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RAMANUJAN LOCAL SYSTEMS ON GRAPHS

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1. INTRODUCTION

Ramanujan graphs were defined in [10] as graphs whose adjacency matrices have eigenvalues satisfying some "best possible" bounds. Such graphs possess many interesting properties. Lubotzky *et al.* [10] gave examples coming from integral quaternions. The inequalities there follow from deep results in algebraic geometry. It is well known that the adjacency matrix of a graph is an analogue of the laplacian on a riemannian manifold, and the Ramanujan graphs of [10] are analogues of locally symmetric spaces. Our starting point is the observation that in the geometric context one is frequently led to study the extra structure of a local system on a manifold, and certain local systems present themselves naturally in the locally symmetric case.

In this paper we will define and construct examples of Ramanujan local systems. Section 2 begins with generalities on local systems. We then define the laplacian on a local system and study its basic properties. Related notions have been studied by Chung and Sternberg [2, Section 3] and by Forman [6]. We then introduce the concept of a Ramanujan local system.

Section 3 begins with the results we need about definite quaternion algebras over $\mathbf{Q}$ and the action of $\mathrm{GL}_2(\mathbf{Q}_p)$ on the $p+1$ regular tree for a prime p. Using these we construct graphs with Ramanujan local systems on them. The examples of Lubotzky *et al.* are a special case. In fact, the examples in [10] are given as Cayley graphs of PSL_2 of a finite field. Our graphs and local systems are initially constructed as quotients of an infinite tree, but we also give a finite description of them through a generalization of Cayley graphs.

2. LOCAL SYSTEMS

In this section we define and study local systems on graphs. Our definition is a slight variation on the usual one in algebraic topology (see, e.g. [12]), but the references we know do not quite have what we need. We have therefore opted for a brief exposition of the general theory, in which proofs are usually not given or are merely sketched.

2.1. Graphs

For a graph Gr we let Ver(Gr) denote the vertices and $\mathrm{Ed}^0(\mathrm{Gr})$ the oriented edges of Gr. An oriented edge $e \in \mathrm{Ed}^0(\mathrm{Gr})$ has initial vertex $o(e)$, terminal vertex $t(e)$, and opposite edge $\bar{e}$, satisfying $\bar{\bar{e}} = e$ and $o(\bar{e}) = t(e)$. We allow $o(e) = t(e)$ but forbid $\bar{e} = e$. The edges of Gr are $\mathrm{Ed}(\mathrm{Gr}) = \{\{e, \bar{e}\} \mid e \in \mathrm{Ed}^0(\mathrm{Gr})\}$. The oriented star $\mathrm{St}^0(v)$ of a vertex v is the set of oriented edges terminating in v. The valency κ_v of a vertex v of Gr is $|\mathrm{St}^0(v)|$. A graph is locally finite if the valencies of its vertices are finite.

2.2. *Local systems*

A local system $\mathscr{L}$ of rank r on a graph Gr consists of

1. An r-dimensional $\mathbf{C}$-vector space $\mathscr{L}(v)$ for each $v \in \mathrm{Ver(Gr)}$. We call $\mathscr{L}(v)$ the fiber of $\mathscr{L}$ at v.
2. An invertible linear transformation $\mathscr{L}_e : \mathscr{L}(o(e)) \to \mathscr{L}(t(e))$ for each $e \in \mathrm{Ed}^0(\mathrm{Gr})$. The $\mathscr{L}_e$s are called *transition maps* and we impose the condition $\mathscr{L}_{\bar{e}} = \mathscr{L}_e^{-1}$ for all e.

A metric on $\mathscr{L}$ is a collection of definite hermitian inner products $\langle\, , \rangle_v$ on each $\mathscr{L}(v)$, such that each $\mathscr{L}_e$ is an isometry: $\langle \mathscr{L}_e u, \mathscr{L}_e u' \rangle_{t(e)} = \langle u, u' \rangle_{o(e)}$ for any $u, u' \in \mathscr{L}_{o(e)}$.

3. A metrized local system is a local system with a metric.

One could also discuss *real* (metrized) local systems. For these the $\mathscr{L}(v)$s would be real vector spaces (equipped with positive-definite inner products). Given such, it is always possible to complexify the space (and extend the inner product to a hermitian one). This being said we shall stick to the more general notion of complex local systems.

The (metrized) local system $\mathscr{T}_V$ all of whose fibers are the same space (with definite product) V with all transition maps the identity is called the trivial local system with fiber V. When $V = \mathbf{C}$ (and the product is the usual one) we call $\mathscr{T} = \mathscr{T}_{\mathbf{C}}$ the trivial (metrized) local system. The zero local system on Gr is the one with all fibers zero.

A local system on a disconnected graph is the same thing as a local system on each connected component. There are the obvious notions of direct sums of (metrized) local systems and sub- (metrized) local systems. In the metrized case any sublocal system $\mathscr{M} \subset \mathscr{L}$ has a direct, orthogonal complement $\mathscr{M}^{\perp}$: set $\mathscr{M}^{\perp}(v) = \mathscr{M}(v)^{\perp}$ for all $v \in \mathrm{Ver(Gr)}$. It is clear that each $\mathscr{L}_e$ maps $\mathscr{M}^{\perp}(o(e))$ to $\mathscr{M}^{\perp}(t(e))$.

If $\mathscr{L}$ is not metrized a sublocal system may fail to have a direct complement.

Definition 2.1. A local system is irreducible if it has no sublocal systems except the zero local system and itself.

The following is clear:

PROPOSITION 2.2. *A metrized local system is a sum of irreducible local systems.*

Next, we have the notion of a map:

Definition 2.3. Let $\mathscr{L}$, $\mathscr{L}'$ be local systems on a graph Gr. A map $\phi : \mathscr{L} \to \mathscr{L}'$ is a collection of maps $\phi_v : \mathscr{L}(v) \to \mathscr{L}'(v)$ for all $v \in \mathrm{Ver(Gr)}$ compatible with the transition maps, in the sense that $\phi_{t(e)} \mathscr{L}_e = \mathscr{L}'_e \phi_{o(e)}$ for all $e \in \mathrm{Ed}^0(\mathrm{Gr})$.

We call ϕ an isomorphism if all the ϕ_v's are isomorphisms. If both $\mathscr{L}$ and $\mathscr{L}'$ are metrized and all the ϕ_v's are isometries we say that ϕ is an isometry. We call a (metrized) local system $\mathscr{L}$ trivial if it is isomorphic (isometric) to any $\mathscr{T}_V$. An isomorphism (isometry) of $\mathscr{L}$ with a trivial local system $\varphi: \mathscr{T}_V \to \mathscr{L}$ is called a *trivialization* of $\mathscr{L}$.

The compatibility condition severely restricts the maps of local systems on a graph:

PROPOSITION 2.4. *Let* Gr *be a connected graph and let* $\mathscr{L}$, $\mathscr{L}'$ *be local systems on* Gr. *If* φ, $\psi: \mathscr{L} \to \mathscr{L}'$ *are maps of local systems and* $\varphi_{v_0} = \psi_{v_0}$ *for some* $v_0 \in \mathrm{Ver(Gr)}$, *then* $\varphi = \psi$.

Proof. Let $S \subset \mathrm{Ver(Gr)}$ be the set of vertices on which φ and ψ agree. If $o(e) \in S$ for $e \in \mathrm{Ed}^0(\mathrm{Gr})$, then $t(e) \in S$ since

$$\varphi_{t(e)} = \mathscr{L}'_e \varphi_{o(e)} \mathscr{L}_e^{-1} = \mathscr{L}'_e \psi_{o(e)} \mathscr{L}_e^{-1} = \psi_{t(e)}.$$

As Gr is connected and $v_0 \in S$, we get $S = \mathrm{Ver(Gr)}$.

Now suppose $f: \mathrm{Gr}' \to \mathrm{Gr}$ is a map of graphs (sending vertices to vertices and edges to edges). A local system $\mathscr{L}$ on Gr induces by pull back a local system $\mathscr{L}' = f^*\mathscr{L}$ on Gr′:

1. $\mathscr{L}'(v') = \mathscr{L}(f(v'))$ for any $v' \in \mathrm{Ver(Gr')}$,
2. $\mathscr{L}'_{e'} = \mathscr{L}_{f(e')}$ for any $e' \in \mathrm{Ed}^0(\mathrm{Gr}')$.

It is clear that this construction is functorial: a map φ of local systems on Gr induces a map between their pull backs on Gr′. □

2.3. *Normal forms and spanning trees*

In this section Gr is a connected graph. Recall that a spanning tree for Gr is a tree $T \subseteq \mathrm{Gr}$ which contains all vertices of Gr.

Definition 2.5. A (metrized) local system $\mathscr{L}$ on Gr is in normal form relative to a spanning tree $T \subseteq \mathrm{Gr}$ if

1. All $\mathscr{L}_v$s are the same vector space V.
2. For any oriented edge e of T, $\mathscr{L}_e = \mathrm{Id}_V$.

We introduce the following notation. For v_1, $v_2 \in \mathrm{Ver(Gr)}$ denote by $P_T(v_1, v_2)$ the (unique) shortest oriented path from v_1 to v_2 in T. For a local system $\mathscr{L}$ on Gr and a path $P = \{e_i\}_{i=1}^n$ in Gr denote by $\mathscr{L}_P: \mathscr{L}(o(e_1)) \to \mathscr{L}(t(e_n))$ the composition $\mathscr{L}_{e_n} \cdots \mathscr{L}_{e_1}$.

THEOREM 2.6. *Let* $\mathscr{L}$ *be a (metrized) local system on* Gr *and let* $T \subseteq \mathrm{Gr}$ *be a spanning tree. Then* $\mathscr{L}$ *has an isomorphic (isometric) normal form relative to* T.

Proof. Choose $v_0 \in \mathrm{Ver(Gr)}$ and set $V = \mathscr{L}(v_0)$. Define a local system $\mathscr{L}'$ on Gr by

1. $\mathscr{L}'(v) = V$ (as inner product spaces for all $v \in \mathrm{Ver(Gr)}$).
2. $\mathscr{L}'_e = \mathscr{L}_{P_T(t(e), v_0)} \mathscr{L}_e \mathscr{L}_{P_T(v_0, o(e))}$ if $e \in \mathrm{Ed}^0(\mathrm{Gr})$.

The path $P_T(v, v_0)$ is opposite to $P_T(v_0, v)$, so that $\mathscr{L}_{P_T(v,v_0)} = \mathscr{L}^{-1}_{P_T(v_0,v)}$. From this it follows that $\mathscr{L}'_e = \mathrm{Id}_V$ for $e \in \mathrm{Ed}^0(T)$. If $\mathscr{L}$ is metrized it is clear that each $\mathscr{L}'_e$ is an isometry. This proves the proposition. □

COROLLARY 2.7. *Suppose that* Gr *is a tree. Then any local system* $\mathscr{L}$ *on* Gr *is trivial.*

There is in fact a unique trivialization $\varphi: \mathscr{T}_{\mathscr{L}(v_0)} \to \mathscr{L}$ inducing the identity on the fibers at v_0.

The proof of Theorem 2.6 gives a more precise result: For a spanning tree $T \subseteq \mathrm{Gr}$, a vertex $v_0 \in \mathrm{Ver}(\mathrm{Gr})$, and a local system $\mathscr{L}$ on Gr, let $\mathrm{NF}_{T,v_0}(\mathscr{L})$ be the local system in normal form relative to T constructed in the theorem. In particular, all its fibers are $\mathscr{L}(v_0)$. It is clear that NF_{T,v_0} is functorial, in the sense that a map $\varphi: \mathscr{L} \to \mathscr{M}$ induces a map $\mathrm{NF}_{T,v_0}(\varphi): \mathrm{NF}_{T,v_0}(\mathscr{L}) \to \mathrm{NF}_{T,v_0}(\mathscr{M})$ by $\mathrm{NF}_{T,v_0}(\varphi)_v = \varphi_{v_0}$. For a local system $\mathscr{L}$ in normal form relative to T, $\mathrm{NF}_{T,v_0}(\mathscr{L}) = \mathscr{L}$.

2.4. Zero cochains and sections

Definition 2.8. (1) A zero cochain of a local system $\mathscr{L}$ on Gr is a collection $\{s(v)\}_{v \in \mathrm{Ver(Gr)}}$ of vectors $s(v) \in \mathscr{L}(v)$. The space $\prod_{v \in \mathrm{Ver(Gr)}} \mathscr{L}(v)$ of all zero cochains of $\mathscr{L}$ will be denoted $C^0(\mathrm{Gr}, \mathscr{L})$.

(2) A section of $\mathscr{L}$ is a zero cochain s satisfying the compatibility conditions $\mathscr{L}_e s(o(e)) = s(t(e))$ for all $e \in \mathrm{Ed}^0(\mathrm{Gr})$. The set of sections is a subspace of $C^0(\mathrm{Gr}, \mathscr{L})$ denoted by $\Gamma(\mathrm{Gr}, \mathscr{L})$.

A map of Gr-local systems $\varphi: \mathscr{L} \to \mathscr{L}'$ induces a map on zero cochains and on sections by $\varphi(s)(v) = \varphi_v(s(v))$. If $\mathscr{L}$ has finite rank r and Ver(Gr) is finite, $C^0(\mathrm{Gr}, \mathscr{L})$ has dimension $r|\mathrm{Ver(Gr)}|$.

The sections of $\mathscr{L}$ can be interpreted as maps of the trivial system $\mathscr{T}$ to $\mathscr{L}$:

LEMMA 2.9. *For a map of local systems* $\alpha: \mathscr{T} \to \mathscr{L}$ *define a zero cochain* $s_\alpha \in C^0(\mathrm{Gr}, \mathscr{L})$ *by* $s_\alpha(v) = \alpha_v(1)$. *Then* $s_\alpha \in \Gamma(\mathrm{Gr}, \mathscr{L})$ *and* $\alpha \mapsto s_\alpha$ *defines a bijection between the set of maps of local systems* $\alpha: \mathscr{T} \to \mathscr{L}$ *and* $\Gamma(\mathrm{Gr}, \mathscr{L})$.

Proof. The inverse map sends a section $s \in \Gamma(\mathrm{Gr}, \mathscr{L})$ to the map $\alpha_s: \mathscr{T} \to \mathscr{L}$ given by $(\alpha_s)_v(z) = zs(v)$ for all $v \in \mathrm{Ver(Gr)}$ and $z \in \mathbf{C}$. We leave the details to the reader. □

PROPOSITION 2.10. *Suppose* $\mathscr{L}$ *is a local system on a connected graph* Gr. *If* s, $s' \in \Gamma(\mathrm{Gr}, \mathscr{L})$ *agree at some point* $v_0 \in \mathrm{Ver(Gr)}$ *they are equal.*

A proof can be given following that of Proposition 2.4. Alternatively, the proposition actually follows from Proposition 2.4 using Lemma 2.9.

Definition 2.11. When Gr is finite and $\mathscr{L}$ is a metrized local system on Gr we define a definite hermitian inner product on $C^0(\mathrm{Gr}, \mathscr{L})$ by

$$\langle r, s \rangle_0 = \sum_{v \in \mathrm{Ver(Gr)}} \langle r(v), s(v) \rangle_v.$$

Definition 2.12. Let $\mathscr{L}$ be a local system on Gr. The star operator

$$S = S_{\mathscr{L}}: C^0(\mathrm{Gr}, \mathscr{L}) \to C^0(\mathrm{Gr}, \mathscr{L})$$

is the linear map associating to a zero cochain $r \in C^0(\mathrm{Gr}, \mathscr{L})$ the zero cochain $s = S_{\mathscr{L}}(r) \in C^0(\mathrm{Gr}, \mathscr{L})$ given by

$$s(v) = \sum_{e \in \mathrm{St}^0(v)} \mathscr{L}_e(r(o(e))).$$

Example 2.13. The star operator $S_{\mathscr{T}}$ for the trivial local system $\mathscr{T}$ is essentially the star operator on the graph [11]. Let $\{s_v\}_{v\in \mathrm{Ver(Gr)}}$ be the natural basis for $C^0(\mathrm{Gr}, \mathscr{T})$, with s_v the zero cochain $s_v(w) = \delta_{vw}$ (Kronecker δ) for any $v, w \in \mathrm{Ver(Gr)}$. In terms of this basis $S_{\mathscr{T}}$ is simply the *adjacency matrix* of Gr.

This example suggests that the eigenvalues of star operators on local systems might be interesting.

2.5. *One cochains, d, and δ*

We shall now define the group of one cochains $C^1(\mathrm{Gr}, \mathscr{L})$ of a (metrized) local system $\mathscr{L}$ on a graph Gr. This will require additional notation. For each $e\in \mathrm{Ed}^0(\mathrm{Gr})$ set $\mathscr{L}(e) = \mathscr{L}(o(e))$ (with inner product $\langle\, ,\rangle_e = \langle\, ,\rangle_{o(e)}$). Now define linear isomorphisms (isometries) $\mathscr{L}_{o,e}: \mathscr{L}(e) \to \mathscr{L}(o(e))$, $\mathscr{L}_{t,e}: \mathscr{L}(e) \to \mathscr{L}(t(e))$, and $\mathscr{L}_{\bar{e},e}: \mathscr{L}(e) \to \mathscr{L}(\bar{e})$ by $\mathscr{L}_{o,e} = \mathrm{Id}$, $\mathscr{L}_{t,e} = \mathscr{L}_e$ and $\mathscr{L}_{\bar{e},e} = \mathscr{L}_e$ for all $e\in \mathrm{Ed}^0(\mathrm{Gr})$. Also set $\mathscr{L}_{e,o} = \mathscr{L}_{o,e}^{-1}$ and $\mathscr{L}_{e,t} = \mathscr{L}_{t,e}^{-1}$. These maps satisfy obvious compatibilities, for example, $\mathscr{L}_{e,\bar{e}} = \mathscr{L}_{\bar{e},e}^{-1}$, $\mathscr{L}_{o,\bar{e}} \circ \mathscr{L}_{\bar{e},e} = \mathscr{L}_{t,e}$, and $\mathscr{L}_e \mathscr{L}_{o,e} = \mathscr{L}_{t,e}$ for all $e\in \mathrm{Ed}^0(\mathrm{Gr})$. Such data are natural for maps of local systems on Gr.

Definition 2.14. Let $\mathscr{L}$ be a local system on Gr. With notation as above, a one cochain s of $\mathscr{L}$ is a collection of $\{s(e)\}_{e\in \mathrm{Ed}^0(\mathrm{Gr})}$ with $s(e)\in \mathscr{L}(e)$ satisfying $s(\bar{e}) = -\mathscr{L}_{\bar{e},e}(s(e))$.

The set $C^1(\mathrm{Gr}, \mathscr{L})$ of one cochains is a subspace of $\Pi_{e\in \mathrm{Ed}^0(\mathrm{Gr})} \mathscr{L}(e)$. If Gr is finite and $\mathscr{L}$ has rank n, then $\dim C^1(\mathrm{Gr}, \mathscr{L}) = n|\mathrm{Ed(Gr)}|$. In this case and assuming $\mathscr{L}$ is metrized, there is a definite hermitian inner product $\langle\, ,\rangle_1$ on $C^1(\mathrm{Gr}, \mathscr{L})$ given by

$$\langle r, s\rangle_1 = \frac{1}{2} \sum_{e\in \mathrm{Ed}^0(\mathrm{Gr})} \langle r(e), s(e)\rangle_e. \tag{1}$$

In fact, if we choose an orientation on the edges—for each $e\in \mathrm{Ed(Gr)}$ we choose an $\tilde{e}\in \mathrm{Ed}^0(\mathrm{Gr})$ so that $\{\tilde{e}, \bar{\tilde{e}}\} = e$—we have also $\langle r, s\rangle_1 = \sum_{e\in \mathrm{Ed(Gr)}} \langle r(\tilde{e}), s(\tilde{e})\rangle_{\tilde{e}}$, because

$$\langle r(\bar{\tilde{e}}), s(\bar{\tilde{e}})\rangle_{\bar{\tilde{e}}} = \langle -\mathscr{L}_{\bar{\tilde{e}},\tilde{e}}(r(\tilde{e})),\ -\mathscr{L}_{\bar{\tilde{e}},\tilde{e}}(s(\tilde{e}))\rangle_{\bar{\tilde{e}}} = \langle r(\tilde{e}), s(\tilde{e})\rangle_{\tilde{e}}$$

since $\mathscr{L}_{\bar{\tilde{e}},\tilde{e}}$ is unitary.

Definition 2.15. (1) Let $\mathscr{L}$ be a local system on a graph Gr. Define the *coboundary operator* $d: C^0(\mathrm{Gr}, \mathscr{L}) \to C^1(\mathrm{Gr}, \mathscr{L})$ by

$$ds(e) = \mathscr{L}_{e,t}(s(t(e))) - \mathscr{L}_{e,o}(s(o(e)))$$

(that ds belongs to $C^1(\mathrm{Gr}, \mathscr{L})$ is a straightforward check).

(2) Suppose Gr is locally finite. Define $\delta: C^1(\mathrm{Gr}, \mathscr{L}) \to C^0(\mathrm{Gr}, \mathscr{L})$ by

$$\delta s(v) = \sum_{e\in \mathrm{St}^0(v)} \mathscr{L}_{t,e}(s(e)).$$

Maps on Gr-local systems $\varphi: \mathscr{L} \to \mathscr{L}'$ induce natural maps $\varphi_*^i: C^i(\mathrm{Gr}, \mathscr{L}) \to C^i(\mathrm{Gr}, \mathscr{L}')$ commuting with d and δ (i.e. $d_{\mathscr{L}'}\varphi_*^0 = \varphi_*^1 d_{\mathscr{L}}$ and $d_{\mathscr{L}'}\varphi_*^1 = \varphi_*^0 \delta_{\mathscr{L}}$). Likewise, if $f: \mathrm{Gr}' \to \mathrm{Gr}$ is a map of graphs and $\mathscr{L}$ is a local system on Gr we get natural maps $f^*: C^i(\mathrm{Gr}, \mathscr{L}) \to C^i(\mathrm{Gr}', f^*\mathscr{L})$ commuting with d. We leave the details to the reader.

Remark 2.16. The "additional notation" $\mathscr{L}(e)$, $\mathscr{L}_{e,\bar{e}}$, $\mathscr{L}_{o,e}$, and $\mathscr{L}_{t,e}$ introduced amounts to one natural choice (out of several) of an "enhancement" of $\mathscr{L}$ to a local system on the barycentric subdivision of Gr. An alternative approach would be to include this extra data in the definition of a local system.

The first proposition following is clear, and the second follows from a straightforward computation.

PROPOSITION 2.17. *The kernel of d on $C^0(\mathrm{Gr}, \mathscr{L})$ is $\Gamma(\mathrm{Gr}, \mathscr{L})$.*

PROPOSITION 2.18. *Let $\mathscr{L}$ be a metrized local system over a finite graph* Gr. *Then d and δ are adjoint operators. In other words, for $r \in C^0(\mathrm{Gr}, \mathscr{L})$ and $s \in C^1(\mathrm{Gr}, \mathscr{L})$ we have $\langle dr, s\rangle_1 = \langle r, \delta s\rangle_0$.*

Definition 2.19. Let $\mathscr{L}$ be a local system on a locally finite graph Gr. The laplacian $\square$ on zero cochains is the map $\square = \delta d\colon C^0(\mathrm{Gr}, \mathscr{L}) \to C^0(\mathrm{Gr}, \mathscr{L})$.

Recall that $S = S_{\mathscr{L}}$ denotes the star operator on $C^0(\mathrm{Gr}, \mathscr{L})$ and κ_v is the valence of a vertex $v \in \mathrm{Ver}(\mathrm{Gr})$.

PROPOSITION 2.20. *With* Gr *locally finite and $\mathscr{L}$ as above we have*

$$\square s(v) = \kappa_v s(v) - Ss(v)$$

for any $s \in C^0(\mathrm{Gr}, \mathscr{L})$ and $v \in \mathrm{Ver}(\mathrm{Gr})$.

Proof. We have

$$\begin{aligned}\square s(v) = \delta ds(v) &= \sum_{e \in \mathrm{St}^0(v)} \mathscr{L}_{t,e}(ds(e)) \\ &= \sum_{e \in \mathrm{St}^0(v)} [s(v) - \mathscr{L}_{t,e}(s(o(e)))] = \kappa_v s(v) - Ss(v)\end{aligned}$$

as claimed. $\square$

2.6. The spectrum of the laplacian on a metrized local system and the Ramanujan property

Throughout this section we will assume that Gr is a finite graph and that $\mathscr{L}$ is a metrized local system on Gr.

Definition 2.21. The *spectrum* of $\mathscr{L}$ is the set of eigenvalues of $\square = \square_{\mathscr{L}}$.

It is clear that the spectrum of a direct sum of local systems is the union of their spectra. Also the spectrum of a local system on a disconnected graph is the union of the spectra of the restrictions to the connected components. Therefore, in studying the spectra of local systems we may assume that the graph is connected and that the system is irreducible.

Set $\kappa = \kappa(\mathrm{Gr}) = \max_{v \in \mathrm{Ver}(\mathrm{Gr})} \kappa_v$.

LEMMA 2.22. (1) *The operator norm of $S = S_{\mathscr{L}}$ is bounded by κ.*
(2) *The operator norm of $\square = \square_{\mathscr{L}}$ is bounded by 2κ.*

Proof. For $s \in C^0(\mathrm{Gr}, \mathscr{L})$ we have

$$\begin{aligned}\|Ss\|_0^2 &= \sum_{v\in \mathrm{Ver(Gr)}} \|Ss(v)\|_v^2 = \sum_{v \in \mathrm{Ver(Gr)}} \left\| \sum_{e\in \mathrm{St}^0(v)} \mathscr{L}_e(s(o(e))) \right\|_v^2 \\ &\leqslant \sum_{v\in \mathrm{Ver(Gr)}} \kappa_v \sum_{e\in \mathrm{St}^0(v)} \|s(o(e))\|_{o(e)}^2 \\ &\leqslant \kappa \sum_{e\in \mathrm{Ed}^0(\mathrm{Gr})} \|s(o(e))\|_{o(e)}^2 \\ &= \kappa \sum_{v\in \mathrm{Ver(Gr)}} \kappa_v \|s(v)\|_v^2 \leqslant \kappa^2 \|s\|_0^2\end{aligned}$$

proving (1).

For (2) we use

$$\begin{aligned}\|\square s\|_0^2 &= \sum_{v\in \mathrm{Ver(Gr)}} \|\kappa_v s(v) - Ss(v)\|_v^2 \\ &\leqslant 2 \sum_{v\in \mathrm{Ver(Gr)}} \kappa_v^2 \|s(v)\|_v^2 + \|Ss(v)\|_v^2 \\ &\leqslant 2(\kappa^2 + \|S\|^2)\|s\|_0^2 \leqslant 4\kappa^2 \|s\|_0^2\end{aligned}$$

by part (1), showing (2). □

PROPOSITION 2.23. (1) *The laplacian* $\square = \square_{\mathscr{L}}$ *is a nonnegative operator, hence diagonalizable.*

(2) *The spectrum of* $\mathscr{L}$ *is contained in* $[0, 2\kappa]$.

(3) *The kernel of* $\square$ *is* $\Gamma(\mathrm{Gr}, \mathscr{L})$.

Proof. For $s \in C^0(\mathrm{Gr}, \mathscr{L})$ we have

$$\langle \square s, s\rangle_0 = \langle \delta ds, s\rangle_0 = \langle ds, ds\rangle_1 \geqslant 0.$$

Hence, $\square$ is a nonnegative operator, proving (1). Moreover, $\square s = 0$ if and only if $ds = 0$, establishing (3). By (1) the spectrum of $\mathscr{L}$ is contained in $[0, \infty)$ and the rest of part (2) follows from the previous lemma. □

Proposition 2.23(3) explains the lowest eigenvalue of $\square$. At the other extreme of the spectrum we have the following.

THEOREM 2.24. (1) *Assume that the graph* Gr *is connected,* κ*-regular, and bipartite. Let* $\mathrm{Ver(Gr)} = \mathrm{Ver}^0 \coprod \mathrm{Ver}^1$ *be a bipartition. For* $s \in C^0(\mathrm{Gr}, \mathscr{L})$, *define* $\bar{s} \in C^0(\mathrm{Gr}, \mathscr{L})$ *by* $\bar{s}(v) = (-1)^i s(v)$ *if* $v \in \mathrm{Ver}^i$. *Then* $s \mapsto \bar{s}$ *gives a bijection from the* $\kappa - \lambda$ *eigenspace of* $\square$ *to the* $\kappa + \lambda$ *eigenspace. In particular, the multiplicities of* $\kappa - \lambda$ *and* $\kappa + \lambda$ *in the spectrum of* $\mathscr{L}$ *are equal.*

(2) *Assume that* Gr *is connected and that the* 2κ*-eigenspace of* $\square$ *is nonzero. Then* Gr *is* κ*-regular and bipartite, and* $\Gamma(\mathrm{Gr}, \mathscr{L}) \neq 0$.

Proof. (1) If $\square s = (\kappa - \lambda)s$, $Ss = \lambda s$. Hence, for $v \in \mathrm{Ver}^i$ we have

$$\begin{aligned}S\bar{s}(v) &= \sum_{e\in \mathrm{St}^0(v)} \mathscr{L}_e(\bar{s}(o(e))) = (-1)^{i+1} \sum_{e\in \mathrm{St}^0(v)} \mathscr{L}_e(s(o(e))) \\ &= (-1)^{i+1} Ss(v) = (-1)^{i+1}\lambda s(v) = -\lambda \bar{s}(v)\end{aligned}$$

so $\square S(\bar{s}) = (\kappa - S)(\bar{s}) = (\kappa + \lambda)\bar{s}$, as required.

(2) If $\square s = 2\kappa s$, we get $2\kappa\|s\|_0^2 = \langle \square s, s\rangle_0 = \|ds\|_1^2$. Hence,

$$
\begin{aligned}
2\kappa\|s\|_0^2 &= \frac{1}{2}\sum_{e\in \mathrm{Ed}^0(\mathrm{Gr})} \|\mathscr{L}_{e,t}(s(t(e))) - \mathscr{L}_{e,o}(s(o(e)))\|_e^2 \\
&\leqslant \sum_{e\in \mathrm{Ed}^0(\mathrm{Gr})} \|\mathscr{L}_{e,t}(s(t(e)))\|_e^2 + \|\mathscr{L}_{e,o}(s(o(e)))\|_e^2 \\
&= 2\sum_{e\in \mathrm{Ed}^0(\mathrm{Gr})} \|s(t(e))\|_{t(e)}^2 = 2\sum_{v\in \mathrm{Ver}(\mathrm{Gr})} \kappa_v\|s(v)\|_v^2 \leqslant 2\kappa\|s\|_0^2. \qquad (2)
\end{aligned}
$$

Therefore, equality holds throughout. This implies that for all $e\in \mathrm{Ed}^0(\mathrm{Gr})$ we have

$$\mathscr{L}_{e,t}(s(t(e))) = -\mathscr{L}_{e,o}(s(o(e))) \quad \text{or equivalently, } \mathscr{L}_e(s(o(e))) = -s(t(e)). \qquad (3)$$

Now suppose $s(v_0) \neq 0$ for some $v_0 \in \mathrm{Ver}(\mathrm{Gr})$. If Gr were not bipartite there would be a path of odd length m beginning and ending at v_0. Applying (3) successively along this path we would get $s(v_0) = (-1)^m s(v_0) = -s(v_0)$, a contradiction.

Hence Gr is bipartite. Choose a bipartition $\mathrm{Ver}(\mathrm{Gr}) = \mathrm{Ver}^0 \coprod \mathrm{Ver}^1$ and set $r(v) = (-1)^i s(v)$ if $v\in \mathrm{Ver}^i$. From (3) we see that $0 \neq r \in \Gamma(\mathrm{Gr}, \mathscr{L})$. Finally, since Gr is connected, $r(v) \neq 0$ for all $v\in \mathrm{Ver}(\mathrm{Gr})$. The last inequality in (2) shows that $\kappa = \kappa_v$ for all $v\in \mathrm{Ver}(\mathrm{Gr})$. This completes the proof. □

Definition 2.25. Let Gr be a finite, κ-regular graph and let $\mathscr{L}$ be a metrized local system on Gr. Set

$$\mu(\mathscr{L}) = \mu(\mathrm{Gr}, \mathscr{L}) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of the star operator } S_{\mathscr{L}}, |\lambda| \neq \kappa\}.$$

We say that $\mathscr{L}$ is a Ramanujan local system if $\mu(\mathscr{L}) \leqslant 2\sqrt{\kappa - 1}$.

In case $\mathscr{L}$ is the trivial local system, Definition 2.25 coincides with the notion of a Ramanujan graph in [10]. In the irreducible nontrivial case *all* eigenvalues must satisfy the bound $2\sqrt{\kappa-1}$. The general case can be reduced to these two by Proposition 2.2. In the next section we will give examples of irreducible nontrivial Ramanujan local systems.

2.7. Local systems and equivariant cochains

Let V be a vector space over $\mathbf{C}$. For simplicity, we assume that V is finite dimensional. A representation (ρ, V) of a group Γ on V is a homomorphism $\rho : \Gamma \to \mathrm{GL}(V)$. If V is equipped with a hermitian inner product $\langle\ ,\ \rangle$, we say ρ is unitary if

$$\langle \rho(g)v_1, \rho(g)v_2\rangle = \langle v_1, v_2\rangle \quad \text{for any } g\in\Gamma \text{ and } v_1, v_2 \in V.$$

The notions of direct sums, subrepresentations, irreducibility, orthogonal complements, morphisms, and isomorphisms of representations are routinely defined. For example, a morphism $\phi : (\rho, V) \to (\rho', V')$ of representations of Γ is a linear map $\phi : V \to V'$ which intertwines the action: $\phi\rho(g) = \rho'(g)\phi$ for all $g\in\Gamma$.

Let (ρ, V) be a (unitary) representation of Γ and let X be a graph on which Γ acts freely, namely without fixed points and without reversing edges. The quotient $\mathrm{Gr} = \Gamma\backslash X$ is then a graph, and we define a (metrized) local system $\mathscr{L} = \Gamma\backslash(X \times (\rho, V))$ on Gr as follows. Choose lifts $\tilde{v}\in X$ for each $v\in \mathrm{Ver}(\mathrm{Gr})$. Let $p : X \to \mathrm{Gr}$ denote the projection.

1. For each $v\in \mathrm{Ver}(\mathrm{Gr})$ set $\mathscr{L}(v) = V$.
2. For $e\in \mathrm{Ed}^0(\mathrm{Gr})$ there exists a unique $\tilde{e}\in \mathrm{Ed}^0(X)$ satisfying $o(\tilde{e}) = \widetilde{o(e)}$ and $p(\tilde{e}) = e$.

There exists furthermore a unique $\gamma\in\Gamma$ such that $t(\tilde{e}) = \widetilde{\gamma t(e)}$. Set $\mathscr{L}_e = \rho(\gamma) : V \to V$.

In some cases this construction can be reversed. For example, suppose $\mathscr{L}$ is a local system on a connected graph Gr and $\Gamma = \pi_1(\mathrm{Gr}, v_0)$ is the fundamental group of Gr relative to some $v_0 \in \mathrm{Ver}(\mathrm{Gr})$. We define a representation $R(\mathscr{L}) = (\rho_{\mathscr{L}}, V_{\mathscr{L}})$ of $\pi_1(\mathrm{Gr}, v_0)$ as follows. Set $V_{\mathscr{L}} = \mathscr{L}(v_0)$. Let $[P]$ be the class in $\pi_1(\mathrm{Gr}, v_0)$ of an oriented path $P = \{e_1, \dots, e_n\}$ in Gr starting and ending at v_0. Then set $\rho_{\mathscr{L}}([P]) = \mathscr{L}_P(= \mathscr{L}_{e_n} \cdots \mathscr{L}_{e_1})$.

The assignment $R: \mathscr{L} \mapsto R(\mathscr{L}) = (\rho_{\mathscr{L}}, V_{\mathscr{L}})$ is clearly functorial. In fact, it is an equivalence of categories, reversing the previous construction:

Proposition 2.26. *Let X be the universal cover of* Gr. *Then*

1. $R(\pi_1(\mathrm{Gr}, v_0)\backslash(X \times (\rho, V))) = (\rho, V)$ *for any representation* (ρ, V) *of* $\pi_1(\mathrm{Gr}, v_0)$.
2. *Fix the liftings* $\tilde{v} \in \mathrm{Ver}(X)$ *of the vertices of* Ver(Gr). *Then there is a natural isomorphism* $\pi_1(\mathrm{Gr}, v_0)\backslash(X \times R(\mathscr{L})) \simeq \mathscr{L}$ *for any local system* $\mathscr{L}$ *on* Gr.

We leave the routine verification to the reader.

It follows tautologically from the proposition that the notions we have constructed for local systems can be associated to a representation of $\pi_1(\mathrm{Gr}, v_0)$ (when Gr is connected). Here is an example.

Proposition 2.27. *Let (ρ, V) be a representation of* $\pi_1(\mathrm{Gr}, v_0)$, *and let X be a universal cover of* Gr. *Put* $\mathscr{L} = \pi_1(\mathrm{Gr}, v_0)\backslash(X \times (\rho, V))$. *Then*

$$\Gamma(\mathrm{Gr}, \mathscr{L}) = V^{\rho} = \{v \in V \mid \rho(g)v = v \text{ for all } g \in \pi_1(\mathrm{Gr}, v_0)\}.$$

For general X and Γ it is possible to define the C^is, $i = 0, 1$; d; δ; and $\square$ of a metrized local system $\mathscr{L} = \Gamma\backslash(X \times (\rho, V))$ directly in terms of X, Γ, and (ρ, V). For this it is convenient to use the language of equivariant cochains (see [1]).

Definition 2.28. The spaces of equivariant i cochains ($i = 0, 1$) for X, Γ, and (ρ, V) are the spaces

$$\begin{aligned} C^0_\Gamma(\Delta, V) &= C^0_\Gamma(X, (\rho, V)) \\ &= \{f: \mathrm{Ver}(X) \to V \mid f(\gamma v) = \rho(\gamma) f(v) \text{ for all } v \in \mathrm{Ver}(X), \gamma \in \Gamma\} \\ C^1_\Gamma(\Delta, V) &= C^1_\Gamma(X, (\rho, V)) \\ &= \{f: \mathrm{Ed}^0(X) \to V \mid f(\bar{e}) = -f(e) \text{ and } f(\gamma e) = \rho(\gamma) f(e) \text{ for all } e \in \mathrm{Ed}^0(X), \\ &\qquad \gamma \in \Gamma\}. \end{aligned}$$

Define $d': C^0_\Gamma(\Delta, V) \to C^1_\Gamma(\Delta, V)$ by

$$d'f(e) = f(t(e)) - f(o(e))$$

(one checks that $d'f$ is in $C^1_\Gamma(\Delta, V)$). Similarly, if X is locally finite, the star operator $S': C^0_\Gamma(\Delta, V) \to C^0_\Gamma(\Delta, V)$ and $\delta': C^1_\Gamma(\Delta, V) \to C^0_\Gamma(\Delta, V)$ are defined by

$$S'f_0(v) = \sum_{e \in \mathrm{St}^0(v)} f_0(o(e)) \quad \text{and} \quad \delta' f_1(v) = \sum_{e \in \mathrm{St}^0(v)} f_1(e).$$

Assume that the quotient $\mathrm{Gr} = \Gamma\backslash X$ is a finite graph and that (ρ, V) is unitary. Choose representatives $\tilde{e} \in \mathrm{Ed}^0(\Delta)$ and $\tilde{v}$ for all $e \in \mathrm{Ed}^0(\mathrm{Gr})$ and $v \in \mathrm{Ver}(\mathrm{Gr})$. Define an inner product on $C^0_\Gamma(\Delta, V)$ by

$$\langle s, s'\rangle_0' = \sum_{v \in \mathrm{Ver}(\Delta)} \langle s(\tilde{v}), s'(\tilde{v})\rangle.$$

This is independent of choices. Likewise a well-defined inner product on $C^1_\Gamma(\Delta, V)$ is obtained by

$$\langle s, s'\rangle_1' = \frac{1}{2} \sum_{e \in \mathrm{Ed}^0(\mathrm{Gr})} \langle s(\tilde{e}), s'(\tilde{e})\rangle.$$

We have the routine adjointness relationship $\langle dr, s\rangle_1' = \langle r, \delta s\rangle_0'$ for all $r \in C^0_\Gamma(\Delta, V)$ and $s \in C^1_\Gamma(\Delta, V)$. Define $\square' = \square_0' = \delta' d'$.

Unsurprisingly, all these notions coincide with the corresponding ones from the previous sections. More precisely, for $\mathrm{Gr} = \Gamma\backslash X$ and $\mathscr{L} = \Gamma\backslash(X \times (\rho, V))$ define $\alpha^0 : C^0_\Gamma(\Delta, V) \to C^0(\mathrm{Gr}, \mathscr{L})$ by $\alpha^0(s)(v) = s(\tilde{v})$ for $s \in C^0_\Gamma(\Delta, V)$ and $v \in \mathrm{Ver}(\mathrm{Gr})$. Likewise define $\alpha^1 : C^1_\Gamma(\Delta, V) \to C^1(\mathrm{Gr}, \mathscr{L})$ by $\alpha^1(s)(e) = s(\tilde{e})$ for $s \in C^1_\Gamma(\Delta, V)$ and $e \in \mathrm{Ed}^0(\mathrm{Gr})$.

Proposition 2.29. *The maps α^i, $i = 0, 1$, are natural isomorphisms. One has $\alpha^1 d' = d\alpha^0$ and if* Gr *is locally finite also $\alpha^0\delta' = \delta\alpha^1$. If $\mathscr{L}$ is metrized and* Gr *is finite, the inner products $\langle\,,\rangle_0$ and $\langle\,,\rangle_0'$ correspond, as do the inner products $\langle\,,\rangle_1$ and $\langle\,,\rangle_1'$.*

The routine proof is left to the reader.

Corollary 2.30. *The two laplacians $\square$ and $\square'$ correspond. In particular, they have the same spectrum. The same holds for the corresponding star operators S and S'.*

3. THE EXAMPLES

In this section we will construct examples of graphs Gr and for each integer $r \geqslant 0$ an irreducible Ramanujan local system $\mathscr{L}_r$ of rank $r + 1$ on Gr. If $r = 0$, $\mathscr{L}_r$ is trivial, but not otherwise. Our graphs will include as special cases the graphs of [10].

We shall now briefly recall the construction in [10] and describe our $\mathscr{L}_r$'s in this case. Let $p \neq q$ be primes with $p \equiv q \equiv 1 \pmod 4$. Let H be the group $\mathrm{PGL}_2(\mathbf{F}_q)$ if the Legendre symbol $(\frac{p}{q})$ is -1, and $\mathrm{PSL}_2(\mathbf{F}_q)$ if $(\frac{p}{q}) = 1$. The graph Gr will be the Cayley graph of H for a specific set of generators, to be described below.

Consider representations of p as a sum of four squares of integers, $p = a^2 + b^2 + c^2 + d^2$, with $a > 0$ odd b, c, d even. To each such representation ξ associate the matrix

$$q(\xi) = \begin{bmatrix} z & -\bar{w} \\ w & \bar{z} \end{bmatrix} \quad \text{with } z = a + \sqrt{-1}c,\ w = b + \sqrt{-1}d.$$

Let Ξ be the set of all such ξ's. Fix some $\sqrt{-1} \in \mathbf{F}_q$, and let $\bar{q}(\xi)$ be the resulting image of $q(\xi)$ in $\mathrm{PGL}_2(\mathbf{F}_q)$. By our assumption $\bar{q}(\xi) \in H$. The graphs of [10] are the Cayley graphs $C(H, \Xi)$ of H relative to the (symmetric) set of generators Ξ. Thus, the vertices of $C(H, \Xi)$ are $\{v_h^{\mathrm{LPS}}\}_{h \in H}$, and the oriented edges of $C(H, \Xi)$ are $\{e_{\xi,h}^{\mathrm{LPS}}\}$ for $\xi \in \Xi$ and $h \in H$. One sets $\overline{e_{h,\xi}^{\mathrm{LPS}}} = e_{\bar{q}(\xi)h,\bar{\xi}}^{\mathrm{LPS}}$, where $\bar{\xi} = (a, -b, -c, -d)$ is the conjugate representation of p if $\xi = (a, b, c, d)$. We have $o(e_{h,\xi}^{\mathrm{LPS}}) = v_h^{\mathrm{LPS}}$ and $t(e_{h,\xi}^{\mathrm{LPS}}) = v_{\bar{q}(\xi)h}^{\mathrm{LPS}}$. To define the systems $\mathscr{L}_r$ notice that $q'(\xi) = p^{-1/2}q(\xi)$ is a unitary matrix for each $\xi \in \Xi$. Consider $\mathbf{C}^{r+1}$ as the space of homogeneous polynomials of degree r in two variables x, y. Then $q'(\xi)$ acts on $\mathbf{C}^2$, hence on $\mathbf{C}^{r+1}$. Up to a scalar there is unique definite hermitian inner product on $\mathbf{C}^{r+1}$ such that all the resulting $(r+1)\times(r+1)$ matrices $\mathrm{Symm}^r q'(\xi)$ are unitary. We set $\mathscr{L}_r(v) = \mathbf{C}^{r+1}$ with this inner product for all $v \in \mathrm{Ver}\, C(H, \Xi)$. For $e = e_{h,\xi}^{\mathrm{LPS}}$ set $\mathscr{L}_{r,e} = \mathrm{Symm}^r q'(\xi)$.

In the following we shall first describe a general construction of graphs with local systems coming from the $(p+1)$-regular tree and a discrete subgroup of $\mathrm{SL}_2(\mathbf{Q}_p)$. We shall construct examples of (a slight generalization of) such graphs using the arithmetic of quaternions algebras over $\mathbf{Q}$. Then we shall show that the graphs of [10] are special cases of ours and that our local systems have the Ramanujan property.

3.1. Graphs on local systems attached to discrete subgroups of $\mathrm{SL}_2(\mathbf{Q}_p)$

We shall now describe certain κ-regular graphs and local systems on them, where $\kappa = p+1$ with p a prime. There is a well-known description of the κ-regular tree $\Delta = \Delta_\kappa$ as a homogeneous space for $G_p = \mathrm{GL}_2(\mathbf{Q}_p)$ (see [11] for details). Let $K_p = \mathrm{GL}_2(\mathbf{Z}_p)$, let $\mathbf{Z}_p$ be the subgroup of scalar matrices in G_p, and let

$$I_p = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in K \mid c \equiv 0 \pmod p \right\}$$

be the standard Iwahori subgroup of K_p. Then $\mathrm{Ver}(\Delta) = G_p/K_pZ_p$ and $\mathrm{Ed}^0(\Delta) = G_p/I_pZ_p$. The element

$$\beta_p = \begin{bmatrix} 0 & 1 \\ p & 0 \end{bmatrix} \in G_p$$

normalizes I_p, and for an oriented edge $e = gI_pZ_p \in \mathrm{Ed}^0(\Delta)$ one has $\bar{e} = g\beta_p I_p Z_p$, $o(e) = gK_pZ_p$, and $t(e) = g\beta_p K_p Z_p$.

Let Γ be a discrete subgroup of $\mathrm{SL}_2(\mathbf{Q}_p)$. The graph Δ is bipartite with bipartition $\mathrm{Ver}(\Delta) = \mathrm{Ver}^0(\Delta) \coprod \mathrm{Ver}^1(\Delta)$ given by

$$\mathrm{Ver}^i(\Delta) = \{gKZ \in \mathrm{Ver}(\Delta) \mid \mathrm{val}_p(\det g) \equiv i \pmod 2\},$$

where val_p is the p-adic valuation and $i = 0, 1$. Since the $\mathrm{SL}_2(\mathbf{Q}_p)$-action preserves each $\mathrm{Ver}^i(\Delta)$, the quotient graph $\mathrm{Gr} = \Gamma\backslash\Delta$ is bipartite as well.

Next choose an embedding $\mathbf{Q}_p \hookrightarrow \mathbf{C}$. Then Γ acts on $\mathbf{C}^2$ via the representation $\rho : \Gamma \hookrightarrow \mathrm{SL}_2(\mathbf{C})$. Hence, Γ acts on $\mathbf{C}^{r+1} \simeq \mathrm{Symm}^r \mathbf{C}^2$ via its action on homogeneous degree r polynomials on the dual $\mathbf{C}^2$. This gives representations $\rho_r : \Gamma \to \mathrm{SL}_{r+1}(\mathbf{C})$. If Γ acts freely on Δ we get local systems $\mathscr{L}_r = \Gamma\backslash(\Delta \times \mathbf{C}^{r+1})$ on Gr. Here $\mathscr{L}_0$ is the trivial local system.

When Γ is co-compact in $\mathrm{SL}_2(\mathbf{Q}_p)$, two things happen. Firstly, Gr is a finite graph, because its set of vertices $\Gamma\backslash G_p/K_pZ_p$ is compact and discrete and hence finite, and similarly its set of edges is finite. In addition, $\rho(\Gamma)$ is Zariski dense in $\mathrm{SL}_2(\mathbf{Q}_p)$, because otherwise its Zariski closure would be finite, or contained in a normalizer of a torus, or contained in a Borel subgroup. Co-compactness then gives that this Zariski closure must be a Borel subgroup, but a Borel subgroup has no discrete co-compact subgroups because of its unipotent radical. Notice that when $\rho(\Gamma)$ is Zariski dense in $\mathrm{SL}_2(\mathbf{Q}_p)$, then each ρ_r, and hence each $\mathscr{L}_r$, is irreducible.

To construct such Γs we now recall the notion of quaternion algebras.

3.2. Quaternion algebras

A quaternion algebra over a field F is a four-dimensional associative F-algebra with 1 whose center is F (see [13]). The standard example is $\mathrm{Mat}_{2\times 2}(F)$. If B is quaternion algebra over F, then $B_{F'} = B \otimes_F F'$ is a quaternion algebra over F' for any field extension F'/F. If $\bar{F}$ is an algebraic closure of F, then $B_{\bar{F}}$ is isomorphic to $\mathrm{Mat}_{2\times 2}(\bar{F})$, so that B can be embedded into $\mathrm{Mat}_{2\times 2}(\bar{F})$. The reduced trace $\mathrm{Tr}\, b$ and reduced norm $\mathrm{Nm}\, b$ of an element $b \in B$ are the trace and determinant of the corresponding element in $\mathrm{Mat}_{2\times 2}(\bar{F})$. They are well defined and F-valued, with Tr F-linear and Nm multiplicative. The conjugate of a quaternion q is $q^* = \mathrm{Tr}(q) - q$. It satisfies $q^*q = qq^* = \mathrm{Nm}(q)$ and $(q_1q_2)^* = q_2^*q_1^*$.

From now on B is a quaternion algebra over $\mathbf{Q}$. We say that B is definite if $B \otimes \mathbf{R}$ is isomorphic to the Hamilton quaternions $\mathbf{H}$. We say that B is ramified at a prime ℓ if $B \otimes \mathbf{Q}_\ell \not\simeq \mathrm{Mat}_{2\times 2}(\mathbf{Q}_\ell)$. The set of ramified primes is finite, and their product is the discriminant $\mathrm{Disc}\, B$ of B. Any square-free integer can be the discriminant of a quaternion algebra over $\mathbf{Q}$, and the isomorphism type of B is uniquely determined by its discriminant. The number of ramified primes is odd if and only if B is definite.

An order in B is a subring with 1 of B which is a free abelian group of rank 4. Each order is contained in a maximal order.

Example 3.1. Let d be the product of m distinct primes $p_1, \dots, p_m$, all congruent to 3 modulo 4. Define an algebra $B = B(-1, -d)$ over $\mathbf{Q}$ with basis $1 = \hat{1}, \hat{\imath}, \hat{\jmath}, \hat{k}$ satisfying $\hat{\jmath}^2 = -1$, $\hat{\imath}^2 = \hat{k}^2 = -d$, $\hat{\imath}\hat{\jmath} = -\hat{\jmath}\hat{\imath} = \hat{k}$, and $1x = x1 = x$ for all $x \in B$. The Hilbert symbol criterion says that B is ramified at ℓ if and only if the Hilbert symbol $(-1, -d)_\ell$ is -1. Using it one checks that $\mathrm{Disc}\, B = d$ or $2d$ according to whether m is odd or even. The algebra B is definite for each d:

$$B(-1, -d) \otimes \mathbf{R} \simeq B(-1, -1) \otimes \mathbf{R} \simeq \mathbf{H}.$$

Viewing $\mathbf{Q}(\hat{\imath}) \simeq \mathbf{Q}(\sqrt{-d})$ as a subfield of $\mathbf{C}$, an embedding of B into $\mathrm{Mat}_{2\times 2}(\mathbf{C})$ is given by

$$q = a_1 + a_2\hat{\imath} + a_3\hat{\jmath} + a_4\hat{k} \mapsto \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix} \quad \text{with } z = a_1 + a_2\hat{\imath} \text{ and } w = a_3 + a_4\hat{\imath}.$$

We have $\mathrm{Tr}\, q = 2a_1$ and $\mathrm{Nm}\, q = a_1^2 + a_3^2 + d(a_2^2 + a_4^2)$. Set $E = \mathbf{Q}(\hat{\imath}) \simeq \mathbf{Q}(\sqrt{-d})$. The ring of integers $\mathcal{O}_E$ is $\mathbf{Z}[\sqrt{-d}]$ if m is even and $\mathbf{Z}[(1 + \sqrt{-d})/2]$ if m is odd. Then $\mathcal{O}_E + \hat{\jmath}\mathcal{O}_E$ is an order, which is maximal if m is odd. If m is even a maximal order is obtained by adjoining to $\mathcal{O}_E + \hat{\jmath}\mathcal{O}_E$ the element $(1 + \hat{\imath} + \hat{\jmath} + \hat{k})/2$.

3.3. Graphs and local systems attached to definite quaternion algebras

Let B be a definite quaternion algebra over $\mathbf{Q}$, let $\mathcal{M} \subset B$ be a maximal order, and let $p \nmid \mathrm{Disc}\, B$ be a prime. Then $\mathcal{M}[p^{-1}] = \{mp^n \mid n \in \mathbf{Z}, m \in \mathcal{M}\}$ is a maximal $\mathbf{Z}[p^{-1}]$-order of B. If $N \geqslant 1$ is an integer prime to p, then the natural map

$$\mathcal{M}/N\mathcal{M} \to \mathcal{M}[p^{-1}]/N\mathcal{M}[p^{-1}]$$

is an isomorphism. If $(N, \mathrm{Disc}\, B) = 1$ then $\mathcal{M}/N\mathcal{M} \simeq \mathrm{Mat}_{2\times 2}(\mathbf{Z}/N\mathbf{Z})$.

Let $\bar{K}$ be the quotient of the group of units $(\mathcal{M}/N\mathcal{M})^\times$ by the image of $p^{\mathbf{Z}} \subset \mathcal{M}[p^{-1}]$, let $\gamma \mapsto \bar{\gamma}$ denote the reduction map $\Gamma = \mathcal{M}[p^{-1}]^\times \to \bar{K}$, and let $\Gamma[N] \subset \mathcal{M}[p^{-1}]^\times$ be the kernel of reduction. Embed B in $B \otimes \mathbf{Q}_p \simeq \mathrm{Mat}_{2\times 2}(\mathbf{Q}_p)$ and in $B \otimes \mathbf{R} \simeq \mathbf{H} \subset \mathrm{Mat}_{2\times 2}(\mathbf{C})$. Then Γ can be viewed as a subgroup of $\mathrm{GL}_2(\mathbf{Q}_p) \times \mathbf{H}^\times$, and hence it acts on Δ (via the first factor), on $\bar{K}$ (via reduction modulo N), on $\mathbf{C}^2$ (via the second factor), and on $\mathrm{Symm}^r \mathbf{C}^2 \simeq \mathbf{C}^{r+1}$. Set $\rho_r(\gamma) = (\mathrm{Nm}\,\gamma)^{-r/2}\mathrm{Symm}^r\gamma$. Then $(\rho_r, \mathbf{C}^{r+1})$ is a unitary representation for any $r \geqslant 0$. Viewing $\Delta \times \bar{K}$ as a disconnected graph, we now define a graph Gr and metrized local systems $\mathscr{L}_r$ ($r \geqslant 0$) on it as explained in Section 2.7:

1. $\mathrm{Gr} = \mathrm{Gr}(B, N, p) = \Gamma \backslash (\Delta \times \bar{K})$,
2. $\mathscr{L}_r = \Gamma \backslash (\Delta \times \bar{K} \times (\rho_r, \mathbf{C}^{r+1}))$.

For the last formula to actually define a local system on Gr, the following condition is necessary and sufficient:

($*$) If an element $\gamma \in \Gamma$ does not act freely on $\Delta \times \bar{K}$, it acts as the identity on $\Delta \times \bar{K} \times \mathbf{C}^{r+1}$.

We therefore assume this condition is satisfied in what follows. Here is a way to insure this. Suppose that $\gamma \in \Gamma$ fixes a point of $\Delta \times \bar{K}$. Then some positive power of γ lies in the center and γ maps trivially to $\bar{K}$, so that γ is a power of p modulo $N\mathscr{M}[p^{-1}]$. Since $p \in \Gamma$ acts trivially on $\Delta \times \bar{K} \times \mathbf{C}^{r+1}$, we may assume, after replacing γ by its product with a power of p, that $\gamma \equiv 1$ modulo N. Now assume $N \geqslant 3$, and choose a prime q so that a power $q^t \geqslant 3$ divides N. In the q-adic topology the exponential function converges on $N\mathscr{M} \otimes \mathbf{Z}_q$ and maps it bijectively onto $1 + N\mathscr{M} \otimes \mathbf{Z}_q$. It follows that γ is in the center, so $\gamma = \pm$ (a power of p). Only the case $\gamma = -1$ needs to be considered, and -1 fails to satisfy condition (*) if and only if r is odd but -1 is a power of p modulo N. In conclusion, we get a local system if $N \geqslant 3$ and if r is even if -1 is a power of p modulo N.

3.4. *First properties of the arithmetic graphs* Gr(B, N, p)

In this section we will relate more closely the graphs $\mathrm{Gr} = \mathrm{Gr}(B, N, p)$ to the graphs of Section 2.1. We will identify their connected components and give a criterion for them to be bipartite. Set $\Gamma^+ = \{\gamma \in \Gamma \mid \mathrm{Nm}\,\gamma = 1\}$. Put $\Gamma^+[N] = \{\gamma \in \Gamma^+ \mid \gamma \equiv 1 \ (\mathrm{mod}\ N\mathscr{M}[p^{-1}])\}$.

LEMMA 3.2. (1) *If p has even order in $(\mathbf{Z}/N\mathbf{Z})^\times$ then $p^{\mathbf{Z}}\Gamma^+[N] = \Gamma[N]$.*
(2) *If p has odd order in $(\mathbf{Z}/N\mathbf{Z})^\times$ then $p^{\mathbf{Z}}\Gamma^+[N]$ is of index 2 in $\Gamma[N]$.*

Proof. Clearly, $p^{\mathbf{Z}}\Gamma^+[N]$ is the subgroup of elements of $\Gamma[N]$ whose norm has even p-adic valuation. Hence, its index in $\Gamma[N]$ is at most 2, and is 2 if and only if there is an element γ in $\Gamma[N]$ of odd p-adic valuation, say $2j - 1$. Then $\gamma \equiv p^m (\mathrm{mod}\, N\mathscr{M}[p^{-1}])$ for some integer m since γ reduces to 1 in $\bar{K}$. Taking norms gives $p^{2j-1} \equiv p^{2m} (\mathrm{mod}\, N)$ so the order of p modulo N is odd. Conversely, suppose $p^{2j-1} \equiv 1\,(\mathrm{mod}\, N)$ and let $\gamma \in \Gamma$ have norm p. (It is known such an element exists: see, for example, [9, Ch. 3.1].) Then $\mathrm{Nm}(p^{-j}\gamma) \equiv 1\,(\mathrm{mod}\, N)$. By the Eichler–Kneser strong approximation theorem there exists an element $\gamma_1 \in \Gamma^+$ satisfying $\gamma_1 \equiv p^{-j}\gamma (\mathrm{mod}\, N\mathscr{M}[p^{-1}])$, so that $\gamma_1^{-1}\gamma \in \Gamma[N]$. Since $\mathrm{Nm}(\gamma_1^{-1}\gamma) = \mathrm{Nm}(\gamma)$ has odd p-adic valuation, we are done. □

The group $\bar{K}$ acts on $\mathrm{Gr} = \Gamma \backslash (\Delta \times \bar{K})$ by right multiplication on the second factor. Since the norms of elements in Γ are powers of p, it follows that sending $(x, k) \in \Delta \times \bar{K}$ to $\mathrm{Nm}(k) \in (\mathbf{Z}/N\mathbf{Z})^\times/\langle p \rangle$ induces a well-defined map, still denoted Nm, from Gr to $(\mathbf{Z}/N\mathbf{Z})^\times/\langle p \rangle$.

PROPOSITION 3.3. (1) *The connected components of* Gr *are precisely the fibres of* Nm: $\mathrm{Gr} \to (\mathbf{Z}/N\mathbf{Z})^{\times}/\langle p\rangle$, *and* $\bar{K}$ *permutes them via its natural action. In particular, they are all isomorphic.*

(2) *Suppose p has even (respectively odd) order in* $(\mathbf{Z}/N\mathbf{Z})^{\times}$. *Then each component of* Gr *is bipartite (respectively not bipartite) and isomorphic to* $\Gamma^{+}[N]\backslash\Delta$ *(respectively* $\Gamma[N]\backslash\Delta$*).*

Proof. The connected components of Gr are the images modulo Γ of $\Delta \times \bar{\Gamma}k_0$ for any $k_0 \in \bar{K}$, and the reduction $\bar{\Gamma}$ of Γ consists of all elements of norm 1 in $(\mathbf{Z}/N\mathbf{Z})^{\times}/\langle p\rangle$. This proves the first part of (1). The rest of part (1) is clear. For part (2) notice that the connected component of Gr which is covered by $\Delta \times \{1\} \subset \Delta \times \bar{K}$ is isomorphic to $\Gamma[N]\backslash\Delta$. If p has even order in $(\mathbf{Z}/N\mathbf{Z})^{\times}$, then it is also isomorphic to $\Gamma^{+}[N]p^{\mathbf{Z}}\backslash\Delta$ or $\Gamma^{+}[N]\backslash\Delta$ since $p^{\mathbf{Z}}$ acts trivially on Δ. As was explained in Section 3.1 the graph $\Gamma^{+}[N]\backslash\Delta$ is bipartite. On the other hand, if p has odd order modulo N then $\Gamma[N]\backslash\Delta$ cannot be bipartite: if it were we could lift the bipartition to a bipartition of Δ which is preserved by $\Gamma[N]$. The bipartition on a connected graph being unique (up to order), this lifted bipartition would be the same as the one in Section 3.1. But an element of $\mathrm{GL}_2(\mathbf{Q}_p)$ whose determinant has odd p-adic valuation does not preserve this bipartition. Since such elements exist in $\Gamma[N]$ we get a contradiction, completing the proof of part (2).

3.5. The local system $\mathscr{L}_r$

In this section we will prove the following:

THEOREM 3.4. (1) $\mathscr{L}_r$ *is irreducible for each* $r \geqslant 0$ *and nontrivial for* $r \geqslant 1$.

(2) $\mathscr{L}_r$ *satisfies the Ramanujan property for each* $r \geqslant 0$.

Proof. (1) By the Eichler–Kneser strong approximation theorem Γ is Zariski dense in $\mathrm{GL}_2(\mathbf{C})$. By the standard representation theory of GL_2, each representation Symm^r of Γ is irreducible. Therefore the ρ_r, and hence $\mathscr{L}_r$, are irreducible.

(2) Fix $r \geqslant 0$. We shall prove the Ramanujan property in a series of steps. Starting with the action of $S = S_{\mathscr{L}_r}$ on $C^0(\mathrm{Gr}, \mathscr{L}_r)$, we will describe in each step a certain space (or spaces) and an operator on it, and show that any eigenvalue of the previous operator is also an eigenvalue of the new one. Eventually, this will relate the eigenvalues of S on $C^0(\mathrm{Gr}, \mathscr{L}_r)$ to the eigenvalues of the Hecke operator T_p on cusp forms of weight $r+2$ and level dividing $N \operatorname{Disc} B$. The standard bounds for these eigenvalues (due to Eichler, Shimura, Weil, Igusa, and Deligne) will then imply the Ramanujan property.

Step 1: *The local Hecke operator.* The local system $\mathscr{L}_r$ on $\mathrm{Gr} = \Gamma\backslash(\Delta \times \bar{K})$ corresponds to the representation ρ_r. By Corollary 2.30 the eigenvalues of S on $C^0(\mathrm{GR}, \mathscr{L}_r)$ are the same as those of S' on $C^0_{\Gamma}(\Delta \times \bar{K}, (\rho_r, \mathbf{C}^{r+1}))$. Let us write the set of elements in $\mathscr{M}_p = \mathscr{M} \otimes \mathbf{Z}_p$ whose norm has p-adic valuation 1 as a disjoint union of left classes $\coprod_{\ell} \alpha_{\ell}\mathscr{M}_p^{\times}$. It is well known that there are $p+1$ such classes (see, e.g. [9, Ch. 1.2]). Moreover, under the identification $\mathrm{Ver}(\Delta) \simeq B_p^{\times}/Z_p\mathscr{M}_p^{\times}$, the vertices adjacent to $v = g\mathscr{M}_p^{\times}Z_p$ are precisely $\{g\alpha_{\ell}\mathscr{M}_p^{\times}Z_p\}_{\ell}$. It follows that $S'\phi(g\mathscr{M}_p^{\times}Z_p, k) = \sum_{\ell}\phi(g\alpha_{\ell}\mathscr{M}_p^{\times}Z_p, k)$ for any $k \in \bar{K}$, $g \in B_p^{\times}$, and $\phi \in C^0_{\Gamma}(\Delta \times \bar{K}, (\rho_r, \mathbf{C}^{r+1}))$. In general, let V be any space on which $B_p^{\times}$ acts and let $V^{\mathscr{M}_p^{\times}}$ denote the space of $\mathscr{M}_p^{\times}$ invariants. Define the Hecke operator $\tilde{T}_p = \tilde{T}_{p,V}: V^{\mathscr{M}_p^{\times}} \to V^{\mathscr{M}_p^{\times}}$ by $\tilde{T}_p v = \sum_{\ell}\alpha_{\ell}v$ for any $v \in V^{\mathscr{M}_p^{\times}}$. This is well defined, and we see that $S' = \tilde{T}_{p,V}$ for $V = C^0_{\Gamma}(\Delta \times \bar{K}, (\rho_r, \mathbf{C}^{r+1}))$.

Step 2: *Adèlization.* Let $\hat{\mathbf{Z}} = \prod_{q \text{ prime}} \mathbf{Z}_q$ be the profinite completion of $\mathbf{Z}$ and $\mathbf{Z}^p = \prod_{q \neq p} \mathbf{Z}_q$. Set $\hat{\mathscr{M}} = \mathscr{M} \otimes \hat{\mathbf{Z}}$, $\hat{\mathscr{M}}^p = \mathscr{M} \otimes \mathbf{Z}^p$. Let $\mathbf{A}$ denote the adèles of $\mathbf{Q}$, $\mathbf{A}^f$ the finite adèles, and $\mathbf{A}^{f,p}$ the adèles without the p component. Set $B_{\mathbf{A}}^{\times} = (B \otimes \mathbf{A})^{\times}$, $B^{\times,f} = (B \otimes \mathbf{A}^{f})^{\times}$, $B^{\times,f,p} = (B \otimes \mathbf{A}^{f,p})^{\times}$. For an element $b \in B^{\times}$ denote by $b_{\mathbf{A}}$, b^{f}, $b^{f,p}$, b_p, and b_{∞}, respectively, its images in $B_{\mathbf{A}}^{\times}$, $B^{\times,f}$, $B^{\times,f,p}$, B_p, and $B_{\infty}^{\times} = (B \otimes \mathbf{R})^{\times}$. Then $b_{\mathbf{A}} = b^{f} b_{\infty} = b^{f,p} b_p b_{\infty}$. Let $\mathscr{B}_r(B^{\times})$ denote the space of continuous maps $\phi : B^{\times,f} \to \mathbf{C}^{r+1}$ satisfying $\phi(b^{f}x) = \rho_r(b_{\infty})\phi(x)$ for any $b \in B^{\times}$ and $x \in B^{\times,f}$. The group $B^{\times,f}$ acts on $\mathscr{B}_r(B^{\times})$ by right translation: $x\phi(y) = \phi(yx)$ for $x, y \in B^{\times,f}$ and $\phi \in \mathscr{B}_r(B^{\times})$, so that $\tilde{T}_p\phi(x) = \sum_{\ell} \phi(x\alpha_{\ell})$ for $\phi \in \mathscr{B}_r(B^{\times})^{\mathscr{M}_p^{\times}}$, $x \in B^{\times,f}$. Let $U(N) \subset \hat{\mathscr{M}}^{\times}$ be the kernel of reduction to $\bar{K}$, defined by

$$\hat{\mathscr{M}}^{\times} \to (\hat{\mathscr{M}}/N\hat{\mathscr{M}})^{\times} \simeq (\mathscr{M}/N\mathscr{M})^{\times} \to \bar{K}.$$

Then $\tilde{T}_p$ maps $\mathscr{B}_r(B^{\times})^{U(N)}$ to itself. By the Eichler–Kneser strong approximation theorem the inclusion $B_p^{\times} \times \hat{\mathscr{M}}^{p,\times} \hookrightarrow B^{f}$ induces bijections

$$\Gamma\backslash(\mathrm{Ver}(\Delta) \times \bar{K}) \simeq \Gamma\backslash((B_p^{\times}/\mathscr{M}_p^{\times}Z_p) \times \bar{K}) \simeq \Gamma\backslash(B_p^{\times} \times \hat{\mathscr{M}}^{p,\times})/U(N) \simeq B^{\times}\backslash B^{\times,f}/U(N).$$

In the middle equality we can indeed drop the p-part Z_p of the center Z^{f}, since $p^{f} = p_p p^{f,p}$ is in Γ and $p^{f,p}$ is in $U(N)$. It follows that the restriction map $\mathrm{res} : \mathscr{B}_r(B^{\times})^{U(N)} \to C_{\Gamma}^{0}(\Delta \times \bar{K}, (\rho_r, \mathbf{C}^{r+1}))$ given by $\mathrm{res}\,\phi(g_p\mathscr{M}_p^{\times}Z_p, kU(N)) = \phi(g_p k)$ for any $\phi \in \mathscr{B}_r(B^{\times})^{U(N)}$, $g_p \in B_p^{\times}$, and $k \in \hat{\mathscr{M}}^{p,\times}$ is an isomorphism, and each eigenvalue of S is an eigenvalue of $\tilde{T}_p$ on $\mathscr{B}_r(B^{\times})^{U(N)}$.

Step 3: *Decomposition under the center.* Decomposing $\mathscr{B}_r(B^{\times})^{U(N)}$ to character spaces under the action of the center Z^{f} gives $\mathscr{B}_r(B^{\times})^{U(N)} = \oplus_{\omega}\mathscr{B}_r(B^{\times}, \omega)^{U(N)}$. Here ω ranges over the (complex-valued) characters ω of Z^{f} and $\mathscr{B}_r(B^{\times}, \omega)$ is the set of $\phi \in \mathscr{B}_r(B^{\times})$ satisfying $\phi(zx) = \omega(z)\phi(x)$ for any $z \in \mathbf{A}^{\times,f} \simeq Z^{f}$ and $x \in B^{\times,f}$. In fact the only characters ω which can participate are those of conductor dividing N which are trivial on $p^{f,p}$, because $p^{f,p}$ and the principal congruence subgroup of level N are in $U(N)$. Extend each ω to $\mathbf{A}^{\times}$ by setting $\omega(\mathbf{Q}^{\times}\mathbf{R}^{>0}) = 1$. Then ω is a uniquely defined Hecke character. The corresponding Dirichlet character $\varepsilon = \varepsilon_{\omega}$ (see, e.g. [7, Ch. 3.A]) has conductor dividing N and is trivial on p. Denote the set of such Hecke characters ω by $\Omega = \Omega(B^{\times}, N, p)$. Then it follows that each eigenvalue of S occurs as an eigenvalue of $\tilde{T}_p$ on $\mathscr{B}_r(B^{\times}, \omega)^{U(N)}$ for some $\omega \in \Omega$.

Step 4: *Automorphic forms on* $B^{\times}$. Let $\mathscr{A}(B^{\times}, \omega)$ be the space of automorphic forms on $B^{\times}$ with central character ω defined by Jacquet–Langlands [8, Definition 10.2, p. 330]. Since B is definite, the quotient $B^{\times}\backslash B_{\mathbf{A}}^{\times}/Z_{\mathbf{A}}$ is compact. This implies that $\mathscr{A}(B^{\times}, \omega)$ is the space of continuous maps $\psi : B_{\mathbf{A}}^{\times} \to \mathbf{C}$ which are left $B^{\times}$-invariant and right $B_{\infty}^{\times}$-finite, satisfying $\psi(zx) = \omega(z)\psi(x)$ for any $x \in B_{\mathbf{A}}^{\times}$ and z in the center of $B_{\mathbf{A}}^{\times}$. In [9, Ch. 1.1] we showed how to realize explicitly $\mathscr{B}_r(B^{\times}, \omega)$ as a specific subspace of $\mathscr{A}(B^{\times}, \omega)$ characterized by conditions at ∞ (e.g. the ∞-type must be ρ_r). This realization is $B^{f,\times}$ equivariant, and it follows that eigenvalues of S appear as eigenvalues of $\tilde{T}_p$ on $\mathscr{A}_r(B^{\times}, \omega)^{U(N)}$, where $\mathscr{A}_r(B^{\times}, \omega)$ denotes the subspace of $\mathscr{A}(B^{\times}, \omega)$ characterized by ∞-type ρ_r.

Step 5: *Automorphic forms on* GL_2. The Eichler/Shimizu/Jacquet-Langlands/Arthur correspondence shows that an eigenvalue λ of $\tilde{T}_p$ on $\mathscr{A}(B^{\times}, \omega)^{\mathscr{M}_p^{\times}}$ either satisfies $\lambda^2 = (p+1)^2\omega(p_p)$, or is an eigenvalue of $\tilde{T}_p$ on the space $\mathscr{A}_0(\mathrm{GL}_{2/\mathbf{Q}}, \omega)^{K_p}$ of cusp forms on $\mathrm{GL}_2(\mathbf{A})$ with the same central character ω. This is essentially proven in [8, Ch. 16] (with some unproven analytic statements later supplied by Arthur). For details how this follows from [8], see [7, Ch. 10.B] or [9, Ch. 4.1]. We take $\omega \in \Omega$, so that $\omega(p) = 1$. The last reference discusses the case $\lambda^2 = (p+1)^2$ (loc. cit., Proposition 4.5 and the formulas preceding Theorem 4.8). These λ's we shall call the *banal* eigenvalues. Actually, a more precise result is proven: each nonbanal eigenvalue of $\tilde{T}_p$ on $\mathscr{A}_r(B^{\times}, \omega)^{U(N)}$ appears as the eigenvalue of $\tilde{T}_p$ on a cusp form ϕ on GL_2, whose ∞-type is the discrete series of lowest weight $r+2$ (see [7, 10.7] or [9, Theorem 4.1 and Lemma 4.2]). Moreover, for any N' let $U'(N')$ be the kernel of reduction of $\mathrm{GL}_2(\hat{\mathbf{Z}})$ to $\mathrm{GL}_2(\mathbf{Z}/N'\mathbf{Z})/\langle p\rangle$. Then $\phi \in \mathscr{A}_0(\mathrm{GL}_{2/\mathbf{Q}}, \omega)^{U'(N')}$ with $N' = N(\mathrm{Disc}\, B)^m$ for some $m \geqslant 0$. (In fact, one can take $m = 1$; for $(N, \mathrm{Disc}\, B) = 1$ this is shown in [9, Theorem 4.1].)

Step 6: *Classical modular forms.* Let $\phi \in \mathscr{A}_0(\mathrm{GL}_{2/\mathbf{Q}}, \omega)^{U'(N')}$ be a cusp form belonging to the discrete series of lowest weight $r + 2$ which is an eigenform for $\tilde{T}_p$. Without changing the eigenvalue we may assume that ϕ is of lowest weight. Then for any $k \in \mathrm{GL}_2(\hat{\mathbf{Z}})$ the function $f_{k,\phi}: \mathscr{H} = \{z \in \mathbf{C} \mid \mathrm{Im}\, z > 0\} \to \mathbf{C}$ defined by

$$f_{k,\phi}\left(\frac{a\sqrt{-1}+b}{c\sqrt{-1}+d}\right) = (c\sqrt{-1}+d)^{r+2}\phi\left(k\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

for any

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbf{R})$$

is well defined, holomorphic, and belongs to the space $S_{r+2}(\Gamma[N'])$ of holomorphic cusp forms of weight $r+2$ for $\Gamma[N']$ (cf. [7, Proposition 3.1]). The map $\phi \mapsto \{f_{k,\phi} \mid k \in \mathrm{GL}_2(\hat{\mathbf{Z}})/U'(N')\}$ is injective. Let T_p be the classical pth Hecke operator on classical modular forms. Then the action of $\tilde{T}_p$ corresponds under the above injection to $p^{-r/2}T_p$ (cf. [7, Lemma 3.7]). Hence, the nonbanal eigenvalues of S are $p^{-r/2}\lambda$, with λ an eigenvalue of T_p on a cusp form f of weight $r + 2$ and level N'.

Step 7: *The Weil conjectures and the Ramanujan–Petersson bounds.* By the results of Eichler–Shimura–Igusa and Shimura-Deligne (see [3]) the Weil conjecture on the size of the eigenvalues of Frobenius implies the Ramanujan–Petersson bound $|\lambda| \leqslant 2p^{(r+1)/2}$ for such T_p eigenvalues λ. This conjecture of Weil was proved by Weil for $r = 0$ and by Deligne in general (see [4]). The graphs Gr being κ-regular with $\kappa = p + 1$ we get that if λ is a nonbanal eigenvalue of S on $C^0(\mathrm{Gr}, \mathscr{L}_r)$, then $|p^{r/2}\lambda| \leqslant 2p^{(r+1)/2}$, or $|\lambda| \leqslant 2\sqrt{p} = 2\sqrt{\kappa - 1}$. This proves the Ramanujan property.

3.6. A finite description of (Gr, $\mathscr{L}_r$)

We shall now give a finite description of (Gr, $\mathscr{L}_r$). We will see that our Gr's generalize the graphs of [10]. They also generalize the graphs and local systems of Eichler–Brandt for maximal orders [5].

Recall (see, e.g. [13] or [5]) that a fractional ideal of $\mathscr{M}$ is a nonzero finitely generated left $\mathscr{M}$-submodule of B. Two fractional ideals I_1, I_2 are equivalent if $I_2 = I_1 b$ for some $b \in B$. The class number h of B is the number of the equivalence classes. It is finite and independent of $\mathscr{M}$. By the strong approximation theorem there exist left $\mathscr{M}$-ideals $I_1, \ldots, I_h$ representing the ideal classes of $\mathscr{M}$ such that each index $[\mathscr{M}: I_j]$ is a power of p. Fix such $I_1, \ldots, I_h$ with $I_1 = \mathscr{M}$ and let $\mathscr{M}_j = \{b \in B \mid I_j b \subset I_j\}$ be their right orders. Then $\mathscr{M}_j \subset \{b \in B \mid \mathscr{M} b \subset \mathscr{M}[p^{-1}]\} = \mathscr{M}[p^{-1}]$. For $1 \leq i, j \leq h$ set $X_{ji} = \{b \in B \mid pI_i \subset I_j b \subset I_i\}$. As before $X_{ji} \subset \mathscr{M}[p^{-1}]$; also $\mathscr{M}_j^\times X_{ji} \mathscr{M}_i^\times = X_{ji}$ and $X_{ij} = pX_{ji}^{-1}$. Let K_j be the image of $\mathscr{M}_j$ in $\bar{K}$.

We define a graph Gr′ and local systems $\mathscr{L}'_r$, $r \geqslant 0$, on it as follows.

(1) For each $1 \leqslant i \leqslant h$ and right coset $K_i k \in K_i \backslash \bar{K}$ there is a vertex $v_{i,K_i k}$ of Gr′, said to be of type i.

(2) For $1 \leqslant i, j \leqslant h$ let $Y_{ji} = X_{ji} \times_{\mathscr{M}_i^\times} \bar{K}$ be the quotient of $X_{ji} \times \bar{K}$ by the equivalence relation $(xm, k) \sim (x, \bar{m}k)$ for all $x \in X_{ji}$, $m \in \mathscr{M}_i^\times$, and $k \in \bar{K}$. Let $[x, k]$ denote the class of $(x, k) \in X_{ji} \times \bar{K}$ in Y_{ji}. For each $1 \leqslant i, j \leqslant h$ and $[x, k] \in Y_{ji}$ put an edge $e_{i,j,[x,k]}$, said to be of

type (i, j), in $\mathrm{Ed}^0(\mathrm{Gr}')$. For $e = e_{i,j,[x,k]}$ set $o(e) = v_{i,K_ik}$ $t(e) = v_{j,K_j\bar{x}k}$, and $\bar{e} = e_{j,i,[x^*,\bar{x}k]}$. The compatibilities $\bar{\bar{e}} = e$ and $o(\bar{e}) = t(e)$ are easily checked (the norm of an element in Γ is a power of p, hence acts trivially on $\bar{K}$).

(3) For any $1 \leqslant i \leqslant h$ choose a set $K^i \subset \bar{K}$ of representatives for $K_i \backslash \bar{K}$. For $v \in \mathrm{Ver}(\mathrm{Gr}')$ set $\mathscr{L}'(v) = \mathbf{C}^{r+1}$. For $e \in \mathrm{Ed}^0(\mathrm{Gr})$ write $e = e_{i,j,[x,k]}$ with $k \in K^i$. Put $\bar{x}k = \bar{m}_j k'$ with $m_j \in \mathscr{M}_j^\times$ and $k' \in K^j$. Define $\mathscr{L}'_{r,e} = \rho_r(m_j^{-1}x)$. A straightforward check shows this gives a metrized local system on Gr'.

Proposition 3.5. *There exists a $\bar{K}$-equivariant isomorphism* $(\mathrm{Gr}, \mathscr{L}_r) \simeq (\mathrm{Gr}', \mathscr{L}'_r)$.

Proof. (1) The tree Δ from Section 3.1 has the following equivalent description. Let $\mathscr{I}$ be the set of left-$\mathscr{M}$-fractional ideals contained in $\mathscr{M}[p^{-1}]$ and containing $p^i\mathscr{M}$ for some $i \in \mathbf{Z}$. Define an equivalence relation on $\mathscr{I}$ by $I \sim p^iI$ for any $i \in \mathbf{Z}$, and denote the equivalence class of I by $[I]$. Let $\mathscr{J} = \{(I, J) \in \mathscr{I} \times \mathscr{I} \mid pI \subsetneq J \subsetneq I\}$, with equivalence relation $(I, J) \sim (p^iI, p^iJ)$ for any $i \in \mathbf{Z}$.

Since B is split at p we may identify $B_p = \mathrm{Mat}_{2\times 2}(\mathbf{Q}_p)$, and we may even assume that $\mathscr{M}_p = \mathrm{Mat}_{2\times 2}(\mathbf{Z}_p)$, so $\mathscr{M}_p^\times = \mathrm{GL}_2(\mathbf{Z}_p) = K_p$. We map $B_p^\times$ to $\mathscr{I}$ by $b \mapsto \mathscr{M}_p b^* \cap \mathscr{M}[p^{-1}]$. By the Eichler–Kneser strong approximation theorem this gives an isomorphism of $B_p^\times$-sets $B_p^\times/K_p \simeq \mathscr{I}$ and hence $(\mathscr{I}/\sim) \simeq B_p^\times/K_p^\times Z_p = \mathrm{Ver}(\Delta)$. We denote the $\mathbf{Z}_p$-lattice $b\mathbf{Z}_p^2$ corresponding to the fractional ideal $I = \mathscr{M}_p b^* \cap \mathscr{M}[p^{-1}]$ by $L(I)$, and the vertex corresponding to $[I]$ by v_I or $v_{[I]}$. The $\{I_j \mid j = 1, \ldots, h\}$ are representatives for the Γ orbits on $\mathscr{I}/\sim$, and the stabilizer of v_{I_j} in Γ is $\mathscr{M}_j^\times$. Hence, the vertices of $\Gamma\backslash(\Delta \times \bar{K})$ are represented by $\{(v_{I_i}, k) \in \Delta \times \bar{K} \mid 1 \leqslant i \leqslant h\}$, where, for each i, k runs over some set of representatives for $K_i \backslash K$ in K. This puts the vertices of $\Gamma\backslash\Delta \times \bar{K}$ in evident bijection with the vertices of Gr'.

(2) Similarly $(\mathscr{J}/\sim) \simeq \mathrm{Ed}^0(\Delta)$, and we let $e_{I,J} \in \mathrm{Ed}^0(\Delta)$ denote the edge corresponding to the class $[I, J]$ of $(I, J) \in \mathscr{J}$. Then $(I, J) \in \mathscr{J}$ if and only if $L(I)$ is a sublattice of index p in $L(J)$. This shows that the map from the disjoint union $\coprod_j \mathscr{M}_j^\times X_{ji}$ to the set of vertices adjacent to v_{I_i} sending $\mathscr{M}_j^\times x$ to v_{I_jx} is a bijection. The oriented star of $(v_{I_i}, k) \in \mathrm{Ver}(\Delta \times \bar{K})$ is therefore the disjoint union $\coprod_j \{(e_{I_jx,I_i}, k) \mid x \in \mathscr{M}_j^\times \backslash X_{ji}\}$. The disjoint union $\coprod_i \coprod_{k \in K^i} \mathrm{St}^0((v_{I_i}, k))$ is therefore a set of representatives in $\mathrm{Ed}^0(\Delta \times \bar{K})$ for $\Gamma\backslash\mathrm{Ed}^0(\Delta \times \bar{K})$. It follows that the map sending such an (e_{I_jx,I_i}, k) to $e_{i,j,[x,k]}$ is a bijection from $\mathrm{Ed}^0(\Delta \times \bar{K})$ to $\mathrm{Ed}^0(\mathrm{Gr}')$, which is clearly compatible with the graph structure maps o, t, and $e \mapsto \bar{e}$.

(3) Lastly, let $\tilde{e} = (e_{I_jx,I_i}, k) \in \mathrm{Ed}^0(\Delta \times \bar{K})$ be as before, let e be its image in the quotient graph Gr, and let $e' \in \mathrm{Ed}^0(\mathrm{Gr}')$ be the corresponding edge. Then for $u \in \mathbf{C}^{r+1}$ we have $\mathscr{L}_{\tilde{e}}(v_{I_i}, k, u) = (v_{I_jx}, k, u)$. To compute $\mathscr{L}_{r,e}(u)$ (here we identify $\mathscr{L}_r(o(e)) \simeq \mathbf{C}^{r+1}$), we assume as before that k belongs to K^i, we write $\bar{x}k = \bar{m}_j k'$ with $k' \in K^j$ and $m_j \in \mathscr{M}_j^\times$. Then

$$(v_{I_jx}, k, u) = x^*(v_{I_j}, \overline{(x^*)^{-1}}k, \rho_r((x^*)^{-1})u) = x^*(v_{I_j}, \bar{x}k, \rho_r(x)u)$$

because $(x^*)^{-1} = \mathrm{Nm}(x)^{-1}x$, and $\mathrm{Nm}(x)$ acts trivially on both K and $\mathbf{C}^{r+1}$. Hence,

$$(v_{I_jx}, k, u) = x^* m_j (v_{I_j}, k', \rho_r(m_j^{-1}x)u),$$

showing that $\mathscr{L}_{r,e}(u) = \rho_r(m_j^{-1})u$, which equals $\mathscr{L}'_{r,e}(u)$ by definition.

Corollary 3.6. *The graph and local systems* $(\mathrm{Gr}', \mathscr{L}'_r)$ *are independent of the* I_i*s.*

Of course, this can be seen directly. We leave this to the reader.

Remark. The graphs of [10] are a special case of our construction. Take B of discriminant 2. The class number is one, and if $\mathcal{M} \subset B$ is a maximal order, the sequence

$$1 \to \{\pm 1\} \to \mathcal{M}^\times \to (\mathcal{M}/2\mathcal{M})^\times \to 1$$

is exact. Let $p \neq q$ be odd primes and take $N = 2q$. Then

$$(\mathcal{M}/N\mathcal{M})^\times \simeq (\mathcal{M}/q\mathcal{M})^\times \times (\mathcal{M}/2\mathcal{M})^\times \simeq \mathrm{GL}_2(\mathbf{F}_q) \times \mathcal{M}^\times/\{\pm 1\}.$$

Let Z_0 be the group of scalar matrices generated by -1 and p in $\mathrm{GL}_2(\mathbf{F}_q)$. Then the vertices of $\mathrm{Gr} = \mathrm{Gr}(B, 2q, p)$ are in bijection with $\langle p \rangle \mathcal{M}^\times \backslash (\mathcal{M}/N\mathcal{M})^\times \simeq Z_0 \backslash \mathrm{GL}_2(\mathbf{F}_q)$. The only set X_{ij} is $X_{1,1}$ consisting of elements x of norm p in $\mathcal{M}$. It is easily checked that up to units each element of $X_{1,1}$ has a unique representative $x = a + b\hat{\imath} + c\hat{\jmath} + d\hat{k}$ with b, c, $d \in 2\mathbf{Z}$, $a \in \mathbf{Z}$ positive and odd. Then the oriented edges of Gr are in bijection with pairs $(g, \xi) \in G \times \Xi$. It follows from Proposition 3.3 that the Cayley graphs $C(H, \Xi)$ of [10], described in the beginning of this section, are the connected components of our graphs.

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