1}\otimes\mathbf{1} \arrow[d, "m_1"] \\
B \arrow[r, "\varepsilon"'] & \mathbf{1}
\end{tikzcd}$$

Fig. 1. The definition of bialgebras in $\mathfrak{M}$.

Obviously, (B, Δ, ε) is a coalgebra both in $(\mathfrak{M}_B, \otimes, \mathbf{1})$ and $({}_B\mathfrak{M}_B, \otimes, \mathbf{1})$. Of course, in both cases, B is regarded as a left and a right B-module via the multiplication on B.

1.4. To each coalgebra (C, Δ, ε) in $(\mathcal{M}, \otimes, \mathbf{1}, a, l, r)$ one associates a class of monomorphisms

$$^C\mathcal{I}^C := \{g \in {}^C\mathcal{M}^C \mid \exists f \text{ in } \mathcal{M} \text{ s.t. } fg = \mathrm{Id}\}.$$

Recall that C is *coseparable* whenever the comultiplication Δ cosplits in ${}^C\mathcal{M}^C$. We say that C is *formally smooth* in $\mathcal{M}$ if Coker Δ_C is ${}^C\mathcal{I}^C$-injective. For other characterizations and properties of coseparable and formally smooth coalgebras the reader is referred to [AMS2] and [Ar2]. In the same papers one can find different equivalent definitions of separable functors.

1.5. Let $(F, \phi_0, \phi_2) : (\mathcal{M}, \otimes, \mathbf{1}, a, l, r) \to (\mathcal{M}', \otimes', \mathbf{1}', a', l', r')$ be a monoidal functor between two monoidal categories, where $\phi_2(U, V) : F(U \otimes V) \to F(U) \otimes' F(V)$, for any $U, V \in \mathcal{M}$ and $\phi_0 : \mathbf{1}' \to \mathbf{F(1)}$. If (C, Δ, ε) is a coalgebra in $\mathcal{M}$ then $(C', \Delta_{C'}, \varepsilon_{C'}) := (F(C), \Delta_{F(C)}, \varepsilon_{F(C)})$ is a coalgebra in $\mathcal{M}'$, with respect to the comultiplication and the counit given by

$$\Delta_{F(C)} := \phi_2^{-1}(C, C)F(\Delta), \qquad \varepsilon_{F(C)} := \phi_0^{-1}F(\varepsilon).$$

Let us consider the functor $F' : {}^C\mathcal{M}^C \to {}^{C'}\mathcal{M}'^{C'}$ that associates to $(M, {}^C\rho_M, \rho_M^C)$ the object $(F(M), {}^{C'}\rho_{F(M)}, \rho_{F(M)}^{C'})$, where

$${}^{C'}\rho_{F(M)} := \phi_2^{-1}(C, M)F({}^C\rho_M), \qquad \rho_{F(M)}^{C'} := \phi_2^{-1}(M, C)F(\rho_M^C).$$

The proposition below is a restatement of [Ar2, Proposition 2.21], from which we have kept only the part that we need to prove Theorem 1.7.

Proposition 1.6. *Let $\mathcal{M}$, $\mathcal{M}'$, C, C', F and F' be as in* (1.5). *We assume that $\mathcal{M}$ and $\mathcal{M}'$ are coabelian monoidal categories.*

(a) *If C is coseparable in $\mathcal{M}$ then C' is coseparable in $\mathcal{M}'$; the converse is true whenever F' is separable.*
(b) *Assume that F preserves cokernels. If C is formally smooth as a coalgebra in $\mathcal{M}$ then C' is formally smooth as a coalgebra in $\mathcal{M}'$; the converse is true whenever F' is separable.*

Now we can prove one of the main results of this section.

Theorem 1.7. *Let B be a Hopf algebra in a braided abelian and coabelian monoidal category $(\mathfrak{M}, \otimes, \mathbf{1}, c)$. We have that*:

(a) *B is coseparable in $(\mathfrak{M}_B, \otimes, \mathbf{1})$ if and only if B is coseparable in $(\mathfrak{M}, \otimes, \mathbf{1})$.*
(b) *B is formally smooth as a coalgebra in $(\mathfrak{M}_B, \otimes, \mathbf{1})$ if and if B is formally smooth as a coalgebra in $(\mathfrak{M}, \otimes, \mathbf{1})$.*

Proof. We apply Proposition 1.6 in the case when $\mathcal{M} := (\mathfrak{M}_B, \otimes, \mathbf{1})$ and $\mathcal{M}' := (\mathfrak{M}, \otimes, \mathbf{1})$, which are coabelian categories. We take $(F, \phi_0, \phi_2) : (\mathfrak{M}_B, \otimes, \mathbf{1}) \to (\mathfrak{M}, \otimes, \mathbf{1})$ to be the forgetful functor from $\mathfrak{M}_B$ to $\mathfrak{M}$, where $\phi_0 = \mathrm{Id}_{\mathbf{1}}$ and, for any $U, V \in \mathfrak{M}_B$, we have $\phi_2(U, V) = \mathrm{Id}_{U \otimes V}$. We also take F' to be the forgetful functor from ${}^B\mathfrak{M}^B_B$ to ${}^B\mathfrak{M}^B$. Since $(\mathfrak{M}, \otimes, \mathbf{1})$ is an abelian monoidal category, then the functor $(-) \otimes B : \mathfrak{M} \to \mathfrak{M}$ is additive and right exact. Hence F preserves cokernels, see [Ar1, Theorem 3.6]. Thus, in view of Proposition 1.6, to conclude the proof of the theorem, it is enough to show that F' is a separable functor.

For each $(M, {}^B\rho_M, \rho^B_M) \in {}^B\mathfrak{M}^B$ we define $(M^{\mathrm{co}(B)}, i_{M^{\mathrm{co}(B)}})$ to be the equalizer of the maps

$$\rho^B_M : M \to M \otimes B \quad \text{and} \quad (M \otimes u_B) r_M^{-1} : M \to M \otimes B.$$

Since B is right flat ($\mathfrak{M}$ is an abelian monoidal category), we can apply the dual version of [Ar1, Proposition 3.3] to show that $(M^{\mathrm{co}(B)}, i_{M^{\mathrm{co}(B)}})$ inherits from M a natural left B-comodule structure ${}^B\rho_{M^{\mathrm{co}(B)}} : M^{\mathrm{co}(B)} \to B \otimes M^{\mathrm{co}(B)}$. As a matter of fact, with respect to this comodule structure, $M^{\mathrm{co}(B)}$ is the kernel of $\rho^B_M - (M \otimes u_B) r_M{}^{-1}$ in the category ${}^B\mathfrak{M}$. We obtain a functor $F'' : {}^B\mathfrak{M}^B \to {}^B\mathfrak{M}$ defined by:

$$F''\big(M, {}^B\rho_M, \rho^B_M\big) = \big(M^{\mathrm{co}(B)}, {}^B\rho_{M^{\mathrm{co}(B)}}\big).$$

Then $F'' \circ F'$ associates to $(M, \mu^B_M, {}^B\rho_M, \rho^B_M)$ the left B-comodule $(M^{\mathrm{co}(B)}, {}^B\rho_{M^{\mathrm{co}(B)}})$ in $\mathfrak{M}$. By the dual version of [BD1, Proposition 3.6.3], it follows that $F'' \circ F'$ is a monoidal equivalence. Therefore $F'' \circ F'$ is a separable functor and hence F' is separable too. □

Remark 1.8. By applying Theorem 1.7 to the category of vector spaces, we recover (4) ⇔ (5) in [AMS1, Theorem 2.26] and (a) ⇔ (b) in [Ar1, Proposition 7.27].

The formal smoothness or coseparability of B in the category of right modules over itself plays a fundamental role in the construction of a weak projection in Theorem 1.14.

A convenient way to check that a Hopf algebra B is coseparable in ${}_B\mathfrak{M}$ is to show that B has a total integral in $\mathfrak{M}$. This characterization of coseparable Hopf algebras will be proved next.

Definition 1.9. Let B be a Hopf algebra in a braided abelian and coabelian monoidal category $(\mathfrak{M}, \otimes, \mathbf{1}, c)$. A morphism $\lambda : B \to \mathbf{1}$ in $\mathfrak{M}$ is called a (left) *total integral* if it satisfies the relations:

$$r_B(B \otimes \lambda)\Delta = u\lambda, \tag{1}$$

$$\lambda u = \mathrm{Id}_{\mathbf{1}}. \tag{2}$$

1.10. In order to simplify the computation we will use the diagrammatic representation of morphisms in a braided category. For details on this method the reader is referred to [Ka, XIV.1]. On the first line of pictures in Fig. 2 are included the basic examples: the representation of a morphism $f : V \to W$ (downwards, the domain up) and the diagrams of $f' \circ f''$, $g' \otimes g''$ and $c_{V,W}$. The last four diagrams denote respectively the multiplication, the comultiplication, the unit and the counit of a bialgebra B in $\mathfrak{M}$. The graphical representation of associativity, existence of unit, coassociativity, existence of counit, compatibility between multiplication and comultiplication, the fact that ε is a morphism of algebras and the fact that u is a morphism of coalgebras can be found also in Fig. 2 (second line). The last two pictures on the same line are equivalent to the definition of a total integral. The fact that the right-hand side of the last equality is empty means that we can remove the left-hand side in any diagrams that contains it.

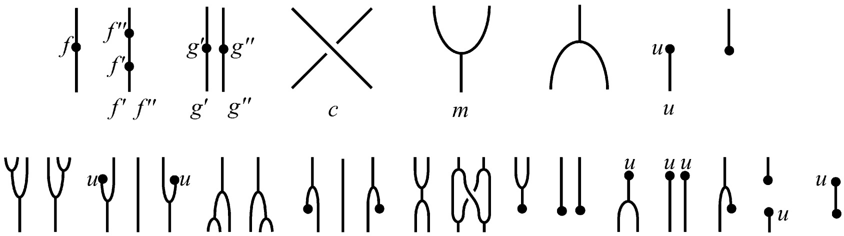

Fig. 2. Diagrammatic representation of morphisms in $\mathfrak{M}$.

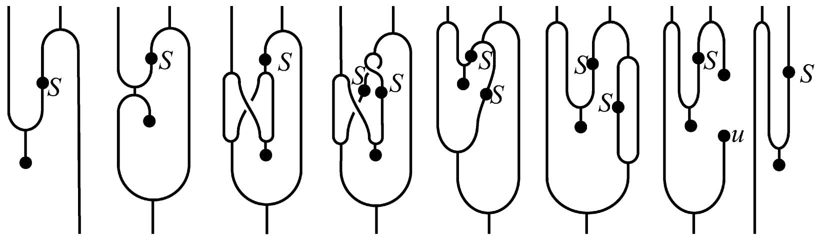

Fig. 3. The proof of relation (3).

Proposition 1.11. *A Hopf algebra B in an abelian and coabelian braided monoidal category $\mathfrak{M}$ is coseparable in $\mathfrak{M}_B$ if and only if it has a total integral. In this case, B is formally smooth as a coalgebra in $\mathfrak{M}_B$.*

Proof. We first assume that there is a total integral $\lambda : B \to \mathbf{1}$. Let us show that:

$$l_B(\lambda m \otimes B)(B \otimes S \otimes B)(B \otimes \Delta) = r_B(B \otimes \lambda m)(B \otimes B \otimes S)(\Delta \otimes B). \tag{3}$$

The proof is given in Fig. 3. The first equality follows by relation (1) and the definition of u. The second relation is a consequence of the fact that B is a bialgebra in $\mathfrak{M}$, so $\Delta m = (m \otimes c_{B,B} \otimes m)(\Delta \otimes \Delta)$. The third equation follows by the fact that the antipode S is an anti-morphism of coalgebras, i.e. $\Delta S = (S \otimes S)c_{B,B}\Delta$. For the fourth equality we used that the braiding is a functorial morphisms (thus m, S and λ can by pulled along the string over and under any crossing). The last two equalities follow $m(S \otimes B)\Delta = u\varepsilon$ and the properties of ε and u.

We now define $\theta : B \otimes B \to B$ by $\theta(x \otimes y) = l_B(\lambda m \otimes B)(B \otimes S \otimes B)(B \otimes \Delta)$. We have to prove that θ is a section of Δ in the category of B-bicomodules in $\mathfrak{M}_B$. Note that the category of B-bicomodules in $\mathfrak{M}_B$ is ${}^B\mathfrak{M}_B^B$. An object $M \in \mathfrak{M}$ is in ${}^B\mathfrak{M}_B^B$ if it is a right B-module and a B-bicomodule such that M is a Hopf module both in ${}^B\mathfrak{M}_B$ and $\mathfrak{M}_B^B$.

Let us show that θ is a B-bicolinear section of Δ. Taking into account relation (3), we prove that θ is left B-colinear in the first equality from Fig. 4. The fact that θ is right B-colinear is proved in the second equality of the same figure. In both of them, we used that Δ is coassociative and that the comodule structures on B and $B \otimes B$ are defined by Δ, $B \otimes \Delta$ and $\Delta \otimes B$. To show that θ is a section of Δ we use that $\lambda u = \mathrm{Id}_{\mathbf{1}}$, see the last sequence of equalities from Fig. 4.

It remains to prove that θ is right B-linear. This is done in Fig. 5. The first equality was obtained by using the fact that Δ is a morphism of algebras and that S is an anti-morphism of algebras, i.e. $mS = (S \otimes S)c_{B,B}m$. To get the second equality we pulled S and Δ under a crossing (this is possible because the braiding is functorial). For the third equality we used associativity

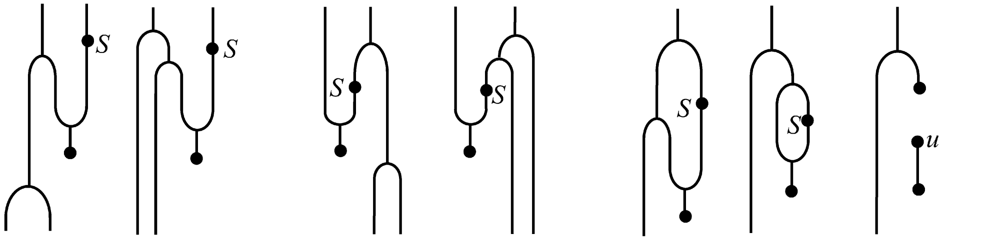

Fig. 4. The map θ is a B-bicolinear section of Δ.

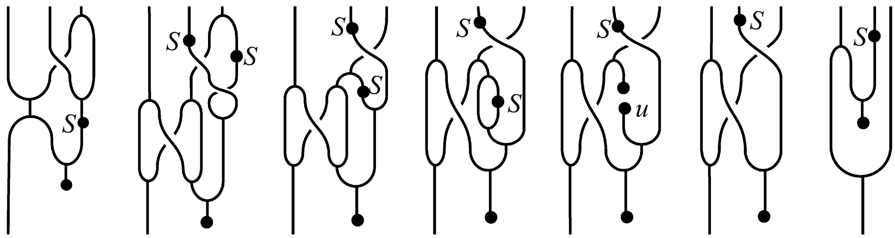

Fig. 5. The map θ is B-linear.

and coassociativity. The fourth and the fifth equalities follow by the definitions of the antipode, unit and counit. To deduce the sixth equality we pulled m and λ over the crossing.

Conversely, let $\theta : B \otimes B \to B$ be a section of Δ, which is a morphism of B-bicomodules in $\mathfrak{M}_B$. Let $\lambda := \varepsilon\theta(B \otimes u)r_B^{-1}$. Since θ is a morphism of right B-comodules it follows that

$$\Delta\theta(B \otimes u)r_B^{-1} = \left[\theta(B \otimes u)r_B^{-1} \otimes u\right]r_B^{-1}.$$

Then, by applying $\varepsilon \otimes B$, we get $\theta(B \otimes u)r_B^{-1} = u\lambda$. As θ is B-colinear, we have:

$$\Delta\theta(B \otimes u)r_B^{-1} = (B \otimes \theta)(\Delta \otimes u)r_B^{-1}.$$

Therefore, by the definition of λ, we get (1). Since θ is a section of Δ we deduce that $\lambda u = \mathrm{Id}_{\mathbf{1}}$. $\square$

Definition 1.12. Let E be a bialgebra in a braided monoidal category $((\mathfrak{M}, \otimes, \mathbf{1}), c)$. Let H be a Hopf subalgebra of E. Following [Scha, Definition 5.1], we say that $\pi : E \to H$ is a (*right*) *weak projection* (onto H) if it is a right H-linear coalgebra homomorphism such that $\pi\sigma = \mathrm{Id}_H$, where $\sigma : H \to E$ is the canonical morphism.

1.13. Let C be a coalgebra in a coabelian monoidal category $\mathfrak{M}$. Following [AMS2, 4.7], as in the case of vector spaces we can introduce the wedge product of two subobjects X, Y of C in $\mathfrak{M}$:

$$\left(X \wedge_C Y, i^C_{X\wedge Y}\right) := \mathrm{Ker}\left[(p_X \otimes p_Y) \circ \Delta_C\right],$$

where $p_X : C \to C/X$ and $p_Y : C \to C/Y$ are the canonical quotient maps.

We will simply write $\wedge$ instead of $\wedge_C$ if there is no danger of confusion.

Let $\delta : D \to C$ be a monomorphism which is a coalgebra homomorphism in $\mathfrak{M}$. Denote by (L, p) the cokernel of δ in $\mathfrak{M}$. Regard D as a C-bicomodule via δ and observe that L is a C-bicomodule and p is a morphism of bicomodules. Let

$$\left(D^{\wedge^n}, \delta_n\right) := \operatorname{Ker}\left(p^{\otimes n} \Delta_C^{n-1}\right)$$

for any $n \in \mathbb{N} \setminus \{0\}$. Note that $(D^{\wedge^1}, \delta_1) = (D, \delta)$ and $(D^{\wedge^2}, \delta_2) = D \wedge_C D$. In order to simplify the notations we set $(D^{\wedge^0}, \delta_0) = (0, 0)$.

Now, since $\mathfrak{M}$ has left exact tensor functors and since $p^{\otimes n} \Delta_C^{n-1}$ is a morphism of C-bicomodules (as a composition of morphisms of C-bicomodules), we get that $D^{\wedge^n}$ is a coalgebra and $\delta_n : D^{\wedge^n} \to C$ is a coalgebra homomorphism for any $n > 0$ and hence for any $n \in \mathbb{N}$.

Direct systems of Hochschild extensions in coabelian monoidal categories were defined in [AMS2, Definition 4.11].

Theorem 1.14. *Let $(\mathfrak{M}, \otimes, \mathbf{1}, c)$ be a braided monoidal category. Assume that $\mathfrak{M}$ is semisimple (i.e. every object is projective) abelian and that both the functors $X \otimes (-)$ and $(-) \otimes X$ from $\mathfrak{M}$ to $\mathfrak{M}$ are additive for every $X \in \mathfrak{M}$. Let A be bialgebra in $\mathfrak{M}$ and let B be a sub-bialgebra of A in $\mathfrak{M}$. Let $\sigma : B \to A$ denote the canonical inclusion. Assume that A is the direct limit (taken in $\mathfrak{M}$) of $(B^{\wedge^n})_{n \in \mathbb{N}}$, the B-adic coalgebra filtration. If B has an antipode (i.e. it is a Hopf algebra in $\mathfrak{M}$) and B is formally smooth as a coalgebra in $\mathfrak{M}$ (e.g. B is coseparable in $\mathfrak{M}$), then A has a right weak projection onto B.*

Proof. Clearly each additive functor $T : \mathfrak{M} \to \mathfrak{M}$ preserves split short exact sequences. Since $\mathfrak{M}$ is semisimple, every short exact sequence is split so that T is an exact functor. In particular $(\mathfrak{M}, \otimes, \mathbf{1})$ is both an abelian and a coabelian monoidal category. Let Δ and ε be respectively the comultiplication and counit of A. We denote $B^{\wedge^{n+1}}$ by A_n, where $A_0 = B$. We will denote the canonical projection onto A/A_n by p_n. One can regard B and A as a coalgebras in $\mathfrak{M}_B$, the latter object being a right B-bimodule via σ. By induction, it follows that $A_n \in \mathfrak{M}_B$, for every $n \in \mathbb{N}$, as $A_{n+1} = \operatorname{Ker}[(p_n \otimes p_0)\Delta]$ and by induction hypothesis p_n is B-linear (of course Δ is a morphism of right B-modules, since σ is a bialgebra map). Therefore, $(B^{\wedge^{n+1}})_{n \in \mathbb{N}}$ is the B-adic filtration on A in $\mathfrak{M}_B$. We want to prove that the canonical injections $A_n \to A_{n+1}$ split in $\mathfrak{M}_B$ so that $(A_n)_{n \in \mathbb{N}}$ is a direct system of Hochschild extensions in $\mathfrak{M}_B$. Indeed, it is enough to show that the canonical projection $A_{n+1} \to A_{n+1}/A_n$ has a section in $\mathfrak{M}_B$. By [AMS3, Lemma 2.19] it follows that $A_{n+1}/A_n = (A_n \wedge B)/B$ has a canonical right comodule structure $\rho_n : A_{n+1}/A_n \to A_{n+1}/A_n \otimes B$, which is induced by the comultiplication of A. If $\mu_n : A_{n+1}/A_n \otimes B \to A_{n+1}/A_n$ denotes the right B-action then A_{n+1}/A_n is a right–right Hopf module, that is:

$$\rho_n \mu_n = (\mu_n \otimes m_B)(M \otimes c_{M,B} \otimes B)(\rho_n \otimes \Delta_B).$$

The structure theorem for Hopf modules works for Hopf algebras in abelian braided categories, thus $\mathfrak{M}_B^B \simeq \mathfrak{M}$. In fact, the proof of [Sw, Theorem 4.11] can be easily written using the graphical calculus explained above, so it holds in an arbitrary abelian braided category (note that a similar result, in a braided category with splitting idempotents can be found in [BGS]). The equivalence of categories is established by the functor that associates to $V \in \mathfrak{M}$ the Hopf module $V \otimes B$ (with the coaction $V \otimes \Delta_B$ and action $V \otimes m_B$). We deduce that there is a $V \in \mathfrak{M}$ such that $A_{n+1}/A_n \simeq V \otimes B$ (isomorphism in $\mathfrak{M}_B^B$). Thus, to prove that the inclusion $A_n \to A_{n+1}$ splits in $\mathfrak{M}_B$, is sufficient to show that $V \otimes B$ is projective in $\mathfrak{M}_B$, for any object V in $\mathfrak{M}$. But, by [AMS2, Proposition 1.6], we have the adjunction:

$$\mathrm{Hom}_{\mathfrak{M}_B}(V \otimes B, X) \simeq \mathrm{Hom}_{\mathfrak{M}}(V, X), \quad \forall X \in \mathfrak{M}_B.$$

Since, by assumption, $\mathfrak{M}$ is semisimple, we deduce that $V \otimes B$ is projective in $\mathfrak{M}_B$, and hence that $(A_n)_{n\in\mathbb{N}}$ is a directed system of Hochschild extensions in $\mathfrak{M}_B$. As B is formally smooth as a coalgebra in $\mathfrak{M}$, by Theorem 1.7, it is also formally smooth as a coalgebra in $\mathfrak{M}_B$. Since A is the direct limit of $(A_n)_{n\in\mathbb{N}}$, we conclude the proof by applying [AMS2, Theorem 4.16] to the case $\mathcal{M} = \mathfrak{M}_B$. $\square$

As a consequence of Theorem 1.14 we recover the following result.

Corollary 1.15. *(See [Ar2, Theorem 9.16].) Let K be a field and let A be a bialgebra in $\mathcal{M}_K$ (ordinary bialgebra). Suppose that B is a sub-bialgebra of A with antipode. Assume that B is formally smooth as a coalgebra and that* $\mathrm{Corad}(A) \subseteq B$. *Then A has a right weak projection onto B.*

Recall that, for a right H-comodule (M, ρ), the subspace of coinvariant elements $M^{\mathrm{co}(H)}$ is defined by setting $M^{\mathrm{co}(H)} = \{m \in M \mid \rho(m) = m \otimes 1\}$. If A is an algebra in $\mathcal{M}^H$ then $A^{\mathrm{co}(H)}$ is a subalgebra of A.

When H is a cosemisimple coquasitriangular Hopf algebra, then $\mathcal{M}^H$ is a semisimple braided monoidal category. Note that bialgebras in $\mathcal{M}^H$ are usual coalgebras, so we can speak about the coradical of a bialgebra in this category.

Corollary 1.16. *Let H be a cosemisimple coquasitriangular Hopf algebra and let A be a bialgebra in $\mathcal{M}^H$. Let B denote the coradical of A. Suppose that B is a sub-bialgebra of A (in $\mathcal{M}^H$) with antipode. If $B \subseteq A^{\mathrm{co}(H)}$ then there is a right weak projection $\pi : A \to B$ in $\mathcal{M}^H$.*

Proof. Since $B \subseteq A^{\mathrm{co}(H)}$ it follows that $c_{B,B}$ is the usual flip map and B is an ordinary cosemisimple Hopf algebra. A Hopf algebra is cosemisimple if and only if there is $\lambda : B \to K$ such that (1) and (2) hold true (see e.g. [DNR, Exercise 5.5.9]). Obviously λ is a morphism of H-comodules, as $B \subseteq A^{\mathrm{co}(H)}$. The conclusion follows by Theorem 1.14. $\square$

2. Weak projections onto a braided Hopf algebra

Our main aim in this section is to characterize bialgebras in a braided monoidal category with a weak projection onto a Hopf subalgebra.

2.1. Throughout this section we will keep the following assumptions and notations.

(1) $(\mathfrak{M}, \otimes, \mathbf{1}, c)$ is an abelian and coabelian braided monoidal category;
(2) $(A, m_A, u_A, \Delta_A, \varepsilon_A)$ is a bialgebra in $\mathfrak{M}$;

(3) $(B, m_B, u_B, \Delta_B, \varepsilon_B)$ is a sub-bialgebra of A that has an antipode S_B (in particular B is a Hopf algebra in $\mathfrak{M}$);
(4) $\sigma : B \to A$ denotes the canonical inclusion (of course, σ is a bialgebra morphism);
(5) $\pi : A \to B$ is a right weak projection onto B (thus π is a morphism of coalgebras in $\mathfrak{M}_B$, where A is a right B-module via σ, and $\pi\sigma = \mathrm{Id}_B$);
(6) We define the following three endomorphisms (in $\mathfrak{M}$) of A:

$$\Phi := \sigma S_B \pi, \qquad \Pi_1 := \sigma\pi, \qquad \Pi_2 := m_A(A \otimes \Phi)\Delta_A.$$

Our characterization of A as a generalized crossed product is based on the work of Bespalov and Drabant [BD3]. We start by proving certain properties of the operators Φ, Π_1 and Π_2. They will be used later on to show that the conditions in [BD3, Proposition 4.6] hold true.

Lemma 2.2. *Under the assumptions and notations in* (2.1)*, Π_1 is a coalgebra homomorphism such that $\Pi_1\Pi_1 = \Pi_1$ and*

$$m_A(\Pi_1 \otimes \Pi_1) = \Pi_1 m_A(\Pi_1 \otimes \Pi_1). \tag{4}$$

Proof. Obviously Π_1 is a coalgebra homomorphism as σ and π are so. Trivially Π_1 is an idempotent, as $\pi\sigma = \mathrm{Id}_B$. Furthermore, we have

$$m_A(\Pi_1 \otimes \Pi_1) = m_A(\sigma\pi \otimes \sigma\pi) = \sigma m_B(\pi \otimes \pi) = \sigma\pi\big[\sigma m_B(\pi \otimes \pi)\big] = \Pi_1 m_A(\Pi_1 \otimes \Pi_1),$$

so the lemma is proved. $\square$

Lemma 2.3. *Under the assumptions and notations in* (2.1) *we have*:

$$\Delta_A \circ \Pi_2 = (m_A \otimes A) \circ (A \otimes \Phi \otimes \Pi_2) \circ (A \otimes c_{A,A} \circ \Delta_A) \circ \Delta_A. \tag{5}$$

Proof. See Fig. 6 on p. 189. The first equality is directly obtained from the definition of Π_2. The second equation follows by the compatibility relation between the multiplication and the comultiplication of a bialgebra in a braided monoidal category. For the third equality we used the definition of $\Phi := \sigma S_B \pi$, that π and σ are coalgebra homomorphisms and the fact that S_B is an anti-homomorphism of coalgebras in $\mathfrak{M}$, i.e. $\Delta S_B = (S_B \otimes S_B)c\Delta$. The fourth relation followed by coassociativity, while the last one was deduced (in view of naturality of the braiding) by dragging down one of the comultiplication morphisms over the crossing and by applying the definition of Π_2. $\square$

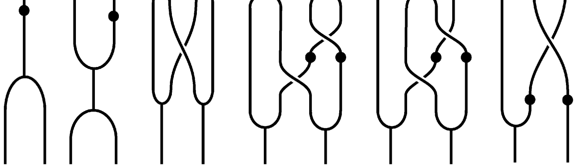

Fig. 6. The proof of Eq. (5).

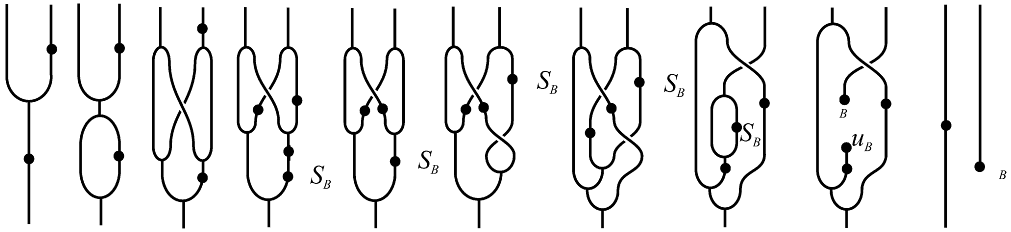

Fig. 7. The proof of Eq. (8).

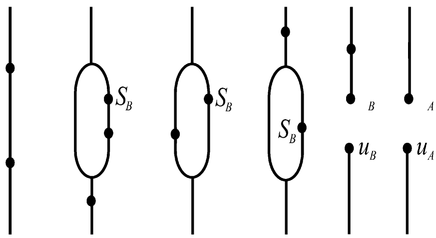

Fig. 8. The proof of Eq. (6).

Lemma 2.4. *Under the assumptions and notations in* (2.1) *we have*:

$$\pi \circ \Pi_2 = u_B \circ \varepsilon_A. \tag{6}$$

Proof. See Fig. 8 on p. 190. By the definition of Π_2 we have the first relation. The second one follows by the fact π is right B-linear (recall that the action of B on A is defined by σ). To deduce the third equality we use that π is a morphism of coalgebras, while the last relations follow immediately by the properties of the antipode, unit, counit and π. □

Lemma 2.5. *Under the assumptions and notations in* (2.1) *we have*:

$$(A \otimes \pi) \circ \Delta_A \circ \Pi_2 = (\Pi_2 \otimes u_B) \circ r_A^{-1}. \tag{7}$$

Proof. See Fig. 9 on p. 191. For the first equality we used relation (5). The second one follows by (6), while the third one is a consequence of the compatibility relation between the counit and the braiding and the compatibility relation between the counit and the comultiplication. The last equation is just the definition of Π_2. □

Lemma 2.6. *Under the assumptions and notations in* (2.1) *we have*:

$$\Pi_2 m_A(A \otimes \sigma) = \Pi_2 r_A(A \otimes \varepsilon_B) = r_A(\Pi_2 \otimes \varepsilon_B). \tag{8}$$

Proof. The proof can be found in Fig. 7 on p. 190. The first and the second equalities are implied by the definition of Π_2 and, respectively, the compatibility relation between multiplication and comultiplication in a bialgebra. For the next relation one uses the definition of Φ and that σ is a morphism of coalgebras. The fourth relation holds as π is right B-linear (the B-action on A is defined via σ). The fifth and the sixth equalities follow as σ is a morphism of coalgebras and

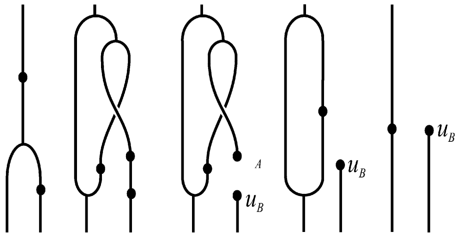

Fig. 9. The proof of Eq. (7).

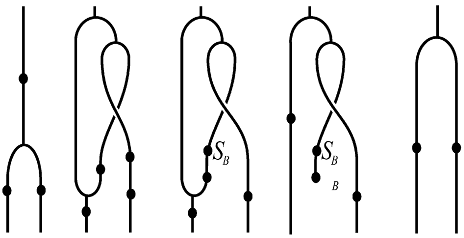

Fig. 10. The proof of Eq. (9).

S_B is an anti-morphism of coalgebras and, respectively, by associativity in A. As the braiding is functorial (so $\Delta_B(B \otimes \sigma S_B)$ can be dragged under the braiding) and σ is a morphism of algebras we get the seventh equality. To get the eighth relation we used the definition of the antipode, while the last one is implied by the properties of the unit and counit in a Hopf algebra, and the definition of Π_2. $\square$

Lemma 2.7. *Under the assumptions and notations in* (2.1)*, Π_2 is an idempotent such that*:

$$(\Pi_2 \otimes \Pi_2)\Delta_A = (\Pi_2 \otimes \Pi_2)\Delta_A \Pi_2. \tag{9}$$

Proof. Let us first prove that

$$\Pi_2 \Pi_2 = \Pi_2, \tag{10}$$

that is Π_2 is an idempotent. We have

$$\begin{aligned}\Pi_2\Pi_2 &= m_A(A \otimes \sigma S_B \pi)\Delta_A \Pi_2 = m_A(A \otimes \sigma S_B)(A \otimes \pi)\Delta_A \Pi_2 \\ &\overset{(7)}{=} m_A(A \otimes \sigma S_B)(\Pi_2 \otimes u_B) r_A^{-1}.\end{aligned}$$

Since σ and S_H are unital morphism and the right unity constraint r is functorial we get

$$\Pi_2\Pi_2 = m_A(A \otimes u_A)(\Pi_2 \otimes \mathbf{1}) r_A^{-1} = m_A(A \otimes u_A) r_A^{-1} \Pi_2 = \Pi_2.$$

For the proof of Eq. (9) see Fig. 10 on p. 191. In that figure, we get the first equality by using (5). The next two relations are consequences of the definition of Φ, relation (10) and relation (8). Finally, to obtain the last equality we use the properties of the antipode and of the counit, together with the fact that $\varepsilon_B \pi = \varepsilon_A$. $\square$

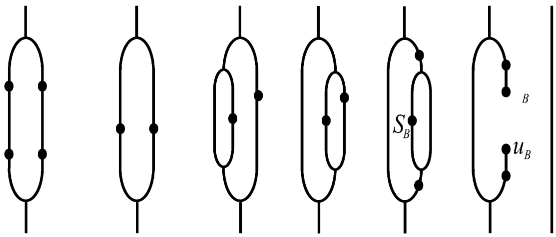

Fig. 11. The proof of Eq. (12).

Lemma 2.8. *Under the assumptions and notations in* (2.1) *we have*:

$$\Pi_1 u_A = u_A \quad \text{and} \quad \varepsilon_A \Pi_2 = \varepsilon_A. \tag{11}$$

Proof. The first relation is easy: $\Pi_1 u_A = \sigma\pi u_A = \sigma\pi\sigma u_B = \sigma u_B = u_A$. To prove the second one we perform the following computation:

$$\varepsilon_A \Pi_2 = \varepsilon_A m_A(A \otimes \sigma S_B \pi)\Delta_A = m_{\mathbf{1}}(\varepsilon_A \otimes \varepsilon_A)(A \otimes \sigma S_B \pi)\Delta_A.$$

Therefore, by the properties of the counit of a Hopf algebra in braided monoidal category, we get

$$\varepsilon_A \Pi_2 = r_{\mathbf{1}}(\varepsilon_A \otimes \mathbf{1})(A \otimes \varepsilon_A)\Delta_A = \varepsilon_A r_A(A \otimes \varepsilon_A)\Delta_A = \varepsilon_A,$$

so the lemma is completely proved. □

Lemma 2.9. *Under the assumptions and notations in* (2.1)*, the homomorphisms* $m_A \circ (\Pi_2 \otimes \Pi_1)$ *and* $(\Pi_2 \otimes \Pi_1) \circ \Delta_A$ *split the idempotent* $\Pi_2 \otimes \Pi_1$.

Proof. We have to prove that

$$m_A(\Pi_2 \otimes \Pi_1)(\Pi_2 \otimes \Pi_1)\Delta_A = \mathrm{Id}_A, \tag{12}$$

$$(\Pi_2 \otimes \Pi_1)\Delta_A m_A(\Pi_2 \otimes \Pi_1) = \Pi_1 \otimes \Pi_2. \tag{13}$$

The proof of relation (12) is shown in Fig. 11 on p. 192. The first three equalities are simple consequences of the fact Π_1 and Π_2 are idempotents, of the definitions of these homomorphisms and of (co)associativity in A. For the fourth relation we used the definition of Φ and that π is a morphism of coalgebras and σ is a morphism of algebras. The last two relations follow by the definitions of the antipode, unit and counit in a Hopf algebra, together with $\varepsilon_B \pi = \varepsilon_A$ and $\sigma u_B = u_A$.

The proof of relation (13) is shown in Fig. 12 on p. 193. The first two equalities immediately follow by the compatibility relation between multiplication and comultiplication on A, the fact that Π_1 is a coalgebra homomorphism and $\Pi_1 = \sigma\pi$. By using (8) and $\pi m_A(A \otimes \sigma) = m_B(\pi \otimes B)$, that is π is right B-linear, we get the third relation. The relation $\varepsilon_B \pi = \varepsilon_A$, the fact that the braiding is functorial and the properties of the counit are used to obtain the fourth equality. The fifth one is implied by (7), while to prove the last equality one uses $\Pi_2 \Pi_2 = \Pi_2$, the definition of the unit and the definition of Π_1. □

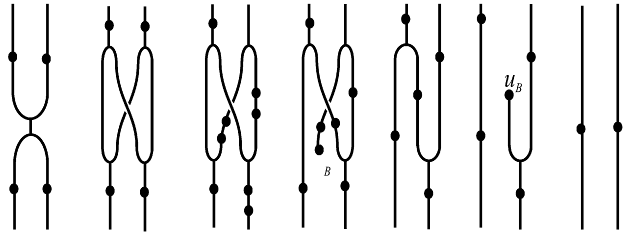

Fig. 12. The proof of Eq. (13).

Lemma 2.10. *Under the assumptions and notations in* (2.1)*, let*

$$(R, i) := \mathrm{Eq}\big[(A \otimes \pi)\Delta_A, (A \otimes u_B)r_A^{-1}\big].$$

Then there exists a unique morphism $p : A \to R$ *such that*

$$ip = \Pi_2 \quad \textit{and} \quad pi = \mathrm{Id}_R.$$

Proof. First of all, we have

$$(A \otimes \pi) \circ \Delta_A \circ \Pi_2 \overset{(7)}{=} (\Pi_2 \otimes u_H) \circ r_A^{-1} = (A \otimes u_H) \circ (\Pi_2 \otimes \mathbf{1}) \circ r_A^{-1} = (A \otimes u_H) \circ r_A^{-1} \circ \Pi_2.$$

Thus, by the universal property of the equalizer, there is a unique morphism $p : A \to R$ such that $ip = \Pi_2$. We have

$$\begin{aligned} ipi = \Pi_2 i &= m_A(A \otimes \sigma S_B \pi)\Delta_A i = m_A(A \otimes \sigma S_B)(A \otimes \pi)\Delta_A i \\ &= m_A(A \otimes \sigma S_B)(A \otimes u_B)r_A^{-1} i = m_A(A \otimes u_A)r_A^{-1} i = i. \end{aligned}$$

Since i is a monomorphism we get $pi = \mathrm{Id}_R$ so that i and p split the idempotent Π_2. □

2.11. Before proving one of the main results of this paper, Theorem 2.12, we introduce some more notations and terminology. First of all the object R, that we introduced in Lemma 2.10, will be called *the diagram of* A. Note that R is the 'coinvariant subobject' of A with respect to the right B-coaction induced by the coalgebra homomorphism π.

We now associate to the weak projection π the following data:

$$m_R : R \otimes R \to R, \quad m_R := pm_A(i \otimes i);$$

$$u_R : \mathbf{1} \to R, \quad u_R := pu_A;$$

$$\Delta_R : R \to R \otimes R, \quad \Delta_R = (p \otimes p)\Delta_A i;$$

$$\varepsilon_R : R \to \mathbf{1}, \quad \varepsilon_R = \varepsilon_A i;$$

$$\xi : R \otimes R \to B, \quad \xi := \pi m_A(i \otimes i);$$

$${}^B\mu_R : B \otimes R \to R, \quad {}^B\mu_R := pm_A(\sigma \otimes i);$$

$$\mu_R^B : R \otimes B \to R, \quad \mu_R^B := pm_A(i \otimes \sigma);$$

$$ {}^{B}\rho_R : R \to B \otimes R, \quad {}^{B}\rho_R := (\pi \otimes p)\Delta_A i; $$
$$ \mu_B^R : B \otimes R \to B, \quad \mu_B^R := \pi m_A(\sigma \otimes i). $$

Theorem 2.12. *We keep the assumptions and notations in* (2.1) *and* (2.11).

(1) *The diagram R is a coalgebra with comultiplication Δ_R and counit ε_R, and p is a coalgebra homomorphism.*
(2) *The morphisms $m_A(i \otimes \sigma)$ and $(p \otimes \pi)\Delta_A$ are mutual inverses, so that $R \otimes B$ inherits a bialgebra structure which is the cross product bialgebra $R \rtimes B$ defined by*

$$
\begin{aligned}
m_{R \rtimes B} = {} & (R \otimes m_B)(m_R \otimes \xi \otimes B)\big(R \otimes {}^{B}\mu_R \otimes R \otimes R \otimes m_B\big)(R \otimes B \otimes c_{R,R} \otimes R \otimes B \otimes B) \\
& \times \big(R \otimes {}^{B}\rho_R \otimes R \otimes R \otimes B \otimes B\big)(\Delta_R \otimes \Delta_R \otimes B \otimes B)\big(R \otimes {}^{B}\mu_R \otimes \mu_B^R \otimes B\big) \\
& \times (R \otimes B \otimes c_{B,R} \otimes R \otimes B)(R \otimes \Delta_B \otimes \Delta_R \otimes B), \\
u_{R \rtimes B} = {} & (u_R \otimes u_B)\Delta_{\mathbf{1}}, \\
\Delta_{R \rtimes B} = {} & (R \otimes m_B \otimes R \otimes B)(R \otimes B \otimes c_{R,B} \otimes B)\big(R \otimes {}^{B}\rho_R \otimes B \otimes B\big)(\Delta_R \otimes \Delta_B), \\
\varepsilon_{R \rtimes B} = {} & m_{\mathbf{1}}(\varepsilon_R \otimes \varepsilon_B).
\end{aligned}
$$

Proof. By the previous lemmata, Π_1 and Π_2 fulfill the requirements of the right-hand version of [BD3, Proposition 4.6(2)]. Thus (1) and (3) of the same result hold. In our case it can be checked that:

$$ (B_1, p_1, i_1) = (B, \sigma, \pi) \quad \text{and} \quad (B_2, p_2, i_2) = (R, i, p). $$

The explicit form of $m_{R \rtimes B}$ and $\Delta_{R \rtimes B}$ is a right-hand version of the one in the fourth box of diagrams in [BD3, Table 2, p. 480]. $\square$

We are now going to investigate a particular case of the above theorem. Namely, when A is cocommutative and ξ is trivial, we will show that A is the double cross product of a matched pair (see definitions below).

Definition 2.13. Let $(R, m_R, u_R, \Delta_R, \varepsilon_R)$ and $(B, m_B, u_B, \Delta_B, \varepsilon_B)$ be bialgebras in a braided abelian and coabelian monoidal category $(\mathfrak{M}, \otimes, \mathbf{1}, c)$. Following [Maj, Definition 7.2.1, p. 298], we say that (R, B) defines a *matched pair of bialgebras*, if there exist morphisms

$$ \triangleright : B \otimes R \to R \quad \text{and} \quad \triangleleft : B \otimes R \to B $$

satisfying the seven conditions below:

(1) $(R, \Delta_R, \varepsilon_R, \triangleright)$ is a left B-module coalgebra;
(2) $(B, \Delta_B, \varepsilon_B, \triangleleft)$ is a right R-module coalgebra;
(3) $\triangleleft(u_B \otimes R) = u_B \varepsilon_R l_R$;
(4) $\triangleright(B \otimes u_R) = u_R \varepsilon_B r_B$;
(5) $m_B(\triangleleft \otimes B)(B \otimes \triangleright \otimes \triangleleft)(B \otimes \Delta_{B \otimes R}) = \triangleleft(m_R \otimes B)$;

(6) $m_R(R \otimes \triangleright)(\triangleright \otimes \triangleleft \otimes R)(\Delta_{B\otimes R} \otimes R) = \triangleright(B \otimes m_R)$;
(7) $(\triangleleft \otimes \triangleright)\Delta_{B\otimes R} = c_{R,B}(\triangleright \otimes \triangleleft)\Delta_{B\otimes R}$.

In this case, for sake of shortness, we will say that $(R, B, \triangleright, \triangleleft)$ *is a matched pair of bialgebras in* $(\mathfrak{M}, \otimes, \mathbf{1}, c)$.

2.14. Let $(R, B, \triangleright, \triangleleft)$ be a matched pair. By [BD2, Corollary 2.17], we get that:

$$m_{R\bowtie B} = (m_R \otimes m_B)(R \otimes \triangleright \otimes \triangleleft \otimes B)(R \otimes B \otimes c_{B,R} \otimes R \otimes B)(R \otimes \Delta_B \otimes \Delta_R \otimes B),$$

$$u_{R\bowtie B} = (u_R \otimes u_B)\Delta_{\mathbf{1}},$$

$$\Delta_{R\bowtie B} = (R \otimes c_{R,B} \otimes B)(\Delta_R \otimes \Delta_B),$$

$$\varepsilon_{R\bowtie B} = m_{\mathbf{1}}(\varepsilon_R \otimes \varepsilon_B)$$

defines a new bialgebra $R \bowtie B$, that is called *the double cross product bialgebra*. It can be obtained as a particular case of the cross product bialgebra $R \rtimes B$ by setting ${}^B\mu_R = \triangleleft$, $\mu_B^R = \triangleright$ and taking ξ, ${}^B\rho_R$ and μ_B^R to be trivial in (2.11).

Theorem 2.15. *Let $\sigma : B \to A$ and $i : R \to A$ be bialgebra morphisms in a braided monoidal category $(\mathfrak{M}, \otimes, \mathbf{1}, c)$ such that $\Phi := m_A(i \otimes \sigma)$ is an isomorphism in $\mathfrak{M}$. Let $\Psi := \Phi^{-1}\Theta$, where $\Theta : B \otimes R \to A$ is defined by $\Theta := m_A(\sigma \otimes i)$.*

Consider the homomorphisms $\triangleright : B \otimes R \to R$ and $\triangleleft : B \otimes R \to B$ defined by:

$$\triangleright := r_R(R \otimes \varepsilon_B)\Psi, \qquad \triangleleft := l_B(\varepsilon_R \otimes B)\Psi. \tag{14}$$

Then $(R, B, \triangleright, \triangleleft)$ is a matched pair and $A \simeq R \bowtie B$.

Proof. We will follow the proof of [Maj, Theorem 7.2.3]. It is easy to see that the proofs of relations [Maj, (7.10)] and [Maj, (7.11)] work in a braided monoidal category, as they can be done in a diagrammatic way. Therefore, we have:

$$(R \otimes m_B)(\Psi \otimes B)(B \otimes \Psi) = \Psi(m_B \otimes R), \qquad \Psi(B \otimes u_R)r_B^{-1} = (u_R \otimes B)l_B^{-1}, \tag{15}$$

$$(m_R \otimes B)(R \otimes \Psi)(\Psi \otimes R) = \Psi(B \otimes m_R), \qquad \Psi(u_B \otimes R)l_R^{-1} = (R \otimes u_B)r_R^{-1}. \tag{16}$$

For example the first relation in (15) is proved in Fig. 13. The first equivalence there holds since $\Phi = m_A(i \otimes \sigma)$ is by assumption an isomorphism. The second and the third equivalences are consequences of associativity in A and of relation $\Phi\Psi = \Theta$. Since the last equality is obviously true by associativity in A, the required relation is proved. The second relation in (15) follows by the computation performed in Fig. 14. The first equality holds since $\Phi\Psi = \Theta$, while the second follows by the fact that i and σ are homomorphisms of algebras and by the definition of the unit in an algebra. To get the second relation in (15) we use the fact that Φ is an isomorphism.

As in the proof of [Maj, Theorem 7.2.3], by applying $l_B(\varepsilon_R \otimes B)$ and $r_R(R \otimes \varepsilon_R)$ respectively to (16) and (15), we get that $\triangleright$ defines a left action of B on R and $\triangleleft$ defines a right action of R on B. Indeed, by applying $\varepsilon_R \otimes B$ to the second relation in (15) it is easy to see that $\triangleleft$ is unital. The second axiom that defines a right action is checked in Fig. 15.

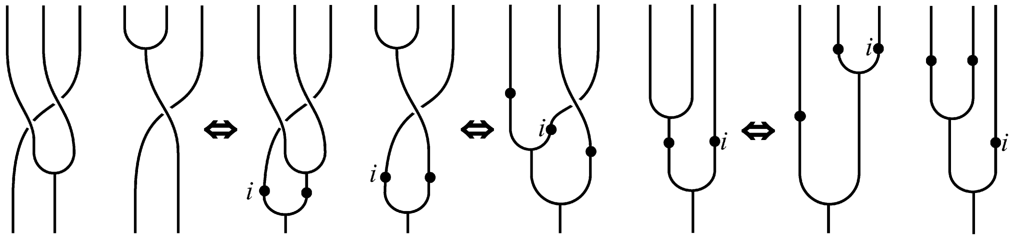

Fig. 13. The proof of the first equation in (15).

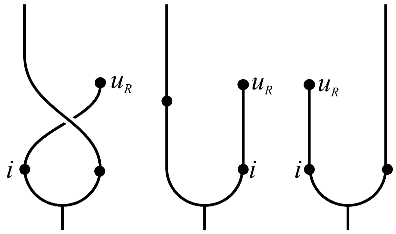

Fig. 14. The proof of the second equation in (15).

Furthermore, by applying $l_B(\varepsilon_R \otimes B)$ and $r_R(R \otimes \varepsilon_R)$ respectively to (15) and (16), we get

$$\lhd(m_B \otimes R) = m_B(\lhd \otimes B)(B \otimes \Psi), \tag{17}$$

$$\rhd(B \otimes m_R) = m_R(R \otimes \lhd)(\Psi \otimes R). \tag{18}$$

For the proof of (17) see Fig. 16. We now want to check that $\Theta : B \otimes R \to A$ is a coalgebra homomorphism, where the coalgebra structure on $B \otimes R$ is given by:

$$\Delta_{B\otimes R} := (B \otimes c_{B,R} \otimes R)(\Delta_B \otimes \Delta_R), \qquad \varepsilon_{B\otimes R} := m_{\mathbf{1}}(\varepsilon_B \otimes \varepsilon_R).$$

Indeed, we have

$$\begin{aligned}
\Delta_A\Theta &= (m_A \otimes m_A)(A \otimes c_{A,A} \otimes A)(\Delta_A \otimes \Delta_A)(\sigma \otimes i) \\
&= (m_A \otimes m_A)(A \otimes c_{A,A} \otimes A)(\sigma \otimes \sigma \otimes i \otimes i)(\Delta_B \otimes \Delta_R) \\
&= (m_A \otimes m_A)(\sigma \otimes i \otimes \sigma \otimes i)(B \otimes c_{B,R} \otimes R)(\Delta_B \otimes \Delta_R) \\
&= (\Theta \otimes \Theta)(B \otimes c_{B,R} \otimes R)(\Delta_B \otimes \Delta_R) = (\Theta \otimes \Theta)\Delta_{B\otimes R},
\end{aligned}$$

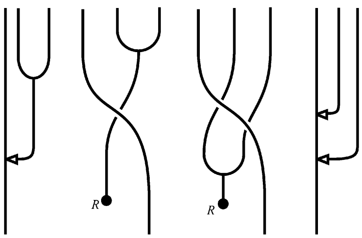

Fig. 15. B is a right R-module with respect to $\lhd$.

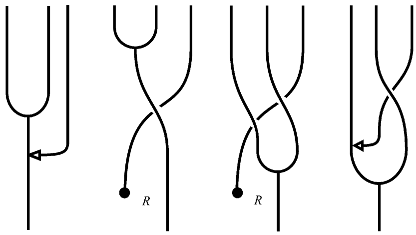

Fig. 16. The proof of Eq. (17).

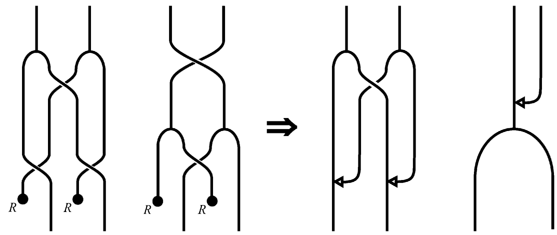

Fig. 17. The proof of Eq. (20).

and $\varepsilon_A \Theta = m_{\mathbf{1}}(\varepsilon_A \otimes \varepsilon_A)(\sigma \otimes i) = m_{\mathbf{1}}(\varepsilon_B \otimes \varepsilon_R) = \varepsilon_{B\otimes R}$. In a similar way, by interchanging B and R, we can prove that Φ is a homomorphism of coalgebras. Thus $\Psi = \Phi^{-1}\Theta$ is a coalgebra homomorphism too, so

$$\Delta_{R\otimes B}\Psi = (\Psi \otimes \Psi)\Delta_{B\otimes R} \quad \text{and} \quad \varepsilon_{R\otimes B}\Psi = \varepsilon_{B\otimes R}. \tag{19}$$

By applying $l_B(\varepsilon_R \otimes B) \otimes l_B(\varepsilon_R \otimes B)$ to both sides of the first equality in (19) we get the first relation in Fig. 17. By the properties of ε_B and ε_R we get the second relation in the same figure, that is we have:

$$\Delta_B \triangleleft = (\triangleleft \otimes \triangleleft)\Delta_{B\otimes R}. \tag{20}$$

As $\varepsilon_B \triangleleft = \varepsilon_B l_B(\varepsilon_R \otimes B)\Psi = \varepsilon_{R\otimes B}\Psi = \varepsilon_{B\otimes R}$ we have proved that $(B, \Delta_B, \varepsilon_B, \triangleleft)$ is a right R-module coalgebra. Similarly one can prove that $(R, \Delta_R, \varepsilon_R, \triangleright)$ is a left B-module coalgebra.

By applying $r_R(R \otimes \varepsilon_H) \otimes l_H(\varepsilon_R \otimes H)$ and $l_H(\varepsilon_R \otimes H) \otimes r_R(R \otimes \varepsilon_H)$ respectively to both sides of the first equality in (19) (see e.g. Fig. 18) one can prove the relations:

$$\Psi = (\triangleright \otimes \triangleleft)\Delta_{B\otimes R} \quad \text{and} \quad c_{R,B}\Psi = (\triangleleft \otimes \triangleright)\Delta_{B\otimes R}. \tag{21}$$

By the two relations of (21) we deduce

$$c_{R,B}(\triangleright \otimes \triangleleft)\Delta_{B\otimes R} = (\triangleleft \otimes \triangleright)\Delta_{B\otimes R}. \tag{22}$$

By applying $\varepsilon_R \otimes B$ to the both sides of (15) we get the first equation in Fig. 19. By the definition of the right action of R on B we get the relation in the middle of that figure. By using the first

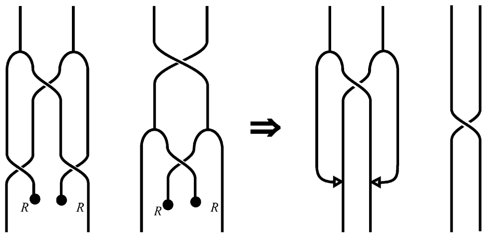

Fig. 18. The proof of Eq. (21).

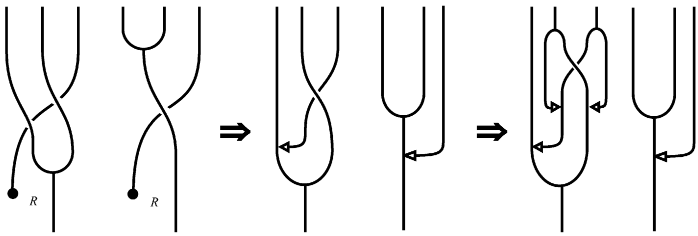

Fig. 19. The proof of Eq. (23).

equation in (21) we get the last equality in Fig. 19. Therefore we have proved the following equation:

$$m_B(\triangleleft \otimes B)(B \otimes \triangleright \otimes \triangleleft)(B \otimes \Delta_{B\otimes R}) = \triangleleft(m_R \otimes B). \tag{23}$$

The relation (6) from Definition 2.13 can be proved similarly. Finally, by composing both sides of (16) by $\varepsilon_R \otimes B$ to the left and by $u_B \otimes B \otimes u_R$ to the right we get:

$$\triangleleft(u_B \otimes R) = u_B \varepsilon_R l_R. \tag{24}$$

The details of the proof are given in Fig. 20. Analogously one can prove relation (4) from Definition 2.13. In conclusion, we have proved that $(R, B, \triangleleft, \triangleright)$ is a matched pair and that Φ is a morphism of coalgebras.

It remains to prove that Φ is an isomorphism of algebras. Obviously Φ is an unital homomorphism. By (21) it follows that $m_{R\bowtie B} = (m_R \otimes m_B)(R \otimes \Psi \otimes B)$. Since i and σ are morphisms of algebras and m_A is associative we get

$$\begin{aligned}\Phi m_{R\bowtie B} &= m_A(i \otimes \sigma)(m_R \otimes m_B)(R \otimes \Psi \otimes B) = m_A(m_A \otimes A)\big[i \otimes m_A(i \otimes \sigma)\Psi \otimes \sigma\big] \\ &= m_A(m_A \otimes A)(i \otimes \Phi\Psi \otimes \sigma) = m_A(m_A \otimes A)(i \otimes \Theta \otimes \sigma) \\ &= m_A(m_A \otimes A)\big[i \otimes m_A(\sigma \otimes i) \otimes \sigma\big] = m_A(\Phi \otimes \Phi).\end{aligned}$$

Trivially $\Phi u_{R\bowtie B} = u_A$ since i and σ are unital homomorphism and $m_A(u_A \otimes u_A)\Delta_{\mathbf{1}} = u_A$, so the theorem is proved. $\square$

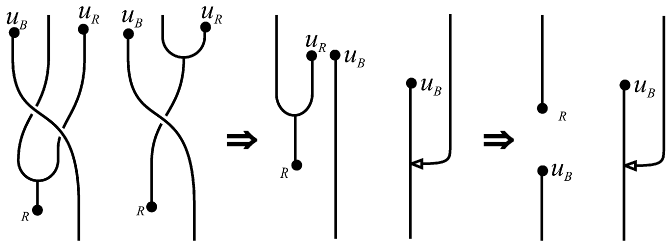

Fig. 20. The proof of Eq. (24).

Theorem 2.16. *We keep the assumptions and notations in* (2.1) *and* (2.11). *We also assume that A is cocommutative and ξ is trivial, i.e.* $c_{A,A}\Delta_A = \Delta_A$ *and* $\xi = u_B m_{\mathbf{1}}(\varepsilon_R \otimes \varepsilon_R)$. *Then*

$$(R, B, \triangleright, \triangleleft)$$

is a matched pair of bialgebras such that $A \simeq R \bowtie B$, where

$$\triangleright := {}^B\mu_R = pm_A(\sigma \otimes i) : B \otimes R \to R \quad \textit{and} \quad \triangleleft := \mu_B^R = \pi m_A(\sigma \otimes i) : B \otimes R \to B. \tag{25}$$

Proof. Since ξ is trivial, by [BD3, Proposition 3.7(5)] it follows that R is an algebra and $i : R \to A$ is an algebra homomorphism. Our aim now is to show that $i : R \to A$ is a coalgebra homomorphism too. In view of [BD3, Proposition 3.7(8)] it is enough to prove that

$${}^B\rho_R = (u_B \otimes R)l_R^{-1} \quad \text{and} \quad (\pi \otimes \pi)\Delta_A i = (u_B \otimes u_B)\Delta_{\mathbf{1}}\varepsilon_R. \tag{26}$$

Since π is a coalgebra homomorphism, the second equality follows by [BD3, Proposition 3.7(6)]. Let us prove the first one. Indeed, as R is the equalizer of $(A \otimes \pi)\Delta_A$ and $(A \otimes u_B)r_A^{-1}$, we have

$$\begin{aligned}(p \otimes \pi)\Delta_A i &= (p \otimes B)(A \otimes \pi)\Delta_A i = (p \otimes B)(A \otimes u_B)r_A^{-1} i = (R \otimes u_B)(p \otimes \mathbf{1})r_A^{-1} i \\ &= (R \otimes u_B)r_R^{-1} pi = (R \otimes u_B)r_R^{-1}.\end{aligned}$$

Therefore

$$\begin{aligned}{}^B\rho_R &= (\pi \otimes p)\Delta_A i = (\pi \otimes p)c_{A,A}\Delta_A i = c_{A,A}(p \otimes \pi)\Delta_A i = c_{A,A}(R \otimes u_B)r_R^{-1} \\ &= (u_B \otimes R)c_{R,\mathbf{1}}r_R^{-1} = (u_B \otimes R)l_R^{-1}.\end{aligned}$$

Hence (26) is proved and, in consequence, it follows that i is a morphism of bialgebras. By Theorem 2.12(2) the morphisms $\Phi = m_A(i \otimes \sigma)$ and $(p \otimes \pi)\Delta_A$ are mutual inverses. Thus we can apply Theorem 2.15. In our case

$$\Psi = \Phi^{-1}\Theta = (p \otimes \pi)\Delta_A m_A(\sigma \otimes i).$$

In view of (14), it follows that

$$\triangleright = r_R(R \otimes \varepsilon_B)(p \otimes \pi)\Delta_A m_A(\sigma \otimes i) = pr_A(A \otimes \varepsilon_B)\Delta_A m_A(\sigma \otimes i) = pm_A(\sigma \otimes i).$$

In a similar way we get

$$\triangleright = \pi m_A(\sigma \otimes i). \qquad \square$$

Remark 2.17. By applying Theorem 2.16 to the category of right comodules over a cosemisimple coquasitriangular Hopf algebra, one can prove [AMS1, Theorem 6.14] and hence [AMS1, Theorem 6.16] which is a generalization of the so-called Cartier–Gabriel–Kostant Theorem.

2.18. We keep the assumptions and the notations in (2.1) and (2.11). We take A to be a cocommutative bialgebra in $\mathfrak{M}$ with trivial cocycle ξ. Thus, by our results, A is the double cross product of a certain matched pair $(R, B, \triangleright, \triangleleft)$, where the actions $\triangleright$ and $\triangleleft$ are defined by relations (25). Our aim now is to investigate those bialgebras A as above which, in addition, have the property that the right action $\triangleleft : B \otimes R \to B$ is trivial. We will see that in this case the left action $\triangleright : B \otimes R \to R$ is the adjoint action. More precisely, we have $i\triangleright = \mathrm{ad}$, where ad is defined by:

$$\mathrm{ad} = m_A(m_A \otimes A)(\sigma \otimes i \otimes \sigma S_B)(B \otimes c_{B,R})(\Delta_B \otimes R).$$

Moreover, A can be recovered from R and B as the 'bosonization' $R\#B$, that is A is the smash product algebra between R and B, and as a coalgebra A is isomorphic to the tensor product coalgebra $R \otimes B$. Recall that the multiplication and the comultiplication on $R\#B$ are given:

$$m_{R\#B} = (m_R \otimes m_B)(R \otimes \triangleright \otimes R \otimes B)(R \otimes B \otimes c_{B,R} \otimes B)(R \otimes \Delta_B \otimes R \otimes B),$$

$$u_{R\#B} = (u_R \otimes u_B)\Delta_{\mathbf{1}},$$

$$\Delta_{R\#B} = (R \otimes c_{R,B} \otimes B)(\Delta_R \otimes \Delta_B),$$

$$\varepsilon_{R\#B} = m_{\mathbf{1}}(\varepsilon_R \otimes \varepsilon_B).$$

Proposition 2.19. *We keep the assumptions and notations in* (2.1) *and* (2.11). *We also assume that A is cocommutative and ξ is trivial.*

(a) *The action $\triangleleft : B \otimes R \to B$ is trivial if and only if π is left B-linear.*
(b) *If $\triangleleft$ is trivial then the left action $\triangleright : B \otimes R \to R$ is the adjoint action.*
(c) *If $\triangleleft$ is trivial then $A \simeq R\#B$, where B acts on R by the left adjoint action.*

Proof. Since (R, i) is the equalizer of $(A \otimes \pi)\Delta_A$ and $(A \otimes u_B)r_A^{-1}$ we get

$$(A \otimes \pi)\Delta_A i = (A \otimes u_B)r_A^{-1} i.$$

By applying $\varepsilon_R \otimes B$ to the both sides of this relation we get $\pi i = u_B \varepsilon_R$. Now we can prove (a). If we assume that π is left B-linear, i.e. $\pi m_A(\sigma \otimes A) = m_B(B \otimes \pi)$, then it follows that

$$\triangleleft = \pi m_A(\sigma \otimes i) = m_B(B \otimes \pi i) = m_B(B \otimes u_B \varepsilon_R) = r_B(B \otimes \varepsilon_R).$$

This means that $\triangleleft$ is trivial. Conversely, let us assume that $\pi m_A(\sigma \otimes i) = r_B(B \otimes \varepsilon_R)$. In order to prove that $\pi m_A(\sigma \otimes A) = m_B(B \otimes \pi)$ we compute $\pi m_A(\sigma \otimes \Phi)$. We get

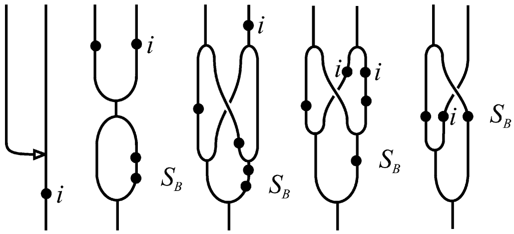

Fig. 21. The proof of $i \triangleright = \mathrm{ad}$.

$$\pi m_A(\sigma \otimes \Phi) = \pi m_A\big[\sigma \otimes m_A(\sigma \otimes i)\big] = \pi m_A(\sigma \otimes i)(m_B \otimes R) = r_B(B \otimes \varepsilon_R)(m_B \otimes R)$$
$$= m_B\big[B \otimes \pi m_A(\sigma \otimes i)\big] = m_B(B \otimes \pi \Phi).$$

Since $B \otimes \Phi$ is an isomorphism we deduce the required equality.

(b) The proof of $i \triangleright = \mathrm{ad}$ is given in Fig. 21. The definition of the action $\triangleright$ together with $ip = m_A(A \otimes \Phi)\Delta_A$ and $\Phi = \sigma S_B \pi$ yield the first equality. The next one is obtained by applying the compatibility relation between m_A and Δ_A and the fact that σ is a morphism of coalgebras. By the first part of the proposition, π is left B-linear. Thus we have the third equality. By using $\pi i = u_B \varepsilon_R$ and the properties of the unit and counit we conclude the proof of $i \triangleright = \mathrm{ad}$.

(c) We already know that $\triangleright$ is induced by the left adjoint action. Obviously, if the right action $\triangleleft$ is trivial then $m_{R \bowtie B} = m_{R\#B}$, where $m_{R\#B}$ is defined in (2.18). □

Remark 2.20. An occurrence of the situation described by Proposition 2.19 is contained in [AMS1, Lemma 6.16] where again the category is that of right comodules over a cosemisimple coquasitriangular Hopf algebra.

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