Computers and Mathematics with Applications 59 (2010) 1184–1190

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

The constant variation formulae for singular fractional differential systems with delay[☆]

Jiang Wei

School of Mathematical Science, Anhui University, Hefei, Anhui, 230039, PR China

ARTICLE INFO

Keywords:
Singular fractional differential systems with delay
$\alpha - \delta$ function
Fundamental solution
Constant variation formula

ABSTRACT

This paper considers the Caputo singular fractional differential systems with delay, and the Riemann–Liouville singular fractional differential systems with delay. A new function $\alpha - \delta$ is defined. By the $D-$inverse matrix and $\alpha - \delta$ function, two fundamental solutions are given. The constant variation formulae for singular fractional differential systems with delay are obtained.

1. Introduction

Recently, fractional differential systems have gained scholar's attention [1–9]. In some practical systems, such as economic systems, biological systems, space-light industry systems and so on, due to the transmission of the signal or the mechanical transmission, we must study time delay [4,5,10–14]. In [13–17], we see that singular differential systems have obtained considerable importance due to their applications in various sciences.

The systems we consider will be singular Caputo fractional differential systems with delay:

$$\begin{cases} E({}^cD^\alpha x(t)) = Ax(t) + Bx(t-1) + f(t), & t \geq 0, \\ x(t) = \varphi(t), & -1 \leq t \leq 0 \end{cases} \tag{1}$$

and singular Riemann–Liouville fractional differential systems with delay

$$\begin{cases} E(D^\alpha x(t)) = Ax(t) + Bx(t-1) + f(t), & t \geq 0, \\ x(t) = \varphi(t), & -1 \leq t \leq 0 \end{cases} \tag{2}$$

where $x(t) \in R^n$ is a state vector; $A, B, E \in R^{n\times n}$ are constant matrices; $E \in R^{n\times n}$ is a singular matrix; $\varphi(t)$ is the initial control function; $0 \leq \alpha < 1, f(t), \varphi(t) \in R^n$; ${}^cD^\alpha x(t)$ denotes α order-Caputo fractional derivative; $D^\alpha x(t)$ denotes α order-Riemann–Liouville fractional derivative.

This paper considers the Caputo singular fractional differential systems with delay and the Riemann–Liouville singular fractional differential systems with delay. A new function $\alpha - \delta$ function is defined. By the $D-$inverse matrix and $\alpha - \delta$ function, two fundamental solutions are given. The constant variation formulae for singular fractional differential systems with delay are obtained.

☆ This research has been supported by the National Nature Science Foundation of China (No. 10771001), the Key Program of Ministry of Education of China (No. 205068) and the Innovation Group Program of Anhui University.

E-mail address: jiangwei@mars.ahu.edu.cn.

0898-1221/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2009.07.010

2. Preliminaries

Let us start with some definitions and preliminaries.

Definition 1. Riemann–Liouville's fractional integral of order $\alpha > 0$ for a function $f : R^+ \to R$ is defined as

$$D^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\theta)^{\alpha-1}f(\theta)\mathrm{d}\theta.$$

Definition 2. Caputo's fractional derivative of order α $(0 \le m \le \alpha < m+1)$ for a function $f : R^+ \to R$ is defined as

$${}^cD^{\alpha}f(t) = \frac{1}{\Gamma(m-\alpha+1)}\int_0^t \frac{f^{(m+1)}(\theta)}{(t-\theta)^{\alpha-m}}\mathrm{d}\theta.$$

Definition 3. Riemann–Liouville's fractional derivative of order α $(0 \le m \le \alpha < m+1)$ for a function $f : R^+ \to R$ is defined as

$$D^{\alpha}f(t) = \frac{1}{\Gamma(m-\alpha+1)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m+1}\int_0^t \frac{f(\theta)}{(t-\theta)^{\alpha-m}}\mathrm{d}\theta.$$

Remark. From [6] we have that for $0 \le m \le \alpha < m+1$

$$D^{\alpha}f(t) = {}^cD^{\alpha}f(t) + \sum_{k=0}^{m}\frac{f^{(k)}(0)}{\Gamma(1+k-\alpha)}t^{k-\alpha}.$$

Specially, when $0 \le \alpha < 1$,

$$D^{\alpha}f(t) = {}^cD^{\alpha}f(t) + \frac{f(0)}{\Gamma(1-\alpha)}t^{-\alpha}.$$

From [1] we have that for $0 \le m \le \alpha < m+1$ the Laplace transformation of $D^{-\alpha}f(t)$, ${}^cD^{\alpha}f(t)$ and $D^{\alpha}f(t)$.

$$L(D^{-\alpha}f(t)) = \lambda^{-\alpha}L[f(t)].$$

$$L({}^cD^{\alpha}f(t)) = \lambda^{\alpha}L[f(t)] - \sum_{k=0}^{m}\lambda^{\alpha-k-1}f^{(k)}(0).$$

$$L(D^{\alpha}f(t)) = \lambda^{\alpha}L[f(t)] - \sum_{k=0}^{m}\lambda^{k}[D^{\alpha-k-1}f(0)].$$

Specially, when $0 \le \alpha < 1$, we have

$$L(D^{-\alpha}f(t)) = \lambda^{-\alpha}L[f(t)].$$
$$L({}^cD^{\alpha}f(t)) = \lambda^{\alpha}L[f(t)] - \lambda^{\alpha-1}f(0).$$
$$L(D^{\alpha}f(t)) = \lambda^{\alpha}L[f(t)] - D^{\alpha-1}f(0).$$

Definition 4. If $\det(\lambda E - A) \not\equiv 0$, we call matrix couple (A, E) regular. If (A, E) is regular, we call system (1) regular.
From [1], it is evident that if (E, A) is regular, the system (1) is solvable.

From [11], we have

Lemma 1. *For any square matrix E, the Drazin inverse matrix E^d exists and is unique, and if the Jordan normalized form is*

$$E = T\begin{pmatrix} J_1 & 0 \\ 0 & J_0 \end{pmatrix}T^{-1}.$$

Here J_0 is a nilpotent matrix, J_1 and T is an invertible matrix. Then

$$E^d = T\begin{pmatrix} J_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}T^{-1}.$$

3. Constant variation formula of system (1)

In this section we give the constant variation formula of the singular Caputo fractional differential systems with delay. We separate system (1) into three systems

$$\begin{cases} E({}^cD^\alpha x(t)) = Ax(t) + Bx(t-1) + EE^d f(t), & t \geq 0 \\ x(t) \equiv 0, & -1 \leq t \leq 0, \end{cases} \tag{3}$$

$$\begin{cases} E({}^cD^\alpha x(t)) = Ax(t) + Bx(t-1) + (I - EE^d) f(t), & t \geq 0 \\ x(t) \equiv 0, & -1 \leq t \leq 0, \end{cases} \tag{4}$$

and

$$\begin{cases} E({}^cD^\alpha x(t)) = Ax(t) + Bx(t-1), & t \geq 0 \\ x(t) = \varphi(t), & -1 \leq t \leq 0. \end{cases} \tag{5}$$

It is easy to see that

Lemma 2. *Assume that $x_1(t), x_2(t), x_3(t)$ is the solution of system (3)–(5) respectively, we have that $x(t) = x_1(t)+x_2(t)+x_3(t)$ is the solution of system (1).*

Definition 5. Let $X(t) \in R^{n\times n}$, and satisfy

$$\begin{cases} E({}^cD^\alpha X(t)) = AX(t) + BX(t-1), \\ X(t) = \begin{cases} EE^d, & t = 0, \\ 0, & -1 \leq t < 0 \end{cases} \end{cases} \tag{6}$$

then $X(t)$ is called the corresponding fundamental solution of system (3). We also call $X(t)$ the first fundamental solution of system (1), the first fundamental solution for short.

To give the corresponding fundamental solution of system (4), we define a new function:

Definition 6. Let α $(0 \leq \alpha < 1)$, the function

$$\delta^\alpha(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\delta(\theta)}{(t-\theta)^\alpha} \mathrm{d}\theta$$

is called $\alpha - \delta$ function, here $\delta(t)$ is the δ-function.

Lemma 3. *The Laplace transformation of $\alpha - \delta$ function is*

$$L(\delta^\alpha(t)) = \frac{1}{\lambda^{(1-\alpha)}}.$$

Proof.

$$\begin{aligned} L(\delta^\alpha(t)) &= L\left(\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\delta(\theta)}{(t-\theta)^\alpha} \mathrm{d}\theta\right) \\ &= \frac{1}{\Gamma(1-\alpha)} L\left(t^{-\alpha}\right) L\left(\delta(t)\right) \\ &= \frac{1}{\Gamma(1-\alpha)} L\left(t^{-\alpha}\right) \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^{+\infty} t^{-\alpha} \mathrm{e}^{-\lambda t} \mathrm{d}t. \end{aligned}$$

Let $\xi = \lambda t$, we have

$$\begin{aligned} L(\delta^\alpha(t)) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{+\infty} \left(\frac{\xi}{\lambda}\right)^{-\alpha} \mathrm{e}^{-\xi} \frac{1}{\lambda} \mathrm{d}\xi \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{1}{\lambda^{(1-\alpha)}} \int_0^{+\infty} \xi^{(1-\alpha)-1} \mathrm{e}^{-\xi} \mathrm{d}\xi \\ &= \frac{1}{\lambda^{(1-\alpha)}}. \quad \square \end{aligned}$$

Definition 7. Let $Y(t) \in R^{n\times n}$, and satisfy

$$\begin{cases} E({}^{c}D^{\alpha}Y(t)) = AY(t) + BY(t-1) + (I - EE^{d})\delta^{\alpha}(t), \\ Y(t) = \begin{cases} I - EE^{d}, & t = 0, \\ 0, & -1 \le t < 0 \end{cases} \end{cases} \tag{7}$$

then $Y(t)$ is called the corresponding fundamental solution of system (4). We also call $Y(t)$ as the second fundamental solution of system (1), the second fundamental solution for short. There $\delta^{\alpha}(t)$ is the $\alpha - \delta$ function.

Let $H(\lambda) = \lambda^{\alpha}E - A - \mathrm{e}^{-\lambda}B$ and L^{-1} denote inverse transformation of Laplace-transformation, from (6) we have

$$X(t) = L^{-1}[\lambda^{\alpha-1}H^{-1}(\lambda)EEE^{d}]. \tag{8}$$

From (7), we have

$$(\lambda^{\alpha}E - A - \mathrm{e}^{-\lambda}B)L[Y(t)] = \lambda^{\alpha-1}(E + I)(I - EE^{d}),$$
$$H(\lambda)L[Y(t)] = \lambda^{\alpha-1}(E + I)(I - EE^{d}),$$

that is

$$Y(t) = L^{-1}[\lambda^{\alpha-1}H^{-1}(\lambda)(E + I)(I - EE^{d})]. \tag{9}$$

Theorem 1. *Assume that (E, A) is regular, $X(t)$ is the first fundamental solution, we have the solution of system* (3)

$$x(t) = \int_0^t ({}^{c}D^{1-\alpha}X)(t-\theta)E^{d}f(\theta)d\theta + E^{d}D^{-\alpha}[f(t)].$$

Proof. Take the Laplace transformation for system (3), we have

$$\begin{aligned} L[x(t)] &= H^{-1}(\lambda)EE^{d}L[f(t)] \\ &= \lambda^{1-\alpha}\lambda^{\alpha-1}H^{-1}(\lambda)EEE^{d}E^{d}L[f(t)] \\ &= \lambda^{1-\alpha}L[X(t)]E^{d}L[f(t)] \\ &= (L[({}^{c}D^{1-\alpha}X)(t)] + \lambda^{-\alpha}EE^{d})E^{d}L[f(t)] \\ &= L[({}^{c}D^{1-\alpha}X)(t)]E^{d}L[f(t)] + E^{d}L[D^{-\alpha}f(t)]. \end{aligned}$$

That is

$$x(t) = \int_0^t ({}^{c}D^{1-\alpha}X)(t-\theta)E^{d}f(\theta)\mathrm{d}\theta + E^{d}D^{-\alpha}[f(t)]. \quad \square$$

Theorem 2. *Assume that (E, A) is regular, $Y(t)$ is the second fundamental solution, we have the solution of system* (4)

$$x(t) = \int_0^t ({}^{c}D^{1-\alpha}Y)(t-\theta)(I + E(I - EE^{d}))^{-1}f(\theta)\mathrm{d}\theta + (I - EE^{d})(I + E(I - EE^{d}))^{-1}D^{-\alpha}[f(t)].$$

Proof. From Lemma 1, we have that if $E = T^{-1}\begin{pmatrix} J_1 & 0 \\ 0 & J_0 \end{pmatrix}T$,

$$I - EE^{d} = T^{-1}\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}T,$$

$$E + I = T^{-1}\begin{pmatrix} J_1 + I & 0 \\ 0 & J_0 + I \end{pmatrix}T,$$

$$(E + I)(I - EE^{d}) = T^{-1}\begin{pmatrix} 0 & 0 \\ 0 & J_0 + I \end{pmatrix}T.$$

$$\begin{aligned} I - EE^{d} &= T^{-1}\begin{pmatrix} 0 & 0 \\ 0 & J_0 + I \end{pmatrix}TT^{-1}\begin{pmatrix} I & 0 \\ 0 & J_0 + I \end{pmatrix}^{-1}T \\ &= (E + I)(I - EE^{d})(I + E(I - EE^{d}))^{-1}. \end{aligned}$$

Take the Laplace transformation for system (4), we have

$$\begin{aligned} L[x(t)] &= H^{-1}(\lambda)(I-EE^d)L[f(t)] \\ &= H^{-1}(\lambda)(E+I)(I-EE^d)(I+E(I-EE^d))^{-1}L[f(t)] \\ &= \lambda^{1-\alpha}\lambda^{\alpha-1}H^{-1}(\lambda)(E+I)(I-EE^d)(I+E(I-EE^d))^{-1}L[f(t)] \\ &= \lambda^{1-\alpha}L[Y(t)](I+E(I-EE^d))^{-1}L[f(t)] \\ &= (L[({}^cD^{1-\alpha}Y)(t)]+\lambda^{-\alpha}(I-EE^d))(I+E(I-EE^d))^{-1}L[f(t)] \\ &= L[({}^cD^{1-\alpha}Y)(t)](I+E(I-EE^d))^{-1}L[f(t)]+\lambda^{-\alpha}(I-EE^d)(I+E(I-EE^d))^{-1}L[f(t)] \\ &= L[({}^cD^{1-\alpha}Y)(t)](I+E(I-EE^d))^{-1}L[f(t)]+(I-EE^d)(I+E(I-EE^d))^{-1}L[D^{-\alpha}f(t)]. \end{aligned}$$

That is

$$x(t)=\int_0^t ({}^cD^{1-\alpha}Y)(t-\theta)(I+E(I-EE^d))^{-1}f(\theta)\mathrm{d}\theta+(I-EE^d)(I+E(I-EE^d))^{-1}D^{-\alpha}[f(t)]. \quad \square$$

From Lemma 2, Theorems 1 and 2, we have the constant variation formula for singular fractional differential systems with delay (1).

Theorem 3. *Assume that $x(t,\varphi(t),0)$ is the solution of system (5), we have that the solution of system (1) $x(t,\varphi(t),f(t))$ can be written as*

$$\begin{aligned} x(t) &= \int_0^t ({}^cD^{1-\alpha})[X(t-\theta)E^d+Y(t-\theta)(I+E(I-EE^d))^{-1}]f(\theta)\mathrm{d}\theta \\ &\quad + (E+(I-EE^d))^{-1}D^{-\alpha}[f(t)]+x(t,\varphi(t),0), \end{aligned} \tag{10}$$

there $X(t)$ is the first fundamental solution, $Y(t)$ is the second fundamental solution.

Proof.

$$\begin{aligned} x(t) &= \int_0^t ({}^cD^{1-\alpha}X)(t-\theta)E^df(\theta)\mathrm{d}\theta+E^dD^{-\alpha}[f(t)]+\int_0^t ({}^cD^{1-\alpha}Y)(t-\theta)(I+E(I-EE^d))^{-1}f(\theta)\mathrm{d}\theta \\ &\quad +(I-EE^d)(I+E(I-EE^d))^{-1}D^{-\alpha}[f(t)]+x(t,\varphi(t),0) \\ &= \int_0^t ({}^cD^{1-\alpha})[X(t-\theta)E^d+Y(t-\theta)(I+E(I-EE^d))^{-1}]f(\theta)\mathrm{d}\theta \\ &\quad +(E^d+(I-EE^d)(I+E(I-EE^d))^{-1})D^{-\alpha}[f(t)]+x(t,\varphi(t),0) \end{aligned}$$

$$\begin{aligned} E^d+(I-EE^d)(I+E(I-EE^d))^{-1} &= T^{-1}\begin{pmatrix} J_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}T+T^{-1}\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & J_0+I \end{pmatrix}^{-1}T \\ &= T^{-1}\begin{pmatrix} J_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}T+T^{-1}\begin{pmatrix} 0 & 0 \\ 0 & (J_0+I)^{-1} \end{pmatrix}T \\ &= T^{-1}\begin{pmatrix} J_1^{-1} & 0 \\ 0 & (J_0+I)^{-1} \end{pmatrix}T \\ &= T^{-1}\begin{pmatrix} J_1 & 0 \\ 0 & J_0+I \end{pmatrix}^{-1}T \\ &= T^{-1}\left(\begin{pmatrix} J_1 & 0 \\ 0 & J_0 \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}\right)^{-1}T \\ &= (E+(I-EE^d))^{-1}. \end{aligned}$$

That is

$$\begin{aligned} x(t) &= \int_0^t ({}^cD^{1-\alpha})[X(t-\theta)E^d+Y(t-\theta)(I+E(I-EE^d))^{-1}]f(\theta)\mathrm{d}\theta \\ &\quad +(E+(I-EE^d))^{-1}D^{-\alpha}[f(t)]+x(t,\varphi(t),0). \quad \square \end{aligned}$$

4. Constant variation formula of system (2)

In this section, we give the constant variation formula of the singular Riemann–Liouville fractional differential systems with delay (2).

From Section 1, we have

$$D^{\alpha}x(t) = {}^{c}D^{\alpha}x(t) + \frac{\varphi(t)(0)}{\Gamma(1-\alpha)}t^{-\alpha}.$$

Take it to system (2), system (2) will become

$$\begin{cases} E({}^{c}D^{\alpha}x(t)) = Ax(t) + Bx(t-1) + f(t) - \dfrac{\varphi(0)}{\Gamma(1-\alpha)}t^{-\alpha}, & t \geq 0, \\ x(t) = \varphi(t), & -1 \leq t \leq 0. \end{cases} \tag{11}$$

Theorem 4. *Assume that $x(t, \varphi(t), 0)$ is the solution of system (5), we have that the solution $x(t, \varphi(t), f(t))$ of the singular Riemann–Liouville fractional differential systems with delay (2) can be written as*

$$\begin{aligned} x(t) = & \int_0^t ({}^{c}D^{1-\alpha})[X(t-\theta)E^d + Y(t-\theta)(I + E(I - EE^d))^{-1}][f(\theta)]\mathrm{d}\theta \\ & + (E + (I - EE^d))^{-1}D^{-\alpha}[f(t)] + x(t, \varphi(t), 0) \\ & - \frac{\varphi(0)}{\Gamma(1-\alpha)}\int_0^t ({}^{c}D^{1-\alpha})[X(t-\theta)E^d + Y(t-\theta)(I + E(I - EE^d))^{-1}][\theta^{-\alpha}]\mathrm{d}\theta - \varphi(0)(E + (I - EE^d))^{-1}, \end{aligned}$$

there $X(t)$ is the first fundamental solution, $Y(t)$ is the second fundamental solution.

Proof. From Theorem 3, (11), we have

$$\begin{aligned} x(t) = & \int_0^t ({}^{c}D^{1-\alpha})[X(t-\theta)E^d + Y(t-\theta)(I + E(I - EE^d))^{-1}]\left[f(\theta) - \frac{\varphi(0)}{\Gamma(1-\alpha)}\theta^{-\alpha}\right]\mathrm{d}\theta \\ & + (E + (I - EE^d))^{-1}D^{-\alpha}\left[f(t) - \frac{\varphi(0)}{\Gamma(1-\alpha)}t^{-\alpha}\right] + x(t, \varphi(t), 0) \\ = & \int_0^t ({}^{c}D^{1-\alpha})[X(t-\theta)E^d + Y(t-\theta)(I + E(I - EE^d))^{-1}][f(\theta)]\mathrm{d}\theta + (E + (I - EE^d))^{-1}D^{-\alpha}[f(t)] + x(t, \varphi(t), 0) \\ & - \frac{\varphi(0)}{\Gamma(1-\alpha)}\int_0^t ({}^{c}D^{1-\alpha})[X(t-\theta)E^d + Y(t-\theta)(I + E(I - EE^d))^{-1}][\theta^{-\alpha}]\mathrm{d}\theta \\ & - \frac{\varphi(0)}{\Gamma(1-\alpha)}(E + (I - EE^d))^{-1}D^{-\alpha}[t^{-\alpha}]. \end{aligned}$$

For

$$D^{-\alpha}[t^{-\alpha}] = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\theta)^{\alpha-1}\theta^{-\alpha}\mathrm{d}\theta,$$

let $\xi = \frac{\theta}{t}$, we have

$$\begin{aligned} D^{-\alpha}[t^{-\alpha}] &= \frac{1}{\Gamma(\alpha)}\int_0^1 (1-\xi)^{\alpha-1}\xi^{-\alpha}\mathrm{d}\xi = \frac{1}{\Gamma(\alpha)}B(\alpha, 1-\alpha) \\ &= \frac{1}{\Gamma(\alpha)}\frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(\alpha + (1-\alpha))} = \Gamma(1-\alpha). \end{aligned}$$

There $B(a, b)$ is the $\beta - function$ [1]. So, we have

$$\begin{aligned} x(t) = & \int_0^t ({}^{c}D^{1-\alpha})[X(t-\theta)E^d + Y(t-\theta)(I + E(I - EE^d))^{-1}][f(\theta)]\mathrm{d}\theta + (E + (I - EE^d))^{-1}D^{-\alpha}[f(t)] + x(t, \varphi(t), 0) \\ & - \frac{\varphi(0)}{\Gamma(1-\alpha)}\int_0^t ({}^{c}D^{1-\alpha})[X(t-\theta)E^d + Y(t-\theta)(I + E(I - EE^d))^{-1}][\theta^{-\alpha}]\mathrm{d}\theta - \varphi(0)(E + (I - EE^d))^{-1}. \quad \square \end{aligned}$$

From Theorems 3 and 4, it is easy to prove the following theorem.

Theorem 5. *Assume that $x(t, \varphi(t), f(t))$ is the solution of the singular Caputo fractional differential systems with delay (1), $\bar{x}(t, \varphi(t), f(t))$ is the solution of the singular Riemann–Liouville fractional differential systems with delay (2), we have*

$$\begin{aligned} \bar{x}(t, \varphi(t), f(t)) = & \; x(t, \varphi(t), f(t)) - \frac{\varphi(0)}{\Gamma(1-\alpha)}\int_0^t ({}^{c}D^{1-\alpha})[X(t-\theta)E^d + Y(t-\theta)(I + E(I - EE^d))^{-1}][\theta^{-\alpha}]\mathrm{d}\theta \\ & - \varphi(0)(E + (I - EE^d))^{-1}, \end{aligned}$$

there $X(t)$ is the first fundamental solution, $Y(t)$ is the second fundamental solution.

Acknowledgement

The author is very grateful to Professor Zheng Zuxiu for his helpful and valuable comments and suggestions for improving this paper.

References

[1] Shantanu Das, Functional Fractional Calculus for System Identification and Controls, Springer-Verlag, Berlin, Heidelberg, 2008.
[2] I. Podlubny, Fractional Differential Equations, San Diego Academic Press, 1999.
[3] K.S. Miller, B. Boss, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley and Sons, New York, 1993.
[4] V. Lakshmikantham, Theory of fractional functional differential equations, Nonlinear Analysis 69 (10) (2008) 3337–3343.
[5] Yong Zhou, Existence and uniqueness of fractional functional differential equations with unbounded delay, International Journal of Dynamical Systems and Differential Equations 4 (1) (2008) 239–244.
[6] B. Bonilla, M. Rivero, J.J. Trujillo, On systems of linear fractional differential equations with constant coefficients, Applied Mathematics and Computation 187 (1) (2007) 68–78.
[7] Aytac Arikoglu, Ibrahim Ozkol, Solution of fractional differential equations by using differential transform method, Chaos, Solitons, Fractals 34 (5) (2007) 1473–1481.
[8] Rabha W. Ibrahim, Shaher Momani, On the existence and uniqueness of solutions of a class of fractional differential equations, Journal of Mathematical Analysis and Applications 334 (1) (2007) 1–10.
[9] Daniela Araya, Carlos Lizama, Almost automorphic mild solutions to fractional differential equations, Nonlinear Analysis 69 (11) (2008) 3692–3705.
[10] Jack K. Hale, Sjoerd M. Verduyn Lunel, Introduction to Functional Differential Equations, Springer-Verlag, Berlin, New York, 1992.
[11] Jiang Wei, The Degenerate Differential Systems with Delay, AnHui University Press, Hefei, 1998.
[12] Thomas Erneux, Applied Delay Differential Equations, Acta Mathematica Scientia, Springer Science-Business Media, LLC, 2009.
[13] Jiang Wei, Eigenvalue and stability of singular differential delay systems, Journal of Mathematical Analysis and Applications 297 (2004) 305–316.
[14] Jiang Wei, WenZhong Song, Controllability of singular systems with control delay, Automatica 37 (2001) 1873–1877.
[15] Ralph Byers, Peter Kunkel, Volker Mehrmann, Regularization of Linear Descriptor systems with variable coefficients, SIAM Journal on Control and Optimization 35 (1) (1997) 117–133.
[16] L. Dai, Singular Control Systems, Springer-Verlag, Berlin, New York, 1989.
[17] S.L. Campbell, Singular Systems of Differential Equation (II), Pitman, Sanfrancisco, London, Melbourne, 1982.