Dirct and inverse theorems for Bernstein polynomials with inner singularities

Wen-ming Lu ¹, Lin Zhang *

Institute of Mathematics, Hangzhou Dianzi University, Hangzhou 310037, PR China

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1. Introduction

The set of all continuous functions, defined on the interval I, is denoted by C(I). For any f ∈ C0[0, 1], the corresponding Bernstein polynomials are defined as follows:
\[ B_n(f, x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}, \]
where \( p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \)
\[ \text{for } k = 0, 1, 2, \ldots, n, \ x \in [0, 1]. \]

Let \( \tilde{w}(x) = |x - \xi|^a, \ 0 < \xi < 1, \ a > 0 \) and \( C_a := \{ f \in C[0, 1] \setminus \{0\} : \lim_{x \to \xi} \tilde{w}(f(x)) = 0 \} \). The norm in \( C_a \) is defined by \( \|f\|_{C_a} := \|\tilde{w}(f)\| = \sup_{0 < x \leq 1} \|\tilde{w}(f)(x)\| \).

Define \( W^2_{\varphi} := \{ f \in C_0 : \|f\|_{C_a} < \infty \} \).
\( W^2_{\varphi} := \{ f \in C_0 : \|f\|_{C_a} < \infty \} \).

For \( f \in C_a \), the weighted modulus of smoothness is defined by \( \omega^2_{\varphi}(f, t) := \sup_{0 < h \leq t} \varphi(x) \).

Recently Felten showed the following two theorems in [1]:

**Theorem A.** Let \( \varphi(x) = \sqrt{x(1-x)} \) and let \( \phi: [0, 1] \to R, \phi \neq 0 \) be an admissible step-weight function of the Ditzian–Totik modulus of smoothness [4] such that \( \varphi^2 \) and \( \varphi^2/\phi^2 \) are concave. Then, for \( f \in C[0, 1] \) and \( 0 < x < 2 \),
\[ |B_n(f, x) - f(x)| \leq \omega^2_{\varphi}(f, n^{-1/2} \varphi(x))/\phi(x). \]

**Theorem B.** Let \( \varphi(x) = \sqrt{x(1-x)} \) and let \( \phi: [0, 1] \to R, \phi \neq 0 \) be an admissible step-weight function of the Ditzian–Totik
Let $x$ satisfies the following conditions:

$$|B_n(x) - f(x)| = O \left( \left( n^{-1/2} \frac{\phi(x)}{\phi(x)} \right)^x \right)$$

implies $\omega_n^f(t, 1) = O(t^\epsilon)$.

Approximation properties of Bernstein polynomials have been studied very well [2–5]. In order to approximate the functions with singularities, Della Vecchia et al. [3] introduced some kinds of modified Bernstein polynomials. Throughout the paper, $C$ denotes a positive constant independent of $n$ and $x$, which may be different in different cases.

Let $\phi: [0, 1] \to R, \phi \not= 0$ be an admissible step-weight function of the Ditijan–Totik modulus of smoothness, that is, $\phi$ satisfies the following conditions:

(I) For every proper subinterval $[a, b] \subseteq [0, 1]$ there exists a constant $C_1 = C(a, b) > 0$ such that $C_1^{-1} \leq \phi(x) \leq C_1$, for $x \in [a, b]$.

(II) There are two numbers $\beta(0) \geq 0$ and $\beta(1) \geq 0$ for which $\phi(x) \sim \begin{cases} x^{\beta(0)}, & \text{as } x \to 0^+, \\
(1 - x)^{\beta(1)}, & \text{as } x \to 1^- . \end{cases} (X \sim Y$ means $C^{-1} Y \leq X \leq CY$ for some $C$).

Combining conditions (I) and (II) on $\phi$, we can deduce that $C^{-1} \phi_2(x) \leq \phi(x) \leq C\phi_2(x), x \in [0, 1]$, where $\phi_2(x) = x^{\beta(0)}(1 - x)^{\beta(1)}$.

2. The main results

Let

$$\psi(x) = \begin{cases} 10x^3 - 15x^4 + 6x^5, & 0 < x < 1, \\
0, & x \leq 0, \\
1, & x \geq 1. \end{cases}$$

Obviously, $\psi$ is non-decreasing on the real axis, $\psi \in C^3([-\infty, +\infty), \psi(0) = 0, \psi(1) = 0, i = 0, 1, 2, \psi(1) = 1$. Further, let

$$x_1 = \frac{[n^3_2 - 2n]}{n}, x_2 = \frac{[n^3_2 - \sqrt{n}]}{n}, x_3 = \frac{[n^3_2 + \sqrt{n}]}{n}, x_4 = \frac{[n^3_2 + 2n]}{n},$$

and

$$\bar{\psi}_1(x) = \psi \left( \frac{x - x_2}{x_2 - x_1} \right), \bar{\psi}_2(x) = \psi \left( \frac{x - x_4}{x_4 - x_3} \right).$$

Consider

$$P(x) := \frac{x - x_3}{x_4 - x_3} f(x) + \frac{x_1 - x}{x_1 - x_4} f(x_4),$$

the linear function joining the points $(x_1, f(x_1))$ and $(x_4, f(x_4))$. And let

$$T_n(f)(x) := T_n(x) = f(x)(1 - \bar{\psi}_1(x)) + \bar{\psi}_1(x)(1 - \bar{\psi}_2(x))P(x).$$

From the above definitions it follows that

$$T_n(f)(x) = \begin{cases} f(x), & x \in [0, x_1] \cup [x_4, 1], \\
f(x)(1 - \bar{\psi}_1(x)) + \bar{\psi}_1(x)P(x), & x \in [x_1, x_2], \\
P(x), & x \in [x_2, x_3], \\
P(x)(1 - \bar{\psi}_2(x)) + \bar{\psi}_2(x)f(x), & x \in [x_3, x_4]. \end{cases}$$

Evidently, $T_n$ is a positive linear polynomials which depends on the functions values $f(k/n), 0 < k/n \leq x_2$ or $x_3 < k/n \leq 1$. It reproduces linear functions, and $T_n \in C^2([0, 1])$ provided $f \in W^2_0$. Now for every $f \in C_\psi$ define the Bernstein type polynomials

$$\overline{B}_n(f)(x) := B_n(T_n(f), x)$$

and

$$\overline{T}_n(f)(x) := T_n(B_n(f), x) = \sum_{k=n(b_1, b_2, \ldots, b_k)} p_{x_k}(x)f \left( \frac{k}{n} \right) + \sum_{x_k \in \mathbb{N} \cap \{n_x \cap \mathbb{N} \}} p_{x_k}(x)P \left( \frac{k}{n} \right)$$

and

$$\overline{W}(n)(x) := \overline{B}_n(f, x) = O \left( \left( \frac{n^{-1} \phi_1^1(1 - x) \phi_2^2(1 - x)}{\phi_1^1(1 - x) \phi_2^2(1 - x)} \right)^{x_0} \right) \equiv \omega_n^f(t, \psi) = O(t^\epsilon).$$

3. Lemmas

Lemma 1. [7] For any non-negative real $u$ and $v$, we have

$$\sum_{k=1}^{n+1} \left( \frac{k}{n} \right)^{u-v} \left( 1 - \frac{k}{n} \right) = C \left( 1 - x \right)^{-v}.$$  \hspace{1cm} (3.1)

Lemma 2. [3] For any $\alpha \geq 0$, $f \in C_\psi$, we have

$$\|\overline{W}^\alpha(f)\| \leq C \|\overline{W}\|.$$  \hspace{1cm} (3.2)

Lemma 3. [6] Let $\min \{\beta(0), \beta(1)\} \geq \frac{1}{2}$, then for $0 < t < \frac{1}{2}$ and $t < x < 1 - t$, we have

$$\overline{W}(n)(x) = \omega_n^f(t, \psi) = O(t^\epsilon).$$
\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi^{-1}(x + \sum_{k=1}^{n} u_k) du_1 du_2 \leq Cr^2 \phi^{-2}(x). \tag{3.3}
\]

**Proof.** From the definition of \( \phi(x) \), it is enough to prove (3.3) for \( t < x \leq \frac{1}{2} \) since the proof for \( \frac{1}{2} < x < 1 - t \) is very similar. Obviously, we have

\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{x + \sum_{k=1}^{n} u_k} du_1 du_2 \leq Cr^2 x^{-1}.
\]

Therefore, by the Hölder inequality, we have

\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi^{-1}(x + \sum_{k=1}^{n} u_k) du_1 du_2 \\
\leq C(16/7)^{2(1)} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{x + \sum_{k=1}^{n} u_k} \frac{1}{x + \sum_{k=1}^{n} u_k} du_1 du_2 \\
\leq C(16/7)^{2(1)} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{x + \sum_{k=1}^{n} u_k} \frac{1}{x + \sum_{k=1}^{n} u_k} du_1 du_2 \\
\leq C(16/7)^{2(1)} r^2 x^{-2(1)}. \quad \Box
\]

**Lemma 4** [3]. If \( y \in R, \) then

\[
\sum_{k=0}^{n} p_{n,k}(x) \mid k - nx \mid \leq Cr^2 \phi(x). \tag{3.4}
\]

**Lemma 5.** Let \( A_n(x) := \tilde{w}(x) \sum_{k=0}^{n} p_{n,k}(x) \). Then \( A_n(x) \leq Cn^\frac{1}{2} \) for \( 0 < \xi < 1 \) and \( x > 0 \).

**Proof 2.** If \( |x - \xi| \leq \frac{\sqrt{a}}{2n} \), then the statement is trivial. Hence assume \( 0 < \xi < \frac{1}{2} (\zeta + \frac{\sqrt{a}}{2n}) \) (the case \( \xi + \frac{\sqrt{a}}{2n} < x < 1 \) can be treated similarly). Then for a fixed \( x \) the maximum of \( p_{n,k}(x) \) is attained for \( k = k_n := \lfloor n\xi - \sqrt{n} \rfloor \). By using Stirling’s formula, we get

\[
p_{n,k_n}(x) \leq C \left( \frac{\xi}{\sqrt{n}} \right)^{k_n} \left( 1 - \frac{k_n - nx}{k_n} \right)^{n-k_n} \left( \frac{\sqrt{n}}{\xi} \right)^{k_n} \leq C \frac{n^{k_n}}{\sqrt{n}} \left( \frac{\sqrt{n}}{\xi} \right)^{k_n} \left( \frac{\sqrt{n}}{\xi} \right)^{n-k_n} \leq C \frac{n^{k_n}}{\sqrt{n}} \left( \frac{\sqrt{n}}{\xi} \right)^{n-k_n}.
\]

Now from the inequalities

\[
k_n - nx = \lfloor n\xi - \sqrt{n} \rfloor - nx > n(\xi - x) - \sqrt{n} - 1 \geq \frac{1}{2} n(\xi - x),
\]

and

\[1 - u \leq e^{-u} - \frac{u}{2}, \quad 1 + u \leq e^u, \quad u \geq 0,
\]

it follows that the second inequality is valid. To prove the first one we consider the function \( \lambda(u) = e^{-u} - \frac{u}{2} + u - 1 \). Here \( \lambda(0) = 0, \lambda'(u) = -(1 + u)e^{-u} - \frac{u}{2} + 1, \lambda''(0) = 0, \lambda''(u) = u(u+2)e^{-u} - \frac{u}{2} \geq 0 \), whence \( \lambda(u) \geq 0 \) for \( u \geq 0 \). Hence

\[
p_{n,k_n}(x) \leq \frac{C}{\sqrt{n}} \left( \frac{\xi}{\sqrt{n}} \right)^{k_n} \left( \frac{\sqrt{n}}{\xi} \right)^{n-k_n} \exp \left\{ k_n \left( \frac{k_n - nx}{k_n} - \frac{1}{2} \left( \frac{k_n - nx}{k_n} \right)^{2} \right) + k_n - nx \right\} \leq \frac{C}{\sqrt{n}} \exp \left\{ - \frac{(k_n - nx)^2}{2k_n} \right\} \leq e^{-Cn(\xi-x)^2}.
\]

Thus \( A_n(x) \leq C(\xi - x)^2 e^{-Cn(\xi-x)^2} \). An easy calculation shows that here the maximum is attained when \( \zeta - x = \frac{\sqrt{n}}{2n} \) and the lemma follows. \( \Box \)

**Lemma 6.** For \( 0 < \xi < 1, \beta > 0, \) we have

\[
\tilde{w}(x) \sum_{|k-nx| < \sqrt{n}} |k-nx|^\beta p_{n,k}(x) \leq Cn^{\frac{1}{2}} \phi(x). \tag{3.5}
\]

**Proof 3.** By (3.4) and the Lemma 5, we have

\[
\tilde{w}(x) \sum_{|k-nx| < \sqrt{n}} |k-nx|^\beta p_{n,k}(x) \leq Cn^{\frac{1}{2}} \phi(x). \quad \Box
\]

**Lemma 7.** For any \( \alpha > 0, f \in W^2_{\alpha}, \min\{\beta(0), \beta(1)\} \geq \frac{1}{2}, \) we have

\[
\tilde{w}(x)|f(x) - P(f, x)|_{\alpha, x_t} \leq C \left( \frac{\delta_n(x)}{\sqrt{n} \phi(x)} \right)^2 \| \tilde{w} f'' \|. \tag{3.6}
\]

**Proof**. If \( x \in [x_t, x_4], \) for any \( f \in W^2_{\alpha}, \) we have

\[
f(x_t) = f(x) + f'(x)(x_t - x) + \int_{x_t}^{x} (t - x)f''(t)dt,
\]

\[
f(x_4) = f(x) + f'(x)(x_4 - x) + \int_{x}^{x_4} (t - x)f''(t)dt,
\]

\[
\delta_n(x) \sim \frac{1}{\sqrt{n}}, \quad n = 1, 2, \ldots.
\]

So

\[
\tilde{w}(x)|f(x) - P(f, x)|_{\alpha, x_t} \leq \tilde{w}(x) \int_{x_t}^{x} \frac{x - x_t}{x_t - x_4} \int_{x_t}^{x} |(t - x_4)| f''(t)dt \leq \tilde{w}(x) \int_{x_t}^{x} \frac{x - x_t}{x_t - x_4} \int_{x_t}^{x} |(t - x_4)| f''(t)dt,
\]

\[
I_1 \leq C \| \tilde{w} f'' \| \| (x - x_t)(x - x_4) \| \int_{x_t}^{x} f''(t)dt,
\]

\[
\leq C \left( \frac{\phi(x)}{\sqrt{n}\phi(x)} \right)^2 \| \tilde{w} f'' \| \leq C \left( \frac{\delta_n(x)}{\sqrt{n} \phi(x)} \right)^2 \| \tilde{w} f'' \|.
\]

Analogously, we have

\[
I_2 \leq C \left( \frac{\delta_n(x)}{\sqrt{n} \phi(x)} \right)^2 \| \tilde{w} f'' \|.
\]
Now the lemma follows from combining these results together. □

**Lemma 8.** If \(f \in W^2_0\), \(\min\{\beta(0),\beta(1)\} \geq \frac{1}{2}\), then
\[
\|\hat{w}\tilde{\phi}^2 \mathcal{T}_n\| = O(\|\hat{w}\tilde{\phi}^2 f''\|). \tag{3.7}
\]

**Proof.** Again, it is sufficient to estimate \((\hat{w}\tilde{\phi}^2 \mathcal{T}_n)\) for \(x \in [x_3, x_4]\), and the same as \(x \in [x_1, x_2]\). For \(x \in [x_2, x_3]\), \(\mathcal{T}_n(x) = 0\), while for \(x \in [0, x_1] \cup [x_4, 1]\), \(\mathcal{T}_n(x) = f(x)\). Thus for \(x \in [x_3, x_4]\), then \(\mathcal{T}_n(x) = P(x) + \psi_2(f(x) - P(x))\) and
\[
\mathcal{T}_n(x) = n\psi^\nu \left[ n\tilde{\phi}(x-x_3) \right] (f(x) - P(x))
+ 2n^2\psi^\nu \left[ n\tilde{\phi}(x-x_3) \right] (f(x) - P(x))^2
+ \psi \left[ n\tilde{\phi}(x-x_3) \right] f'(x)
:= I_1(x) + I_2(x) + I_3(x).
\]

From the proof of Lemma 7, we have
\[
|\hat{w}(x)\tilde{\phi}^2(x)I_1(x)| = O(n\tilde{\phi}^2(x)\tilde{\phi}(x-x_3)|\hat{w}(x)|f(x) - P(x)|)
= O\left( n\tilde{\phi}^2(x) \cdot \left( \frac{\phi(x)}{\sqrt{n\tilde{\phi}(x)}} \right)^2 \|\hat{w}\tilde{\phi}^2 f''\| \right)
= O(\|\hat{w}\tilde{\phi}^2 f''\|).
\]

For \(I_2(x)\), it is obvious that
\[
|\hat{w}(x)\tilde{\phi}^2(x)I_2(x)| = O(\|\hat{w}\tilde{\phi}^2 f''\|).
\]

Finally
\[
|\hat{w}(x)\tilde{\phi}^2(x)I_3(x)| = O\left( n\tilde{\phi}(x)\tilde{\phi}(x-x_3)|f(x) - P(x)| \right)
= O\left( n\tilde{\phi}(x-x_3) \cdot \int_{x_1}^{x_4} f'(t)dt \right)
= O\left( n\tilde{\phi}(x-x_3) \cdot \int_{x_1}^{x_4} f'(u)du \right)
= O\left( n\tilde{\phi}(x-x_3) \cdot \int_{x_1}^{x_4} f'(u)du \right)
= O(\|\hat{w}\tilde{\phi}^2 f''\|).
\]

By (3.2), we have
\[
A_1(x) = n\tilde{\phi}^2(x)\hat{w}(x)|\mathcal{B}_n(f, x)| \leq Cn^2\|\hat{w}\tilde{\phi}^2 f''\|. \tag{4.2}
\]

and
\[
A_2 = \hat{w}(x)\tilde{\phi}^4(x) \left[ \sum_{|k| \leq 1} k - nx \right] \|\hat{B}_n(x)\| p_{n,k}(x)
+ \sum_{x_2 < k/n < x_3} k - nx \|P_{\left(\frac{k}{n}\right)}(x)\| p_{n,k}(x) := \sigma_1 + \sigma_2.
\]

thereof \(A := [0,x_2] \cup [x_3,1]\). If \(\frac{x_3}{2} \in A\), when \(\frac{\hat{w}(x)}{\sqrt{n\tilde{\phi}(x)}} \leq C(1 + n^2)\)
\(k - nx\), we have \(k - nx \leq \sqrt{n}\), by (3.4), then
\[
\sigma_1 \leq C\|\tilde{\phi}\|\|\tilde{\phi}\|f(x) \sum_{k=0}^{n} p_{n,k}(x)(k - nx)\|1 + n^2[k - nx]\|
= C\|\tilde{\phi}\|\|\tilde{\phi}\|f(x) \sum_{k=0}^{n} p_{n,k}(x)(k - nx)
+ Cn^{2}\|\tilde{\phi}\|\|\tilde{\phi}\|f(x) \sum_{k=0}^{n} p_{n,k}(x)[k - nx]^{1+u}
\leq Cn^2\|\tilde{\phi}\|\|\tilde{\phi}\|f(x) + Cn^2\|\tilde{\phi}\|\|\tilde{\phi}\|f(x) \leq Cn^2\|\tilde{\phi}\|f(x).
\]

For \(\sigma_2\), \(P\) is a linear function. We note \(\|P_{\left(\frac{k}{n}\right)}\| \leq \max (|P(x)|, |P(x_1)|) := P(a)\). If \(x \in [x_1, x_4]\), we have \(\tilde{w}(x) \leq \tilde{w}(a)\). So, if \(x \in [x_3, x_4]\), by (3.4), then
\[
\sigma_2 \leq C\|\tilde{w}(a)\|P(a)\|\tilde{\phi}\|f(x) \sum_{k=0}^{n} p_{n,k}(x)(k - nx) \leq Cn^2\|\tilde{\phi}\|f(x).
\]

If \(x \notin [x_3, x_4]\), then \(\tilde{w}(a) > n^{\frac{1}{2}}\), by (3.5), we have
\[
\sigma_2 \leq C\|\tilde{w}(x)\|P(a)\|\tilde{\phi}\|f(x) \sum_{x_2 < k/n < x_3} p_{n,k}(x)(k - nx) \leq Cn^2\|\tilde{\phi}\|f(x).
\]
\[
\leq Cn^2\|\tilde{w}\|f(x).
\]

So
\[
A_2 \leq Cn^2\|\tilde{\phi}\|f(x). \tag{4.3}
\]

Similarly
\[
A_3 \leq Cn^2\|\tilde{\phi}\|f(x). \tag{4.4}
\]

It follows from combining with (4.1)-(4.4) that the inequality is proved.

**Case 2.** When \(x \in [0, \frac{1}{4}]\) (The same as \(x \in [1 - \frac{1}{4}, 1]\)), by (4.4), then
\[
\mathcal{B}_n(f, x) = n(n - 1) \sum_{k=0}^{n-1} \hat{w}(k/n)p_{n-2,k}(x).
\]

We have
\[
|\hat{w}(x)\mathcal{B}_n(f, x)| \leq Cn^2\tilde{w}(x) \sum_{k=0}^{n-1} \hat{w}(k/n)p_{n-2,k}(x)
= Cn^2\tilde{w}(x) \left[ \sum_{k=0}^{n-1} p_{n-2,k}(x) \right] \hat{B}_n(k/n)
+ \sum_{x_2 \leq k/n \leq x_3} p_{n-2,k}(x) \hat{B}_n(k/n).
\]
We can deal with it in accordance with Case 1, and prove it immediately, then the theorem is done. □

4.2. Proof of Theorem 2

(1) We prove the first inequality of Theorem 2.

Case I. If 0 ≤ ϕ(x) ≤ 1/n, by (2.2), we have

\[ |\bar{w}(x)\phi^2(x) T^n(f, x)| = \phi^2(x) \cdot \frac{\phi(x)}{\phi^2(x)} |\bar{w}(x) T^n(f, x)| \leq C_n |\bar{w}|. \]

Case II. If \( \phi(x) > \frac{1}{n} \), by [4], we have

\[ T^n_n(f, x) = T^n_n(T_n a, x) = (\phi^2(x))^\frac{1}{2} \sum_{k=0}^{n} \left( x - \frac{k}{n} \right)^n \binom{n}{k} p_{\phi}(k/n) \leq C_n \phi^2(x)^{1/2}. \]

So

\[ |\bar{w}(x)\phi^2(x) T^n(f, x)| \leq C\bar{w}(x)\phi^2(x) \sum_{i=0}^{n} \left( \frac{n}{\phi^2(x)} \right)^{1/2} \sum_{k=0}^{n} \left| x - \frac{k}{n} \right| \binom{n}{k} p_{\phi}(k/n) \]

\[ = C\bar{w}(x)\phi^2(x) \sum_{i=0}^{n} \left( \frac{n}{\phi^2(x)} \right)^{1/2} \sum_{k=0}^{n} \left| x - \frac{k}{n} \right| \binom{n}{k} p_{\phi}(k/n) \]

\[ + C\bar{w}(x)\phi^2(x) \sum_{i=0}^{n} \left( \frac{n}{\phi^2(x)} \right)^{1/2} \left( x - \frac{k}{n} \right) \binom{n}{k} p_{\phi}(k/n) \]

\[ \times \sum_{x_i \leq k/n \leq x} \left( x - \frac{k}{n} \right) p_{\phi}(k/n) := \sigma_1 + \sigma_2. \]

where \( A := [0, x] \cup [x, 1] \). Working as in the proof of Theorem 1, we can get \( \sigma_1 \leq C_n \bar{w}|\bar{w}| \), \( \sigma_2 \leq C_n |\bar{w}| \). By bringing these facts together, we can immediately get the first inequality of Theorem B.

(2) If \( f \in W^2_n \), by (2.1), then

\[ |\bar{w}(x)\phi^2(x) T^n(f, x)| \leq n^2\bar{w}(x)\phi^2(x) \sum_{i=0}^{n-2} \left( \frac{n}{\phi^2(x)} \right)^{1/2} \sum_{k=0}^{n-2} \left| x - \frac{k}{n} \right| \binom{n}{k} p_{\phi-2}(k/n) \]

\[ = n^2\bar{w}(x)\phi^2(x) \sum_{i=0}^{n-2} \left( \frac{n}{\phi^2(x)} \right)^{1/2} \sum_{k=0}^{n-2} \left| x - \frac{k}{n} \right| \binom{n}{k} p_{\phi-2}(k/n) \]

\[ + n^2\bar{w}(x)\phi^2(x) \sum_{i=0}^{n-2} \left( \frac{n}{\phi^2(x)} \right)^{1/2} \binom{n}{k} p_{\phi-2}(0) \]

\[ + n^2\bar{w}(x)\phi^2(x) \sum_{i=0}^{n-2} \left( \frac{n}{\phi^2(x)} \right)^{1/2} \binom{n}{k} p_{\phi-2}(n-2) \]

\[ := I_1 + I_2 + I_3. \]

By [4], if 0 < k < n - 2, we have

\[ \left| \frac{\Delta^2_x T_n(k/n)}{\phi^2(x)} \right| \leq C_n^{-1} \int_0^1 \| T_n(u) \| du. \]  

If k = 0, we have

\[ \left| \frac{\Delta^2_x T_n(0)}{\phi^2(x)} \right| \leq C \int_0^1 u |T_n(u)| du. \]

Similarly

\[ \left| \frac{\Delta^2_x T_n(0)}{\phi^2(x)} \right| \leq C \int_0^1 u |T_n(u)| du. \]  

4.3. Proof of Theorem 3

4.3.1. The direct theorem

We know

\[ T_n(t) = T_n(x) + T_n(t - x) + \int_x^t (t-u) T_n(u) du. \]

\[ B_n(t - x, x) = 0. \]
According to the definition of $W_{\phi}^2$, by (4.13) and (4.14), for any $g \in W_{\phi}^2$, we have $B_{\eta}(g, x) = B_{\eta}(T_{\eta}(g, x))$, then
\[
\tilde{w}(x) | T_{\eta}(g, x) - B_{\eta}(T_{\eta}(g, x)) | = \tilde{w}(x) | B_{\eta}(R_{\eta}(T_{\eta}(g, x), x)) |.
\] (4.15)
thereof $R_{\eta}(T_{\eta}(g, x), x) = \int_0^1 (1 - u) \tilde{G}_{\eta}(u) \, du$.
\[
\tilde{w}(x) | T_{\eta}(g, x) - B_{\eta}(T_{\eta}(g, x)) | \\
\leq C \tilde{w}(x) \sum_{k=0}^{n-1} p_{\eta,k}(x) \int_x^\infty \frac{|k - u|}{\tilde{w}(u) \phi^2(u)} \, du \\
+ C \tilde{w}(x) \sum_{k=0}^{n-1} p_{\eta,k}(x) \int_0^x u \tilde{G}_{\eta}(u) \, du \\
+ C \tilde{w}(x) \sum_{k=0}^{n-1} p_{\eta,k}(x) \int_0^1 (1 - u) \tilde{G}_{\eta}(u) \, du := I_1 + I_2 + I_3.
\] (4.16)

If $u$ between $\frac{x}{2}$ and $x$, we have
\[
\left| \frac{\tilde{w}(x) - u}{\tilde{w}(u)} \right| \leq \frac{|x|}{\tilde{w}(x)} \leq \frac{|x|}{\phi^2(x)}.
\] (4.17)

By (3.4) and (4.17), then
\[
I_1 \leq C \left| \tilde{w} \tilde{G}_{\eta}^2 \right| \left| \tilde{w}(x) \sum_{k=0}^{n-1} p_{\eta,k}(x) \int_x^\infty \frac{|k - u|}{\tilde{w}(u) \phi^2(u)} \, du \\
\leq C \left| \tilde{w} \tilde{G}_{\eta}^2 \right| \left| \tilde{w}(x) \sum_{k=0}^{n-1} p_{\eta,k}(x) \int_x^\infty \frac{|k - u|}{\tilde{w}(u) \phi^2(u)} \, du \right|.
\]
\[
\leq C \eta^{-2} \left| \tilde{w} \tilde{G}_{\eta}^2 \right| \left| \phi^2(x) \sum_{k=0}^{n-1} p_{\eta,k}(x)(k - nx)^2 \right| \\
\leq C \eta^{-2} \left| \tilde{w} \tilde{G}_{\eta}^2 \right| \left| \phi^2(x) \right| \left| \phi^2(x) \right| \\
\leq C \left( \frac{\delta_{\phi}(x)}{\sqrt{\eta \phi(x)}} \right)^2 \left| \tilde{w} \tilde{G}_{\eta}^2 \right|.
\] (4.18)

For $I_2$, when $u$ between $\frac{x}{2}$ and $x$, we let $k = 0$, then $\frac{|x|}{\tilde{w}(x)} \leq \frac{1}{n \phi(x)}$, and
\[
I_2 \leq C \left| \tilde{w} \tilde{G}_{\eta}^2 \right| \left| \tilde{w}(x) \sum_{k=0}^{n-1} p_{\eta,k}(x) \int_0^x (u - \tilde{w}) \phi^{-2}(u) \, du \\
\leq C(n\phi)(1 - x)^{-1} \cdot n^{-2} \phi^2(x) \left| \tilde{w} \tilde{G}_{\eta}^2 \right| \\
\leq C \left( \frac{\delta_{\phi}(x)}{\sqrt{\eta \phi(x)}} \right)^2 \left| \tilde{w} \tilde{G}_{\eta}^2 \right|.
\] (4.19)

Similarly, we have
\[
I_3 \leq C \left( \frac{\delta_{\phi}(x)}{\sqrt{\eta \phi(x)}} \right)^2 \left| \tilde{w} \tilde{G}_{\eta}^2 \right|.
\] (4.20)

By bringing (4.18)-(4.20), we have
\[
\tilde{w}(x) | T_{\eta}(g, x) - B_{\eta}(T_{\eta}(g, x)) | \leq C \left( \frac{\delta_{\phi}(x)}{\sqrt{\eta \phi(x)}} \right)^2 \left| \tilde{w} \tilde{G}_{\eta}^2 \right|.
\] (4.21)

By (3.6) and (4.21), when $g \in W_{\phi}^2$, then
\[
\tilde{w}(x) | g(x) - \tilde{B}_{\eta}(g, x) | \leq \tilde{w}(x) | g(x) - \tilde{B}_{\eta}(g, x) | + \tilde{w}(x) | \tilde{B}_{\eta}(g, x) - B_{\eta}(g, x) | \\
\leq \tilde{w}(x) | g(x) - P(g) |_{[x_1, x_2]} + C \left( \frac{\delta_{\phi}(x)}{\sqrt{\eta \phi(x)}} \right)^2 \left| \tilde{w} \tilde{G}_{\eta}^2 \right| \\
\leq C \left( \frac{\delta_{\phi}(x)}{\sqrt{\eta \phi(x)}} \right)^2 \left| \tilde{w} \tilde{G}_{\eta}^2 \right|.
\] (4.22)

For $f \in C_\gamma$, we choose proper $g \in W_{\phi}^2$, by (3.2) and (4.22), then
\[
\tilde{w}(x) | f(x) - \tilde{B}_{\eta}(g, x) | \leq \tilde{w}(x) | f(x) - g(x) | + \tilde{w}(x) | \tilde{B}_{\eta}(f - g, x) | \\
+ \tilde{w}(x) | g(x) - \tilde{B}_{\eta}(g, x) | \\
\leq C \left( \tilde{w} | f(x) | + \left( \frac{\delta_{\phi}(x)}{\sqrt{\eta \phi(x)}} \right)^2 \left| \tilde{w} \tilde{G}_{\eta}^2 \right| \right). \\
\] (4.23)

\[ 4.3.2. \text{The inverse theorem} \]

The main-part $K$-functional is given by
\[
K_{\phi}(f, t) = \sup_{0 \leq a \leq t} \inf \{ \| \tilde{w}(g - f) \| + t \| \tilde{w} \tilde{G}_{\eta}^2 \| : g \in A.C._{\phi} \}.
\]

By [4], we have
\[
C^{-1} K_{\phi}(f, t) \leq C_{\phi} t \leq CK_{\phi}(f, t).
\] (4.24)

\[ \text{Proof} \]

Let $\delta > 0$, by (4.23), we choose proper $g$ so that
\[
\tilde{w}(x) | \Delta_{\phi}^2 g(x) | \leq C \Delta_{\phi}^2 t \Delta_{\phi}^2 \tilde{w}, \quad \left| \tilde{w} \tilde{G}_{\eta}^2 \right| \leq C \tilde{w} \tilde{G}_{\eta}^2. \\
\] (4.25)

then
\[
\tilde{w}(x) | \Delta_{\phi}^2 g(x) | \leq \left| \tilde{w}(x) \Delta_{\phi}^2 \left( f(x) - \tilde{B}_{\eta}(g, x) \right) \right| \\
+ \left| \tilde{w}(x) \Delta_{\phi}^2 \tilde{B}_{\eta}(f - g, x) \right| + \left| \tilde{w}(x) \Delta_{\phi}^2 \tilde{B}_{\eta}(g, x) \right| \\
\leq \sum_{j=0}^\infty C_{\phi}^2 \left( n^{-1} \delta_{\phi}(x) (1 - j) \phi(x) \right)^2 \phi(x) \left| \tilde{w} \tilde{G}_{\eta}^2 \right| \\
+ \sum_{j=0}^\infty \int_{B_{\eta}(j)} \int_{B_{\eta}(j)} \tilde{w}(x) \tilde{B}_{\eta}(f - g, x + \sum_{k=1}^\infty u_k) \, du \, dv \\
+ \sum_{j=0}^\infty \int_{B_{\eta}(j)} \int_{B_{\eta}(j)} \tilde{w}(x) \tilde{B}_{\eta}(g, x + \sum_{k=1}^\infty u_k) \, du \, dv \\
:= J_1 + J_2 + J_3.
\] (4.26)

Obviously
\[
J_1 \leq C \left( \left| \left( n^{-1} \phi^2(x) \delta_{\phi}(x) \right)^2 \right| \right). \\
\] (4.27)

By (2.2) and (4.24), we have
\[
J_2 \leq Cn^2 \left| \tilde{w}(f - g) \right| \left| \tilde{w} \tilde{G}_{\eta}^2 \right| \left| \tilde{w} \tilde{G}_{\eta}^2 \right| \leq C\delta^2 \tilde{w} \tilde{G}_{\eta}^2. \\
\] (4.28)

By the second inequality of (4.12) and (4.24), we have
\[ J_2 \leq Cn\|\tilde{w}(f-g)\| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi^{-1}\left(x + \sum_{k=1}^{\infty} u_k \right) du_1 \, du_2 \]
\[ \leq C n^2 \varphi^2(x)\varphi^{-2}(x)\|\tilde{w}(f-g)\| \leq C n^2 \varphi^2(x)\varphi^{-2}(x) \omega_2^\omega(f,\delta). \]

By the second inequality of (2.3), (3.3) and (4.24), we have

\[ J_3 \leq C\|\tilde{w}\varphi^2g\| \|\tilde{w}(x)\| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi^{-1}\left(x + \sum_{k=1}^{\infty} u_k \right) \varphi^{-2} \]
\[ \times \left(x + \sum_{k=1}^{\infty} u_k \right) \, du_1 \, du_2 \]
\[ \leq C h^2 \|\tilde{w}\varphi^2g\| \leq C h^2 \delta^{-2} \omega_2^\omega(f,\delta). \]

Now, by (4.25)-(4.29), there exists a constant \( M > 0 \) so that

\[ \left| \tilde{w}(x) \Delta^n_{bf}(f) \right| \leq C \left( n^{\frac{1}{2}} \frac{\delta_n(x)}{\phi(x)} \right)^n + \min \left\{ n \frac{\varphi^2(x)}{\varphi^2(x)} n^2 \varphi^2(x) \right\} \]
\[ \times h^2 \omega_2^\omega(f,\delta) + h^2 \delta^{-2} \omega_2^\omega(f,\delta) \]
\[ \leq C \left( n^{\frac{1}{2}} \frac{\delta_n(x)}{\phi(x)} \right)^n + h^2 M \left( n^{\frac{1}{2}} \frac{\varphi(x)}{\phi(x)} + n^{\frac{1}{2}} n^{-1/2} \right)^{-2} \]
\[ \times \omega_2^\omega(f,\delta) + h^2 \delta^{-2} \omega_2^\omega(f,\delta) \]
\[ \leq C \left( n^{\frac{1}{2}} \frac{\delta_n(x)}{\phi(x)} \right)^n + h^2 M \left( n^{\frac{1}{2}} \frac{\varphi(x)}{\phi(x)} \right)^{-2} \]
\[ \times \omega_2^\omega(f,\delta) + h^2 \delta^{-2} \omega_2^\omega(f,\delta). \]

When \( n \geq 2 \), we have

\[ n^{\frac{1}{2}} \delta_n(x) < (n-1)^{\frac{1}{2}} \delta_n(x) \leq \sqrt{2} n^{\frac{1}{2}} \delta_n(x). \]

Choosing proper \( x, \delta, n \in \mathbb{N} \), so that

\[ n^{\frac{1}{2}} \frac{\delta_n(x)}{\phi(x)} \leq \delta < (n-1)^{\frac{1}{2}} \frac{\delta_n(x)}{\phi(x)}. \]

Therefore

\[ \left| \tilde{w}(x) \Delta^n_{bf}(f) \right| \leq C \left\{ \delta^n + h^2 \delta^{-2} \omega_2^\omega(f,\delta) \right\}. \]

Which implies

\[ \omega_2^\omega(f,t)_a \leq C \left\{ \delta^n + h^2 \delta^{-2} \omega_2^\omega(f,\delta) \right\}. \]

So, by Berens–Lorentz lemma in [4], we get

\[ \omega_2^\omega(f,t)_a \leq C t^n. \]

We can obtain the similar results when the Bernstein polynomials have no singularities. Now, we can consider the combinations of Bernstein Polynomials with inner singularities as Theorem 3 with countable or uncountable singularities.

References