Hamilton cycles in 5-connected line graphs

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Abstract
A conjecture of Carsten Thomassen states that every 4-connected line graph is hamiltonian. It is known that the conjecture is true for 7-connected line graphs. We improve this by showing that any 5-connected line graph of minimum degree at least 6 is hamiltonian. The result extends to claw-free graphs and to Hamilton-connectedness.

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1. Introduction
Is there a positive constant $C$ such that every $C$-connected graph is hamiltonian? Certainly not, as shown by the complete bipartite graphs $K_{n,n+1}$, where $n$ is large. The situation may change, however, if the problem is restricted to graphs not containing a specified forbidden induced subgraph. For instance, for the class of claw-free graphs (those not containing an induced $K_{1,3}$), Matthews and Sumner [18] conjectured the following in 1984.

Conjecture 1 (Matthews and Sumner). Every 4-connected claw-free graph is hamiltonian.

The class of claw-free graphs includes all line graphs. Thus, Conjecture 1 would in particular imply that every 4-connected line graph is hamiltonian. This was stated at about the same time as a separate conjecture by Thomassen [23].

Conjecture 2 (Thomassen). Every 4-connected line graph is hamiltonian.

Although formally weaker, Conjecture 2 was shown to be equivalent to Conjecture 1 by Ryjáček [21]. Several other statements are known to be equivalent to these conjectures, including the Dominating Cycle Conjecture [5,6]; for more work related to these equivalences, see also [2,11,12].

Conjectures 1 and 2 remain open. The best general result to date in the direction of Conjecture 2 is due to Zhan [26] and Jackson (unpublished).
Theorem 3 (Zhan; Jackson). Every 7-connected line graph is hamiltonian.

In fact, the result of [26] shows that any 7-connected line graph $G$ is Hamilton-connected — it contains a Hamilton path from $u$ to $v$ for each choice of distinct vertices $u, v$ of $G$.

For 6-connected line graphs, hamiltonicity has been proved only for restricted classes of graphs [9,25]. Many papers investigate the Hamiltonian properties of other special types of line graphs; see, e.g., [15,16] and the references given therein.

The main result of the present paper is the following improvement of Theorem 3.

Theorem 4. Every 5-connected line graph with minimum degree at least 6 is hamiltonian.

This provides a partial result towards Conjecture 2. Furthermore, the theorem can be strengthened in two directions: it extends to claw-free graphs by a standard application of the results of [21], and it remains valid if ‘hamiltonian’ is replaced by ‘Hamilton-connected’.

One of the ingredients of our method is an idea used (in a simpler form) in [10] to give a short proof of the characterization of graphs with $k$ disjoint spanning trees due to Tutte [24] and Nash-Williams [19] (the ‘tree-packing theorem’). It may be helpful to consult [10] as a companion to Section 5 of the present paper.

The paper is organized as follows. In Section 2, we recall the necessary preliminary definitions concerning graphs and hypergraphs. Section 3 introduces several notions related to quasigraphs, a central concept of this paper. Here, we also state our main result on quasitrees with tight complement (Theorem 5). Sections 4–7 elaborate the theory needed for the proof of this theorem, which is finally given in Section 8. Sections 9 and 10 explain why quasitrees with tight complement are important for us, by exhibiting their relation to connected eulerian subgraphs of a graph. This relation is used in Section 10 to prove the main result of this paper, which is Theorem 4 and its corollary for claw-free graphs. In Section 11, we outline a way to further strengthen this result by showing that graphs satisfying the assumptions of Theorem 4 are in fact Hamilton-connected. Closing remarks are given in Section 12.

The end of each proof is marked by □. In proofs consisting of several claims, the end of the proof of each claim is marked by △.

2. Preliminaries

All the graphs considered in this paper are finite and may contain parallel edges but no loops. The vertex set and the edge (multi)set of a graph $G$ is denoted by $V(G)$ and $E(G)$, respectively. For background on graph theory and any terminology which is not explicitly introduced, we refer the reader to [4].

A hypergraph $H$ consists of a vertex set $V(H)$ and a (multi)set $E(H)$ of subsets of $V(H)$ that are called the hyperedges of $H$. We will be dealing exclusively with 3-hypergraphs, that is, hypergraphs each of whose hyperedges has cardinality 2 or 3. Multiple copies of the same hyperedge are allowed. Throughout this paper, any hypergraph is assumed to be a 3-hypergraph unless stated otherwise. Furthermore, the symbol $H$ will always refer to a 3-hypergraph with vertex set $V$. For $k \in \{2, 3\}$, a $k$-hyperedge is a hyperedge of cardinality $k$.

To picture a 3-hypergraph, we will represent a vertex by a solid dot, a 2-hyperedge by a line as usual for graphs, and a 3-hyperedge $e$ by three lines joining each vertex of $e$ to a point which is not a solid dot (see Fig. 1).

In our argument, 3-hypergraphs are naturally obtained from graphs by replacing each vertex of degree 3 by a hyperedge consisting of its neighbours. Conversely, we may turn a 3-hypergraph $H$ into a graph $Gr(H)$: for each 3-hyperedge $e$ of $H$, we add a vertex $v_e$ and replace $e$ by three edges joining $v_e$ to each vertex of $e$.

As in the case of graphs, the hypergraph $H$ is connected if for every nonempty proper subset $X \subseteq V$, there is a hyperedge of $H$ intersecting both $X$ and $V - X$. If $H$ is connected, then an edge-cut in $H$ is any inclusionwise minimal set of hyperedges $F$ such that $H - F$ is disconnected. For any integer $k$, the hypergraph $H$ is $k$-edge-connected if it is connected and contains no edge-cuts of cardinality less than $k$. The degree of a vertex $v$ is the number of hyperedges incident with $v$. 
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The quasigraph \ \pi\ is\ an\ acyclic\ (or\ a\ quasiforest)\ if\ \pi^*\ is\ a\ forest;\ \pi\ is\ a\ quasitree\ if\ \pi^*\ is\ a\ tree.\ Furthermore,\ we\ define\ \pi\ to\ be\ a\ quasicycle\ if\ \pi^*\ is\ an\ union\ of\ a\ cycle\ and\ a\ (possibly\ empty)\ set\ of\ isolated\ vertices.\ The\ hypergraph\ \pi\ is\ acyclic\ if\ there\ exists\ no\ quasicycle\ in\ \pi.
If\ e\ is\ a\ hyperedge\ of\ \pi,\ then\ \pi - e\ is\ the\ quasigraph\ obtained\ from\ \pi\ by\ changing\ the\ value\ at\ e\ to\ \emptyset.\ The\ complement\ \overline{\pi}\ of\ \pi\ is\ the\ spanning\ subhypergraph\ of\ \pi\ comprised\ of\ all\ the\ hyperedges\ of\ \pi\ not\ used\ by\ \pi.\ Since\ \pi\ includes\ the\ information\ about\ its\ underlying\ hypergraph\ \pi,\ it\ makes\ sense\ to\ speak\ about\ its\ complement\ without\ specifying\ \pi\ (although\ \pi\ sometimes\ do\ specify\ it\ for\ emphasis).\ Note\ that\ \overline{\pi}\ is\ not\ a\ quasigraph.
3. Quasigraphs
A basic notion in this paper is that of a quasigraph. It is a generalization of tree representations and forest representations used, e.g., in [7].
Recall from Section 2 that \ H is a 3-hypergraph on vertex set V. A quasigraph in \ H is a pair (H, \pi), where \pi is a function assigning to each hyperedge e of H a set \pi(e) ⊆ e which is either empty or has cardinality 2. The value \pi(e) is called the representation of e under \pi. Usually, the underlying hypergraph is clear from the context, and we simply speak about a quasigraph \pi. Quasigraphs will be denoted by lowercase Greek letters.
In this section, \pi will be a quasigraph in \ H. Considering all the nonempty sets \pi(e) as graph edges, we obtain a graph \pi^* on V. The hyperedges e with \pi(e) ≠ \emptyset are said to be used by \pi. The set of all such hyperedges of \ H is denoted by E(\pi). The edges of the graph \pi^*, in contrast, are denoted by E(\pi^*) as expected. We emphasize that, by definition, \pi^* spans all the vertices in V.
To picture \ \pi,\ we\ use\ a\ bold\ line\ to\ connect\ the\ vertices\ of\ \pi(e)\ for\ each\ hyperedge\ e\ used\ by\ \pi.\ An\ example\ of\ a\ quasigraph\ is\ shown\ in\ Fig. 2.
The quasigraph \ \pi\ is\ an\ acyclic\ (or\ a\ quasiforest)\ if\ \pi^*\ is\ a\ forest;\ \pi\ is\ a\ quasitree\ if\ \pi^*\ is\ a\ tree.\ Furthermore,\ we\ define\ \pi\ to\ be\ a\ quasicycle\ if\ \pi^*\ is\ an\ union\ of\ a\ cycle\ and\ a\ (possibly\ empty)\ set\ of\ isolated\ vertices.\ The\ hypergraph\ \pi\ is\ acyclic\ if\ there\ exists\ no\ quasicycle\ in\ \pi.
If\ e\ is\ a\ hyperedge\ of\ \pi,\ then\ \pi - e\ is\ the\ quasigraph\ obtained\ from\ \pi\ by\ changing\ the\ value\ at\ e\ to\ \emptyset.\ The\ complement\ \overline{\pi}\ of\ \pi\ is\ the\ spanning\ subhypergraph\ of\ \pi\ comprised\ of\ all\ the\ hyperedges\ of\ \pi\ not\ used\ by\ \pi.\ Since\ \pi\ includes\ the\ information\ about\ its\ underlying\ hypergraph\ \pi,\ it\ makes\ sense\ to\ speak\ about\ its\ complement\ without\ specifying\ \pi\ (although\ \pi\ sometimes\ do\ specify\ it\ for\ emphasis).\ Note\ that\ \overline{\pi}\ is\ not\ a\ quasigraph.
Fig. 2. A quasigraph $\rho$ in the hypergraph of Fig. 1.

Fig. 3. An illustration to the definition of the $\pi$-section at $X$.

How to define an analogue of the induced subgraph for quasigraphs? Let $X \subseteq V$. At first sight, a natural choice for the underlying hypergraph of a quasigraph induced by $\pi$ on $X$ is $H[X]$. It is clear how to define the value of the quasigraph on a hyperedge $e \cap X$, except if $|e| = 3$ and $|e \cap X| = 2$ (see Fig. 3(a)). In particular, if $\pi(e)$ intersects both $X$ and $V - X$, then $e \cap X$ will not be used by the induced quasigraph; furthermore, it is (at least for our purposes) not desirable to include $e \cap X$ in the complement of the induced quasigraph either. This brings us to the following replacement for $H[X]$ (cf. Fig. 3(b)).

The $\pi$-section of $H$ at $X$ is the hypergraph $H[X]^\pi$ defined as follows:

- $H[X]^\pi$ has vertex set $X$,
- its hyperedges are the sets $e \cap X$, where $e$ is a hyperedge of $H$ such that $|e \cap X| \geq 2$ and $\pi(e) \subseteq X$.

The quasigraph $\pi$ in $H$ naturally determines a quasigraph $\pi[X]$ in $H[X]^\pi$, defined by

$$(\pi[X])(e \cap X) = \pi(e),$$

where $e \in E(H)$ and $e \cap X$ is any hyperedge of $H[X]^\pi$. We refer to $\pi[X]$ as the quasigraph induced by $\pi$ on $X$. Let us stress that whenever we speak about the complement of $\pi[X]$, it is – in accordance with the definition – its complement in $H[X]^\pi$.

The ideal quasigraphs for our purposes in the later sections of this paper would be quasitrees with connected complement. It turns out, however, that this requirement is too strong, and that the following weaker property will suffice. The quasigraph $\pi$ has tight complement (in $H$) if one of the following holds:

(a) $\pi$ is connected, or
(b) there is a partition $V = X_1 \cup X_2$ such that for $i = 1, 2$, $X_i$ is nonempty and $\pi[X_i]$ has tight complement (in $H[X_i]^\pi$); furthermore, there is a hyperedge $e \in E(\pi)$ such that $\pi(e) \subseteq X_1$ and $e \cap X_2 \neq \emptyset$.

The definition is illustrated in Fig. 4.

Our main result regarding quasitrees in hypergraphs is the following.

**Theorem 5.** Let $H$ be a 4-edge-connected 3-hypergraph. If no 3-hyperedge in $H$ is included in any edge-cut of size 4, then $H$ contains a quasitree with tight complement.
particular, \(H\) is narrow and wide partitions \(P\) has tight complement in \(H\). Consequently, there is a hyperedge \(f\) complement in \(P\) then there is a trivial partition. Assume that \(|V| > 1\), the assertion is trivial. Assume that \(|V| = 1\), and that \(P\) is a nontrivial partition of \(V\); we aim to prove that \(P\) is not \(\pi\)-narrow. Consider the two cases in the definition of tight complement. If \(\pi\) is connected (Case (a)), then there is a \(\mathcal{P}\)-crossing hyperedge \(e\) of \(\pi\). Since \(\pi(e) = \emptyset\) is not \(\mathcal{P}\)-crossing, \(\pi\) is not \(\pi\)-narrow.

In Case (b), there is a partition \(V = X_1 \cup X_2\) into nonempty sets such that each \(\pi[X_i]\) has tight complement in \(H[X_i]\). Suppose that \(\mathcal{P}[X_1]\) is nontrivial. By the induction hypothesis, it is not \(\pi[X_i]\)-narrow. Consequently, there is a hyperedge \(f\) of \(H[X_1]\) contained in \(\pi[X_1]\) and such that \(\pi(f) \subseteq P \cap X_1\), where \(P \in \mathcal{P}\). It follows that \(\mathcal{P}\) is not \(\pi\)-narrow as claimed.

By symmetry, we may assume that both \(\mathcal{P}[X_1]\) and \(\mathcal{P}[X_2]\) are trivial. Since \(\mathcal{P}\) is nontrivial, it must be that \(\mathcal{P} = \{X_1, X_2\}\). Case (b) of the definition of tight complement ensures that there is a hyperedge \(e \in E(\pi)\) such that \(\pi(e) \subseteq X_1\) and \(e \cap X_2 \neq \emptyset\). Since \(e\) is \(\mathcal{P}\)-crossing and \(\pi(e)\) is not, \(\mathcal{P}\) is not \(\pi\)-narrow. This finishes the proof of the ‘only if’ part.

The ‘if’ direction will be proved by contradiction. Suppose that \(V\) admits no nontrivial \(\pi\)-narrow partition, but \(\pi\) does not have tight complement in \(H\). Let \(\mathcal{R}\) be a coarsest possible partition of \(V\) such that each \(\pi[X]\) for \(X \in \mathcal{R}\), has tight complement in \(H[X]\). (To see that at least one partition with this property exists, consider the partition of \(V\) into singletons.) Since \(\mathcal{R}\) is nontrivial by assumption, there is an \(\mathcal{R}\)-crossing hyperedge \(e\) of \(H\) with \(\pi(e) \subseteq R_1\), where \(R_1\) is some class of \(\mathcal{R}\). Since \(e\) is \(\mathcal{R}\)-crossing, it intersects another class \(R_2\) of \(\mathcal{R}\). By the definition, \(\pi[R_1 \cup R_2]\) has tight complement in \(H[R_1 \cup R_2]\) \(\pi\), which contradicts the maximality of \(\mathcal{R}\). \(\Box\)

4. Narrow and wide partitions

We begin this section by modifying the definition of a \(\pi\)-narrow partition of \(V\). If \(\pi\) is a quasigraph in \(H\), then a partition \(\mathcal{P}\) of \(V\) is \(\pi\)-wide if for every hyperedge \(e\) of \(H\), \(\pi(e)\) is a subset of a class of \(\mathcal{P}\). (In particular, \(\pi(e)\) is not \(\mathcal{P}\)-crossing for any \(\mathcal{P}\)-crossing hyperedge \(e\).) An example of a \(\pi\)-wide partition...
Fig. 5. Positive and negative parts.

is shown in Fig. 5(a) below. Again, the trivial partition is $\pi$-wide for any $\pi$. Lemma 6 has the following easier analogue.

**Lemma 7.** If $\pi$ is a quasigraph in $H$, then $\pi^*$ is connected if and only if there is no nontrivial $\pi$-wide partition of $V$.

**Proof.** We begin with the ‘only if’ direction. Suppose that $\mathcal{P}$ is a nontrivial partition of $V$. Since $\pi^*$ is a connected graph with vertex set $V$, there is an edge $\pi(e)$ of $\pi^*$ crossing $\mathcal{P}$. This shows that $\mathcal{P}$ is not $\pi$-wide.

Conversely, suppose that $\pi^*$ is disconnected, and let $\mathcal{P}$ be the partition of $V$ whose classes are the vertex sets of components of $\pi^*$. Let $e$ be a hyperedge of $H$. We claim that $\pi(e)$ is not $\mathcal{P}$-crossing. This is certainly true if $e \notin E(\pi)$. In the other case, $\pi(e)$ is an edge of $\pi^*$ and both of its endvertices must be contained in the same component of $\pi^*$, which proves the claim. We conclude that $\mathcal{P}$ is a (nontrivial) $\pi$-wide partition of $V$. □

It is interesting that both the class of $\pi$-narrow partitions and the class of $\pi$-wide partitions are closed with respect to meets in the lattice of partitions:

**Observation 8.** If $\pi$ is a quasigraph in $H$ and $\mathcal{P}$ and $\mathcal{R}$ are $\pi$-narrow partitions, then $\mathcal{P} \wedge \mathcal{R}$ is $\pi$-narrow. Similarly, if $\mathcal{P}$ and $\mathcal{R}$ are $\pi$-wide, then $\mathcal{P} \wedge \mathcal{R}$ is $\pi$-wide.

By Observation 8, for any quasigraph $\pi$ in $H$, there is a unique finest $\pi$-narrow partition of $V$, which will be denoted by $\mathcal{A}_-(\pi; H)$. Similarly, there is a unique finest $\pi$-wide partition of $V$, denoted by $\mathcal{A}_+(\pi; H)$. If the hypergraph is clear from the context, we write just $\mathcal{A}_+(\pi)$ or $\mathcal{A}_-(\pi)$. Lemmas 6 and 7 provide us with a useful interpretation of $\mathcal{A}_+(\pi)$ and $\mathcal{A}_-(\pi)$. It is not hard to show from the latter lemma that the classes of $\mathcal{A}_+(\pi)$ are exactly the vertex sets of components of $\pi^*$. Similarly, by Lemma 6, the classes of $\mathcal{A}_-(\pi)$ are all maximal subsets $X$ of $V$ such that $\pi[X]$ has tight complement in $H[X]^\pi$.

We call the classes of $\mathcal{A}_+(\pi)$ the positive $\pi$-parts of $H$ and the classes of $\mathcal{A}_-(\pi)$ the negative $\pi$-parts of $H$. (See Fig. 5 for an illustration.) The terms ‘positive’ and ‘negative’ are chosen with regard to the terminology of photography, with ‘positive’ used for $\pi$ and ‘negative’ for its complement, in accordance with the above discussion.

We note the following simple corollary of Lemma 6.

**Lemma 9.** Let $\pi$ be a quasigraph in $H$. For $i = 1, 2$, let $X_i \subseteq V$ be such that $\pi[X_i]$ has tight complement in $H[X_i]^\pi$. Then the following holds:

(i) each $X_i$ is contained in a class of $\mathcal{A}_-(\pi)$ (as a subset), and
(ii) if $H$ contains a hyperedge $e$ such that $e$ intersects each $X_i$ and $\pi(e) \subseteq X_1$ (we allow $e \notin E(\pi)$), then $X_1 \cup X_2$ is contained in a class of $\mathcal{A}_-(\pi)$. 
Proof. (i) Clearly, if $\mathcal{P}$ is a $\pi$-narrow partition of $V$, then $\mathcal{P}[X_\pi]$ is $\pi[X_\pi]$-narrow; it follows that $A_-(\pi)[X_\pi] \supseteq A_-(\pi[X_\pi])$. By Lemma 6, $A_-(\pi[X_\pi])$ is trivial. Hence $A_-(\pi)[X_\pi]$ is also trivial. A symmetric argument works for $X_\sigma$.

(ii) It suffices to prove that $\pi[X_1 \cup X_2]$ has tight complement in $H[X_1 \cup X_2]^\pi$. If not, let $\mathcal{P}$ be a nontrivial $\pi[X_1 \cup X_2]$-narrow partition of $X_1 \cup X_2$. By the assumption, each $\mathcal{P}[X_i]$ has to be trivial as it is $\pi[X_i]$-narrow. Thus, $\mathcal{P} = \{X_1, X_2\}$. However, since $\pi(e) \subseteq X_1$, this is not a $\pi[X_1 \cup X_2]$-narrow partition — a contradiction. □

We use the partitions $A_+(\pi)$ and $A_-(\pi)$ to introduce an order on quasigraphs. If $\pi$ and $\sigma$ are quasigraphs in $H$, then we write

$$\pi \preceq \sigma \quad \text{if} \quad A_+(\pi) \leq A_+(\sigma) \quad \text{and} \quad A_-(\pi) \leq A_-(\sigma).$$

Clearly, $\preceq$ is a partial order.

For a set $X \subseteq V$, let us say that two quasigraphs $\pi$ and $\sigma$ in $H$ are $X$-similar if the following holds for every hyperedge $e$ of $H$:

(i) $\pi(e) \subseteq X$ if and only if $\sigma(e) \subseteq X$, and

(ii) $\pi(e) \not\subseteq X$ then $\pi(e) = \sigma(e)$.

Let us collect several easy observations about $X$-similar quasigraphs.

**Observation 10.** If $X \subseteq V$ and quasigraphs $\pi$ and $\sigma$ are $X$-similar, then the following holds:

(i) $H[X]\pi = H[X]\sigma$,

(ii) if $X \in A_+(\pi)$, then $A_+(\sigma) \leq A_+(\pi)$,

(iii) if $X \in A_-(\pi)$, then $A_-(\sigma) \leq A_-(\pi)$.

The following lemma is an important tool which facilitates the use of induction in our argument.

**Lemma 11.** Let $X \subseteq V$ and let $\pi$ and $\sigma$ be $X$-similar quasigraphs in $H$. Then the following holds:

if $\pi[X] \preceq \sigma[X]$, then $\pi \preceq \sigma$.

**Proof.** Note that by Observation 10(i), $H[X]\pi = H[X]\sigma$. We need to prove that

$$A_-(\pi[X]) \leq A_-(\sigma[X]), \quad \text{then} \quad A_-(\pi) \leq A_-(\sigma), \quad (1)$$

and an analogous assertion ($1^+$) with all occurrences of ‘$-$’ replaced by ‘$+$’.

We prove (1). By the definition of $A_-(\sigma)$, (1) is equivalent to the statement that

if every $\sigma[X]$-narrow partition of $X$ is $\pi[X]$-narrow (in $H[X]\pi$),

then every $\sigma$-narrow partition of $V$ is $\pi$-narrow (in $H$).

Assume thus that every $\sigma[X]$-narrow partition is $\pi[X]$-narrow and that $\mathcal{P}$ is a $\sigma$-narrow partition of $V$. For contradiction, suppose that $\mathcal{P}$ is not $\pi$-narrow.

We claim that $\mathcal{P}[X]$ is $\sigma[X]$-narrow in $H[X]\pi$. Let $e \cap X$ be a $\mathcal{P}[X]$-crossing hyperedge of $H[X]\pi$ (where $e \in E(H)$). Then $e$ is $\mathcal{P}$-crossing, and since $\mathcal{P}$ is $\sigma$-narrow, $\sigma(e)$ is $\mathcal{P}$-crossing. By the definition of $H[X]\pi$, $\sigma(e) \subseteq X$ and thus $\sigma(e) = \sigma[X](e \cap X)$ is $\mathcal{P}[X]$-crossing. This proves the claim.

Since every $\sigma[X]$-narrow partition of $X$ is assumed to be $\pi[X]$-narrow, $\mathcal{P}[X]$ is $\pi[X]$-narrow.

On the other hand, $\mathcal{P}$ is not $\pi$-narrow, so there is a $\mathcal{P}$-crossing hyperedge $f$ of $H$ such that $\pi(f)$ is not $\mathcal{P}$-crossing. However, $\sigma(f)$ is $\mathcal{P}$-crossing as $\mathcal{P}$ is $\sigma$-narrow. Thus, $\pi(f) \neq \sigma(f)$, and since $\pi$ and $\sigma$ are $X$-similar, both $\pi(f)$ and $\sigma(f)$ are subsets of $X$. It follows that $\sigma(f)$, and therefore also the hyperedge $f \cap X$ of $H[X]\pi = H[X]\pi$, is $\mathcal{P}[X]$-crossing. We have seen that $\mathcal{P}[X]$ is $\pi[X]$-narrow, and this observation implies that $\pi(f)$ is $\mathcal{P}[X]$-crossing and therefore $\mathcal{P}$-crossing. This contradicts the choice of $f$.

The proof of ($1^+$) is similar to the above but simpler. The details are omitted. □
5. Partition sequences

Besides the order \( \preceq \) introduced in Section 4, we will need another derived order \( \preceq \) on quasigraphs, one that is used in the basic optimization strategy in our proof. Let \( \pi \) be a quasigraph in \( H \). Similarly as in [10], we associate with \( \pi \) a sequence of partitions of \( V \), where each partition is a refinement of the preceding one. Since \( H \) is finite, the partitions ‘converge’ to a limit partition whose classes have a certain favourable property.

Recall from Section 4 that there is a uniquely defined partition of \( V \) into positive \( \pi \)-parts; we will let this partition be denoted by \( \mathcal{P}_0^\pi \). The partition sequence of \( \pi \) is the sequence \( \mathcal{P}^\pi = (\mathcal{P}_0^\pi, \mathcal{P}_1^\pi, \ldots) \), where for even (odd) \( i \geq 1 \), \( \mathcal{P}_i^\pi \) is obtained as the union of partitions of \( X \) into positive (negative, respectively) \( \pi[X] \)-parts of \( H[X]^\pi \) as \( X \) ranges over classes of \( \mathcal{P}_{i-1}^\pi \). (See Fig. 6.) Thus, for instance, for even \( i \geq 2 \) we can formally write \( \mathcal{P}_i^\pi = \bigcup_{X \in \mathcal{P}_{i-1}^\pi} A_+(\pi[X]). \)

Since \( H \) is finite, we have \( \mathcal{P}_k^\pi = \mathcal{P}_{k+2}^\pi \) for large enough \( k \), and we set \( \mathcal{P}_\infty^\pi = \mathcal{P}_k^\pi \).

Let us call a set \( X \subseteq V \) \( \pi \)-solid (in \( H \)) if \( \pi[X] \) is a quasitree with tight complement in \( H[X]^\pi \). By the construction, any class of \( \mathcal{P}_\infty^\pi \) is \( \pi \)-solid.

Let us define a lexicographic order on sequences of partitions: if \((A_0, A_1, \ldots)\) and \((B_0, B_1, \ldots)\) are sequences of partitions of \( V \), write

\[(A_0, A_1, \ldots) \preceq_L (B_0, B_1, \ldots)\]

if there exists some \( i \) such that for \( j < i \), \( A_j = B_j \), while \( A_i \) strictly refines \( B_i \).

We can now define the order \( \preceq \) on quasigraphs as promised. Let \( \pi \) and \( \sigma \) be quasigraphs in \( H \). Define

\[\pi \preceq \sigma \text{ if } \pi \preceq \sigma \text{ and } \mathcal{P}_\infty^\pi \preceq_L \mathcal{P}_\infty^\sigma.\]

If \( \pi \preceq \sigma \) but \( \sigma \npreceq \pi \), we write \( \pi < \sigma \).

From Lemma 11, we can deduce a similar observation regarding the order \( \preceq \) (in which the implication is actually replaced by equivalence).
Lemma 12. Let $X \subseteq V$ and assume that either $X$ is a positive $\pi$-part of $H$, or $\mathcal{P}_0^\pi$ is trivial and $X$ is a negative $\pi$-part of $H$. Let $\pi$ and $\sigma$ be $X$-similar quasigraphs in $H$. Then the following holds:

$$\pi[X] \leq \sigma[X] \quad \text{if and only if} \quad \pi \preceq \sigma.$$  

Proof. We consider two cases depending on whether $X$ is a positive or negative $\pi$-part of $H$.

Case 1: $X$ is a positive $\pi$-part of $H$.

Since $\pi$ and $\sigma$ are $X$-similar, we have

$$\mathbb{P}^\pi = (\mathcal{P}_0^\pi, \mathcal{P}_1^\pi[X] \cup \mathcal{P}_2^\pi[V - X], \mathcal{P}_3^\pi[V - X], \ldots) \quad \text{and}$$

$$\mathbb{P}^\sigma = (\mathcal{P}_0^\sigma, \mathcal{P}_1^\sigma[X] \cup \mathcal{P}_2^\sigma[V - X], \mathcal{P}_3^\sigma[V - X], \ldots). \quad (2)$$

Assume first that $\pi[X] \leq \sigma[X]$. Eqs. (2) imply that for each $i \geq 1$, $\mathcal{P}_i^\pi \leq \mathcal{P}_i^\sigma$. Furthermore, $\pi[X] \leq \sigma[X]$ and Lemma 11 imply that $\pi \preceq \sigma$. In particular,

$$\mathcal{P}_0^\pi = \mathcal{A}_+(\pi) \leq \mathcal{A}_+(\sigma) = \mathcal{P}_0^\sigma$$

so $\mathbb{P}^\pi \preceq \mathbb{P}^\sigma$ and therefore also $\pi \preceq \sigma$.

Conversely, assume that $\pi \preceq \sigma$. The fact that $\mathbb{P}^\pi \preceq \mathbb{P}^\sigma$ together with (2) implies that for $i \geq 1$, $\mathcal{P}_i^\pi[X] \leq \mathcal{P}_i^\sigma[X]$. Recall that $X$ is a positive $\pi$-part of $H$. We claim that $X$ is also a positive $\sigma$-part of $H$; indeed, this follows from the fact that $\mathcal{P}_0^\sigma \leq \mathcal{P}_0^\pi$ and that $\pi$ and $\sigma$ are $X$-similar. This claim implies

$$\mathcal{P}_0^\pi[X] = X = \mathcal{P}_0^\sigma[X] \quad (3)$$

and, consequently, $\mathcal{P}_i^\pi[X] \leq \mathcal{P}_i^\sigma[X]$. It remains to verify that $\pi[X] \leq \sigma[X]$. This follows from (3) and the observation that $\mathcal{P}_1^\pi[X] \leq \mathcal{P}_1^\sigma[X]$. (Here we use the fact that if $\mathcal{P}_0^\pi$ is trivial, then $\mathcal{P}_1^\pi = \mathcal{A}_-($).

Case 2: $\mathcal{P}_0^\pi$ is trivial and $X$ is a negative $\pi$-part of $H$.

In this case, Eqs. (2) are replaced by

$$\mathbb{P}^\pi = \left\{ V, \mathcal{A}_-(\pi[X]) \cup \mathcal{P}_1^\pi[V - X], \mathcal{P}_2^\pi[V - X], \mathcal{P}_3^\pi[V - X], \ldots \right\} \quad \text{and}$$

$$\mathbb{P}^\sigma = \left\{ V, \mathcal{A}_-(\sigma[X]) \cup \mathcal{P}_1^\sigma[V - X], \mathcal{P}_2^\sigma[V - X], \mathcal{P}_3^\sigma[V - X], \ldots \right\}. \quad (4)$$

Assume first that $\pi \preceq \sigma$. Since $X$ is a positive $\pi$-part of $H$, the partition $\mathcal{A}_-(\pi[X])$ appearing in the second term of $\mathbb{P}^\pi$ is trivial. A similar observation holds for $\sigma$ in place of $\pi$. Hence, $\mathbb{P}^\pi$ and $\mathbb{P}^\sigma$ are equal in their first two terms and (4) directly implies that $\mathcal{P}_i^\pi[X] \leq \mathcal{P}_i^\sigma[X]$. Moreover, $\pi[X] \leq \sigma[X]$ is implied by (4) as well. We conclude that $\pi[X] \leq \sigma[X]$.

The converse implication follows from (4) without any further effort. The proof is complete.\[\square\]

Corollary 13. Let $\pi$ and $\sigma$ be $X$-similar quasigraphs in $H$, where $X \in \mathcal{P}_i^\pi$ for some $i$. Then the following holds:

$$\pi[X] \leq \sigma[X] \quad \text{if and only if} \quad \pi \preceq \sigma.$$  

Proof. Follows from Lemma 12 by easy induction.\[\square\]

We conclude this section by a lemma that suggests a relation between $\preceq$-maximal and acyclic quasigraphs. If $\pi$ and $\sigma$ are quasigraphs in $H$, then let us call $\sigma$ a restriction of $\pi$ if for every hyperedge $e$ of $H$, $\sigma(e)$ equals either $\pi(e)$ or $\emptyset$.

Lemma 14. Let $\pi$ be a quasigraph in $H$ and $i \geq 0$. If $\pi[X]$ is acyclic for each $X \in \mathcal{P}_i^\pi$, but $\pi$ itself is not acyclic, then there exists an acyclic restriction $\sigma$ of $\pi$ such that $\sigma \succ \pi$.\[\square\]
In this section, we introduce two concepts related to partitions: contraction and substitution.

Let $\mathcal{P}$ be a partition of $V$. The contraction of $\mathcal{P}$ is the operation whose result is the hypergraph $H/\mathcal{P}$ defined as follows. For $A \subseteq V$, define $A/\mathcal{P}$ as the subset of $\mathcal{P}$ consisting of all the classes $P \in \mathcal{P}$ such that $A \cap P \neq \emptyset$. The hypergraph $H/\mathcal{P}$ has vertex set $\mathcal{P}$ and it hyperedges are all the sets of the form $e/\mathcal{P}$, where $e$ ranges over all $\mathcal{P}$-crossing hyperedges. Thus, $H/\mathcal{P}$ is a 3-hypergraph, possibly with multiple hyperedges. As in the case of induced subhypergraphs, each hyperedge $f$ of $H/\mathcal{P}$ is understood to have an assigned corresponding hyperedge $e$ of $H$ such that $f = e/\mathcal{P}$.

If $\pi$ is a quasigraph in $H$, we define $\pi/\mathcal{P}$ as the quasigraph in $H/\mathcal{P}$ consisting of the hyperedges $e/\mathcal{P}$ such that $\pi(e)$ is $\mathcal{P}$-crossing; the representation is defined by $(\pi/\mathcal{P})(e/\mathcal{P}) = \pi(e)/\mathcal{P}$.

(Contraction is illustrated in Fig. 7.) In keeping with our notation, the complement of $\pi/\mathcal{P}$ in $H/\mathcal{P}$ is denoted by $\pi/\mathcal{P}$. Observe that if $e \in E(H)$, then $e/\mathcal{P}$ is an edge of $\pi/\mathcal{P}$ if and only if $e$ is $\mathcal{P}$-crossing and $\pi(e)$ is not. The following lemma will be useful.
Lemma 15. Let $\mathcal{R} \leq \mathcal{P}$ be partitions of $V$ and $\pi$ be a quasigraph in $H$. If $\gamma / \mathcal{R}$ is a quasicycle in $\pi / \mathcal{R}$, then one of the following holds:

(a) for some $X \in \mathcal{P}$, $\gamma[X] / \mathcal{R}[X]$ is a quasicycle in the complement of $\pi[X] / \mathcal{R}[X]$ in $H[X]^\pi / \mathcal{R}[X],$

(b) $\gamma / \mathcal{P}$ is a nonempty quasigraph in $\pi / \mathcal{P}$ such that $(\gamma / \mathcal{P})^*$ is an eulerian graph (a graph with all vertex degrees even).

Proof. We will use the following two formal equalities whose proof is left to the kind reader as a slightly tedious exercise: for $X \in \mathcal{P}$ and any quasigraph $\sigma$ in $H$,

$\sigma[X] / \mathcal{R}[X] = (\sigma / \mathcal{R})[\mathcal{R}[X]],$

$H[X]^\pi / \mathcal{R}[X] = (H / \mathcal{R})[\mathcal{R}[X]]^\pi / \mathcal{R}.$

Let $\gamma / \mathcal{R}$ be a quasicycle in $\pi / \mathcal{R}$. Suppose that there is $X \in \mathcal{P}$ such that every edge of $(\gamma / \mathcal{R})^*$ is a subset of $\mathcal{R}[X]$. Let $\tilde{\gamma} = (\gamma / \mathcal{R})[\mathcal{R}[X]]$. Thus, $\tilde{\gamma}$ is a quasicycle in $(H / \mathcal{R})[\mathcal{R}[X]]$ and $E(\tilde{\gamma})$ is disjoint from $E((\pi / \mathcal{R})[\mathcal{R}[X]])$. We infer that $\tilde{\gamma}$ is a quasigraph in $(H / \mathcal{R})[\mathcal{R}[X]]^\pi / \mathcal{R}$. Using (6), we find that $\tilde{\gamma}$ is a quasigraph in $H[X]^\pi / \mathcal{R}[X]$. Finally, we use (5) twice (for $\gamma$ and $\pi$) and conclude that condition (a) holds.

Thus, we may assume that the endvertices $Y_1$, $Y_2$ of some edge $\gamma(e)$ of $(\gamma / \mathcal{R})^*$ are classes of $\mathcal{R}$ contained in different classes of $\mathcal{P}$ (say, $X_1$ and $X_2$, respectively). Thus, $\gamma / \mathcal{P}$ is a nonempty quasigraph in $H / \mathcal{P}$. Moreover, $E(\gamma / \mathcal{P})$ is clearly disjoint from $E(\pi / \mathcal{P})$. To verify (b), it remains to prove that $(\gamma / \mathcal{P})^*$ is eulerian. This is immediate from the fact that $(\gamma / \mathcal{P})^*$ can be obtained from the graph $(\gamma / \mathcal{R})^*$ (which consists of a cycle and isolated vertices) by identifying certain sets of vertices (namely those contained in the same class of $\mathcal{P}$).

If $X \subseteq V$ and $\sigma$ is a quasigraph in $H[X]^\pi$, we define the substitution of $\sigma$ into $\pi$ as the operation which produces the following quasigraph $\pi \mid \sigma$ in $H$:

$$(\pi \mid \sigma)(e) = \begin{cases} \pi(e) & \text{if } e \cap X \not\in E(H[X]^\pi), \\ \sigma(e \cap X) & \text{otherwise}. \end{cases}$$

This yields a well-defined represented subhypergraph of $H$ (see Fig. 8). More generally, let $\mathcal{P}$ be a family of disjoint subsets of $V$ and for each $X \in \mathcal{P}$, let $\sigma_X$ be a quasigraph in $H[X]^\pi$. Assume we substitute each $\sigma_X$ into $\pi$ in any order. For distinct $X \in \mathcal{P}$, the hyperedge sets of the hypergraphs $H[X]^\pi$ are pairwise disjoint, since $e \in E(H[X]^\pi)$ only if $|e \cap X| \geq 2$. It follows easily that the resulting hypergraph $\sigma$ in $H$ is independent of the chosen order. This hypergraph will be denoted by

$$\sigma = \pi \mid \{\sigma_X : X \in \mathcal{P}\}.$$

Substitution behaves well with respect to taking induced quasigraphs and contraction.

Lemma 16. Let $\pi$ be a quasigraph in $H$ and $\mathcal{P}$ a partition of $V$. Suppose that for each $X \in \mathcal{P}$, $\sigma_X$ is a quasigraph in $H[X]^\pi$, and define

$$\sigma = \pi \mid \{\sigma_X : X \in \mathcal{P}\}.$$ 

Then the following holds for every $Y \subseteq X \in \mathcal{P}$:

(i) $H[Y]^\sigma = (H[X]^\pi)[Y]^\sigma_X$,

(ii) $\sigma[Y] = \sigma_X[Y]$.

Furthermore,

(iii) $\sigma / \mathcal{P} = \pi / \mathcal{P}$.

Proof. (i) Using the definition of $H[Y]^\sigma$ and the definition of substitution, it is not hard to verify that $e_0 \subseteq V$ is a hyperedge of $H[Y]^\sigma$ if and only if $e_0 = e \cap Y$, where $e$ is a hyperedge of $H$ such that $|e \cap Y| \geq 2$, $\pi(e) \subseteq X$ and $\sigma_X(e \cap X) \subseteq Y$. If we expand the right hand side of the equality in (i) according to these definitions, we arrive at precisely the same set of conditions.
(ii) Both sides of the equation are quasigraphs in \( H[Y]^{\sigma} \). We will check that they assign the same value to a hyperedge \( e \cap Y \) of \( H[Y]^{\sigma} \). For such hyperedges, we have

\[
\sigma[Y](e \cap Y) = \sigma(e) = \sigma_X(e \cap X)
\]

where the second equality follows from the definition of substitution. On the other hand, by part (i), \( e \cap Y \) is a hyperedge of \( (H[X]^{\pi})[Y]^{\sigma_X} \), and thus

\[
\sigma_X[Y](e \cap Y) = \sigma_X(e \cap X).
\]

The assertion follows by comparing (7) and (8).

(iii) Both \( \sigma / \mathcal{P} \) and \( \pi / \mathcal{P} \) are quasigraphs in \( H / \mathcal{P} \). Let \( e / \mathcal{P} \) be a hyperedge of \( H / \mathcal{P} \), where \( e \in E(H) \). Using the definitions of substitution and contraction, one can check that

\[
(\sigma / \mathcal{P})(e / \mathcal{P}) = \begin{cases} 
\pi(e) / \mathcal{P} & \text{if } e \cap X \notin E(H[X]^{\pi}) \text{ and } \pi(e) \text{ is } \mathcal{P}\text{-crossing,} \\
\sigma_X(e) / \mathcal{P} & \text{if } e \cap X \in E(H[X]^{\pi}) \text{ and } \sigma_X(e) \text{ is } \mathcal{P}\text{-crossing,} \\
\emptyset & \text{otherwise.}
\end{cases}
\]

However, the middle case can never occur since \( \sigma_X(e) \subseteq X \) and \( \sigma_X(e) \) is therefore not \( \mathcal{P}\)-crossing. It follows easily that \( (\sigma / \mathcal{P})(e / \mathcal{P}) = (\pi / \mathcal{P})(e / \mathcal{P}) \). \( \square \)

7. The Skeletal Lemma

In this section, we prove a lemma which is a crucial piece of our method. It leads directly to an inductive argument for the existence of a quasitree with tight complement under suitable assumptions, which will be given in Section 8.

If \( \pi \) is a quasigraph in \( H \), then a partition \( \mathcal{P} \) of \( V \) is said to be \( \pi\text{-skeletal} \) if every \( X \in \mathcal{P} \) is \( \pi\text{-solid} \) and the complement of \( \pi / \mathcal{P} \) in \( H / \mathcal{P} \) is acyclic.
Lemma 17 (Skeletal Lemma). Let \( \pi \) be an acyclic quasigraph in \( H \). Then there is an acyclic quasigraph \( \sigma \) in \( H \) such that \( \sigma \geq \pi \) and \( \sigma \) satisfies one of the following:

(a) \( \sigma \succ \pi \), or
(b) there is a \( \sigma \)-skeletal partition \( \delta \).

Proof. We proceed by contradiction. Let the pair \( (\pi, H) \) be a counterexample such that \( H \) has minimal number of vertices; thus, no acyclic quasigraph \( \sigma \geq \pi \) in \( H \) satisfies any of (a) and (b). Note that \( \pi \) is not a quasitree with tight complement (which includes the case \(|V| = 1\)), for otherwise \( \sigma = \pi \) would satisfy condition (b) with \( \delta = \{V\} \).

Claim 1. \( \mathcal{P}_0^{\pi} \) is nontrivial.

Suppose the contrary and note that \( \mathcal{P} := \mathcal{A}_-(\pi) \) is nontrivial. Consider a set \( Y \in \mathcal{P} \) and the acyclic quasigraph \( \pi[Y] \). By the minimality of \( H \), there is a quasigraph \( \sigma_Y \geq \pi[Y] \) in \( H[Y]^\pi \) satisfying condition (a) or (b) (with respect to \( \pi[Y] \) and \( H[Y]^\pi \)). Define

\[
\sigma = \pi | \{ \sigma_Y \mid Y \in \mathcal{P} \}.
\]

By Lemmas 14 and 16(ii), we may assume that \( \sigma \) is acyclic.

Assume first that for some \( Y \in \mathcal{P} \), \( \sigma_Y \succ \pi[Y] \) (case (a) of the lemma). Since \( \sigma[Y] = \sigma_Y \) (Lemma 16(ii)), Lemma 12 implies that \( \sigma \succ \pi \), a contradiction with the choice of \( \pi \).

We conclude that case (b) holds for each \( Y \in \mathcal{P} \), namely that there exists a partition \( \delta_Y \) which is \( \sigma_Y \)-skeletal in \( H[Y]^\pi \). Set

\[
\delta = \bigcup_{Y \in \mathcal{P}} \delta_Y.
\]

We claim that \( \delta \) is \( \sigma \)-skeletal. Let \( Z \in \delta \) and assume that \( Z \subseteq Y \in \mathcal{P} \). Since \( Z \) is \( \sigma_Y \)-solid, and since \( \sigma[Z] = \sigma_Y[Z] \) and \( H[Y]^\pi = (H[Y]^\pi)[Z]^{\sigma_Y} \) by Lemma 16(i)–(ii), \( Z \) is \( \sigma \)-solid.

Suppose that \( \sigma/\delta \) is not acyclic and choose a quasigraph \( \gamma \) in \( H \) such that \( \sigma/\delta \) is a quasicycle in \( \sigma/\delta \). By Lemma 15, \( \gamma/\mathcal{P} \) is a nonempty quasigraph in the complement \( \overline{\pi}/\mathcal{P} \) of \( \pi/\mathcal{P} \) in \( H/\mathcal{P} \). However, by the definition of \( \mathcal{A}_-(\pi) \), every \( \mathcal{P} \)-crossing hyperedge of \( H \) belongs to \( \pi/\mathcal{P} \) and thus cannot be used by \( \gamma/\mathcal{P} \), a contradiction. It follows that \( \sigma/\delta \) is indeed acyclic and \( \delta \) is \( \sigma \)-skeletal. This contradiction with the choice of \( \pi \) concludes the proof of the claim. \( \triangle \)

For each \( X \in \mathcal{P}_0^{\pi} \), \( H[X]^\pi \) has fewer vertices than \( H \). By the minimality of \( H \), there is an acyclic quasigraph \( \rho_X \geq \pi[X] \) in \( H[X]^\pi \). Define

\[
\rho = \pi | \{ \rho_X \mid X \in \mathcal{P}_0^{\pi} \}.
\]

By Lemma 12, \( \rho \geq \pi \). Note that since \( \mathcal{P}_0^{\pi} \) is \( \pi \)-wide, \( \rho^* = \) the disjoint union of the graphs \( \rho_X^* (X \in \mathcal{P}_0^{\pi}) \). Therefore, \( \rho \) is acyclic.

If \( \rho_X \succ \pi[X] \) for some \( X \in \mathcal{P}_0^{\pi} \), then by Lemmas 16(ii) and 12, \( \rho \succ \pi \) and we have a contradiction. Consequently, for each \( X \in \mathcal{P}_0^{\pi} \), there is a \( \rho_X \)-skeletal partition \( \mathcal{R}_X \) (with respect to the hypergraph \( H[X]^\pi \)). We define a partition \( \mathcal{R} \) of \( V \) by

\[
\mathcal{R} = \bigcup_{X \in \mathcal{P}_0^{\pi}} \mathcal{R}_X.
\]

Similarly as in the proof of Claim 1, each \( Y \in \mathcal{R} \) is easily shown to be \( \rho \)-solid. An important difference in the present situation, however, is that \( \mathcal{R} \) may not be \( \rho \)-skeletal as there may be quasicycles in \( \overline{\rho}/\mathcal{R} \).

Any such quasicycle \( \gamma' \) can be represented by a quasigraph \( \gamma \) in \( H \) such that \( \gamma' = \gamma/\mathcal{R} \).

Thus, let \( \gamma \) be a quasigraph in \( H \) such that \( \gamma/\mathcal{R} \) is a quasicycle in \( \overline{\rho}/\mathcal{R} \). By Lemma 15, there are two possibilities:

(a) for some \( X \in \mathcal{P}_0^{\pi} \), \( \gamma[X]/\mathcal{R}_X \) is a quasicycle in the complement of \( \rho[X]/\mathcal{R}_X \) in \( H[X]^\rho/\mathcal{R}_X \), or
(b) \( \gamma/\mathcal{P}_0^{\pi} \) is a nonempty quasigraph in the complement of \( \rho/\mathcal{P}_0^{\pi} \) in \( H/\mathcal{P} \) such that \( (\gamma/\mathcal{P}_0^{\pi})^* \) is an eulerian graph.
the choice of distinct components of that definition proves (11). If each $H$ by symmetry, we may assume that First of all, we claim that $u$ (see Fig. 9). Considering the role of the hyperedge $e$ more than one vertex of $(\gamma / f_y)$ is used by $\rho$, then this class will be denoted by $Y_{\gamma}$ and we will say that the chosen hyperedge $f_y$ is a connector for $Y_{\gamma}$.

Claim 2. For each quasicycle $\gamma / R$ in $\overline{\rho / R}$, the hyperedge $f_y$ is used by $\rho$.

Suppose to the contrary that $\gamma(f_y) = u_1u_2$, where each $u_i (i = 1, 2)$ is contained in a different class $X_i$ of $\mathcal{P}^0_\rho$. By Lemma 11 and Observation 10(ii), $\mathcal{P}^\pi_0 = \mathcal{P}^\rho_0$. Let $\sigma$ be the quasigraph in $H$ defined by

$$\sigma(e) = \begin{cases} \pi(e) & \text{if } e \neq f_y, \\ u_1u_2 & \text{otherwise} \end{cases}$$

(see Fig. 9). Considering the role of the hyperedge $e$, we see that

$$\mathcal{P}^\rho_0 < \mathcal{P}^\sigma_0. \quad (10)$$

Next, we would like to prove that

$$\mathcal{A}_-(\rho) \leq \mathcal{A}_-(\sigma). \quad (11)$$

First of all, we claim that $u_1$ and $u_2$ are contained in the same class of $\mathcal{A}_-(\sigma)$. Let the vertices on the unique cycle in $(\gamma / R)^* = (T_1, \ldots, T_k)$ in this order, where each $T_i$ is a class of $R$, $u_1 \in T_1$ and $u_2 \in T_k$. By symmetry, we may assume that $f_y \cap T_i = 1$ (i.e., $T_1$ is the only class of $R$ which may contain more than one vertex of $f_y$).

By Lemma 16(i)–(ii), together with the fact that each $Y \in R$ is $\rho_X$-solid (where $Y \subseteq X \subseteq \mathcal{P}^\pi_0$), each $T_i (i = 1, \ldots, k)$ is $\rho$-solid. Thus, $T_i$ is also $\sigma$-solid for $i \geq 2$. Let $T_1'$ be the negative $\sigma[T_1]$-part of $H[T_1]^\rho$ containing $u_1$.

For $i = 1, \ldots, k - 1$, let $e_i$ be the hyperedge of $E(\gamma)$ such that $\gamma(e_i) / R = T_iT_{i+1}$ (choosing $e_1 \neq f_y$ if $k = 2$). Let $T = T_1' \cup T_2 \cup \cdots \cup T_k$. Using the minimality of $H$ and Lemma 9(ii), it is easy to prove that $T$ is a subset of a class, say $Q$, of $\mathcal{A}_-(\sigma)$. Note that $Q$ contains $u_1$ and $u_2$ as claimed.

If (11) is false, then the unique vertex of $f_y - \{u_1, u_2\}$ is necessarily contained in a class of $\mathcal{A}_-(\sigma)$ distinct from $Q$. In that case, however, $\mathcal{A}_-(\sigma)$ is not $\sigma$-narrow as $\sigma(f_y) \subseteq Q$. This contradiction with the definition proves (11).

By (10) and (11), $\pi \preceq \rho < \sigma$. Moreover, $\sigma$ is acyclic, since $\rho$ is acyclic and $\sigma(f_y)$ has endvertices in distinct components of $\rho^\pi$. Thus, $\sigma$ satisfies condition (a) in the statement of the lemma, contradicting the choice of $\pi$. \( \Delta \)
For any $Y \in \mathcal{R}$, let $\text{conn}(Y)$ be the set of all connectors for $Y$, and write

$$\text{conn}_2(Y) = \{ f \cap Y : f \in \text{conn}(Y) \}.$$ 

Note that for any connector $f$ for $Y$, $f \cap Y$ is a 2-hyperedge of $\rho[Y]$.

Let us describe our strategy in the next step in intuitive terms (see Fig. 10 for an illustration). We want to modify $\rho$ within the classes of $\mathcal{R}$ and 'free' one of the hyperedges $f_\gamma$ from $\rho$, which would enable us to apply the argument from the proof of Claim 2 and reach a contradiction. If no such modification works, we obtain a quasigraph $\sigma$ and a partition $\delta$ which refines $\mathcal{R}$. The effect of the refinement is to 'destroy' all quasicycles $\gamma / \mathcal{R}$ in $\rho / \mathcal{R}$ by making the representation $\rho(f_\gamma)$ of each associated connector $f_\gamma$ $\delta$-crossing. Thanks to this, it will turn out that $\delta$ is $\sigma$-skeletal as required to satisfy condition (b).

Thus, let $Y \in \mathcal{R}$ and set

$$\bar{H}_Y = H[Y]^{\rho} - \text{conn}_2(Y),$$

$$\bar{\rho}_Y = \rho[Y] - \text{conn}_2(Y)$$

(we allow $\text{conn}_2(Y) = \emptyset$) and observe that $\bar{\rho}_Y$ is an acyclic quasigraph in $\bar{H}_Y$. Let $\sigma_Y$ be a $\leq$-maximal acyclic quasigraph in $\bar{H}_Y$ such that $\sigma_Y \geq \bar{\rho}_Y$. We define a quasigraph $\tau_Y$ in $H[Y]^{\rho}$ by

$$\tau_Y(e) = \begin{cases} e & \text{if } e \in \text{conn}_2(Y), \\ \sigma_Y(e) & \text{otherwise}. \end{cases}$$

Claim 3. For all $Y \in \mathcal{R}$,

$$\mathcal{A}_+(\sigma_Y; \bar{H}_Y) = \mathcal{A}_+(ar{\rho}_Y; \bar{H}_Y).$$

From $\sigma_Y \geq \bar{\rho}_Y$, we know that the left hand side in the statement of the claim is coarser than (or equal to) the right hand side. Suppose that for some $Y \in \mathcal{R}$, $\mathcal{A}_+(\sigma_Y; \bar{H}_Y)$ is strictly coarser than $\mathcal{A}_+(ar{\rho}_Y; \bar{H}_Y)$. Then we can choose vertices $u_1, u_2 \in Y$ which are contained in different classes $U_1, U_2$, respectively, of $\mathcal{A}_+(ar{\rho}_Y; \bar{H}_Y)$, but in the same class $U$ of $\mathcal{A}_+(\sigma_Y; \bar{H}_Y)$. Since $Y$ is $\rho$-solid, the graph $\rho[Y]^*$ contains a path $P$ joining $u_1$ to $u_2$. The choice of $u_1$ and $u_2$ implies the following:

(A1) $P$ contains the edge $f_\gamma \cap Y \in \text{conn}_2(Y)$ for some quasicycle $\gamma$, and

(A2) all the edges of $E(P) \cap \text{conn}_2(Y)$ are contained in a cycle in $(\rho[\sigma_Y])^*$.

We choose a quasicycle $\gamma$ satisfying (A1) and let $\tau$ be the quasigraph in $H$ obtained as

$$\tau = (\rho[\tau_Y] - f_\gamma) \cap Y.$$ 

By (A2) and the fact that $\rho[Y]$ is connected, $\tau[Y]$ is connected as well. Since $\sigma_Y$ has tight complement in $\bar{H}_Y$, $\tau[Y]$ has tight complement in $H[Y]^{\rho}$ (the two complements coincide). Thus, $\tau$ is $\tau$-solid. By Corollary 13, $\tau \geq \rho$. By Lemma 14 and the fact that $\rho \geq \pi$, we may assume that $\tau$ is acyclic.

Since $\rho$ and $\tau$ are $Y$-similar, we have

$$\rho / \mathcal{R} = \tau / \mathcal{R}.$$ 

In particular, the quasicycle $\gamma$ in $\rho / \mathcal{R}$ (associated with $f_\gamma$) is also a quasicycle in $\tau / \mathcal{R}$. As $f_\gamma$ is not used by $\tau$ (and $\tau \geq \rho$), we can repeat the argument used in the proof of Claim 2, namely add $f_\gamma$ (with a suitable representation) to $\tau$ and reach a contradiction with the choice of $\pi$. △

We will now construct a $\sigma$-skeletal partition of $V$. Let $Y \in \mathcal{R}$. By the choice of $H$ and the maximality of $\sigma_Y$, there is a $\sigma_Y$-skeletal partition $\delta_Y$ of $Y$ (in $\bar{H}_Y$). We define a quasigraph $\sigma$ in $H$ and a partition $\delta$ of $V$ by

$$\sigma = \rho \{ \tau_Y : Y \in \mathcal{R} \},$$

$$\delta = \bigcup_{Y \in \mathcal{R}} \delta_Y.$$
(a) Bold lines show the quasigraph $\rho$, the dotted regions are the positive $\rho_Y$-parts of $\tilde{H}_Y$.

(b) The dotted regions here are the positive $\sigma_Y$-parts of $\tilde{H}_Y$. If the partition is strictly coarser than in (a), we can 'free' a suitable connector $f_\gamma$ and use it as before.

(c) Otherwise, we obtain a finer partition $\delta$ (darkest grey regions) such that $\rho(f_\gamma)$ is $\delta$-crossing for each $\gamma$.

Fig. 10. An illustration to the proof of Claim 3 and the following part of the proof. We use similar conventions as in Fig. 9.

We aim to show that $\delta$ is $\sigma$-skeletal. Let $Z \in \delta$ and suppose that $Z \subseteq Y \subseteq X$, where $X \in \mathcal{P}_{\pi}^0$ and $Y \in \mathcal{R}$. Since $\sigma[Z] = \sigma_Y[Z]$ and $\delta_Y$ is $\sigma_Y$-skeletal, $\sigma[Z]$ is a quasitree.

To show that the complement of $\sigma[Z]$ in $H[Z]^{\sigma}$ is tight, we use Lemma 16(i):

$$H[Z]^{\sigma} = (H[Y]^{\rho})[Z]^{\sigma_Y} = \tilde{H}_Y[Z]^{\sigma_Y} = \tilde{H}_Y[Z]^{\sigma_Y}.$$  \hfill (12)

Here, the second and the third equality follows from Claim 3 which implies that any connector for $Y$ intersects two classes of $\mathcal{A}_+(\sigma_Y; \tilde{H}_Y)$. From (12) and the fact that $\sigma_Y[Z]$ has tight complement in $\tilde{H}_Y[Z]^{\sigma_Y}$, it follows that $\sigma[Z]$ has tight complement as well.

It remains to prove that $\sigma/\delta$ is acyclic. Suppose, for the sake of a contradiction, that $\gamma$ is a quasigraph in $H$ such that $\gamma/\delta$ is a quasicycle in $\sigma/\delta$. Note that the complement of $\tau_Y/\delta_Y$ in $H[Y]^{\rho}$ is the same as the complement of $\sigma_Y/\delta_Y$ in $\tilde{H}_Y$, and hence acyclic. By Lemma 15, $\gamma/\mathcal{R}$ is a nonempty quasigraph in $\rho/\mathcal{R}$ with $(\gamma/\mathcal{R})^* \text{ eulerian}.$

Let $\delta$ be a restriction of $\gamma$ such that $\delta/\mathcal{R}$ is a quasicycle in $\rho/\mathcal{R}$. Every such quasicycle has an associated hyperedge $f_\delta$ which is a connector for a class $Y_\delta \in \mathcal{R}$ (Claim 2). In particular, $f_\delta$ is used by $\rho$. By the fact that $f_\delta$ intersects two classes of $\mathcal{A}_+(\sigma_Y; \tilde{H}_Y)$, $\rho(f_\delta)$ is $\delta$-crossing. This implies that $\sigma(f_\delta)$ is $\delta$-crossing, which contradicts the assumption that $\gamma/\delta$ is a quasicycle in $\sigma/\delta$. The proof is complete. \qed
8. Proof of Theorem 5

We can now prove our main result regarding spanning trees in hypergraphs, announced in Section 3 as Theorem 5:

**Theorem.** Let \( H \) be a 4-edge-connected 3-hypergraph. If no 3-hyperedge of \( H \) is included in any edge-cut of size 4, then \( H \) contains a quasitree with tight complement.

**Proof.** Let \( \pi \) be a \( \leq \)-maximal acyclic quasigraph in \( H \). By the Skeletal Lemma (Lemma 17), there exists a \( \pi \)-skeletal partition \( P \) of \( V \). For the sake of a contradiction, suppose that \( \pi \) is not a quasitree with tight complement. In particular, \( P \) is nontrivial.

Assume that \( H/P \) has \( n \) vertices (that is, \( |P| = n \)) and \( m \) hyperedges. For \( k \in \{2, 3\} \), let \( m_k \) be the number of \( k \)-hyperedges of \( \pi/P \). Similarly, let \( \overline{m_k} \) be the number of \( k \)-hyperedges of \( \pi/\overline{P} \). Thus, \( m = m_2 + m_3 + \overline{m_2} + \overline{m_3} \).

Since \( \pi/\overline{P} \) is acyclic, the graph \( G(\pi/\overline{P}) \) (defined in Section 2) is a forest. As \( G(\pi/\overline{P}) \) has \( n + \overline{m_2} \) vertices and \( \overline{m_2} + 3\overline{m_3} \) edges, we find that

\[
\overline{m_2} + 2\overline{m_3} \leq n - 1. \tag{13}
\]

Since \( P \) is \( \pi \)-solid and \( \pi \) is an acyclic quasigraph, we know that \( m_2 + m_3 \leq n - 1 \). Moreover, by the assumption that \( \pi \) is not a quasitree with a tight complement, either this inequality or (13) is strict. Summing the two, we obtain

\[
m + \overline{m_3} \leq 2n - 3. \tag{14}
\]

We let \( n_4 \) be the number of vertices of \( H/P \) of degree 4, and \( n_{5+} \) be the number of the other vertices. Since \( n \geq 2 \) and \( H \) is 4-edge-connected, we have \( n = n_4 + n_{5+} \). By double counting,

\[
4n_4 + 5n_{5+} \leq 2(m_2 + \overline{m_2}) + 3(m_3 + \overline{m_3}) = 2m + m_3 + \overline{m_3}. \tag{15}
\]

The left hand side equals \( 4n + n_{5+} \). Using (14), we find that

\[
4n + n_{5+} \geq 2m + 2\overline{m_3} + n_{5+} + 6.
\]

Combining with (15), we obtain

\[
m_3 \geq \overline{m_3} + n_{5+} + 6. \tag{16}
\]

We show that \( m_3 \leq n_{5+} \). Let \( T' = (\pi/P)^* \) be the forest on \( P \) which represents \( \pi/P \). In each component of \( T' \), choose a root and direct the edges of \( T' \) away from it. To each 3-hyperedge \( e \in E(\pi/P) \), assign the head \( h(e) \) of the arc \( \pi(e) \). By the assumptions of the theorem, no edge-cut of size 4 contains a 3-hyperedge, so \( h(e) \) is a vertex of degree at least 5. At the same time, since each vertex is the head of at most one arc in the directed forest, it gets assigned to at most one hyperedge. The inequality \( m_3 \leq n_{5+} \) follows. This contradiction to inequality (16) proves that \( \pi \) is a quasitree with tight complement. \( \square \)

9. Even quasitrees

In the preceding sections, we were busy looking for quasitrees with tight complement in hypergraphs. In this and the following section, we will explain the significance of such quasitrees for the task of finding a Hamilton cycle in the line graph of a given graph.

Let \( \pi \) be a quasitree in \( H \). For a set \( X \subseteq V \), we define a number \( \Phi_{\pi}(X) \in \{0, 1\} \) by

\[
\Phi_{\pi}(X) \equiv \sum_{v \in X} d_{\pi^*}(v) \pmod{2}.
\]

Observe that \( \Phi_{\pi}(X) = 0 \) if and only if \( X \) contains an even number of vertices whose degree in the tree \( \pi^* \) is odd.

For \( X \subseteq V \), we say that \( \pi \) is even on \( X \) if for every component \( K \) of \( \overline{\pi} \) whose vertex set is a subset of \( X \), it holds that \( \Phi_{\pi}(V(K)) = 0 \). If \( \pi \) is even on \( V \), then we just say \( \pi \) is even.

The main result of this section is the following:
Lemma 18. If \( \pi \) is a quasitree in \( H \) with tight complement, then there is a quasigraph \( \rho \) in \( H \) such that \( E(\rho) = E(\pi) \) and \( \rho \) is an even quasitree in \( H \).

Lemma 18 is a direct consequence of the following more technical statement (to derive Lemma 18, set \( X = V \)):

Lemma 19. Let \( \pi \) be a quasitree in \( H \) and \( X \subseteq V \). Assume that \( \Phi_\pi(X) = 0 \) and \( \pi \) has tight complement in \( H[X]^{\pi} \). Then there is a quasitree \( \rho \) in \( H \) such that \( \pi \) and \( \rho \) are \( X \)-similar, and \( \rho \) is even on \( X \).

Proof. We proceed by induction on \( |X| \). We may assume that \( |X| \geq 2 \), since otherwise the claim is trivially true. Similarly, if \( \overline{\pi}[X] \) is connected, then the assumption \( \Phi_\pi(X) = 0 \) implies that \( \pi \) is even on \( X \). Thus, we assume that \( \overline{\pi}[X] \) is disconnected.

The definition implies that there is a partition \( X = X_1 \cup X_2 \) such that:

- (B1) for each \( i = 1, 2 \), \( \pi[X_i] \) has tight complement in \( H[X_i]^{\pi} \),
- (B2) there is a hyperedge \( e \) intersecting \( X_2 \) with \( \pi(e) \subseteq X_1 \), and
- (B3) for any hyperedge \( f \) intersecting both \( X_1 \) and \( X_2 \), we have \( f \in E(\pi) \).

If \( \Phi_\pi(X_1) = 0 \), then we may use the induction hypothesis with \( X_1 \) playing the role of \( X \). The result is a quasitree \( \rho_1 \) in \( H \) which is even on \( X_1 \) and \( X_1 \)-similar to \( \rho \). In particular, \( \Phi_{\rho_1}(X_1) = 0 \) and hence also \( \Phi_{\rho_1}(X_2) = 0 \). Using the induction hypothesis for \( X_2 \), we obtain a quasitree \( \rho_2 \) in \( H \) which is even on \( X_2 \); furthermore, being \( X_2 \)-similar to \( \rho_1 \), it is even on \( X_1 \) as well. By (B3), the vertex set of every component \( K \) of \( \overline{\pi} \) with \( V(K) \subseteq X \) is a subset of \( X_1 \) or \( X_2 \). Thus, \( \rho := \rho_2 \) is even on \( X \), and clearly \( X \)-similar to \( \pi \).

It remains to consider the case that \( \Phi_\pi(X_1) = 1 \), illustrated in Fig. 11. Here we need to ‘switch’ the representation of \( e \) (the hyperedge from (B2)) as follows. Let \( e = x_1x_2y \), with \( \pi(e) = x_1x_2 \). The removal of the edge \( x_1x_2 \) from \( \pi^* \) splits \( \pi^* \) into two components, each containing one of \( x_1 \) and \( x_2 \). By symmetry, we may assume that \( y \) is contained in the component containing \( x_1 \). We define a new quasigraph \( \pi' \) in \( H \) by

\[
\pi'(e) = \begin{cases} 
x_2y \\
\pi(f) 
\end{cases} \quad \text{if } f = e, \\
\pi(f) \quad \text{otherwise}.
\]

Note that \( \pi' \) is a quasitree and \( \Phi_{\pi'}(X_1) = 0 \). Consequently, we can proceed as before, apply the induction hypothesis and eventually obtain a representation \( \rho \) which satisfies the assertions of the lemma. \( \square \)

10. Hamilton cycles in line graphs and claw-free graphs

We recall two standard results which interpret the connectivity and the hamiltonicity of a line graph in terms of its preimage. The first result is a folklore observation, the second is due to Harary and Nash-Williams [8]. We combine them into one theorem, but before we state them, we recall some necessary terminology.

Let \( G \) be a graph. An edge-cut \( C \) in \( G \) is trivial if it consists of all the edges incident with some vertex \( v \) of \( G \). The graph \( G \) is essentially \( k \)-edge-connected \((k \geq 1) \) if every edge-cut in \( G \) of size less than \( k \) is trivial. A subgraph \( D \) of \( G \) is dominating if \( G - V(D) \) has no edges.
Theorem 20. For any graph $G$ and $k \geq 1$, the following holds:

(i) $L(G)$ is $k$-connected if and only if $G$ is essentially $k$-edge-connected,

(ii) $L(G)$ is hamiltonian if and only if $G$ contains a dominating connected eulerian subgraph $C$.

In a similar spirit, the minimum degree of $L(G)$ equals the minimum edge weight of $G$, where the weight of an edge $e$ is defined as the number of edges incident with $e$ and distinct from it.

Given a set $X$ of vertices of $G$, an $X$-join in $G$ is a subgraph $G'$ of $G$ such that a vertex of $G$ is in $X$ if and only if its degree in $G'$ is odd. (In particular, $\emptyset$-joins are eulerian subgraphs.)

We will need a lemma which has been used a number of times before, either explicitly or implicitly.

Lemma 22. If $\pi$ is an even quasitree in $H$, then there is a quasigraph $\tau$ in $H$ such that $E(\pi)$ and $E(\tau)$ are disjoint, and $\pi^* + \tau^*$ is a connected eulerian subgraph of $Gr(H)$ spanning all vertices in $V$.

Proof. Let $K$ be a component of $\pi$, and let $X$ be the set of vertices of $K$ whose degree in $\pi^*$ is odd. Since $\pi$ is even, $|X|$ is even. Choose a spanning tree $T$ of the (connected) graph $Gr(K)$. Using Lemma 21, choose a subforest $T'$ of $T$ such that for every vertex $w$ of $Gr(K)$, $d_{T'}(w)$ is odd if and only if $w \in X$. In $\pi^* + T'$, all the vertices of $K$ have even degrees. In fact, the same holds for any vertex $v_e$ of $Gr(K)$, where $e$ is a hyperedge of $H$ of size 3: if $e$ is used by $\pi$, then $d_{\pi^* + T'}(v_e) = 2$, and otherwise we have

$$d_{\pi^* + T'}(v_e) = d_{T'}(v_e),$$

which is even since $v_e \not\in X$. In particular, there is a quasigraph $\tau_K$ in $H$ such that $\tau_K^* = T'$.

We apply the above procedure repeatedly, one component of $\pi$ at a time. For this, we need to be sure that a 3-hyperedge $e$ will not be used by $\tau_K$, as well as $\tau_{K_1}$, where $K_1$ and $K_2$ are distinct components of $\pi$. For this, we need to be sure that a 3-hyperedge $e$ will not be used by $\tau_K$, as well as $\tau_{K_1}$, where $K_1$ and $K_2$ are distinct components of $\pi$. For this, we need to be sure that a 3-hyperedge $e$ will not be used by $\tau_K$, as well as $\tau_{K_1}$, where $K_1$ and $K_2$ are distinct components of $\pi$. For this, we need to be sure that a 3-hyperedge $e$ will not be used by $\tau_K$, as well as $\tau_{K_1}$, where $K_1$ and $K_2$ are distinct components of $\pi$. For this, we need to be sure that a 3-hyperedge $e$ will not be used by $\tau_K$, as well as $\tau_{K_1}$, where $K_1$ and $K_2$ are distinct components of $\pi$. 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components of $\pi$. This is clear, however, since $e$ can only be used by $\tau_k$ if $|e \cap V(K)| \geq 2$. Thus, the components of $\pi$ can be treated independently, and we eventually obtain an eulerian subgraph $S$ of $Gr(H)$. Since it contains the tree $\pi^*, S$ spans all of $V$, and since each of the trees $(\tau_k)^*$ contains an edge incident with a vertex in $V$ (unless $(\tau_k)^*$ is edgeless), it follows that $S$ is connected.  \qed

Using Theorem 20, it will be easy to derive our main result (Theorem 4) as a consequence of the following proposition. Let us remark that the proposition is closely related to a conjecture made by Jackson (see [1, Conjecture 4.48]) and implies one of its three versions.

**Proposition 23.** If $G$ is an essentially 5-edge-connected graph with minimum edge weight at least 6, then $G$ contains a connected eulerian subgraph spanning all the vertices of degree at least 4 in $G$.

**Proof.** For the sake of a contradiction, let $G$ be a counterexample with as few vertices as possible. Since the claim is trivially true for a one-vertex graph, we may assume $|V(G)| \geq 2$. For brevity, a good subgraph in a graph $G'$ will be a connected eulerian subgraph spanning all the vertices of degree at least 4 in $G'$.

**Claim 1. The minimum degree of $G$ is at least 3.**

Suppose first that $G$ contains a vertex $v$ of degree 2 with distinct neighbours $w_1$ and $w_2$. If we suppress $v$, the resulting graph $G'$ will be essentially 5-edge-connected. Furthermore, the minimum edge weight of $G'$ is at least 6 unless $G$ is the triangle $v w_1 w_2$ with the edge $w_1 w_2$ of multiplicity 5, which is however not a counterexample to the proposition. By the minimality assumption, $G'$ contains a good subgraph $C'$. It is easy to see that the corresponding subgraph of $G$ is also good.

Suppose then that $G$ contains a vertex $u$ of degree 1 or 2 with a single neighbour $z$. Let $U$ be the set of all the vertices of degree 1 or 2 in $G$ whose only neighbour is $z$. If $V(G) = U \cup \{z\}$, then the Eulerian subgraph consisting of just the vertex $z$ shows that $G$ is not a counterexample to the proposition. Thus, $z$ has a neighbour $x$ outside $U$. In fact, since $G$ is essentially 5-edge-connected, $z$ is incident with at least 5 edges whose other endvertex is not in $U$. Let $e$ be an edge with endvertices $z$ and $x$. Since the degree of $x$ is at least 3, the edge weight of $e$ in $G - U$ is at least 6. This implies that the minimum edge weight of $G - U$ is at least 6. Since the removal of $U$ does not create any new minimal essential edge-cut, $G - U$ is essentially 5-edge-connected. Since the degree of $z$ in $G - U$ is at least 5, any good subgraph in $G - U$ is a good subgraph in $G$. Thus, $G - U$ is a smaller counterexample than $G$, contradicting the minimality of $G$. \hfill \triangle$

**Claim 2. No vertex of degree 3 in $G$ is incident with a pair of parallel edges.**

Suppose that $v$ is a vertex of degree 3 incident with parallel edges $e_1, e_2$. If $v$ has only one neighbour, then any good subgraph of $G - v$ is good in $G$. By the minimality of $G$, $v$ must have exactly two neighbours, say $w$ and $z$, where $w$ is incident with $e_1$ and $e_2$. Let $G'$ be obtained from $G$ by removing $v$ and adding the edge $e_0$ with endvertices $w$ and $z$.

It is easy to see that $G'$ is essentially 5-edge-connected, and that any good subgraph of $G'$ can be modified to a good subgraph of $G$ (as $d_G(w) \geq 6$). We show that the minimal edge weight in $G'$ is at least 6.

Suppose the contrary and let $e$ be an edge of $G'$ of weight less than 6. We have $e \neq e_0$ as the assumptions imply that $d_G(w) \geq 6$ and $d_G(z) \geq 5$, so the weight of any edge with endvertices $w$ and $z$ in $G'$ is at least 8. Thus, $e$ is an edge of $G$.

It must be incident with $w$, for otherwise its weight in $G'$ would be the same as in $G$. Let $u$ be the endvertex of $e$ distinct from $w$. Since $d_G(w) \geq 6$, $w$ is incident in $G'$ with at least 3 edges of $G'$ distinct from $e_0$ and $e$. By the weight assumption, $u$ must be incident with only at most one edge of $G'$ other than $e$, contradicting Claim 1. \hfill \triangle$

Let $H$ be the 3-hypergraph whose vertex set $V$ is the set of all vertices of $G$ whose degree is at least 4; the hyperedges of $H$ are of two kinds:

- the edges of $G$ with both endvertices in $V$,
- 3-hyperedges consisting of the neighbours of any vertex of degree 3 in $G$.

Note that $H$ is well-defined, for any neighbour of a vertex of degree 3 in $G$ must have degree at least 4 (otherwise they would be separated from the rest of the graph by an essential edge-cut of size at
most 4). Furthermore, by Claim 2, any vertex of degree 3 does indeed have three distinct neighbours in \( V \).

In the following two claims, we show that \( H \) satisfies the hypotheses of Theorem 5.

**Claim 3.** The hypergraph \( H \) is 4-edge-connected.

Suppose that this is not the case and \( F \) is an inclusionwise minimal edge-cut in \( H \) with \( |F| \leq 3 \). Let \( A \) be the vertex set of a component of \( H - F \).

Let \( e \in F \). By the minimality of \( G \), \( |e - A| \geq 1 \). We assign to \( e \) an edge \( e' \) of \( G \), defined as follows:

- if \( |e| = 2 \), then \( e' = e \),
- if \( |e| = 3 \) and \( e \cap A = \{u\} \), then \( e' = uv_e \),
- if \( |e| = 3 \), \( |e \cap A| = 2 \) and \( e - A = \{u\} \), then \( e' = uv_e \).

Observe that \( F' := \{ e' : e \in F \} \) is an edge-cut in \( G \). Since \( G \) is 5-edge-connected, \( F' \) must be a trivial edge-cut. This means that a vertex \( v \in V \) has degree 3 in \( H \), a contradiction as \( v \) has degree at least 4 in \( G \) and therefore also in \( H \).  \( \triangle \)

The other claim regards edge-cuts of size 4 in \( H \):

**Claim 4.** No 3-hyperedge of \( H \) is included in an edge-cut of size 4 in \( H \).

Let \( F \) be an edge-cut of size 4 in \( H \). As in the proof of Claim 3, we consider the corresponding edge-cut \( F' \) in \( G \). Since \( G \) is essentially 5-edge-connected, one component of \( G - F' \) consists of a single vertex \( w \) whose degree in \( G \) is 4. Assuming that \( F \) includes a 3-hyperedge \( e \), we find that in \( G \), \( w \) has a neighbour \( v \) of degree 3. Since the weight of the edge \( vw \) is 5, we obtain a contradiction with our assumptions about \( G \).  \( \triangle \)

Since the assumptions of Theorem 5 are satisfied, we can use it to find a quasitree \( \pi \) with tight complement in \( H \). By Lemmas 18 and 22, \( Gr(H) = G \) admits a connected eulerian subgraph spanning the set \( V \). This is what we wanted to find.  \( \square \)

We can now prove our main theorem, stated as Theorem 4 in Section 1:

**Theorem.** Every 5-connected line graph of minimum degree at least 6 is hamiltonian.

**Proof.** Let \( L(G) \) be a 5-connected line graph of minimum degree at least 6. By Theorem 20(i), \( G \) is essentially 5-edge-connected. Furthermore, the minimum edge weight of \( G \) is at least 6. By Proposition 23, \( G \) contains a connected eulerian subgraph \( C \) spanning all the vertices of degree at least 4. By Theorem 20(ii), it is sufficient to prove that \( G - V(C) \) has no edges. Indeed, the vertices of any edge \( e \) in \( G - V(C) \) must have degree at most 3 in \( G \), which implies that \( e \) is incident to at most 4 other edges of \( G \), a contradiction to the minimum degree assumption. Thus, \( L(G) \) is hamiltonian.  \( \square \)

Using the claw-free closure concept developed by Ryjáček [21], Theorem 4 can be extended to claw-free graphs. Let us recall the main result of [21]:

**Theorem 24.** Let \( G \) be a claw-free graph. Then there is a well-defined graph \( cl(G) \) (called the closure of \( G \)) such that the following holds:

- (i) \( G \) is a spanning subgraph of \( cl(G) \),
- (ii) \( cl(G) \) is the line graph of a triangle-free graph,
- (iii) the length of a longest cycle in \( G \) is the same as in \( cl(G) \).

**Corollary 25.** Every 5-connected claw-free graph \( G \) of minimum degree at least 6 is hamiltonian.

**Proof.** Apply Theorem 24 to obtain the closure \( cl(G) \) of \( G \). Since \( G \subseteq cl(G) \), the closure is 5-connected and has minimum degree at least 6. Being a line graph, \( cl(G) \) is hamiltonian by Theorem 4. Since \( G \) is a spanning subgraph of \( cl(G) \), property (iii) in Theorem 24 implies that \( G \) is hamiltonian.  \( \square \)

### 11. Hamilton-connectedness

Recall from Section 1 that a graph is Hamilton-connected if for every pair of distinct vertices \( u, v \), there is a Hamilton path from \( u \) to \( v \). The method used to prove Theorem 4 and Corollary 25 can be adapted to yield the following stronger result:
Theorem 26. Every 5-connected claw-free graph of minimum degree at least 6 is Hamilton-connected.

In this section, we sketch the necessary modifications to the argument. For a start, let \( H = L(G) \) be a 5-connected line graph of minimum degree at least 6. By considerations similar to those in the proof of Proposition 23, it may be assumed that the minimum degree of \( G \) is at least 3 and that no vertex of \( G \) is incident with a pair of parallel edges, so we may associate with \( G \) a 3-hypergraph \( H \) just as in that proof. Moreover, \( H \) may again be assumed to satisfy the assumptions of Theorem 5.

Let \( V_{≥4} \subseteq V(G) \) be the set of vertices of degree at least 4 in \( G \).

First, we will need a replacement of Theorem 20(ii) that translates the Hamilton-connectedness of \( H \) to a property of \( G \). A trail \( F \) is a sequence of edges of \( G \) such that each pair of consecutive edges is adjacent in \( G \), and \( F \) contains each edge of \( G \) at most once. We will say that \( F \) spans a set \( Y \) of vertices if each vertex in \( Y \) is incident with an edge of \( F \). A trail is an \((e_1, e_2)\)-trail if it starts with \( e_1 \) and ends with \( e_2 \). Furthermore, an \((e_1, e_2)\)-trail \( F \) is internally dominating if every edge of \( G \) has a common endvertex with some edge in \( F \) other than \( e_1 \) and \( e_2 \). The following fact is well-known (see, e.g., [17]):

Theorem 27. Let \( G \) be a graph with at least 3 edges. Then \( L(G) \) is Hamilton-connected if and only if for any pair of edges \( e_1, e_2 \in E(G) \), \( G \) has an internally dominating \((e_1, e_2)\)-trail.

One way to find an internally dominating \((e_1, e_2)\)-trail (where \( e_1, e_2 \) are edges) is by using a connection to \( X \)-joins as defined in Section 10. For each edge \( e \) of \( G \), fix an endvertex \( u_e \) of degree at least 4 in \( G \) (which exists since \( G \) is essentially 5-edge-connected). If \( e_1 \) and \( e_2 \) are edges, set

\[
X(e_1, e_2) = \begin{cases} \{u_{e_1}, u_{e_2}\} & \text{if } u_{e_1} \neq u_{e_2}, \\ \emptyset & \text{otherwise}. \end{cases}
\]

Suppose now that the graph \( G - e_1 - e_2 \) happens to contain a connected \( X(e_1, e_2) \)-join \( J \) spanning all of \( V_{≥4} \). By the classical observation of Euler, all the edges of \( J \) can be arranged in a trail \( T_J \) whose first edge is incident with \( u_{e_1} \) and whose last edge is incident with \( u_{e_2} \). Adding \( e_1 \) and \( e_2 \), we obtain an \((e_1, e_2)\)-trail \( T \) in \( G \). (If \( u_1 = u_2 \), we use the fact that \( u_1 \) is incident with an edge of \( T_J \).) Since \( G \) contains no adjacent vertices of degree 3, \( T \) is an internally dominating \((e_1, e_2)\)-trail.

Summing up, the Hamilton-connectedness of \( L(G) \) will be established if we can show that for every \( e_1, e_2 \in E(G) \), the graph \( G - e_1 - e_2 \) contains a connected \( X(e_1, e_2) \)-join spanning \( V_{≥4} \).

How to find such \( X(e_1, e_2) \)-joins? Recall that in Section 10, the existence of a connected dominating eulerian subgraph of \( G \) (a connected dominating \( \theta \)-join) was guaranteed by Lemma 22 based on the assumption that \( H \) contains an even quasitree. As shown by Lemma 18, an even quasitree in \( H \) exists whenever \( H \) contains a quasitree with tight complement. A rather straightforward modification of the proofs of these two lemmas (which we omit) leads to the following generalization:

Lemma 28. Let \( H' \) be a 3-hypergraph containing a quasitree \( \pi \) with tight complement, and let \( X \subseteq V(H') \). Then there is a quasitree \( \tau \) such that \( E(\pi) \) and \( E(\tau) \) are disjoint, and \( \pi^* + \tau^* \) is a connected \( X \)-join in \( Gr(H') \) spanning all vertices in \( V(H') \).

Roughly speaking, Lemma 28 will reduce our task to showing that for each pair of edges \( e_1, e_2 \) of \( G \), a suitably defined 3-hypergraph \( H' \) admits a quasitree with tight complement.

Let us define the 3-hypergraph \( H' \) to which Lemma 28 is to be applied. Suppose that \( e_1 \) and \( e_2 \) are given edges of \( G \), and let \( u_i \) (\( i = 1, 2 \)) be the endvertex of \( e_i \) distinct from \( u_i \). We distinguish two cases:

1. if \( e_1 \) and \( e_2 \) have a common vertex of degree 3 (namely, the vertex \( u_1 = u_2 \)), then \( H' \) is obtained from \( H \) by removing the 3-hyperedge corresponding to \( w_1 \);
2. otherwise, \( H' \) is the hypergraph obtained by performing the following for \( i = 1, 2 \):
   1. if \( u_i \) has degree 3, then the 3-hyperedge \( e_{u_i} \) of \( H \) corresponding to \( u_i \) is replaced by the 2-hyperedge \( e_{u_i} - \{u_i\} \);
   2. otherwise, the 2-hyperedge \( e_i \) of \( H \) is deleted.
By Lemma 28 and the preceding remarks, it suffices to show that \( H' \) admits a quasitree with tight complement. To do so, we apply to \( H' \) the proof of Theorem 5, which works well as far as equation (14). However, the inequality (15) may fail since \( H' \) is not necessarily 4-edge-connected. It has to be replaced as follows.

For an arbitrary hypergraph \( H^* \), let \( s(H^*) \) be the sum of all vertex degrees in \( H^* \). Let \( \mathcal{P} \) be the partition of \( V(H') \) obtained in the proof of Theorem 5. Furthermore, let \( n_4^s \) be the number of vertices of degree 4 in \( H/\mathcal{P} \), and let \( n_5^s = n - n_4^s \). (All the symbols such as \( n, m, m_3 \) etc., used in the proof of Theorem 5, are now related to the hypergraph \( H' \) rather than \( H \).)

It is not hard to relate \( s(H') \) to \( s(H) \). Indeed, the operations in cases (1), (2a) and (2b) above decrease the degree sum by 3, 1 and 2, respectively. It follows that \( s(H') \geq s(H) - 4 \) and, in fact,

\[
\begin{align*}
\frac{4n_4^s + 5n_5^s}{2m + m_3 + m_3}.
\end{align*}
\]

This eventually leads to

\[
\begin{align*}
m_3 \geq m_3 + n_5^s + 2
\end{align*}
\]

as a replacement for (16). Thus, the contradiction is much the same as before, since we have (by the same argument as in the old proof) that \( m_3 \leq n_5^s \). This proves Theorem 26 in the case of line graphs.

If \( G \) is a claw-free graph, we will use a closure operation again. However, the claw-free closure described in Section 10 is not applicable, since the closure of \( G \) may be Hamilton-connected even if \( G \) is not. Instead, we use the \( M \)-closure which was defined in [22] and applied there to prove that 7-connected claw-free graphs are Hamilton-connected. Let us list its relevant properties [22, Theorem 9]:

**Theorem 29.** If \( G \) is a connected claw-free graph, then there is a well-defined graph \( cl^M(G) \) with the following properties:

(i) \( G \) is a spanning subgraph of \( cl^M(G) \),
(ii) \( cl^M(G) \) is the line graph of a multigraph \( H \),
(iii) \( cl^M(G) \) is Hamilton-connected if and only if \( G \) is Hamilton-connected.

Using this result (and the fact that parallel edges are allowed throughout our argument), it is easy to prove Theorem 26 just like Corollary 25 is proved using the claw-free closure.

12. Conclusion

We have developed a method for finding dominating eulerian subgraphs in graphs, based on the concept of a quasitree with tight complement. Using this method, we have made some progress on Conjecture 2, although the conjecture itself is still wide open. It is conceivable that a refinement in some part of the analysis may improve the result a bit — perhaps to all 5-connected line graphs. On the other hand, the 4-connected case would certainly require major new ideas. For instance, the preimage \( G \) of a 4-connected line graph may be cubic, in which case we do not even know how to associate a 3-hypergraph with \( G \) in the first place.

As mentioned in Section 1, a simpler variant of our method yields a short proof of the tree-packing theorem of Tutte and Nash-Williams. It is well known that spanning trees in a graph \( G \) are the bases of a matroid, the cycle matroid of \( G \), and thus matroid theory provides a very natural setting for the tree-packing theorem. Interestingly, quasitrees with tight complement do not quite belong to the realm of matroid theory, although quasitrees themselves do. Is there an underlying abstract structure, more general than the matroidal one, which forms the ‘reason’ for the existence of both disjoint spanning trees in graphs, and quasitrees with tight complement in hypergraphs?

It remains a question for further research whether our approach may be useful for other problems on the packing of structures similar to spanning trees, but also lacking their matroidal properties. These include the packing of Steiner trees [13,14] or \( T \)-joins [3,20].
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