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Identical Relations in Finite Groups

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1. INTRODUCTION

The main purpose of this paper is to prove in answer to a question of B. H. Neumann [7]:

THEOREM 1. *If G is a finite group, then there is a finite basis for the identical relations holding in G.*

It will be recalled that a *factor* of a group G is a quotient group H/K, where $1 \leqslant K \lhd H \leqslant G$, and this is *proper* unless $K = 1$ and $H = G$. A *critical group* is a finite group which does not belong to the variety generated by its proper factors, while a *Cross variety* is a variety $\mathfrak{B}$ which satisfies:

(i) $\mathfrak{B}$ has a finite basis for its identical relations;

(ii) finitely generated groups in $\mathfrak{B}$ are finite;

(iii) $\mathfrak{B}$ contains only a finite number of critical groups.

Professor Higman has pointed out that in order to prove Theorem 1 it is sufficient to prove:

THEOREM 2. *If B is a finite group, and $\mathfrak{A}$ is a Cross variety containing all factors of B, then the variety $\mathfrak{U}$ generated by $\mathfrak{A}$ and B is Cross.*

For, since a subvariety of a Cross variety is itself Cross, a simple induction argument gives:

THEOREM 3. *A variety of groups is Cross if and only if it is generated by a finite group.*

Theorem 1 is an immediate consequence of this.

Theorem 2 is trivial if B is contained in $\mathfrak{A}$ and, in particular, if B is not critical. If B is critical, then it has a unique minimal normal subgroup M, and the proof divides into two according as M is or is not abelian. In either case, it is sufficient to prove that $\mathfrak{U}^{(n)}$ satisfies the conditions for a Cross variety, where $\mathfrak{U}^{(n)}$ is the variety satisfying the identical relations of $\mathfrak{U}$ which involve n or fewer variables. By a result of Neumann [7], $\mathfrak{U}^{(n)}$ always has a finite basis for its identical relations, so it remains to prove that it satisfies the other two conditions for a Cross variety.

If M is not abelian, the main difficulty lies in showing that n can be chosen so that finitely generated groups in $\mathfrak{U}^{(n)}$ are finite; it is then relatively easy to show that $\mathfrak{U}^{(n)}$ contains only a finite number of critical groups. The main concept here is that of an *M-subgroup* of a group G; roughly, this is a subgroup isomorphic to M, and such that the automorphisms induced in it by conjugation by elements of its normalizer include the restrictions to M of the inner automorphisms of B. The key result is that, for large enough n, a finitely generated group G in $\mathfrak{U}^{(n)}$ has a normal subgroup N which is a direct product of normal M-subgroups such that G/N belongs to $\mathfrak{A}$.

If, on the other hand, M is abelian, it is easy to choose n so that finitely generated groups in $\mathfrak{U}^{(n)}$ are finite, but much harder to choose n so that $\mathfrak{U}^{(n)}$ contains only a finite number of critical groups. For this purpose we make the following definitions:

The *absolute p-rank* (for a given prime p) of a finite group is the maximum dimension of an absolutely irreducible component of any chief factor of G of p-power order, and the *p-measure* of G is the maximum of the p-ranks of its factors (epimorphic images of subgroups).

The *S-rank* (for a given nonabelian simple group S) of a finite group G is the maximum number of isomorphic copies of S which occur as direct factors of a chief factor of G, and the *S-measure* of G is the maximum of the S-ranks of its factors.

We then prove:

THEOREM 4. *The order of a critical group can be bounded in terms of its exponent, the maximum of its p-measures and S-measures, the maximum of the classes of its Sylow subgroups, and the maximum of the orders of its composition factors.*

Theorem 2 for the case in which B is a critical group with abelian minimal normal subgroup then follows.

Section 2 consists of notation, definitions, and preliminary results, and in Section 3, Theorem 2 is proved for the nonabelian case. Theorem 4 is proved in Section 4.

2. NOTATION, DEFINITIONS, AND PRELIMINARY RESULTS

2.1. The reader is referred to [8] for the definitions of a variety, the variety defined by a set of relations, and the variety generated by a class of groups.

Throughout, groups are denoted by upper case Roman letters, and varieties by upper case Gothic letters. Group elements are denoted by lower case Roman letters. As in [8], the variety generated by a group G is denoted by $\mathfrak{G}$, and the variety defined by those relations of G involving at most n variables is denoted by $\mathfrak{G}^{(n)}$.

If G, H are groups, then $H \leqslant G$, $H < G$, and $H \lhd G$ mean respectively that H is a subgroup of G, that H is a proper subgroup of G, and that H is a normal subgroup of G. The normalizer of H in G is written $N_G(H)$, or more frequently $N(H)$, and its centralizer is written $C_G(H)$, or $C(H)$. The center of H is denoted by $Z(H)$, and its Frattini subgroup by $\Phi(H)$. If S is any subset of H, $\{S\}$ denotes the subgroup of H which is generated by S.

For any two elements g, h of a group, the element $h^{-1}gh$ is written g^h. The commutator $h^{-1}g^{-1}hg$ is written $[h, g]$. Higher commutators are defined inductively by the rule $[g_1, \cdots, g_n, g_{n+1}] = [[g_1, \cdots, g_n], g_{n+1}]$. If H_1, H_2 are subgroups of G, then $[H_1, H_2]$ denotes the subgroup of G generated by all commutators $[h_1, h_2]$ with h_1 in H_1 and h_2 in H_2. As with commutators of group elements, $[H_1, \cdots, H_n, H_{n+1}]$ is defined to be $[[H_1, \cdots, H_n], H_{n+1}]$.

If $w(x_1, \cdots, x_n)$ is a word in the free variables $x_1, \cdots x_n$, then $w(G)$ denotes the subgroup of G generated by all elements of the form $w(g_1, \cdots, g_n)$ where $g_1, \cdots, g_n \in G$.

The reader is reminded that the Frattini subgroup of a finite group G is the intersection of its maximal subgroups. It is also the subgroup which consists of the set of nongenerators of G. The Fitting subgroup of G is its greatest normal nilpotent subgroup. The subgroup T is said to be a partial complement to the normal subgroup H of G if $G = TH$ and $T \cap H$ is a proper subgroup of H. Throughout, "simple" is taken to mean "nonabelian simple."

2.2. The following results of R. Remak [9] about direct products are required.

DEFINITION 2.2.1. Let $G \leqslant H \times K$; then any element g of G is uniquely expressible in the form $g\phi g\psi$, where $g\phi \in H$, $g\psi \in K$. The mappings ϕ, ψ are called the *projections* from G to H, K, and the images $G\phi$, $G\psi$ are called the *projections* of G on H, K.

LEMMA 2.2.2. *If $G \cap H = G\phi$, then $G = (G \cap H) \times (G \cap K)$.*

LEMMA 2.2.3. *$G \cap H \lhd G\phi$, $G \cap K \lhd G\psi$, $G/(G \cap H) \simeq G\psi$, $G/(G \cap K) \simeq G\phi$, and $G\phi/(G \cap H) \simeq G\psi/(G \cap K)$.*

LEMMA 2.2.4. *A normal subgroup of a direct product of simple groups is a (possibly empty) direct product of a number of the factors (where we make the convention that the direct product of the empty set of groups is the trivial group). The decomposition of a direct product of simple groups is unique, and any normal subgroup of such a direct product has a unique normal complement.*

LEMMA 2.2.5. *Let* $H_i \lhd G$ $(i = 1, \cdots, r)$, *and let* $D = \cap_{i=1}^{r} H_i$, *then* $G/D \simeq \bar{G} \leqslant (G/H_1) \times \cdots \times (G/H_r)$.

Other facts about direct products which are needed are:

LEMMA 2.2.6. *If* $N \lhd G \leqslant H \times K$, *and* $C_{G\phi}(N\phi) = 1$, *then* $N \cap H = 1$ *if and only if* $G \cap H = 1$.

Proof. If $G \cap H = 1$, then trivially $N \cap H = 1$. Conversely, suppose $g\phi \in G \cap H, g\phi \neq 1$, and let $n = n\phi n\psi \in N$. Since $N \lhd G$, $(g\phi)^{-1} n\phi n\psi g\phi \in N$, and so

$$(n\phi n\psi)^{-1} (g\phi)^{-1} n\phi n\psi g\phi = [n\phi, g\phi] \in N.$$

But this is also an element of H, and so belongs to $N \cap H$. It cannot be 1 for all n, since $C_{G\phi}(N\phi) = 1$ and $g\phi \neq 1$.

LEMMA 2.2.7. *If* $G = \{D_1, \cdots, D_r\}$, *where each* D_i *is a direct product of simple groups and is normal in* G, *and* $D_i \cap D_j = 1$ $(i \neq j)$, *then*

$$G = D_1 \times \cdots \times D_r .$$

Proof. By induction on r, the lemma being certainly true for $r = 1$. By assumption, then,

$$\{D_2, \cdots, D_r\} = D_2 \times \cdots \times D_r \lhd G.$$

Now $D_1 \cap (D_2 \times \cdots \times D_r)$ is normal in $D_2 \times \cdots \times D_r$, a direct product of simple groups, and so must be a direct product of a number of the factors. But $D_1 \cap D_i = 1$ $(i = 2, \cdots, r)$ and so $D_1 \cap (D_2 \times \cdots \times D_r) = 1$. Thus D_1 and $D_2 \times \cdots \times D_r$ generate their direct product; i.e.,

$$G = D_1 \times \cdots \times D_r ,$$

as required.

LEMMA 2.2.8. *Let* $G/N \simeq H_1 \times \cdots \times H_r$, *where* N *is abelian, and each* H_i *is a direct product of simple groups. Let* F_i *be the subgroup of* G *such that* $F_i/N \simeq H_i$. *If* $F_i = D_i \times N$ $(D_i \simeq H_i)$ *for* $i = 1, \cdots, r$, *then*

$$G = D_1 \times \cdots \times D_r \times N.$$

Proof. Certainly $G = \{D_1, \cdots, D_r, N\}$. Now, $F_i \lhd G$, being the inverse image of a normal subgroup of G/N; thus, for any g in G,

$$g^{-1}D_ig \leqslant F_i = D_i \times N.$$

But the projection of $g^{-1}D_ig$ on N is an abelian homomorphic image of $g^{-1}D_ig$, which is a direct product of simple groups, and so this projection must be 1. Hence $g^{-1}D_ig = D_i$, and $D_i \lhd G$. Also $F_i \cap F_j = N$ (since $(F_i/N) \cap (F_j/N) = 1$) and so $D_i \cap D_j = 1$. It follows from Lemma 2.2.7 that the D_i generate their direct product, which is also normal in G. But $N \cap (D_1 \times \cdots \times D_r)$ is an abelian normal subgroup of a direct product of simple groups, and so is 1. Hence $G = D_1 \times \cdots \times D_r \times N$, as required.

LEMMA 2.2.9. *If G is a finite group with a set of non-nilpotent normal subgroups $M_1, \cdots, M_s$ which together generate their direct product, then there is a subgroup L_s of G such that:*

(i) *G is generated by L_s together with $M_1, \cdots, M_s$;*

(ii) *G is not generated by L_s together with any proper subset of $M_1, \cdots, M_s$.*

Proof. By induction on s. If $s = 1$, then L_1 may be chosen to be any maximal subgroup of G not containing M_1. Such a group exists because M_1 cannot be contained in $\Phi(G)$, which is nilpotent.

If $s > 1$, let $Y = M_1 \times \cdots \times M_{s-1}$, and consider the normal subgroup YM_s/M_s of G/M_s. By induction, there is a subgroup L_{s-1} of G, which contains M_s, and is such that L_{s-1}/M_s and $M_1M_s/M_s, \cdots, M_{s-1}M_s/M_s$ satisfy conditions (i) and (ii) with respect to G/M_s. If $X = L_{s-1} \cap Y$, then in L_{s-1}, M_s and X generate their direct product and so $M_sX/X \simeq M_s$. As M_sX/X is not nilpotent, there is a subgroup L_s of G, containing X, such that L_s/X is a proper subgroup at L_{s-1}/X, and such that:

(i) $L_sM_s/X = L_{s-1}/X$;

(ii) $M_s/X \not\subset L_s/X$.

Evidently $L_sM_s = L_{s-1}$, since $L_s \supset X$. Thus $G = \{L_s, M_1, \cdots, M_s\}$. On the other hand, if M_j ($j \neq s$) is omitted from the set, G is not generated by the remaining groups, because G/M_s is not generated by their homomorphic images. If M_s is omitted, the group generated by the remaining subgroups is, modulo Y, simply

$$L_sY/Y \simeq L_s/(L_s \cap Y) = L_s/(L_{s-1} \cap Y) \neq G/Y.$$

Thus L_s has the required properties, and the truth of the lemma follows.

2.3 *Varieties and Identical Relations*

In [7], Neumann proves the following lemmas:

LEMMA 2.3.1. *Any variety is generated by its finitely generated groups.*

LEMMA 2.3.2. *Those identical relations of a finite group which involve at most n variables possess a finite basis.*

Now, the variety generated by a set of groups $\mathfrak{X}$ is obtained by repeated applications of the operations of taking subgroups, homomorphic images, and unrestricted direct products, but it is shown by Higman [5] that these operations need each be applied once only; in fact, one has:

LEMMA 2.3.4. *A group which belongs to the variety generated by a set of groups $\mathfrak{X}$ is a homomorphic image of a subgroup of a direct product of groups isomorphic to groups in $\mathfrak{X}$.*

Moreover, if $\mathfrak{X}$ is a finite set of finite groups. Higman proves the refinement:

LEMMA 2.3.5. *If G is a finitely generated group belonging to the variety generated by a finite set of finite groups $\mathfrak{X}$ then G is a homomorphic image of a subgroup of a direct product of a finite number of copies of groups in $\mathfrak{X}$.*

As a corollary, one has:

LEMMA 2.3.6. *A finitely generated group in a variety generated by a finite set of finite groups is finite.*

2.4. *Critical Groups*

It has already been mentioned that a critical group has a unique minimal normal subgroup. In this section two further lemmas about critical groups are proved.

LEMMA 2.4.1. *A finite simple group is critical.*

Proof. In a variety generated by a finite set of finite groups all of whose composition factors have order less than k, the composition factors of any finite group have order less than k. But the order of any composition factor of a proper factor of a simple group (which is its own only composition factor) must be less than that of the group itself, and so a simple group cannot belong to the variety generated by its proper factors.

LEMMA 2.4.2. *If the group G has a set of normal subgroups $M_1, \cdots, M_s$ and a subgroup L such that:*

(i) *G is generated by L together with the subgroups $M_1, \cdots, M_s$;*

(ii) *G is not generated by L together with any proper subset of the subgroups $M_1, \cdots, M_s$;*

(iii) *$[M_{\sigma(1)}, \cdots, M_{\sigma(s)}] = 1$ for all permutations σ of the integers $1, 2, \cdots, s$;*

then G is not critical.

Proof. For each j, $(j = 1, \cdots, s)$ let S_j be a set whose elements are in one-one correspondence with the nontrivial elements of M_j. Let S_0 be a set whose elements are in one-one correspondence with the nontrivial elements of L, and assume that the sets $S_0, S_1, \cdots, S_s$ are so chosen that they are mutually disjoint in pairs. Let F be the free group freely generated by the elements of $S_0, S_1, \cdots, S_s$, and let Y be the set of all conjugates of these elements. The notion of a commutator in the elements of Y is defined inductively; y and y^{-1} are commutators for all y in Y; if y, z are commutators, so is $[y, z]$; and nothing else is a commutator. The commutator u involves S_j if it is either a conjugate of a generator of F which belongs to S_j, the inverse of such an element, or is of the form $[y, z]$ where y or z involves S_j. The endomorphism of F which maps each element of S_j onto 1 and leaves S_k $(k \neq j)$ invariant is denoted by π_j. Let $K = \cap_{j=1}^{s} \ker \pi_j$. By a trivial modification of Lemma 3.2 of [5], $K \leqslant \gamma_s(F)$ (the sth term of the lower central series of F). Thus each k in K may be written $k = u_1 u_2 \cdots u_r$ where each u_i is of the form $[y_1, y_2, \cdots, y_s]$ $(y_j \in Y)$. Now, if u_{i+1} involves S_j, and u_i does not, we may write $u_i u_{i+1} = u_{i+1} u_i'$, where $u_i' = u_{i+1}^{-1} u_i u_{i-1}$. In this way, it is seen that the above expression for k may be rearranged so as to take the form

$$k = c_0 c_1 \cdots c_s$$

where c_0 is a product of powers of commutators involving each S_j; c_r is a product of powers of commutators not involving S_r but involving $S_1, S_2, \cdots, S_{r-1}$ $(r = 1, 2, \cdots, s)$. But now $k\pi_s = k\pi_{s-1} = \cdots k\pi_1 = 1$, and so $c_1 = c_2 = \cdots = c_s = 1$. Hence every element of K may be written as a product of powers of commutators which involve each S_j $(j = 1, 2, \cdots, s)$. Since G is generated by L and the M_j, it is an epimorphic image of F under the natural homomorphism α induced by $S_0\alpha = L$, $S_j\alpha = M_j$ $(j = 1, \cdots, s)$. Taken in conjunction with the hypothesis (iii) of the theorem, what we have just proved implies that K is in the kernel of α, so that we can write $\alpha = \gamma\delta$, where γ is the natural map of F on F/K, and δ is a homomorphism of F/K on G. The proof of the theorem is now precisely analogous to that of Theorem 3 of [8]. By trivially modifying Lemma 3.4 of [5], one deduces that there are endomorphisms $\beta_1, \cdots, \beta_t$ of F such that each element g of F may be written

$$g = k(g^{\lambda_1}\beta_1)(g^{\lambda_2}\beta_2) \cdots (g^{\lambda_t}\beta_t) \qquad \text{(A)}$$

where $k \in K$, and each $\lambda_j = \pm 1$ independently of g. From hypothesis (ii) each of the subgroups $F\beta_j\alpha$ ($j = 1, \cdots, t$) is a proper subgroup of G. By means of the identity (A), it may be verified that the mapping θ which maps each element $g\gamma$ of $F\gamma$ on the element with component $g\beta\alpha$ in $F\beta_j\alpha$ in the direct product $\prod_{j=1}^{t} F\beta_j\alpha$ is a homomorphism of $F\gamma$, and that the mapping δ may be factored through θ. This demonstrates that G is an epimorphic image of a subgroup of a direct product of its proper factors, thereby proving the result. The reader is referred to [*8*] for details.

LEMMA 2.4.3. *If the critical group G has a normal nilpotent subgroup N, partially complemented by a subgroup L and a set of subgroups $N_1, \cdots, N_c$ of N satisfying:*

(i) *each N_j contains $L \cap N$ and admits L;*

(ii) *G is generated by the subgroups $L, N_1, \cdots, N_c$;*

(iii) *G is not generated by L together with any proper subset of the subgroups $N_1, \cdots, N_c$;*

then c cannot exceed the nilpotency class of N.

This result was proved in [*8*]. It can, however, be deduced as a special case of 2.4.2 because the subgroups $L, N_1\Phi(N), \cdots, N_c\Phi(N)$ satisfy the conditions of that proposition whenever c exceeds the nilpotency class of N.

In the particular case where $G = N$, i.e., $L = 1$, one may deduce from the Burnside basis theorem:

LEMMA 2.4.4. *If the critical group G is nilpotent of class c, then it may be generated by c elements.*

This result was first proved by D. C. Cross [*1*].

2.5. *The p-Measure and S-Measure of a Finite Group*

The p-measure and S-measure of a finite group are extensions of the concept of the absolute rank of a finite soluble group introduced by G. Higman. A full account of this may be found in [*8*], as may proofs of the results quoted in this subsection. Here we give fuller definitions than those given in the introduction.

If X/Y is an abelian chief factor of a finite group G, it is an elementary abelian p-group for some prime p on which G acts as a group of automorphisms. It may be identified with the additive group of a vector space V over the prime field $\mathscr{F}_p$ and the action of G then gives V the structure of an irreducible representation module for G. If $\mathscr{K}_p$ is any finite extension of $\mathscr{F}_p$, the module $\mathscr{K}_p \otimes V$ is the direct sum of G submodules which are irreducible over $\mathscr{K}_p$ and which are all equivalent under the Galois group of $\mathscr{K}_p$ over $\mathscr{F}_p$. In particular, if $\mathscr{K}_p$ is chosen to be a splitting field for the representation, it follows that each absolutely irreducible component is of the same dimension and this common dimension is defined to be the *absolute p-degree* of the chief factor X/Y of G. If the chief factor X/Y of G is not an elementary abelian p-group, it is defined to have absolute p-degree zero.

LEMMA 2.5.1. *If the chief factor X/Y of G is an elementary abelian p-group and has absolute p-degree d, then the congruence*

$$\Pi\{x^{g_{\sigma(1)}\cdots g_{\sigma(m)}}\}^{\chi(\sigma)} \equiv 1 \qquad \bmod Y$$

is valid for all $x \in X$ and all $g_1, \cdots, g_m \in G$ if and only if $m \geqslant 2d$. (Here $g^{-1}xg$ is written as x^g, χ is the alternating character, and the product is taken over all permutations σ of the integers $1, \cdots, m$.) This result is a trivial modification of Lemma 2.2.3 of [6].

The *absolute p-rank* of a finite group is defined to be the maximum of the absolute p-degrees of its chief factors. Its *p-measure* is the maximum of the absolute p-ranks of its factors (i.e., homomorphic images of subgroups). It is easy to see that if H is a homomorphic image of G then the absolute p-rank of H does not exceed the absolute p-rank of G. On the other hand, by considering a simple group it is seen that the absolute p-rank of a subgroup may exceed that of the group.

LEMMA 2.5.2. *If H, K are finite groups, then the p-measure of $H \times K$ is equal to the maximum of the p-measures of H and K.*

Proof. Consider a subgroup G of $H \times K$. Let ϕ and ψ be defined as in 2.2.1, and let k be the maximum of the p-measures of H and K. Since $G\phi \simeq G/G \cap K$, the p-measure of $G/G \cap K$ is at most k. Again, if X/Y is a chief factor of G such that $G \cap K \geqslant X > Y \geqslant 1$, then X/Y is also a chief factor of $G\psi$, and the representation of G as a group of automorphisms of X/Y is isomorphic to that of $G\psi$. Thus G has p-measure at most k, and the proof of Lemma 2.5.2 is now straightforward.

If S is a finite simple group, and X/Y is a chief factor of a finite group G, X/Y is defined to have *S-degree d* if it is a direct product of d groups isomorphic to S. If X/Y is not a direct product of groups isomorphic to S, it is defined to have S-degree zero. The *S-rank* of a finite group is the maximum of the S-degrees of its chief factors and its *S-measure* is the maximum of the S-ranks of its factors (i.e., homomorphic images of subgroups).

LEMMA 2.5.3. *If H, K are finite groups, then the S-measure of $H \times K$ is equal to the maximum of the S-measures of H and K.*

The proof is as for Lemma 2.5.2.

LEMMA 2.5.4. *If a variety $\mathfrak{V}$ is generated by some finite group G, there is an integer k, depending only on G, such that for each prime p and each finite simple group S, the p-measure and S-measure of any finite group in $\mathfrak{V}$ does not exceed k.*

Proof. By Lemmas 2.5.2, 2.5.3, and 2.3.5 it is sufficient to choose k as the maximum of the p-measures and S-measures of G. Only a finite number of these can be nonzero.

3. Critical Groups with Nonabelian Minimal Normal Subgroups

In this section we prove:

THEOREM 2(A). *If B is a critical group whose minimal normal subgroup is non-abelian, and $\mathfrak{A}$ is a Cross variety containing all proper factors of B, then the variety $\mathfrak{U}$ generated by $\mathfrak{A}$ and B is Cross.*

(We may assume that B does not belong to $\mathfrak{A}$, since the theorem is trivial if it does.)

3.1. *Special Notation and Preliminary Results*

Let A be a finite group which generates $\mathfrak{A}$ (e.g. the direct product of the critical groups of $\mathfrak{A}$) and let $w(x_1, \cdots, x_n) = 1$ be a basis for the identical relations of $\mathfrak{A}$.

Let the minimal normal subgroup of B be M. Let m be the minimum number of generators of M, and let b be the minimum number of generators in a generating set for B which includes a minimum generating set for M.

DEFINITION 3.1.1. A subgroup D of a group G is called an *M-subgroup* if $D \simeq M$, and there exists a subgroup H of G such that $D \lhd H$ and $H/C_H(D) \simeq B$ (H is called an *M-normalizer* of D).

If a group G contains a subgroup K which has as a direct factor an M-subgroup D of G, then D is said to be an *M-factor* of K (in G).

LEMMA 3.1.2. *If D is an M-subgroup of G and $N \lhd G$, then $D \cap N = D$ or 1, and, in the latter case, DN/N is an M-subgroup of G/N.*

Proof. Let H be an M-normalizer of D, so that $D \lhd H$, and $H/C_H(D) \simeq B$. Under this homomorphism the image of D is a normal subgroup of B, isomorphic to D (since $D \cap C_H(D) = 1$), i.e., isomorphic to M. Thus it must be M itself. Since M is the minimal normal subgroup of B, it follows that no proper nontrivial subgroup of D can be normal in H, and hence $D \cap N$ is D or 1. In the latter case D and $H \cap N$ will commute elementwise, and so $H \cap N \leqslant C_H(D)$. Now, $DN/N \simeq D/(D \cap N) = D \simeq M$, and $DN/N \lhd HN/N \leqslant G/N$. Let $\bar{c}$, an element of $C_{HN/N}(DN/N)$, be the image under the homomorphism from G to G/N of c, an element of H, so that $dc = cdn$, $n \in N$, for any d in D. Thus $d^{-1}c^{-1}dc = n$. But $d^{-1}c^{-1}dc \in D$ (since $c \in H$, and $D \lhd H$), and so $d^{-1}c^{-1}dc \in D \cap N = 1$. Hence $c \in C_H(D)$. Conversely, if $c \in C_H(D)$ then certainly its image under the homomorphism belongs to $C_{HN/N}(DN/N)$; i.e., $C_{HN/N}(DN/N) = C_H(D)\, N/N$. Thus, since

$$H \cap N \leqslant C_H(D),$$

$$\begin{aligned}(HN/N)/C_{HN/N}(DN/N) &= (HN/N)/(C_H(D)\, N/N)\\ &\simeq (H/H \cap N)/(C_H(D)/C_H(D) \cap N)\\ &\simeq H/C_H(D) \simeq B.\end{aligned}$$

It follows that DN/N is an M-subgroup of G/N.

LEMMA 3.1.3. *If $G > K = D_1 \times \cdots \times D_r$, where the D_is are M-subgroups of G, and $N \lhd G$, then KN/N is a direct product of M-subgroups of G/N.*

Proof. From Lemma 3.1.2 we have that $D_i \cap N = D_i$ or 1, and, in the latter case, D_iN/N is an M-subgroup of G/N. Since each D_i is a direct product of simple groups, so is K, and so any normal subgroup of K is a direct product of a number of these. Thus $K \cap N$ consists precisely of the direct product of those D_i with which N has intersection D_i. Thus

$$KN/N \simeq K/(K \cap N) \simeq D_1 \times \cdots \times D_s,$$

where these are the D_is with which N has intersection 1. It follows that KN/N is the direct product of the corresponding D_iN/N, and thus has the stated form.

COROLLARY 3.1.4. *If in Lemma 3.1.3 each D_i is normal in G, then $KN = D_1 \times \cdots \times D_s \times N$, where these are the D_is such that $D_i \cap N = 1$.*

Proof. This follows immediately from the fact that $D_1 \times \cdots \times D_s$ and N are normal subgroups of G with trivial intersection.

LEMMA 3.1.5. *Let X and Y be normal subgroups of G, and let D/X be an M-subgroup of G/X, where $D \leqslant XY$. Then $(D \cap Y)/(X \cap Y)$ is an M-subgroup of $G/(X \cap Y)$.*

Proof. Since $X \leqslant D \leqslant XY$, $D = X(D \cap Y)$, and so

$$M \simeq D/X = X(D \cap Y)/X \simeq (D \cap Y)/((D \cap Y) \cap X) = (D \cap Y)/(X \cap Y).$$

Let H/X be an M-normalizer of D/X. Then $D \cap Y \lhd H$, since $D \lhd H$, $Y \lhd G$. Suppose $C_{H/X}(D/X) = U/X$, and $C_{H/(X \cap Y)}((D \cap Y)/(X \cap Y)) = V/(X \cap Y)$.

Let $u \in U$, $d \in D \cap Y$; then, since uX centralizes D/X, $u^{-1}du = dx$, where $x \in X$. But $D \cap Y \lhd H$, and so $x \in D \cap Y$. Hence $x \in X \cap D \cap Y = X \cap Y$. But this means that $u(X \cap Y)$ centralizes $(D \cap Y)/(X \cap Y)$, and so $u \in V$. Hence $U \leqslant V$. Now let $v \in V$, $d \in D$; then $d = d_1 x$, where $d_1 \in D \cap Y$, and $x \in X$. Hence $v^{-1}dv = v^{-1}d_1 v v^{-1} x v = d_1 x_1 v^{-1} x v$, where $x_1 \in X \cap Y$, since $v(X \cap Y)$ centralizes $D \cap Y$. Thus $v^{-1}dv = dx'$, where $x' \in X$ (since $X \lhd G$). Hence vX centralizes D/X, and so $v \in U$, i.e., $V \leqslant U$. It follows that $U = V$, and

$$(H/(X \cap Y))/(C_{H/(X \cap Y)})((D \cap Y)/(X \cap Y)) \simeq H/V$$
$$= H/U \simeq (H/X)/(C_{H/X}(D/X)) \simeq B.$$

Thus $(D \cap Y)/(X \cap Y)$ is an M-subgroup of $G/(X \cap Y)$, with M-normalizer $H/(X \cap Y)$.

3.2. *The Variety* $\mathfrak{U}$

Throughout this Section G is a finitely generated (and thus finite) group in $\mathfrak{U}$.

LEMMA 3.2.1. *In order to show that a property P holds for G it is sufficient to show that 1 has P, and that, if all factors of a group U in $\mathfrak{U}$ have P, then so have all factors of $U \times A$, and all factors of $U \times B$.*

Proof. By Lemma 2.3.5, G is a homomorphic image of a subgroup of a direct product of a finite number of groups isomorphic to A and B. Thus it is sufficient to show that all factors of a direct product $A_1 \times \cdots \times A_r \times B_1 \times \cdots \times B_r$ have P. We proceed by induction on r, the basis of the induction being $r = 0$, since we have assumed that 1 has P. Assume all factors of $A_1 \times \cdots \times A_{r-1} \times B_1 \times \cdots \times B_{r-1} = U$ have P, then so have all factors of $U \times A$, and thus of $U \times A \times B = A_1 \times \cdots \times A_r \times B_1 \times \cdots \times B_r$.

LEMMA 3.2.2. *In applying Lemma 3.2.1, in order to show that all factors of $U \times A$, or of $U \times B$, have P, it is sufficient to consider those factors K/N such that $K \cap A > 1$, or $K \cap B > 1$.*

Proof. By Lemma 2.2.3, $K \cap A = 1$ or $K \cap B = 1$ implies $K \simeq K\phi$ (its projection on U). Thus K/N is isomorphic to a factor of U and so, by assumption, has P.

LEMMA 3.2.3. *$w(G)$ is a finite direct product of M-subgroups of G, each of which is normal in G (and, by Lemma 2.2.4, this expression is unique).*

Proof. This is certainly true for 1, and so it must be shown that if it is true for all factors of U, where $U \in \mathfrak{U}$, then it is true for all factors of $U \times V$, where $V \simeq A$ or $V \simeq B$.

Let $G \simeq K/N$, where $K \leqslant U \times V$. Let ϕ, ψ be the projections from K to U, V; then $w(K)\phi = w(K\phi)$, and $w(K)\psi = w(K\psi)$. Now, unless $K\psi = V \simeq B$, $K\psi \in \mathfrak{A}$, and so $w(K)\psi = w(K\psi) = 1$, and $w(K) = w(K\phi) \times 1$. If $K\psi \simeq B$, then $w(K\psi) = w(B) > 1$, since $B \notin \mathfrak{A}$. Now $w(B) \lhd B$, and so $w(B) > M$. But $w(B/M) = 1$, and so $w(B) \leqslant M$, hence $w(B) = M$. By Lemma 3.2.2 it may be assumed that $K \cap B > 1$. But

$$C_{K\psi}(w(K)\psi) = C_B(M) = 1,$$

and so, by Lemma 2.2.6, $w(K) \cap B > 1$. But $w(K) \lhd K$, and so

$$w(K) \cap B \lhd K\psi = B.$$

Thus $w(K) \cap B = M = w(K)\psi$, and so M is a direct factor of $w(K)$. But $C_K(M) = K \cap U$, and so $K/C_K(M) = K/(K \cap U) \simeq K\psi = B$. Hence M is an M-subgroup of K. Also $M \lhd K\psi$, and so $M \lhd K$. Thus

$$w(K) = w(K\phi) \times M,$$

where M satisfies the given conditions.

Now consider $w(K\phi)$, which we have seen to be a direct factor of $w(K)$. $K\phi$ is a factor of U, and so, by hypothesis, $w(K\phi)$ is a direct product of M-subgroups of $K\phi$, each of which is normal in $K\phi$. Let D be one of these factors, and $H\phi$ be an M-normalizer of D in $K\phi$. Let H be the inverse image of $H\phi$ in K. Then $D \lhd H$, and $h \in C_H(D)$ if and only if $h\phi \in C_{H\phi}(D)$. Thus $H/C_H(D) \simeq H\phi/C_{H\phi}(D) \simeq B$. Also $D \lhd K$, since $D \lhd K\phi$. Thus $w(K\phi)$ is a direct product of M-subgroups of K, and it follows that, in both cases, $w(K)$ has the required form.

Now consider $G \simeq K/N$. If $w(K) = D_1 \times \cdots \times D_r$, by Lemma 3.1.4 we have $w(K)N = D_1 \times \cdots \times D_s \times N$, where $D_1, \cdots, D_s$ are the M-factors of $w(K)$ which have intersection 1 with N. But $w(G) = w(K)N/N$, and, by Lemma 3.1.3, this is a direct product of M-subgroups of G. Clearly, each of these, being a homomorphic image of a normal subgroup of K, will be normal in G, and the truth of the lemma follows.

LEMMA 3.2.4. *If D is an M-subgroup of G, then $D \leqslant w(G)$.*

Proof. $D \lhd H \leqslant G$, and $H/C_H(D) \simeq B$. Thus

$$DC_H(D)/C_H(D) = w(H/C_H(D)) = w(H)\,C_H(D)/C_H(D),$$

and so

$$DC_H(D) = D_1 \times \cdots \times D_s \times C_H(D),$$

where $D_1, \cdots, D_s$ are the factors of $w(H)$ which have intersection 1 with $C_H(D)$. But $DC_H(D) = D \times C_H(D)$, since D has no center, and thus $s = 1$, and $D \times C_H(D) = D_1 \times C_H(D)$. Now, if D had a nontrivial component in $C_H(D)$ this would necessarily be abelian, but it is a homomorphic image of $D\,(\simeq M)$, so this is impossible. Thus $D = D_1 \leqslant w(H) \leqslant w(G)$.

LEMMA 3.2.5. *If $N \lhd G$, and L/N is an M-subgroup of G/N, with M-normalizer H/N, then $L = D \times N$, where D is an M-subgroup of H (and thus of G).*

Proof. As in the above lemma, L/N is one of the direct factors of $w(H/N)$ and so is of the form DN/N, where D is an M-factor of $w(G)$. But $N \lhd H$, $D \lhd H$, and $D \cap N = 1$, thus $L = D \times N$, and has the required form.

LEMMA 3.2.6. *If D is an M-subgroup of G, then the projection of D on any D_i which is an M-factor of $w(G)$ is either D_i or 1.*

Proof. The lemma is certainly true for 1, so we must show that if it is true for all factors of $U\,(\in \mathfrak{U})$, then it is true for $G \simeq K/N$, where $K \leqslant U \times V$, and $V \simeq A$ or $V \simeq B$.

Let the inverse image of D in K be L, then, from Lemma 3.2.5, $L = D' \times N$, where D' is an M-subgroup of K. If D' has projection D_i or 1 on each M-factor of $w(K)$, then, since $w(G) = (D_1 \times \cdots \times D_s \times N)/N$, where $D_1, \cdots, D_s$ are factors of $w(K)$, and $D = (D' \times N)/N$, D will also have this property. Hence it is sufficient to prove the lemma for K. Let ϕ, ψ be the projections from K to U, V. As in Lemma 3.2.3, $w(K) = w(K\phi) \times 1$, or $w(K\phi) \times M$, the latter holding if $K\psi = V = B$. Let H be an M-normalizer of D in K, then $D \cap V \lhd H$, and so is D or 1.

(i) If $D \cap V = D$, then, since $D \leqslant w(K)$, we must have $V = B$, and $D = M$, so that D is itself an M-factor of $w(K)$.

(ii) If $D \cap V = 1$, then $D \simeq D\phi$, and, as in Lemma 3.1.2, $D\phi$ is an M-subgroup of K (with M-normalizer $H\phi$). Thus, by hypothesis, $D\phi$ has projection D_i or 1 on each factor or $w(K\phi)$, and thus so has D. It remains to consider $D\psi$. Since $D \leqslant w(H)$, this will be 1 unless $H\psi = V = B$ (in which case $w(K) = w(K\phi) \times M$). But $D\psi \lhd H\psi$, and $D\psi \leqslant M$. Thus $D\psi = 1$ or $D\psi = M$. Hence, in all cases, D has the required properties.

LEMMA 3.2.7. *If D is an M-subgroup of G, then $C_G(D) \lhd G$.*

Proof. From Lemma 3.2.6, $D \leqslant w(G) = D_1 \times \cdots \times D_r$, and has projection D_i on $D_1, \cdots, D_s$, say, and 1 on the other factors. Let $d = d_1 \cdots d_s$ be an element of D, and let $c \in C_G(D)$, so that $c^{-1}dc = d$. Since $D_i \lhd G$, we have $c^{-1}d_ic = d_i' \in D_i$, and so $d_1' \cdots d_s' = d_1 \cdots d_s$. Hence $d_i' = d_i$. But this will hold for any element d_i of D_i ($i = 1, \cdots, s$), since D has projection D_i on these factors, and so c centralizes each of $D_1, \cdots, D_s$ also. Conversely, any element which centralizes each of these groups will certainly centralize D, so we have $C_G(D) = \bigcap_{i=1}^{s} C_G(D_i)$. But $C_G(D_i) \lhd G$, since $D_i \lhd G$, and so $C_G(D) \lhd G$, as required.

LEMMA 3.2.8. *If $G \geqslant S \rhd D \times N$, where D is an M-subgroup of G, then $D \lhd S$.*

Proof. As in Lemma 3.2.6, $D \leqslant w(G) = D_1 \times \cdots \times D_r$, having projection D_i on $D_1, \cdots, D_s$, say, and 1 on the other factors. Now, $N \leqslant C_G(D)$, and so $N \leqslant C_G(D_i)$ ($i = 1, \cdots, s$). Thus

$$N \cap (D_1 \times \cdots \times D_s) = 1.$$

Let $d \in D$, $s \in S$. Since $D \times N \lhd S$, $s^{-1}ds = d'n$, where $d' \in D$ and $n \in N$, i.e., $n = d'^{-1}s^{-1}ds$. But, since each D_i is normal in G, the right hand side is an element of $D_1 \times \cdots \times D_s$, which has intersection 1 with N. Hence $n = 1$, and $D \lhd S$, as required.

LEMMA 3.2.9. *If D is an M-subgroup of $L \leqslant G$, then any element of G which centralizes D also centralizes every M-factor of $w(L)$ on which D has nontrivial projection.*

Proof. (i) If $L = G$, then the result has already been proved in the course of proving Lemma 3.2.7.

(ii) In the general case, let $D_1, \cdots, D_r$ be the M-factors of $w(G)$, and $E_1, \cdots, E_s$ the M-factors of $w(L)$. E_1 is an M-subgroup of G, so we may suppose it to have projection D_i on D_i for $i = 1, \cdots, t$, and 1 otherwise. Then, for $j > 1$, E_j centralizes E_1, and so, by (i), centralizes $D_1, \cdots, D_t$, so that its projection on D_i must be 1 for $i = 1, \cdots, t$. It follows that we can calculate the projection of an element of $w(L)$ on D_i ($i = 1, \cdots, t$) by first projecting on E_1, and then projecting the result. Thus, if D has projection E_1 on E_1, it has projection D_i on D_i ($i = 1, \cdots, t$). By (i), an element which centralizes D also centralizes $D_1 \times \cdots \times D_t$, and so centralizes E_1, which is contained in $D_1 \times \cdots \times D_t$.

3.3. *The Variety $\mathfrak{U}^{(n)}$*

In this section G is a finitely generated group in $\mathfrak{U}^{(n)}$, where

$$n \geqslant n_0 = \max(a, 2b + 1).$$

LEMMA 3.3.1. *$w(G)$ is generated by a finite number of M-subgroups of G.*

Proof. $G/w(G) \in \mathfrak{A}$, and is finitely generated, and so finite. It follows that $w(G)$ is finitely generated, and so is generated by a finite number of elements of the form $w(g_1, \cdots, g_a)$. Let $L = \{g_1, \cdots, g_a\}$ then L has at most n_0 generators, and so belongs to $\mathfrak{U}$. From Lemma 3.2.3 we have that $w(L)$ is a finite direct product of M-subgroups of L. But these will also be M-subgroups of G, and so each of the finite number of generators of $w(G)$ is contained in the direct product of a finite number of M-subgroups of G, which is itself contained in $w(G)$. It follows that the totality of such subgroups generates $w(G)$.

LEMMA 3.3.2. *Let D be an M-subgroup of G, then there is an M-normalizer H of D in G having only b generators, and $D \leqslant w(H) \leqslant w(G)$.*

Proof. Let H' be an M-normalizer of D in G, and let $a_1, \cdots, a_m$ be generators of D. Consider $H'/C_{H'}(D)$. A set of generators $a_1C_{H'}(D), \cdots a_bC_{H'}(D)$ for this can be chosen so that the first m correspond to the generators of D (because $D \cap C_{H'}(D) = 1$). Let $H = \{a_1, \cdots, a_b\}$. Then $D \lhd H$, $H' = HC_{H'}(D)$, and

$$B \simeq H'/C_{H'}(D) = HC_{H'}(D)/C_{H'}(D) \simeq H/(H \cap C_{H'}(D)) = H/C_H(D).$$

Thus H is also an M-normalizer of D in G, and had only b generators. By the choice of n_0, $H \in \mathfrak{U}$, and so, by Lemma 3.2.4 $D \leqslant w(H) \leqslant w(G)$.

LEMMA 3.3.3. *If D is an M-subgroup of G, then $C_G(D) \lhd G$.*

Proof. Let H be an M-normalizer of D with b generators, and let $g \in G$, $c \in C_G(D)$. If $L = \{H, c, g\}$, then since $b + 2 \leqslant 2b + 1$, $L \in \mathfrak{U}$. But D is an M-subgroup of L, and so, by Lemma 3.2.7, $C_L(D) \lhd L$. But $c \in C_L(D)$, and so $g^{-1}cg \in C_L(D) \leqslant C_G(D)$. Thus $C_G(D) \lhd G$, as required.

LEMMA 3.3.4. *If $G \geqslant S \rhd D \times N$, where D is an M-subgroup of G, then $D \lhd S$.*

Proof. Let H be an M-normalizer of D with b generators, and let $s \in S$. If $L = \{H, s\}$, then $L \in \mathfrak{U}$. But D is an M-subgroup of L, and

$$L \cap S \rhd L \cap (D \times N) = D \times (L \cap N).$$

Thus the conditions of Lemma 3.2.8. are satisfied, and $D \lhd L \cap S$. i.e, $s^{-1}Ds = D$. Thus $D \lhd S$.

LEMMA 3.3.5. *If $G \rhd N$, where N is abelian, and K/N is a direct product* *of M-subgroups of G/N, then $K = E \times N$, where E is a direct product of M-subgroups of G.*

Proof. Let J/N be one of the M-factors of K/N, and H'/N an M-normalizer of J/N having b generators, $a_1N, \cdots, a_bN$. Let $H = \{a_1, \cdots, a_b\}$ Then $H/(H \cap N) \simeq HN/N = H'/N$. Hence H must contain a subgroup L such that $L/(H \cap N)$ is an M-subgroup of $H/(H \cap N)$, corresponding to J/N under this isomorphism. But H, having only b generators, belongs to $\mathfrak{U}$, and so, by Lemma 3.2.5, we have $L = D \times (H \cap N)$, where D is an M-subgroup of H (and thus of H' and G). Also $LN = DN$, and so $DN/N = J/N$. Now suppose H' contains two M-subgroups, D_1, D_2 such that $D_1N/N = D_2N/N = J/N$. Let H_1, H_2 be M-normalizers of D_1, D_2 each having b generators; then, if $P = \{H_1, H_2\}$, $P \in \mathfrak{U}$, and so, by Lemma 3.2.4, $\{D_1, D_2\} \leqslant w(P)$. Now, $w(P)$, as a direct product of M-subgroups of P, has no nontrivial abelian normal subgroup, so that $w(P) \cap N = 1$. Thus $w(P)$ is isomorphic to its image $w(P)N/N$ in the natural map of H' on H'/N. But J/N is the image in this map of both D_1 and D_2. It follows that $D_1 = D_2$. Now, for any n in N, $n^{-1}Dn$ will be an M-subgroup of of H' such that its image in H'/N is J/N. Hence we have $n^{-1}Dn = D$, i.e., $D \lhd DN$. But $D \cap N = 1$, and so $DN = D \times N$. This will hold for each M-factor of K/N, and so, by Lemma 2.2.8, $K = D_1 \times \cdots \times D_r \times N$, where the D_i are M-subgroups of G.

LEMMA 3.3.6. *Let D be an M-subgroup of G, and put $X = C_G(D)$, $Y = C_G(X)$, then XY contains every M-subgroup of G.*

Proof. XY certainly contains D itself. Let D' be any other M-subgroup of G, and let H, H' be M-normalizers of D and D' respectively, each having only b generators. Let $L = \{H, H'\}$; then $L \in \mathfrak{U}$, and $\{D, D'\} \leqslant w(L)$. By Lemma 3.2.6, D has projection D_i or 1 on each M-factor, D_i of $w(L)$. If D has projection 1 on D_i then $D_i \leqslant X$. If it has projection D_i, then $D_i \leqslant Y$; for, let $x \in X$, and consider the group $P = \{L, x\}$. $P \in \mathfrak{U}$, and D, L and P satisfy the conditions of Lemma 3.2.9. But $x \in C_P(D)$, and so $x \in C_P(D_i)$. But this will hold for every x in X, and so $D_i \leqslant C_G(X) = Y$. Hence $w(L) \leqslant \{X, Y\} = XY$ (since, by Lemma 3.3.3, $X \lhd G$). Thus $D' \leqslant XY$.

LEMMA 3.3.7. *If I is an M-subgroup of G, and J is a finite direct product of M-subgroups of G, then the subgroup $\{I, J\}$ which they generate is contained in a finite direct product of M-subgroups of G.*

Proof. Let J be the direct product of s M-subgroups. We proceed by induction on s. If $s = 1$, let H and H' be M-normalizers of I and J respectively each having b generators. Then $L = \{H, H'\} \in \mathfrak{U}$, and $\{I, J\} \leqslant w(L)$, a finite direct product of M-subgroups of L, and thus of G.

Now suppose $s > 1$, and assume the lemma to be true when J is a direct product of fewer than s M-subgroups. Then $J = D_1 \times J_2$, say, and $\{I, D_1\} \leqslant K_1$, $\{I, J_2\} \leqslant K_2$, where K_1 and K_2 are finite direct products of M-subgroups. Let $X = C_G(D_1)$, $Y = C_G(X)$, so that, by Lemma 3.3.3, $X \lhd G$, and thus $Y \lhd G$. Also $J_2 \leqslant X$, $D_1 \leqslant Y$, so that, if $K = \{I, J\}$, then $K \leqslant K_1X$, and $K \leqslant K_2Y$. But, by Lemma 3.3.6, every M-subgroup of G is contained in XY, so that $K_1 \leqslant XY$, $K_2 \leqslant XY$, and $K \leqslant XY$. By Lemma 3.1.3, K_1X/X is a direct product of M-subgroups of G/X. If R/X is one of these factors, then it corresponds in the natural isomorphism between K_1X/X and $(K_1X \cap Y)/(X \cap Y)$ to $(R \cap Y)/(X \cap Y)$, and, by Lemma 3.1.5, this is an M-subgroup of $G/(X \cap Y)$. Thus $(K_1X \cap Y)/(X \cap Y)$, and, by symmetry, $(K_2Y \cap X)/(X \cap Y)$, are direct products of M-subgroups of $G/(X \cap Y)$. But these groups, being contained respectively in $Y/(X \cap Y)$ and $X/(X \cap Y)$, generate their direct product. That is

$$((K_1X \cap Y)(K_2Y \cap X))/(X \cap Y)$$

is a direct product of M-subgroups of $G/(X \cap Y)$. By Lemma 3.3.5, since $X \cap Y$, being the intersection of a group with its centralizer, is abelian,

$$(K_1X \cap Y)(K_2Y \cap X) = E \times (X \cap Y),$$

where E is a direct product of M-subgroups of G. Finally, from $K \leqslant XY$, $K \leqslant K_1X$, $K \leqslant K_2Y$, we have that $K \leqslant (K_1X \cap Y)(K_2Y \cap X)$. Thus $K \leqslant E \times (X \cap Y)$. But K, being generated by simple groups, is its own derived group, so that its projection on $X \cap Y$ is 1. Hence $K \leqslant E$, a group of the required form.

LEMMA 3.3.8. *Let $G > K = \{D_1, \cdots, D_s\}$, where the D_i are M-subgroups of G; then K is a subgroup of a finite direct product of M-subgroups of G.*

Proof. We proceed by induction on s. The lemma is certainly true for $s = 1$. Assume true for $s - 1$, then $K = \{I, L\}$, where $I = D_1$, and $L = \{D_2, \cdots, D_s\} \leqslant J$, a finite direct product of M-subgroups of G. By Lemma 3.3.8 $\{I, J\}$, and thus K, is contained in a finite direct product of M-subgroups of G.

LEMMA 3.3.9. *$w(G)$ is a finite direct product of M-subgroups of G, each of which is normal in G.*

Proof. By Lemma 3.3.1, $w(G)$ is generated by a finite number of M-subgroups of G, and thus, by Lemma 3.3.8, it is contained in a finite direct product of M-subgroups of G. But, by Lemma 3.3.2, each such M-subgroup is itself contained in $w(G)$, and so $w(G)$ must actually be equal to this direct product. The normality of each factor follows from Lemma 3.3.4, since $w(G) < G$.

3.4. *Proof of Theorem* 2(*A*)

It must now be shown that $\mathfrak{U}^{(n)}$, where $n \geqslant n_0 = \max(a, 2b+1)$, satisfies conditions (ii) and (iii) for a Cross variety.

From Lemma 3.3.9 we have that, if G is a finitely generated group in $\mathfrak{U}^{(n)}$, then $w(G)$ is a finite direct product of M-subgroups of G, each of which is normal in G. Since $G/w(G)$ is finite, it follows that G is finite, and hence condition (ii) is satisfied.

If G is critical, then either $w(G) = 1$, in which case $G \in \mathfrak{A}$, which, by hypothesis, is Cross, and so possesses only a finite number of critical groups, or, since a critical group has a unique minimal normal subgroup, $w(G) \simeq M$. In the latter case, since $w(G) \cap C_G(w(G)) = 1$ and $C_G(w(G)) \lhd G$, we must have $C_G(w(G)) = 1$, and so G is isomorphic to a subgroup of the full automorphism group of M. Only a finite number of such groups exist, and the truth of the theorem follows.

4. Critical Groups with Abelian Minimal Normal Subgroups

In this section, we prove:

THEOREM 2(B). *If B is a critical group whose minimal normal subgroup is abelian, and $\mathfrak{A}$ is a Cross variety containing all proper factors of B, then the variety $\mathfrak{U}$ generated by $\mathfrak{A}$ and B is Cross.*

It is assumed that the reader is familiar with the concept of the upper p-series of a p-soluble group. Frequent use is made of the facts that if N_{i-1}, P_i, and N_i are consecutive terms of the upper p-series, such that P_i/N_{i-1} is a p-group and N_i/P_i has order prime to p, then for $i \geqslant 1$, $C(P_i/N_{i-1}) \leqslant P_i$ and $C(N_i/P_i) \leqslant N_i$. Moreover, if F_i/N_{i-1} is the Frattini subgroup of P_i/N_{i-1}, then for $i \geqslant 1$, $C(P_i/F_i) = P_i$.

THEOREM 4. *The order of a critical group G is bounded in terms of its exponent, the maximum of the classes of its Sylow subgroups, the maximum of its various p-measures and S-measures, and the maximum of the orders of its composition factors.*

Let n, c, k, and a respectively denote the exponent, the maximum of the classes of the Sylow subgroups, the maximum of the p-measures and the S-measures, and the maximum of the orders of the composition factors of G; and let N be its unique minimal normal subgroup. If N is not abelian, its order cannot exceed a^k, while $C(N) \cap N = 1$. This means that $C(N) = 1$, and that G can be faithfully represented as a subgroup of the automorphism group of a group whose order is not greater than a^k. Thus when N is non-abelian, Theorem 4 is immediate, and so it is assumed throughout the remainder of this section that N is abelian. If G is nilpotent, it is of class c and exponent n, and by Lemma 2.4.4 it can be generated by c elements. In this case also, Theorem 4 is immediate, so we assume that G is not nilpotent.

The proof of the theorem is conveniently divided into five stages. We begin with an examination of the structure of G.

4.1. The Fitting subgroup of G is nontrivial because N is abelian. Each of its Sylow subgroups is characteristic, and so normal in G. As these intersect trivially, there can only be one such subgroup, otherwise G would not have a unique minimal normal subgroup. Hence the Fitting subgroup is a p-group; for some prime p, and it is clearly the greatest normal p-subgroup of G. Also G has no nontrivial normal p'-subgroup (i.e., one whose order is coprime to p).

Let D/P be the greatest normal p'-subgroup of G/P (this may be trivial) and let K be the normal subgroup of G such that K/P is the centralizer of D/P in G/P. By the Schur-Zassenhaus theorem, D may be written as QP, where Q is a p'-subgroup of G. Evidently, $K \cap QP = Z(Q)\,P$. Let M be the normal subgroup of G generated by all normal subgroups U of G which are contained in K and are such that $UZ(Q)\,P/Z(Q)\,P$ is a minimal normal subgroup of $G/Z(Q)\,P$. Again, it is possible that $M/Z(Q)\,P$ is trivial, as is the case when $K = Z(Q)\,P$. However, as G is assumed not to be nilpotent, QP/P and $M/Z(Q)\,P$ cannot both be trivial.

LEMMA 4.1.1. *If $K/Z(Q)\,P$ is nontrivial, then every minimal normal subgroup of $G/Z(Q)\,P$ contained in it is a direct product of isomorphic simple groups with nontrivial Sylow p-subgroup.*

Proof. Let $M_1/Z(Q)\,P$ be such a normal subgroup. It is required to prove that this cannot be a p-group nor a p'-group. But in these cases the group M_1Q is p-soluble with upper p-series of the form

$$\text{(i)} \qquad 1 \lhd P \lhd QP \lhd M_1Q$$

or of the form

$$\text{(ii)} \qquad 1 \lhd P \lhd M_1Q$$

respectively. The first possibility cannot occur because

$$QP/P \nleqslant M_1/P \leqslant C(QP/P).$$

The second cannot occur because QP/P is the greatest normal p'-subgroup of G/P.

It follows from Lemma 4.1.1 that

$$\frac{M}{Z(Q)\,P} = \prod_{i=1}^{s} \frac{M_i}{Z(Q)\,P} \tag{4.1.2}$$

where each direct factor is a non-abelian minimal normal subgroup of $G/Z(Q)\,P$ with nontrivial Sylow p-subgroup.

4.2. In the paper [8], many lemmas are proved about soluble critical groups, which do not really make use of the full power of the solubility. With very minor modifications, these proofs can be applied to yield lemmas about the critical group G in question. Where such lemmas are needed, the proof is either omitted or only briefly sketched. The reader is then referred to the corresponding lemma of [8].

LEMMA 4.2.1 *There is a subgroup L of G such that* (i) $G = LP$, (ii) $L \cap P \leqslant C(Q)$, *and* (iii) $L \cap P \leqslant \Phi(L)$.

Proof. If $Q \neq 1$, it is a Hall complement to P in QP. Any two such complements are conjugate in QP, and by the usual Frattini argument, $G = N(Q)\,P$. This is obviously true if $Q = 1$. The proof now proceeds as for Lemma 5.2.2 of [8], and L is chosen to be a subgroup of $N(Q)$ which has minimal order with respect to the property $G = LP$.

COROLLARY 4.2.2. (cf. Lemma 5.2.3 of [8]). *If $Q \neq 1$, L is a partial complement to P in G.*

If $R = (L \cap P)\,\Phi(P)$, R admits P because $\Phi(P) \leqslant R$. It admits L because both $L \cap P$ and $\Phi(P)$ do. Thus R is normal in G, contains $\Phi(P)$, and is contained in P.

COROLLARY 4.2.3. *If $Q \neq 1$, R is a proper subgroup of P.*

LEMMA 4.2.4. *If $U_1, \cdots, U_m$ are subgroups of P containing R and admitting L, and which together generate P, then some subset of them, consisting of at most c members, also generates P.*

If $R = P$, the lemma is trivial. Otherwise the proof is as in Lemma 5.2.4 of [8].

4.3. We now begin the task of bounding the order of G. In this part, the third stage of the proof of Theorem 4, we indicate how $|\,Q\,|$ can be bounded in terms of n, k and c. If Q is trivial, there is nothing to prove, and it is assumed in this part that $Q \neq 1$. In this case P/R is nontrivial by Corollary 4.2.3. We therefore obtain a representation of G as a group of automorphisms of P/R by conjugation.

In Theorem 5.3.6 of [8], it is stated that if G is a soluble critical group, and L, P, Q, R, n, and c are defined as above, then $|Q|$ can be bounded in terms of n, c and the absolute rank of G. It may be verified that:

(i) the proof, as it stands, actually shows that $|Q|$ can be bounded in terms of n, c and the maximum of the absolute p-degrees of the chief factors of G between P and R,

(ii) the solubility of G is not required but only that the representation of G as a group of automorphisms of P/R represents Q faithfully and that the image of Q is normal in that of G.

If then C is the kernel of this representation, the bounding of $|Q|$ in terms of the stated invariants becomes equivalent to proving.

LEMMA 4.3.1. $C \cap QP = P$.

Proof. Clearly, this intersection contains P. Assume therefore to the contrary that this intersection strictly contains P. Then there is a normal subgroup X/P of QP/P which induces trivial automorphisms of P/R. Using the Schur- Zassenhaus theorem, write $X = Q_0P$, where $Q_0 \leqslant Q$. By Lemma 4.2.1, $L \cap P \leqslant C(Q)$, so that $L \cap P \leqslant C(Q_0)$. Hence the elements of Q_0 induce trivial automorphisms of $R/\Phi(P)$, and so Q_0 acts trivially on both P/R and $R/\Phi(P)$. Since Q_0 is a p'-group, this implies that Q_0 acts trivially on $P/\Phi(P)$ and that, in turn, implies that $C_{QP}(P) \geqslant Q_0$. This is impossible unless $Q_0 = 1$, contrary to assumption.

4.4. Next, we show that $|M/Z(Q)\,P|$ can be bounded in terms of n, k, c, and a. We assume that it is nontrivial, otherwise there is nothing to prove.

Now (4.2.2)

$$\frac{M}{Z(Q)\,P} = \prod_{i=1}^{s} \frac{M_i}{Z(Q)\,P}$$

where each $M_i/Z(Q)\,P$ is a direct product of isomorphic simple groups with nontrivial Sylow p-subgroups. Each $M_i/Z(Q)\,P$ is a chief factor of $G(Z(Q)\,P$ and its order cannot exceed a^k. Thus it is sufficient to prove that the number s of direct factors is bounded in terms of n, k, and c.

For each j ($j = 1, 2, \cdots, s$) let N_j be a normal subgroup of G which is minimal with respect to the conditions, (i) $N_j \rhd P$, (ii) $N_jZ(Q)\,P = M_j$.

LEMMA 4.4.1. *N_j/P is a minimal normal complement to $Z(Q)P/P$ in M_j/P.*

For otherwise, N_j could not be minimal with respect to the above conditions.

LEMMA 4.4.2. *If H is any normal subgroup of G, then either $N_j \cap H = N_j$ or $N_j \cap H \leqslant Z(Q)\,P$.*

For $Z(Q)\,P \leqslant (N_j \cap H)\,Z(Q)\,P \leqslant N_jZ(Q)\,P = M_j$, and $M_j/Z(Q)\,P$ is a chief factor of G.

LEMMA 4.4.3. *If also $H \geqslant P$, and is such that $N_j \cap H \leqslant Z(Q)\,P$ $(j = 1, \ldots, t)$, then $N_1N_2 \ldots N_t \cap H \leqslant Z(Q)\,P$.*

Proof. Let $U = N_1N_2 \cdots N_t \cap H$. Now,

$$\frac{U}{U \cap Z(Q)\,P} = \frac{UZ(Q)\,P}{Z(Q)\,P} \leqslant \frac{N_1N_2 \cdots N_tZ(Q)\,P}{Z(Q)\,P} = \prod_{j=1}^{t} \frac{M_j}{Z(Q)\,P}$$

Hence $UZ(Q)\,P/Z(Q)\,P$ is either trivial or contains at least one of the normal subgroups $M_j/Z(Q)\,P$. In the latter case, it follows that for such j, each n_j in N_j may be written $n_j = zh$ for z in $Z(Q)\,P$ and h in H, and because $N_j/P \leqslant C(QP/P)$, $n_j'^{-1}n_jn_j' = zn_j'^{-1}hn_j$ (mod P) for all n_j' in N_j. Thus $[n_j, n_j'] \in H$, for all $n_j, n_j' \in N_j$, and as it is assumed that $H \cap N_j \leqslant Z(Q)\,P$, this implies that $[n_j, n_j'] = 1$ (mod $Z(Q)\,P$). By the very nature of N_j, this is a contradiction. Hence $UZ(Q)\,P/Z(Q)\,P$ is trivial and the lemma is proved.

LEMMA 4.4.4. *Suppose that $s > [\log_2 n^kk!]$. Let t be an integer such that $s \geqslant t > [\log_2 n^kk!]$, and let A, B be normal subgroups of G such that $1 \leqslant B \leqslant A \leqslant P$ and such that A/B is elementary abelian and central in P/B. If $n_1, n_2, \cdots, n_t$ is a sequence of elements of G such that $n_j \in N_j$ and each n_j has p'-order modulo $N_j \cap P$, then for all $h \in A$*

$$[h, n_1, n_2, \cdots, n_t] \equiv 1 \pmod{B}.$$

Proof. The group A/B may be identified with the additive group of a vector space V over the field $\mathscr{F}_p$ of p elements. By conjugation, G acts as a group of automorphisms of A/B, and this action gives V the structure of a representation module for G. We choose $\mathscr{K}_p$ to be a finite extension field of $\mathscr{F}_p$ such that every irreducible G-submodule and G-factor module of $\mathscr{K}_p \otimes V$ is absolutely irreducible (e.g., $\mathscr{K}_p$ may be chosen as the extension field obtained by the adjunction of the $|G|$th roots of unity). Let J be the subgroup of G generated by $n_1, n_2, \cdots, n_t$, then $J/J \cap Z(Q)\,P$ is a direct product of t cyclic groups of p'-order, and $J/J \cap P$ is therefore a p'-group. In the usual manner, g is used to denote both the element of the group and the transformation of $\mathscr{K}_p \otimes V$ that it induces.

Each chief factor of G between A and B has absolute p-degree at most k. Consequently the space $\mathscr{K}_p \otimes V$ has a G-composition series

$$\mathscr{K}_p \otimes V = Y_0 \geqslant Y_1 \geqslant \cdots \geqslant Y_v = 0$$

in which each composition factor is of dimension at most k. As $[A, P] \leqslant B$, the spaces Y_i $(i = 0, 1, \cdots, v)$ may be regarded as G/P-modules and therefore, a fortiori, as JP/P-modules. But JP/P is a p'-group, and so for each i, a JP/P-module U_j can be found such that $Y_{i-1} = U_i \oplus Y_i$. In this way, $\mathscr{K}_p \otimes V$ may be written as

$$U_1 \oplus U_2 \oplus \cdots \oplus U_v$$

where each U_i is a JP/P-module of dimension at most k.

Suppose now that the lemma is false. Then for at least one value i

$$U_i(-1 + n_1)(-1 + n_2) \cdots (-1 + n_t) \neq 0$$

and so $U_i(-1 + n_j) \neq 0$ for $j = 1, 2, \cdots, t$. Thus each n_j acts nontrivially on U_i and cannot therefore centralize Y_{i-1}/Y_i. If H is the normal subgroup of G which centralizes Y_{i-1}/Y_i, H contains P but not N_j $(j = 1, 2, \cdots, t)$. By Lemma 4.4.2, $H \cap N_j \leqslant Z(Q)\, P$, and by Lemma 4.4.3,

$$N_1 N_2 \cdots N_t \cap H \leqslant Z(Q)\, P.$$

Thus

$$\left|\frac{N_1 \cdots N_t H}{H}\right| = \left|\frac{N_1 \cdots N_t}{N_1 \cdots N_t \cap H}\right| \geqslant \left|\frac{N_1 \cdots N_t}{N_1 \cdots N_t \cap Z(Q)\, P}\right| = \left|\prod_{j=1}^{t} \frac{M_j}{Z(Q)\, P}\right|.$$

Now as $M_j/Z(Q)\, P$ is not a p-group, it contains a p'-subgroup of order at least 2 and $\prod_{j=1}^{t} M_j/Z(Q)\, P$ contains one of order at least $2^t > n^k k!$. Hence G/H has exponent dividing n, has a p'-subgroup of order exceeding $n^k k!$ and acts faithfully on a vector space of dimension at most k over a field of characteristic p. By Theorem 5 of [6], this is impossible.

The assumption that Lemma 4.4.4 is false thus leads to a contradiction. Hence the lemma is true.

Lemma 4.4.5. *Suppose* $s > [\log_2 n^k k!]$. *Let the integer* t *and the subgroups* A, B *be as in the previous lemma. If* $i_1, i_2, \cdots, i_t$ *be any sequence of distinct integers between* 1 *and* s, *then*

$$[A, N_{i_1}, N_{i_2}, \cdots, N_{i_t}] \leqslant B.$$

Proof. Clearly it is sufficient to consider the case where $i_j = j$ $(j = 1, 2, \cdots, t)$.

As $N_j/N_j \cap Z(Q)\,P \simeq M_j/Z(Q)\,P$, it is a direct product of isomorphic simple groups, N_j therefore has a normal generating set S_j in which every element has p'-order modulo $N_j \cap P$. By the previous lemma, $[A, S_1, S_2, \cdots, S_t] = 1$. The proof of the lemma is now a consequence of the relation

$$[A, N_1, N_2, \cdots, N_t] \equiv [A, S_1, S_2, \cdots, S_t] \bmod B,$$

a proof of which may be found in [*3*].

The group P is a p-group, so its lower central series may be refined to a series

$$P = P_0 \rhd P_1 \rhd \cdots \rhd P_r = 1,$$

whose length is not greater than nc, and is such that each factor P_i/P_{i+1} is elementary abelian and central in P/P_{i+1}.

LEMMA 4.4.6. *If* $s > nc([\log_2 n^k k!] + 1) + 3$, *then for all permutations* σ *of the integers* $1, 2, \cdots, s$

$$[N_{\sigma(1)}, N_{\sigma(2)}, \cdots, N_{\sigma(s)}] = 1.$$

Proof. The groups $N_{\sigma(1)}, N_{\sigma(2)}$ generate their direct product modulo $Z(Q)\,P$, and $N_{\sigma(3)}/P$ is contained in K/P which is the centralizer of QP/P in G/P. Hence $[N_{\sigma(1)}, N_{\sigma(2)}, N_{\sigma(3)}] \leqslant P$. Writing u for $[\log_2 n^k k!] + 1$, it follows from Lemma 4.4.5 that

$$[N_{\sigma(1)}, N_{\sigma(2)}, \cdots N_{\sigma(u+3)}] \equiv 1 \bmod P_1$$

and inductively that

$$[N_{\sigma(1)}, N_{\sigma(2)}, \cdots, N_{\sigma(iu+3)}] \equiv 1 \bmod P_i\,.$$

As $P_{nc} = 1$, this completes the proof of the lemma.

LEMMA 4.4.7. *The order of* $M/Z(Q)\,P$ *can be bounded in terms of* n, k, c *and* a.

Proof. By Lemma 2.2.9, there is a subgroup T of G such that

(i) $Z(Q) \leqslant T$,

(ii) the group $G/Z(Q)\,P$ is generated by $T/Z(Q)\,P$ together with the subgroups $M_j/Z(Q)\,P$ ($j = 1, 2, \cdots, s$),

(iii) the group $G/Z(Q)\,P$ is not generated by $T/Z(Q)\,P$ together with any proper subset of the groups $M_j/Z(Q)\,P$ ($j = 1, 2, \cdots, s$).

As $M_j = N_jZ(Q)\,P$, we see from this that in G

(i) G is generated by T together with $N_1, N_2, \cdots, N_s$

(ii) G is not generated by T together with any proper subset of the subgroups $N_1, N_2, \cdots, N_s$,

i.e., the subgroups $T, N_1, N_2, \cdots, N_s$ of G satisfy the first two of the hypotheses of Lemma 2.4.2. As G is critical, it must fail to satisfy the third, and by Lemma 4.4.6, this can only happen if $s \leqslant nc[\log_2 n^k k!] + 1) + 3$ Hence $M/Z(Q)\,P = \prod_{i=1}^{s} M_i/Z(Q)\,P$ where each $M_i/Z(Q)\,P$ has order at most a^k, and where the value of s is at most $nc([\log_2 n^k k!] + 1) + 3$.

4.5. So to the final stage of the proof of Theorem 4. We first show how the bound for $|Q|$, obtained in Section 4.3, and the bound for $|M/Z(Q)\,P|$, obtained in Section 4.4, lead to a bound for $|KQ/P|$ in terms of n, k, c, and a.

LEMMA 4.5.1. *$|KQ/P|$ is bounded in terms of n, k, c and a.*

Proof. Let $X/Z(Q)\,P$ be the centralizer of $M/Z(Q)\,P$ in $K/Z(Q)\,P$. Clearly $X \lhd G$. Now $M/Z(Q)\,P$ is generated by all minimal normal subgroups of $G/Z(Q)\,P$ in $K/Z(Q)\,P$, and so, if $X/Z(Q)\,P$ is nontrivial it has non-trivial intersection with $M/Z(Q)\,P$. On the other hand, $M/Z(Q)\,P$ is a direct product of non-abelian chief factors of $G/Z(Q)\,P$ and is without center. Hence $X/Z(Q)\,P$ is trivial, and $K/Z(Q)\,P$ may be faithfully represented as a group of automorphisms of $M/Z(Q)\,P$. Since $|M/Z(Q)\,P|$ can be bounded in terms of n, k, c, and a (Lemma 4.4.7) so then can $|K/Z(Q)\,P|$, and combining this bound with the bound for $|Q|$ of 4.3, the lemma follows.

LEMMA 4.5.2. *$|G/KQ|$ is bounded in terms of n, k c, and a.*

For G/K acts faithfully as a group of automorphisms of QP/P by conjugation, and $|Q|$ has been shown to be bounded in terms of these invariants.

COROLLARY 4.5.3. *$|G/P|$ is bounded in terms of n, k, c, and a.*

LEMMA 4.5.4. *$|P|$ is bounded in terms of n, k, c and a.*

Proof. $|P|$ is bounded in terms of n, c, and $|P/\Phi(P)|$ because it is a p-group. We need only bound $|P/\Phi(P)|$ therefore. We first bound $|P/R|$, then $|R/\Phi(P)|$.

Consider P/R. If this is not trivial, define for each g ($g \in P$, $g \notin R$) the subgroup U_g as the intersection of all subgroups of P which admit L and contain g and R. Since $G = LR$ and P/R is elementary abelian, it is clear that $|U_g/R|$ divides $p^{|G/P|}$, for each U_g. The set of all such subgroups generate P and so, by Lemma 4.4.2, some subset of them, containing at most c members also generates P. Hence $|P/R|$ divides $p^{c|G/P|}$, and by Lemma 4.5.3, this is a number which is bounded in terms of n, k, c, and a.

Next, consider $R/\Phi(P)$. Since $G/P \simeq L/L \cap P$, $|L/L \cap P|$ is of bounded order. By Lemma 4.2.1 (iii), $L \cap P \leqslant \Phi(L)$, and so $|L/\Phi(L)|$ is of bounded order. This means that L can be generated by a bounded number of elements. The subgroup $L \cap P$ is of bounded index in L, and so this, too, may be generated by a bounded number of elements. But it is nilpotent of class at most c and its exponent divides n. Thus it has bounded order, and as

$$R = (L \cap P)\, \Phi(P),$$

so then has $R/\Phi(P)$.

This completes the proof of Theorem 4.

4.6. Using Theorem 4, it is possible to prove Theorem 2(B). As before, let A be a finite group which generates $\mathfrak{A}$ and let $w \equiv w(x_1, x_2, \cdots x_a)$ be a basis for the defining relations of $\mathfrak{A}$. Let $C = A \times B$, and choose c to be the maximum of the classes of the Sylow subgroups of C, and k to be the maximum of its various p-measures and S-measures. Let

$$r \geqslant \max(2a, c + 1, a + 2k).$$

The aim is to show that $\mathfrak{C}^{(r)}$ is a Cross variety, as this immediately implies that $\mathfrak{C} = \mathfrak{U}$ is a Cross variety. We observe that as $w(C)$ is an elementary abelian p-group of aboslute p-degree at most k, the variety $\mathfrak{C}^{(r)}$ includes

(i) $w^p = 1$

(ii) $[w(x_1, \cdots, x_a), w(x_{a+1}, \cdots, x_{2a})] = 1$

(iii) $\prod_{\sigma} \{w^{x_{\sigma(1)} \cdots x_{\sigma(2k)}}\}^{\chi(\sigma)} = 1$ (Lemma 2.5.1)

among its defining relations.

Because of Theorem 4, we need to verify that

(a) *there is a finite basis to the relations of* $\mathfrak{C}^{(r)}$

(b) *finitely generated groups in* $\mathfrak{C}^{(r)}$ *are finite,*

(c) *groups in* $\mathfrak{C}^{(r)}$ *have some common finite exponent,*

(d) *there is a bound to the order of the finite simple groups in* $\mathfrak{C}^{(r)}$.

(e) *nilpotent groups in* $\mathfrak{C}^{(r)}$ *are of class at most* c.

(f) *every finite group in* $\mathfrak{C}^{(r)}$ *has* p*-measure and* S*-measure at most* k, *for each prime* p *and each finite simple group* S.

That $\mathfrak{C}^{(r)}$ satisfies (a) is the statement of Lemma 2.3.2, while (b) and (c) are true because of the relations (i) and (ii). The factor group $G/w(G)$ of any group G in $\mathfrak{C}^{(r)}$ belongs to $\mathfrak{A}$ which contains only a finite number of critical groups and, a fortiori (Lemma 2.4.1), only a finite number of finite simple groups. Hence, as $w(G)$ is abelian, (d) is true. For (e), suppose on the contrary that $\mathfrak{C}^{(r)}$ contains a nilpotent group of class greater than c. It must then contain such a group which can be generated by $c + 1$ elements. As $r \geqslant c + 1$, this would belong to $\mathfrak{C}$, which is a contradiction, by Lemma 2.3.5.

We have already remarked that if G belongs to $\mathfrak{C}^{(r)}$ then $G/w(G)$ belongs to $\mathfrak{A}$ and $w(G)$ is an elementary abelian p-group. By Lemmas 2.5.2 and 2.5.3, this means that for all primes q other than p, and for all finite simple groups S, the q-measure and S-measure of the finite group G cannot exceed the corresponding q-measure or S-measure of A. As for the prime p itself, because of the relation (iii), we may apply Lemma 2.5.1 to give that the absolute p-degree of any chief factor of G between $w(G)$ and 1 is at most k. Since the p-rank of $G/w(G)$ is not greater than that of A, we have that the p-rank of any finite group G in $\mathfrak{C}^{(r)}$ is at most k. Hence, for all primes p, and for all finite simple groups S, the p-measure and S-measure of any finite group in $\mathfrak{C}^{(r)}$ is at most k.

This completes the proof that $\mathfrak{C}^{(r)}$ is a Cross variety and so Theorem 2(B) is proved. Theorems 2(A) and 2(B) give Theorem 2, from which Theorems 1 and 3 follow.

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