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Renormalization Approach to the Dimension of Diffusion in Cantorian Space

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Abstract—A simple renormalization group method is used to drive the Hausdorff dimension for a critical Cantorian space. The results reinforce previous ones regarding the role played by the Golden Mean dimension, Cantor triadic set and the Sierpinski gasket. Connection to diffusion and interference are also considered.

1. INTRODUCTION

The possibility that spacial chaos could model micro space-time using Cantorian or fractal geometry has quite recently drawn some strong interest from various schools in the context of quantum mechanics, stochastic physics and questions pertaining to the dimensionality of micro space-time [1,2].

A closely related problem discussed in several previous works [3,4] is, of course, the possibly fundamental role played by Golden Mean Hausdorff dimensions $d = \phi = (\sqrt{5} - 1)/2$ and nearby Cantor sets such as the famous triadic one ($d = \ln 2/\ln 3$). In [5] it was mentioned that "... we are justified in concluding that the triadic set represent the most probable backbone of a typical strange behaviour ...." Similarly in Appendix 2 of [6], we conjectured that "... the Golden Mean represents ... an average of all possible backbone sets laying in the unit interval (0-1) ... " and this was found to be "... reminiscent of Feynman's path integral formulation." Now as for the "Golden Mean conjecture," it was proved recently under fairly general assumptions [4]. It is also obvious that this result is directly related to a theorem by Mauldin concerning random Cantor sets [3]. By contrast, the first conclusion regarding the triadic set ($\ln 2/\ln 3$) was made only reasonably plausible using elementary statistical mechanics [5] as well as the obvious proximity of $\ln 2/\ln 3$ to ϕ. Nonetheless, no analytically clear cut proof was given so far. In what follows, we use an elementary version of the so-called renormalization group method to reinforce the results of the statistical approach of [5,6] regarding the central role played by Cantor's triadic set.

Based on a renormalization transformation which is equivalent to that of a diamond hierarchical lattice, we derive an expectation value for the dimension of a backbone set $d_c^{(0)}$ which is very close numerically to the Hausdorff fractal dimension of the triadic set. We show further that the Sierpinski gasket could be very near indeed to the "prespace" or the "basic elementary stone" of which the "fabric" of two-dimensional micro space is made, and derive an exact explicit relation between the correlation length exponent of this space and its Hausdorff dimension.

It should be appreciated that a successful treatment of Cantorian space using renormalization groups is not in the least surprising since both theories are based on scale invariance and self similarity. The only slight difference is that Cantorian geometry (or fractals) are used normally

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX for objects in real space while renormalization groups* are used for physical quantities as demonstrated by Feigenbaum for chaotic maps for instance. Mandelbrot did surmise quite early that both methods may be equivalent in many cases. In the mean time, this prediction turned out to be indeed the case. An elementary introduction to renormalization may be found in [7], while an authoritative account of renormalization and critical phenomena in physics which is also a very readable one, is that of Wilson *et al.* [8].

2. COARSE-GRAINING TRANSFORMATION

The first step in our present analysis is to define a lattice configuration covering the plan and calculate certain corresponding statistical weights [8]. Subsequently, it is easily reasoned that for a nontrivial critical probability for forming a space cluster, we need four points (Cantorions) and a two-dimensional lattice block of the form #. Now, following the renormalization procedure, our block is supposed to collapse into a so-called super site. In turn, four super sites will build a block which will collapse into another super site and so on. Now we consider three cases. We assume that

(a) two,
(b) three, and
(c) all of the four intersection points of the block

are each occupied by one Cantorion. Consequently, for case (c), for which all corners are occupied, the super site probability found from the multiplication theorem is

$$P_{(c)} = (P)^4. \tag{1}$$

Here P is the probability of finding a Cantorion in any of the intersection points of the lattice. For case (b), a combination of the multiplication and the addition theorem gives

$$P_{(b)} = 4\,P^3(1-P). \tag{2}$$

This is so because we must have in this case three occupied intersections and one empty intersection which gives the probability $P^3(1-P)$, and since there are 4 possible empty intersections, the probability is $4\,P^3(1-P)$. Finally for case (a) the probability is found to be

$$P_{(a)} = 2\,P^2(1-P)^2. \tag{3}$$

This comes from having simultaneously two occupied intersections with probability P^2 and two unoccupied with probability $(1-P)^2$, making the total probability $P^2(1-P)^2$. Since we have two different probabilities for the two unoccupied intersection the addition theorem gives $2P^2(1-P)^2$. The total probability of the super site is, thus, the sum of all probabilities [8], giving

$$P_s = P^4 + 4P^3(1-P) + 2P^2(1-P)^2. \tag{4}$$

3. CRITICAL PROBABILITY

It is relevant to give a physical interpretation of the super site of the (a), (b), and (c) configurations. Case (a) is evidently the minimum requirement of a topologically nonzero space. Case (b) is minimal representation of two-dimensionality, while case (c) is superimposed heuristically to give a nontrivial solution for the critical probability ($P^c \neq 0, P^c \neq 1$) as will be evident later on from looking at equation (6).

*This expression is elegant yet misleading. Renormalization has no inverse. Thus, it is a semi-group, and we rarely make use of any group theoretical properties in this context.

Now P_s, as given by equation (4), is clearly a renormalization transformation leading from a block to a super site *ad infinitum.* Consequently, and in analogy with numerous critical phenomena in physics, if at a certain state we have more empty space than Cantor points, then as we increase our "coarse graining" transformation, we find more and more empty space, and finally, in the limit, we end with absolute nothingness. By contrast, if we have more Cantor points "Cantorions" than empty space, then as we proceed with our coarse graining transformation, we will reach a state of a solid space or classical space continuum. It follows then that at a certain critical state, we will have a dividing line which keeps the space spanned by our Cantorions scale invariant. The corresponding critical probability of this state is obviously the fix point of P_s. Consequently, the critical probability is found from (4) to be

$$(P^c)^4 + 4(P^c)^3(1 - P^c) + 2(P^c)^2(1 - P^c)^2 - (P_s = P^c) = 0. \tag{5}$$

That means

$$(P^c)^4 - 2(P^c)^2 + P^c = 0$$

or

$$(P^c)^3 - 2P^c + 1 = 0. \tag{6}$$

This gives two trivial solutions $(0, 1)$ plus a noteworthy Golden Mean critical probability solution.

$$P^c = \frac{\sqrt{5} - 1}{2}. \tag{7}$$

4. EXPECTATION VALUE AND HAUSDORFF DIMENSION AT CRITICALITY

It is quite interesting to note the similarity of the probability discrete map arising from (4), namely

$$P_{n+1} = 2P_n^2 - P_n^4 = 2P_n^2\left(1 - \frac{P_n^2}{2}\right), \tag{8}$$

with the logistic map [7]. Therefore, it is important to investigate the multipliers λ (eigenvalues) at these critical points. Thus, from

$$\lambda = \frac{d\,P_{n+1}}{d\,P_n} = 4P_n - 4P_n^3, \tag{9}$$

and evaluating for 1 and 0, we find

$$\lambda_{(0)} = 0 \quad \text{and} \quad \lambda_{(1)} = 0,$$

while for $P = \phi$, we have

$$\lambda(\phi) = 4\phi - 4\phi^3 = 4\phi^2 > 1. \tag{10}$$

Consequently, $P = \phi$ is an unstable fix point of the probability map equation[1] (8).

Next we calculate the expectation value of the number of Cantorions in a block based on our critical probability $P^c = \phi$. This is easily found from the weighted probability.

$$\langle P \rangle = \frac{\sum n_i P_i}{\langle n \rangle}. \tag{11}$$

[1]It may be possible to show that the disappearance of interference patern in the two-slit expirment of quantum mechanics is related to such instability point of a probability map of a DNA-like microspace made of Cantorian geodesics.

Consequently,

$$\langle P\rangle = \frac{\{4P^4 + 3[4P^3(1-P)] + 2[2P^2(1-P)^2]\}}{E(n)}. \tag{12}$$

Setting $\langle P\rangle = P^c = \phi$, one finds the expectation value for n to be

$$E(n) = N = 4(\phi + \phi^2 - \phi^3) = 4(1-\phi^3) = 3.055. \tag{13}$$

Compared to N of the Sierpinski gasket namely $N = 3$. In case of $P = 1$, we have of course $E(n) = N = 4$.

In terms of our binary possibilities $(1,0)$, we may define a Hausdorff dimension which is a consequence of the observation scale invariance for $b = 2$ at $P^c = \phi$, namely

$$D = \frac{\ln N}{\ln b} = \frac{\ln 4(1-\phi^3)}{\ln 2} = \frac{\ln 3.055}{\ln 2} = 1.6115 \quad \text{and} \quad \frac{1}{D} = 0.6205. \tag{14}$$

This is very close indeed to the dimension of the Sierpinski gasket $D = \ln 3/\ln 2 = d_c^{(2)}$ and $1/D$ is also close to Cantor triadic set $d_c^{(0)} = \ln 2/\ln 3$. This result confirms to a considerable extent our conjecture, which we stated on several previous occasions, that the triadic set $d_c^{(0)} = \ln 2/\ln 3$ and the Sierpinski gasket $d_c^{(2)} = 1/d_c^{(0)} = \ln 3/\ln 2$ are fundamental to the formation of Cantorian space and the four-dimensionality of micro space-time of bosons [1, 3, 4]. In turn, this Cantorian nature could explain quantum non-locality. Finally, for $N = 4$, one finds of course $D = \ln 4/\ln 2 = 2$, as expected. One should also note that, in deriving the relationship $d_c^{(0)} = 1/d_c^{(2)}$, we made use of the volume interpretation of the Hausdorff dimension following for instance Jonot [9].

5. THE CORRELATION LENGTH EXPONENT—DIMENSION OF DIFFUSION FRONT

Next, we determine the correlation length exponent. This is given by

$$\nu = \frac{1}{k} = \frac{1}{\ln\lambda/\ln^2} = \frac{\ln 2}{\ln\lambda}. \tag{15}$$

For $\lambda(\phi)$, this is

$$\nu = \frac{\ln 2}{\ln\varphi\phi^2} = \frac{1}{[(\ln 8\phi^2)/\ln 2] - 1} = 1.63527. \tag{16}$$

Noting that $\ln 8\phi^2 = \ln 4(1-\phi^3)$, we see clearly from equation (14) and (16) that

$$\nu = \frac{1}{D-1}. \tag{17}$$

The correlation length exponent is thus exactly equal to the reciprocal value of the fractal part of the Hausdorff dimension. One might also mention in passing that $D \simeq 1.611$ given by (14) is equal to the Cartesian product of a line with a two scale Cantor triadic set $\nu_1^D + \nu_2^D$ with $\nu = 0.4$ and $\nu_2 = 0.25$ which was found by Halsly *et al.* to be $D \simeq 0.611$. It might be interesting to calculate now the fractal dimension of the corresponding diffusion front. This is given by Bunde and Gouyet as $d_f = (1+\nu)/\nu$. Consequently, $d_f = D$.

6. CONCLUDING REMARKS—QUANTUM DIFFUSION

The renormalization transformation of a spacially chaotic two-dimensional micro space spanned by two to four Cantorions on a rectangular lattice leads to a Hausdorff dimension almost identical to that of the Sierpinski gasket ($D = \ln^3/\ln^2$) which is believed to be fundamental for many two-dimensional processes [1, 2–6].

Consequently, using the bijection formula, the zero-dimensional set (backbone set) corresponding to this space must have $d_c^{(0)} \simeq \ln^2 / \ln^3$. It should be noted however that the numerical value is nearer to the Golden Mean and thus to Mauldin random Cantor sets theorem. The correlation length exponent of the space was found to be exactly $\nu = 1/(D - 1)$. The analysis is naturally exact only within the usual error of the renormalization group method in which connected points may become disconnected and vice versa.

It is quite possible that the mere fact that lattice calculations give such good results is already an indication for the transfinite character of quantum space-time. It is also quite possible to conclude from the present discrete Cantorian picture that quantum space is a kind of superimposition of many elementary subquantum spaces. Therefore, it may well be a kind of a Borel set as suggested some time ago by Wheeler.

This picture could then account for some of the nonlocal quantum effects such as tunneling. Thinking along these lines it seems to us quite possible to be able to give a mathematical formulation for quantum mechanics as a diffusion process in a Cantorian space-time medium. The double slit experiment could be then interpreted quite realistically as a natural interference between two diffusion equations which are formulated in a manner analogous to the Schroedinger equations. Alternatively, we could view the infinitely many Cantorian geodesics in the sense of Aristoteles and Heisenberg as potentially existing. When the two slits are open, the geodesics flow through both holes, so to speak, and interference take place. The stage is thus set up before any electron passes through. It is then irrelevant whether the electrons are fired one at a time or in large numbers. In all cases, interference takes place. In this sense, interference takes place between different parts of the Cantorian micro space-time itself as it is subdivided and restructured by the experimental set up [10]. In a sense, the presence of holes plays a crucial role for micro space similar to the role played by matter for the large scale structure of space-time. Thus, the phenomenon of quantum interference is best understood in terms of an informational interference of two slits correlated by a DNA-like transfinite (Cantorian) space [10]. Consequently wave collapse is a bifurcational instability similar to that of equation (10).

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