

Global stability of a delayed SIRS epidemic model with saturation incidence and temporary immunity

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ABSTRACT

In this paper, a delayed SIRS epidemic model with saturation incidence and temporary immunity is investigated. The immunity gained by experiencing a disease is temporary, whenever infected the diseased individuals will return to the susceptible class after a fixed period. By analyzing the corresponding characteristic equations, the local stability of an endemic equilibrium and a disease-free equilibrium is discussed. By comparison arguments, it is proved that if the basic reproduction number is less than unity, the disease-free equilibrium is globally asymptotically stable. If the basic reproduction number is greater than unity, by means of an iteration technique, sufficient conditions are obtained for the global asymptotic stability of the endemic equilibrium. Numerical simulations are carried out to illustrate the main theoretical results.

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1. Introduction

Mathematical models describing the population dynamics of infectious diseases have played an important role in better understanding epidemiological patterns and disease control for a long time. Attempts have been made to develop realistic mathematical models for the transmission dynamics of infectious diseases. Many models in the literature represent the dynamics of disease by systems of ordinary differential equations without time delay. However, inclusion of temporal delays in such models makes them more realistic by allowing the description of the effects of disease latency or immunity (see, for example, [1–12]). In [11], Wen and Yang considered the following delayed SIRS epidemic model

$$\begin{aligned}\dot{S}(t) &= \Lambda - \mu_1 S(t) - \beta S(t)I(t) + \gamma e^{-\mu_3 \tau} I(t - \tau), \\ \dot{I}(t) &= \beta S(t)I(t) - (\mu_2 + \gamma)I(t), \\ \dot{R}(t) &= \gamma I(t) - \mu_3 R(t) - \gamma e^{-\mu_3 \tau} I(t - \tau),\end{aligned}\tag{1.1}$$

where $S(t)$ denotes the number of members of a population susceptible to the disease, $I(t)$ the number of infective members and $R(t)$ the number of members who have been removed from the possibility of infection through a temporal immunity. In (1.1), parameters $\mu_1, \mu_2, \mu_3, \beta, \gamma$ and Λ are positive constants, here μ_1, μ_2, μ_3 represent the death rates of susceptibles, infectives and recovered, respectively. Biologically, it is natural to assume that $\mu_1 \leq \min\{\mu_2, \mu_3\}$; Λ is the recruitment rate of the susceptible population, β is the contact rate, and γ is the recovery rate from the infected compartment. The term $\gamma e^{-\mu_3 \tau} I(t - \tau)$ represents that an individual has survived natural death in a recovery pool before becoming susceptible again, where $\tau \geq 0$ is a constant representing the length of the immunity period. Sufficient conditions were derived in [11]

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for the global stability of an endemic equilibrium by using a Lyapunov functional approach. Recently, in [13], Brauer et al. further studied the system (1.1). By analyzing the characteristic equation, the local stability of the endemic equilibrium and the existence of Hopf bifurcations at the endemic equilibrium were carried out.

Incidence rate plays an important role in the modelling of epidemic dynamics. It has been suggested by several authors that the disease transmission process may have a nonlinear incidence rate. This allows one to include behavioral changes and prevent unbounded contact rates (see, for example, [14–16]). In many epidemic models, the bilinear incidence rate βSI and the standard incidence rate $\beta SI/N$ are frequently used. The bilinear incidence rate is based on the law of mass action. This contact law is more appropriate for communicable diseases such as influenza etc., but not for sexually transmitted diseases. It has been pointed out that for the standard incidence rate, it may be a good approximation if the number of available partners is large enough and everybody could not make more contacts than is practically feasible. After studying the cholera epidemic spread in Bari in 1973, Capasso and Serio [14] introduced a saturated incidence rate $g(I)S$ into epidemic models, where $g(I)$ tends to a saturation level when I becomes large, i.e.,

$$g(I) = \frac{\beta I}{1 + \alpha I},$$

where βI measures the infection force of the disease and $1/(1 + \alpha I)$ measures the inhibition effect from the behavioral change of the susceptible individuals when their number increases or from the crowding effect of the infective individuals. This incidence rate seems more reasonable than the bilinear incidence rate βSI , because it includes the behavioral change and crowding effect of the infective individuals and prevents the unboundedness of the contact rate by choosing suitable parameters.

Motivated by the work of Capasso and Serio [14] and Wen and Yang [11], in this paper, we are concerned with the effect of a saturation incidence rate and a time delay describing the temporal immunity period on the global dynamics of epidemic models. To this end, we consider the following delay differential equations

$$\begin{aligned} \dot{S}(t) &= \Lambda - \mu_1 S(t) - \frac{\beta S(t)I(t)}{1 + \alpha I(t)} + \gamma e^{-\mu_3 \tau} I(t - \tau), \\ \dot{I}(t) &= \frac{\beta S(t)I(t)}{1 + \alpha I(t)} - (\mu_2 + \gamma)I(t), \\ \dot{R}(t) &= \gamma I(t) - \mu_3 R(t) - \gamma e^{-\mu_3 \tau} I(t - \tau), \end{aligned} \quad (1.2)$$

where the parameters $\mu_1, \mu_2, \mu_3, \beta, \gamma, \Lambda$ and τ are defined as in (1.1), and α is a positive constant.

The initial conditions for system (1.2) take the form

$$\begin{aligned} S(\theta) &= \phi_1(\theta), & I(\theta) &= \phi_2(\theta), & R(\theta) &= \phi_3(\theta), \\ \phi_1(\theta) &\geq 0, & \phi_2(\theta) &\geq 0, & \phi_3(\theta) &\geq 0, & \theta \in [-\tau, 0], \\ \phi_1(0) &> 0, & \phi_2(0) &> 0, & \phi_3(0) &> 0, \end{aligned} \quad (1.3)$$

where $\Phi = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) \in C([-\tau, 0], \mathbb{R}_{+0}^3)$, the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into \mathbb{R}_{+0}^3 , where $\mathbb{R}_{+0}^3 = \{(x_1, x_2, x_3) : x_i \geq 0, i = 1, 2, 3\}$.

It is well known by the fundamental theory of functional differential equations [17], that system (1.2) has a unique solution $(S(t), I(t), R(t))$ satisfying the initial conditions (1.3). It is easy to show that all solutions of system (1.2) with initial conditions (1.3) are defined on $[0, +\infty)$ and remain positive for all $t \geq 0$.

The organization of this paper is as follows. In the next section, by analyzing the corresponding characteristic equations, the local stability of a disease-free equilibrium and an endemic equilibrium of system (1.2) is established. The existence of Hopf bifurcations at the endemic equilibrium is established. In Section 3, when the basic reproduction number is greater than unity, sufficient conditions are derived for the global stability of the endemic equilibrium of system (1.2) by means of an iteration technique. By comparison arguments, it is proved that if the basic reproduction number is less than unity, the disease-free equilibrium of system (1.2) is globally asymptotically stable. Numerical simulations are carried out to illustrate the main theoretical results. A brief discussion is given in Section 4 to conclude this work.

2. Local stability of equilibria

In this section, we discuss the local stability of a disease-free equilibrium and an endemic equilibrium of system (1.2) by analyzing the corresponding characteristic equations, respectively.

System (1.2) always has a disease-free equilibrium $E_1(\Lambda/\mu_1, 0, 0)$. Further, if $\Lambda\beta > \mu_1(\mu_2 + \gamma)$, then system (1.2) has a unique endemic equilibrium $E^*(S^*, I^*, R^*)$, where

$$\begin{aligned} S^* &= \frac{(\mu_2 + \gamma)[\Lambda\alpha + \mu_2 + \gamma(1 - e^{-\mu_3 \tau})]}{(\alpha\mu_1 + \beta)(\mu_2 + \gamma) - \beta\gamma e^{-\mu_3 \tau}}, \\ I^* &= \frac{\Lambda\beta - \mu_1(\mu_2 + \gamma)}{(\alpha\mu_1 + \beta)(\mu_2 + \gamma) - \beta\gamma e^{-\mu_3 \tau}}, \\ R^* &= \frac{\gamma(1 - e^{-\mu_3 \tau})[\Lambda\beta - \mu_1(\mu_2 + \gamma)]}{\mu_3[(\alpha\mu_1 + \beta)(\mu_2 + \gamma) - \beta\gamma e^{-\mu_3 \tau}]} \end{aligned}$$

Let

$$\mathcal{R}_0 = \frac{\Lambda\beta}{\mu_1(\mu_2 + \gamma)}.$$

\mathcal{R}_0 is called the basic reproduction number. It is easy to show that if $\mathcal{R}_0 > 1$, the endemic equilibrium E^* exists; if $\mathcal{R}_0 < 1$, then E^* is not feasible.

The characteristic equation of system (1.2) at the disease-free equilibrium $E_1(\Lambda/\mu_1, 0, 0)$ takes the form

$$(\lambda + \mu_1)(\lambda + \mu_3) \left(\lambda - \beta \frac{\Lambda}{\mu_1} + \mu_2 + \gamma \right) = 0. \tag{2.1}$$

Clearly, Eq. (2.1) always has two negative roots $\lambda_1 = -\mu_1, \lambda_2 = -\mu_3, \lambda_3 = \beta\Lambda/\mu_1 - \mu_2 - \gamma$. Hence, if $\mathcal{R}_0 < 1$, the equilibrium E_1 is locally asymptotically stable; if $\mathcal{R}_0 > 1, E_1$ is unstable.

The characteristic equation of system (1.2) at the positive equilibrium E^* is of the form

$$(\lambda + \mu_3)[\lambda^2 + p_1(\tau)\lambda + p_0(\tau) + q_0(\tau)e^{-\lambda\tau}] = 0, \tag{2.2}$$

where

$$\begin{aligned} p_0(\tau) &= (\mu_2 + \gamma)(\alpha\mu_1 + \beta) \frac{I^*}{1 + \alpha I^*}, \\ p_1(\tau) &= \mu_1 + \frac{\beta I^*}{1 + \alpha I^*} + (\mu_2 + \gamma) \frac{\alpha I^*}{1 + \alpha I^*}, \\ q_0(\tau) &= -\frac{\beta I^*}{1 + \alpha I^*} \gamma e^{-\mu_3\tau}. \end{aligned} \tag{2.3}$$

Clearly, Eq. (2.2) always has a negative real root $\lambda = -\mu_3$. Other roots of Eq. (2.2) are determined by the following equation

$$\lambda^2 + p_1(\tau)\lambda + p_0(\tau) + q_0(\tau)e^{-\lambda\tau} = 0. \tag{2.4}$$

When $\tau = 0$, Eq. (2.4) becomes

$$\lambda^2 + p_1(0)\lambda + p_0(0) + q_0(0) = 0. \tag{2.5}$$

It is easy to see that

$$p_0(0) + q_0(0) = [\alpha\mu_1(\mu_2 + \gamma) + \mu_2\beta] \frac{I^*}{1 + \alpha I^*} \Big|_{\tau=0} > 0.$$

Hence, if $\mathcal{R}_0 > 1$, the endemic equilibrium E^* of system (1.2) exists and is locally asymptotically stable when $\tau = 0$.

If $i\omega(\omega > 0)$ is a solution of (2.4), separating real and imaginary parts, we derive that

$$\begin{aligned} p_1(\tau)\omega &= q_0(\tau) \sin \omega\tau, \\ \omega^2 - p_0(\tau) &= q_0(\tau) \cos \omega\tau. \end{aligned} \tag{2.6}$$

Squaring and adding the two equations of (2.6), it follows that

$$\omega^4 + (p_1^2(\tau) - 2p_0(\tau))\omega^2 + p_0^2(\tau) - q_0^2(\tau) = 0. \tag{2.7}$$

Clearly, $p_0(\tau) - q_0(\tau) > 0$. It is easy to show that

$$\begin{aligned} p_0(\tau) + q_0(\tau) &= [(\mu_2 + \gamma)(\alpha\mu_1 + \beta) - \beta\gamma e^{-\mu_3\tau}] \frac{I^*}{1 + \alpha I^*} > 0, \\ p_1^2(\tau) - 2p_0(\tau) &= \frac{1}{(1 + \alpha I^*)^2} f(I^*), \end{aligned}$$

where

$$f(I) = [(\alpha\mu_1 + \beta)^2 + \alpha^2(\mu_2 + \gamma)^2]I^2 + 2[\mu_1(\alpha\mu_1 + \beta) - \beta(\mu_2 + \gamma)]I + \mu_1^2.$$

Hence, if $p_1^2 > 2p_0$, Eq. (2.7) has no positive real root. Hence, the endemic equilibrium E^* is always locally asymptotically stable for all $\tau \geq 0$ in this case.

Denote

$$\Delta = -(\mu_2 + \gamma)(\alpha\mu_1 + \beta)[(\alpha\mu_1 - \beta)(\mu_2 + \gamma) + 2\beta\mu_1].$$

Clearly, if $(\alpha\mu_1 - \beta)(\mu_2 + \gamma) + 2\beta\mu_1 > 0$, then $f(I) > 0$, which yields $p_1^2(\tau) - 2p_0(\tau) > 0$. Hence, the positive equilibrium E^* of system (1.1) is locally asymptotically stable for all $\tau \geq 0$ if $(\alpha\mu_1 - \beta)(\mu_2 + \gamma) + 2\beta\mu_1 > 0$.

If $2p_0(\tau) - p_1^2(\tau) = 2\sqrt{p_0^2(\tau) - q_0^2(\tau)}$, Eq. (2.7) has a unique positive root $\omega_+ = [p_0^2(\tau) - q_0^2(\tau)]^{1/4}$.

If $2p_0(\tau) - p_1^2(\tau) > 2\sqrt{p_0^2(\tau) - q_0^2(\tau)}$, Eq. (2.7) has two positive roots $\omega_+(\tau)$ and $\omega_-(\tau)$ satisfying

$$\omega_{\pm}^2(\tau) = \frac{1}{2} \left[2p_0(\tau) - p_1^2(\tau) \pm \sqrt{(2p_0(\tau) - p_1^2(\tau))^2 - 4(p_0^2(\tau) - q_0^2(\tau))} \right]. \tag{2.8}$$

In order to find the τ values of stability switches, for each positive root $\omega(\tau)$ of Eq. (2.7), we define the angle $\theta(\tau) \in (\pi, 2\pi)$ as a solution of

$$\begin{aligned} \sin \theta(\tau) &= \frac{p_1(\tau)\omega(\tau)}{q_0(\tau)}, \\ \cos \theta(\tau) &= \frac{\omega^2(\tau) - p_0(\tau)}{q_0(\tau)}. \end{aligned} \tag{2.9}$$

For each $\omega(\tau)$ satisfying (2.7), we define

$$S_n(\tau) = \tau - \frac{\theta(\tau) + 2n\pi}{\omega(\tau)}, \quad n = 0, \pm 1, \dots \tag{2.10}$$

According to the known work developed by Beretta and Kuang [18] (Section 4), we have the following result.

Theorem 2.1. *Let $\mathcal{R}_0 > 1$ hold. For system (1.2), we have*

- (i) *If $[(\alpha\mu_1 + \beta)^2 + \alpha^2(\mu_2 + \gamma)^2]I^{*2} + 2[\mu_1(\alpha\mu_1 + \beta) - \beta(\mu_2 + \gamma)]I^* + \mu_1^2 > 0$, the endemic equilibrium E^* of system (1.2) is locally asymptotically stable for all $\tau \geq 0$.*
- (ii) *Assume that there is a $\tau^* > 0$ satisfying $S_n(\tau^*) = 0$ for some $n \in N_0$ and that the Eq. (2.4) has a pair of simple and conjugate pure imaginary roots $\lambda = \pm i\omega(\tau^*)$ with $\omega(\tau^*) > 0$.
If $\omega(\tau^*) = \omega_+(\tau^*)$, this pair of simple pure imaginary roots crosses the imaginary axis from left to right (as τ increases) if $\delta_+(\tau^*) > 0$ and from right to left if $\delta_+(\tau^*) < 0$, where*

$$\delta_+(\tau^*) = \text{sign} \left\{ \left. \frac{d(\text{Re } \lambda)}{d\tau} \right|_{\lambda=i\omega_+(\tau^*)} \right\} = \text{sign} \left\{ \left. \frac{dS_n(\tau)}{d\tau} \right|_{\tau=\tau^*} \right\}. \tag{2.11}$$

If $\omega(\tau^) = \omega_-(\tau^*)$, this pair of simple pure imaginary roots crosses the imaginary axis from left to right (as τ increases) if $\delta_-(\tau^*) > 0$ and from right to left if $\delta_-(\tau^*) < 0$, where*

$$\delta_-(\tau^*) = \text{sign} \left\{ \left. \frac{d(\text{Re } \lambda)}{d\tau} \right|_{\lambda=i\omega_-(\tau^*)} \right\} = -\text{sign} \left\{ \left. \frac{dS_n(\tau)}{d\tau} \right|_{\tau=\tau^*} \right\}. \tag{2.12}$$

Corollary 2.1. *If $(\alpha\mu_1 - \beta)(\mu_2 + \gamma) + 2\beta\mu_1 > 0$ holds, the endemic equilibrium E^* of system (1.2) is locally asymptotically stable for all $\tau \geq 0$.*

3. Global asymptotical stability

In this section, we are concerned with the global asymptotic stability of the endemic equilibrium E^* and the disease-free equilibrium E_1 of system (1.2), respectively.

Theorem 3.1. *Let $\mathcal{R}_0 > 1$. If the following holds:*

$$(H1) \quad (\alpha\mu_1 - \beta)(\mu_2 + \gamma) > \alpha\mu_1\gamma e^{-\mu_3\tau},$$

then the positive (endemic) equilibrium $E^(S^*, I^*, R^*)$ of system (1.2) is globally asymptotically stable.*

Proof. Let $(S(t), I(t), R(t))$ be any positive solution of system (1.2) with initial conditions (1.3).

Let

$$\begin{aligned} \bar{S} &= \limsup_{t \rightarrow +\infty} S(t), & \bar{I} &= \limsup_{t \rightarrow +\infty} I(t), \\ \underline{S} &= \liminf_{t \rightarrow +\infty} S(t), & \underline{I} &= \liminf_{t \rightarrow +\infty} I(t). \end{aligned}$$

In the following we claim that $\bar{S} = \underline{S} = S^*$, $\bar{I} = \underline{I} = I^*$.

Letting $N(t) = S(t) + I(t) + R(t)$, it follows from system (1.2) that

$$\frac{dN(t)}{dt} = \Lambda - \mu_1 S(t) - \mu_2 I(t) - \mu_3 R(t) \leq \Lambda - \mu_1 N(t).$$

By comparison we derive that

$$\limsup_{t \rightarrow +\infty} N(t) \leq \frac{\Lambda}{\mu_1},$$

which yields

$$\limsup_{t \rightarrow +\infty} S(t) \leq \frac{\Lambda}{\mu_1} := M_1^S. \tag{3.1}$$

Hence, for any $\varepsilon > 0$, there is a $T_1 > 0$ such that if $t > T_1$, $S(t) < M_1^S + \varepsilon$.

It follows from the second equation of system (1.2) that, for $t > T_1$,

$$\begin{aligned} \dot{I}(t) &\leq \frac{\beta(M_1^S + \varepsilon)I(t)}{1 + \alpha I(t)} - (\mu_2 + \gamma)I(t) \\ &= \frac{I(t)}{1 + \alpha I(t)} [\beta(M_1^S + \varepsilon) - (\mu_2 + \gamma) - \alpha(\mu_2 + \gamma)I(t)]. \end{aligned} \tag{3.2}$$

Note that if $\mathcal{R}_0 > 1$, for any $\varepsilon > 0$, $\beta(M_1^S + \varepsilon) - (\mu_2 + \gamma) > 0$. By comparison we derive that

$$\limsup_{t \rightarrow +\infty} I(t) \leq \frac{\beta(M_1^S + \varepsilon) - (\mu_2 + \gamma)}{\alpha(\mu_2 + \gamma)}.$$

Since this inequality holds true for any $\varepsilon > 0$, we conclude that $\bar{I} \leq M_1^I$, where

$$M_1^I = \frac{\beta M_1^S - (\mu_2 + \gamma)}{\alpha(\mu_2 + \gamma)}. \tag{3.3}$$

Since $\mathcal{R}_0 > 1$ holds, we have $M_1^I > 0$. Hence, for any $\varepsilon > 0$, there is a $T_2 > T_1$ such that if $t > T_2$, $I(t) \leq M_1^I + \varepsilon$.

We derive from the first equation of system (1.2) that, for $t > T_2 + \tau$,

$$\dot{S}(t) \geq \Lambda - \mu_1 S(t) - \frac{\beta(M_1^I + \varepsilon)}{1 + \alpha(M_1^I + \varepsilon)} (M_1^S + \varepsilon).$$

By comparison it follows that

$$\liminf_{t \rightarrow +\infty} S(t) \geq \frac{1}{\mu_1} \left[\Lambda - \frac{\beta(M_1^I + \varepsilon)}{1 + \alpha(M_1^I + \varepsilon)} (M_1^S + \varepsilon) \right].$$

Since this inequality holds true for any $\varepsilon > 0$, we conclude that $\underline{S} \geq N_1^S$, where

$$N_1^S = \frac{1}{\mu_1} \left(\Lambda - \frac{\beta M_1^I M_1^S}{1 + \alpha M_1^I} \right). \tag{3.4}$$

It is easy to show that if (H1) holds, $N_1^S > 0$. Hence, for any $\varepsilon > 0$, there is a $T_3 > T_2 + \tau$ such that if $t > T_3$, $S(t) \geq N_1^S - \varepsilon$.

We derive from the second equation of system (1.2) that, for $t > T_3$,

$$\begin{aligned} \dot{I}(t) &\geq \frac{\beta(N_1^S - \varepsilon)I(t)}{1 + \alpha I(t)} - (\mu_2 + \gamma)I(t) \\ &= \frac{I(t)}{1 + \alpha I(t)} [\beta(N_1^S - \varepsilon) - (\mu_2 + \gamma) - \alpha(\mu_2 + \gamma)I(t)]. \end{aligned} \tag{3.5}$$

By comparison it follows that

$$\liminf_{t \rightarrow +\infty} I(t) \geq \frac{\beta(N_1^S - \varepsilon) - (\mu_2 + \gamma)}{\alpha(\mu_2 + \gamma)}.$$

Since this inequality holds true for any $\varepsilon > 0$, we conclude that $\underline{I} \geq N_1^I$, where

$$N_1^I = \frac{\beta N_1^S - (\mu_2 + \gamma)}{\alpha(\mu_2 + \gamma)}.$$

It is easy to show that if (H1) holds, $N_1^I > 0$. Therefore, for any $\varepsilon > 0$, there is a $T_4 > T_3$ such that if $t > T_4$, $I(t) \geq N_1^I - \varepsilon$.

Again, it follows from the first equation of system (1.2) that for $t > T_4 + \tau$

$$\dot{S}(t) \leq \Lambda - \mu_1 S(t) - \frac{\beta(N_1^I - \varepsilon)(N_1^S - \varepsilon)}{1 + \alpha(N_1^I - \varepsilon)} + \gamma e^{-\mu_3 \tau} (M_1^I + \varepsilon).$$

By comparison we derive that

$$\limsup_{t \rightarrow +\infty} S(t) \leq \frac{1}{\mu_1} \left[\Lambda - \frac{\beta(N_1^I - \varepsilon)(N_1^S - \varepsilon)}{1 + \alpha(N_1^I - \varepsilon)} + \gamma e^{-\mu_3 \tau} (M_1^I + \varepsilon) \right].$$

Since this inequality holds true for any $\varepsilon > 0$ sufficiently small, we conclude that $\bar{S} \leq M_2^S$, where

$$M_2^S = \frac{1}{\mu_1} \left(\Lambda - \frac{\beta N_1^I N_1^S}{1 + \alpha N_1^I} + \gamma e^{-\mu_3 \tau} M_1^I \right).$$

Hence, for any $\varepsilon > 0$, there is a $T_5 > T_4$ such that if $t > T_5$, $S(t) \leq M_2^S + \varepsilon$.

It follows from the second equation of system (1.2) that, for $t > T_5$,

$$\begin{aligned} \dot{I}(t) &\leq \frac{\beta(M_2^S + \varepsilon)I(t)}{1 + \alpha I(t)} - (\mu_2 + \gamma)I(t) \\ &= \frac{I(t)}{1 + \alpha I(t)} [\beta(M_2^S + \varepsilon) - (\mu_2 + \gamma) - \alpha(\mu_2 + \gamma)I(t)]. \end{aligned} \quad (3.6)$$

Note that if (H1) holds, for any $\varepsilon > 0$, $\beta(M_2^S + \varepsilon) - (\mu_2 + \gamma) > 0$. By comparison we derive that

$$\limsup_{t \rightarrow +\infty} I(t) \leq \frac{\beta(M_2^S + \varepsilon) - (\mu_2 + \gamma)}{\alpha(\mu_2 + \gamma)}.$$

Since this inequality holds true for any $\varepsilon > 0$, we conclude that $\bar{I} \leq M_2^I$, where

$$M_2^I = \frac{\beta M_2^S - (\mu_2 + \gamma)}{\alpha(\mu_2 + \gamma)}. \quad (3.7)$$

Since $\mathcal{R}_0 > 1$ holds, we have $M_2^I > 0$. Hence, for any $\varepsilon > 0$, there is a $T_6 > T_5$ such that if $t > T_6$, $I(t) \leq M_2^I + \varepsilon$.

We derive from the first equation of system (1.2) that, for $t > T_6 + \tau$,

$$\dot{S}(t) \geq \Lambda - \mu_1 S(t) - \frac{\beta(M_2^I + \varepsilon)}{1 + \alpha(M_2^I + \varepsilon)} (M_2^S + \varepsilon) + \gamma e^{-\mu_3 \tau} (N_1^I - \varepsilon).$$

By comparison it follows that

$$\liminf_{t \rightarrow +\infty} S(t) \geq \frac{1}{\mu_1} \left[\Lambda - \frac{\beta(M_2^I + \varepsilon)}{1 + \alpha(M_2^I + \varepsilon)} (M_2^S + \varepsilon) + \gamma e^{-\mu_3 \tau} (N_1^I - \varepsilon) \right].$$

Since this inequality holds true for any $\varepsilon > 0$, we conclude that $\underline{S} \geq N_2^S$, where

$$N_2^S = \frac{1}{\mu_1} \left(\Lambda - \frac{\beta M_2^I M_2^S}{1 + \alpha M_2^I} + \gamma e^{-\mu_3 \tau} N_1^I \right).$$

It is easy to show that if (H1) holds, $N_2^S > 0$. Hence, for any $\varepsilon > 0$, there is a $T_7 > T_6 + \tau$ such that if $t > T_7$, $S(t) \geq N_2^S - \varepsilon$.

We derive from the second equation of system (1.2) that, for $t > T_7$,

$$\begin{aligned} \dot{I}(t) &\geq \frac{\beta(N_2^S - \varepsilon)I(t)}{1 + \alpha I(t)} - (\mu_2 + \gamma)I(t) \\ &= \frac{I(t)}{1 + \alpha I(t)} [\beta(N_2^S - \varepsilon) - (\mu_2 + \gamma) - \alpha(\mu_2 + \gamma)I(t)]. \end{aligned} \quad (3.8)$$

By comparison it follows that

$$\liminf_{t \rightarrow +\infty} I(t) \geq \frac{\beta(N_2^S - \varepsilon) - (\mu_2 + \gamma)}{\alpha(\mu_2 + \gamma)}.$$

Since this inequality holds true for any $\varepsilon > 0$, we conclude that $\underline{I} \geq N_2^I$, where

$$N_2^I = \frac{\beta N_2^S - (\mu_2 + \gamma)}{\alpha(\mu_2 + \gamma)}.$$

It is easy to show that if (H1) holds, $N_2^I > 0$. Therefore, for any $\varepsilon > 0$, there is a $T_8 > T_7$ such that if $t > T_8$, $I(t) \geq N_2^I - \varepsilon$. Continuing this process, we obtain four sequences $M_n^S, M_n^I, N_n^S, N_n^I$ ($n = 1, 2, \dots$) such that, for $n \geq 2$,

$$\begin{aligned} M_n^S &= \frac{1}{\mu_1} \left(\Lambda - \frac{\beta N_{n-1}^I N_{n-1}^S}{1 + \alpha N_{n-1}^I} + \gamma e^{-\mu_3 \tau} M_{n-1}^I \right), \\ M_n^I &= \frac{\beta M_n^S - (\mu_2 + \gamma)}{\alpha(\mu_2 + \gamma)}, \\ N_n^S &= \frac{1}{\mu_1} \left(\Lambda - \frac{\beta M_n^I M_n^S}{1 + \alpha M_n^I} + \gamma e^{-\mu_3 \tau} N_{n-1}^I \right), \\ N_n^I &= \frac{\beta N_n^S - (\mu_2 + \gamma)}{\alpha(\mu_2 + \gamma)}. \end{aligned} \tag{3.9}$$

A direct calculation shows that

$$M_2^S = \frac{1}{\mu_1} \left\{ \Lambda - \frac{\beta \Lambda - \mu_1(\mu_2 + \gamma)}{\alpha^2 \mu_1^2 (\mu_2 + \gamma)} [(\alpha \mu_1 - \beta)(\mu_2 + \gamma) - \alpha \mu_1 \gamma e^{-\mu_3 \tau}] \right\}.$$

Clearly, if $\mathcal{R}_0 > 1$ and (H1) hold, $M_2^S < M_1^S$. Accordingly, $M_2^I < M_1^I$.

We derive from (3.3) and (3.4) that

$$N_1^S = \frac{1}{\mu_1} [\Lambda - (\mu_2 + \gamma) M_1^I]. \tag{3.10}$$

It follows from the second and the third equations of (3.9) that

$$N_2^S = \frac{1}{\mu_1} [\Lambda - (\mu_2 + \gamma) M_2^I + \gamma e^{-\mu_3 \tau} N_1^I]. \tag{3.11}$$

A direct calculation shows that if $\mathcal{R}_0 > 1$ and (H1) hold, $N_1^I = (\alpha \mu_1 - \beta)[\beta \Lambda - \mu_1(\mu_2 + \gamma)]/(\alpha \mu_1^2) > 0$. Hence, $N_1^S > 0$ follows from the fourth equation of (3.9). Noting that if $\mathcal{R}_0 > 1$ and (H1) hold, $M_2^I < M_1^I$, we derive from (3.10) and (3.11) that $N_2^S > N_1^S$. By induction, we can show that $M_{n+1}^S < M_n^S, N_{n+1}^S > N_n^S$. Hence, the sequence M_n^S is decreasing and the sequence N_n^S is increasing. Clearly, we have that

$$N_n^S \leq \underline{S} \leq \bar{S} \leq M_n^S, \quad N_n^I \leq \underline{I} \leq \bar{I} \leq M_n^I.$$

Hence, the limits of the sequences M_n^S and N_n^S exist. Denote

$$\bar{S} = \lim_{t \rightarrow +\infty} M_n^S, \quad \underline{S} = \lim_{t \rightarrow +\infty} N_n^S. \tag{3.12}$$

We derive from (3.9) that

$$\begin{aligned} M_{n+1}^S &= \frac{1}{\mu_1} \left[\Lambda - (\mu_2 + \gamma) \frac{\beta N_n^S - (\mu_2 + \gamma)}{\alpha(\mu_2 + \gamma)} + \gamma e^{-\mu_3 \tau} \frac{\beta M_n^S - (\mu_2 + \gamma)}{\alpha(\mu_2 + \gamma)} \right], \\ N_n^S &= \frac{1}{\mu_1} \left[\Lambda - (\mu_2 + \gamma) \frac{\beta M_n^S - (\mu_2 + \gamma)}{\alpha(\mu_2 + \gamma)} + \gamma e^{-\mu_3 \tau} \frac{\beta N_{n-1}^S - (\mu_2 + \gamma)}{\alpha(\mu_2 + \gamma)} \right]. \end{aligned} \tag{3.13}$$

It follows from (3.12) and (3.13) that

$$\bar{S} = \frac{1}{\mu_1} \left[\Lambda - (\mu_2 + \gamma) \frac{\beta \underline{S} - (\mu_2 + \gamma)}{\alpha(\mu_2 + \gamma)} + \gamma e^{-\mu_3 \tau} \frac{\beta \bar{S} - (\mu_2 + \gamma)}{\alpha(\mu_2 + \gamma)} \right], \tag{3.14}$$

and

$$\underline{S} = \frac{1}{\mu_1} \left[\Lambda - (\mu_2 + \gamma) \frac{\beta \bar{S} - (\mu_2 + \gamma)}{\alpha(\mu_2 + \gamma)} + \gamma e^{-\mu_3 \tau} \frac{\beta \underline{S} - (\mu_2 + \gamma)}{\alpha(\mu_2 + \gamma)} \right]. \tag{3.15}$$

(3.14) minus (3.15) gives

$$[(\alpha \mu_1 - \beta)(\mu_2 + \gamma) - \beta \gamma e^{-\mu_3 \tau}](\bar{S} - \underline{S}) = 0. \tag{3.16}$$

Since $\mathcal{R}_0 > 1$ and (H1) hold, we have that

$$(\alpha\mu_1 - \beta)(\mu_2 + \gamma) - \beta\gamma e^{-\mu_3\tau} > 0.$$

It therefore follows from (3.16) that $\bar{S} = \underline{S}$. Accordingly, we derive from (3.14) and (3.15) that

$$\bar{S} = \underline{S} = \frac{(\mu_2 + \gamma)[\Lambda\alpha + \mu_2 + \gamma(1 - e^{-\mu_3\tau})]}{(\alpha\mu_1 + \beta)(\mu_2 + \gamma) - \beta\gamma e^{-\mu_3\tau}} = S^*. \quad (3.17)$$

It follows from (3.9) and (3.17) that $\bar{I} = \underline{I} = I^*$. We therefore have

$$\lim_{t \rightarrow +\infty} S(t) = S^*, \quad \lim_{t \rightarrow +\infty} I(t) = I^*. \quad (3.18)$$

We derive from the third equation of system (1.2) that, for $t > \tau$,

$$R(t) = \int_{t-\tau}^t \gamma I(u) e^{-\mu_3(t-u)} du. \quad (3.19)$$

By L'Hospital's rule, we derive from (3.18) and (3.19) that $\lim_{t \rightarrow +\infty} R(t) = R^*$. Hence, the endemic equilibrium E^* of system (1.2) is globally attractive. By Corollary 2.1, we see that if (H1) holds, E^* is locally asymptotically stable, it therefore is globally asymptotically stable. This completes the proof. \square

Theorem 3.2. *If $\mathcal{R}_0 < 1$, then the disease-free equilibrium $E_1(\Lambda/\mu_1, 0, 0)$ of system (1.2) is globally asymptotically stable.*

Proof. Let $(S(t), I(t), R(t))$ be any positive solution of system (1.2) with initial conditions (1.3).

If $\mathcal{R}_0 < 1$, we may choose $\varepsilon > 0$ sufficiently small satisfying

$$\beta(\Lambda/\mu_1 + \varepsilon) < (\mu_2 + \gamma). \quad (3.20)$$

We derive from (3.1) that

$$\limsup_{t \rightarrow +\infty} S(t) \leq \frac{\Lambda}{\mu_1}. \quad (3.21)$$

Hence, for any $\varepsilon > 0$ satisfying (3.20), there is a $T_1 > 0$ such that if $t > T_1$, $S(t) < \Lambda/\mu_1 + \varepsilon$.

For any $\varepsilon > 0$ satisfying (3.20), it follows from the second equation of system (1.2) that, for $t > T_1$,

$$\begin{aligned} \dot{I}(t) &\leq \frac{\beta(\Lambda/\mu_1 + \varepsilon)I(t)}{1 + \alpha I(t)} - (\mu_2 + \gamma)I(t) \\ &= \frac{I(t)}{1 + \alpha I(t)} [\beta(\Lambda/\mu_1 + \varepsilon) - (\mu_2 + \gamma) - \alpha(\mu_2 + \gamma)I(t)]. \end{aligned} \quad (3.22)$$

Consider the following auxiliary equation

$$\dot{u}(t) = \frac{u(t)}{1 + \alpha u(t)} [\beta(\Lambda/\mu_1 + \varepsilon) - (\mu_2 + \gamma) - \alpha(\mu_2 + \gamma)u(t)]. \quad (3.23)$$

Noting that (3.20) holds, we obtain from (3.23) that

$$\lim_{t \rightarrow +\infty} u(t) = 0.$$

By comparison it follows that

$$\limsup_{t \rightarrow +\infty} I(t) = 0. \quad (3.24)$$

Hence, for any $\varepsilon > 0$ satisfying (3.20), there is a $T_2 > T_1$ such that if $t > T_2$, $I(t) < \varepsilon$.

We obtain from the first equation of system (1.2) that, for $t > T_2 + \tau$,

$$\dot{S}(t) \geq \Lambda - \mu_1 S(t) - \frac{\beta\varepsilon}{1 + \alpha\varepsilon} S(t).$$

By comparison we obtain that

$$\liminf_{t \rightarrow +\infty} S(t) \geq \frac{\Lambda(1 + \alpha\varepsilon)}{\mu_1 + (\beta + \alpha\mu_1)\varepsilon}.$$

Letting $\varepsilon \rightarrow 0$, it follows that

$$\liminf_{t \rightarrow +\infty} S(t) \geq \frac{\Lambda}{\mu_1}.$$

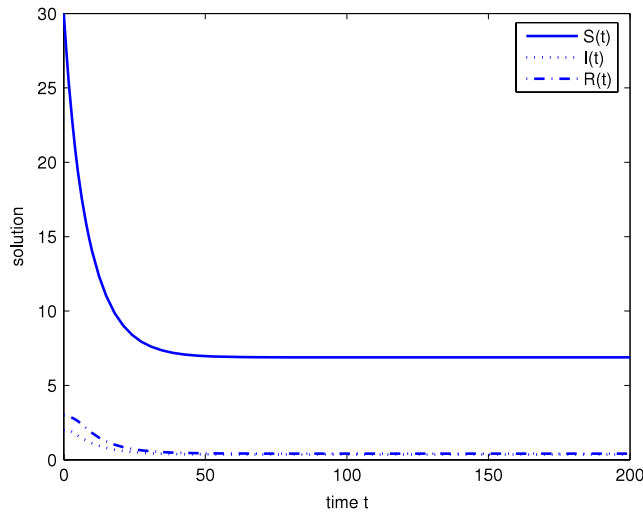


Fig. 1. The temporal solution found by numerical integration of system (1.2) with $\Lambda = 0.8, \alpha = 5, \beta = 0.2, \gamma = 0.3, \mu_1 = 0.1, \mu_2 = 0.2, \mu_3 = 0.1, \tau = 5, (\phi_1, \phi_2, \phi_3) \equiv (30, 2, 3)$.

This, together with (3.21), yields

$$\lim_{t \rightarrow +\infty} S(t) = \frac{\Lambda}{\mu_1}.$$

We derive from the third equation of system (1.2) that, for $t > \tau$,

$$R(t) = \int_{t-\tau}^t \gamma I(u) e^{-\mu_3(t-u)} du. \tag{3.25}$$

By L'Hospital's Rule, it follows from (3.24) and (3.25) that

$$\lim_{t \rightarrow +\infty} R(t) = 0.$$

Noting that if $\mathcal{R}_0 < 1$, the disease-free equilibrium $E_1(\Lambda/\mu_1, 0, 0)$ of system (1.2) is locally asymptotically stable, we conclude that E_1 is globally asymptotically stable. This completes the proof. \square

In the following, we give some examples to illustrate the main theoretical results above.

Example 1. In system (1.2), we choose $\Lambda = 0.8, \alpha = 5, \beta = 0.2, \gamma = 0.3, \mu_1 = 0.1, \mu_2 = 0.2, \mu_3 = 0.1, \tau = 5$. It is easy to show that $\mathcal{R}_0 = 16/5 > 1$, system (1.2) has a unique endemic equilibrium $E^*(6.8845, 0.3508, 0.414)$. By calculation we get $(\alpha\mu_1 - \beta)(\mu_2 + \gamma) - \beta\gamma e^{-\mu_3\tau} = 0.15 - 0.06e^{-0.05} \approx 0.0929 > 0$. Hence, by Theorem 3.1, the endemic equilibrium E^* of system (1.2) is globally asymptotically stable. Numerical simulation illustrates this fact (see, Fig. 1).

Example 2. In system (1.2), let $\Lambda = 0.3, \mu_1 = 0.1, \mu_2 = 0.2, \mu_3 = 0.1, \alpha = 2, \beta = 0.1, \gamma = 0.3, \tau = 15$. It is easy to show that $\mathcal{R}_0 = 0.6 < 1$. By Theorem 3.2, we see that the disease-free equilibrium $E_1(3, 0, 0)$ of system (1.2) is globally asymptotically stable. Numerical simulation illustrates this result (see, Fig. 2).

Example 3. In system (1.2), let $\Lambda = 0.2, \alpha = 1, \beta = 4, \gamma = 5, \mu_1 = 0.1, \mu_2 = 0.2, \mu_3 = 0.1$. It is easy to show that $\mathcal{R}_0 = 40/13 > 1$. System (1.2) has a unique endemic equilibrium $E^*(26(27 - 25e^{-0.1\tau})/(533 - 500e^{-0.1\tau}), 7/(533 - 500e^{-0.1\tau}), 350(1 - e^{-0.1\tau})/(533 - 500e^{-0.1\tau}))$. We choose $\tau = 6.5$ in system (1.2) with the coefficients above. Numerical simulation shows that system (1.2) admits a periodic solution near the endemic equilibrium E^* (see, Fig. 3). As the immunity period τ increases, the amplitude of these oscillations increases correspondingly. Further, if we choose $\tau = 9.5$, the periodic solution disappears and the endemic equilibrium E^* becomes stable (see, Fig. 4).

4. Discussion

In this paper, we formulated an SIRS epidemic model with a saturation incidence rate and a time delay describing the temporary immunity period. The global stability of the disease-free equilibrium and the endemic equilibrium of system (1.2) was studied by comparison arguments and the iteration scheme, respectively. By comparison arguments, it has been shown that if the basic reproduction number $\mathcal{R}_0 < 1$, the disease-free equilibrium $E_1(\Lambda/\mu_1, 0, 0)$ is globally asymptotically stable

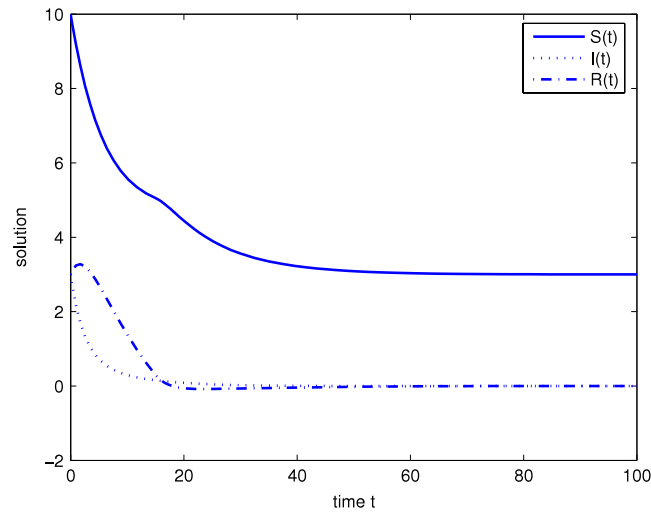


Fig. 2. The temporal solution found by numerical integration of system (1.2) with $\Lambda = 0.3$, $\mu_1 = 0.1$, $\beta = 0.1$, $\alpha = 2$, $\mu_2 = 0.2$, $\gamma = 0.3$, $\mu_3 = 0.1$, $\tau = 15$, $(\phi_1, \phi_2, \phi_3) \equiv (10, 3, 3)$.

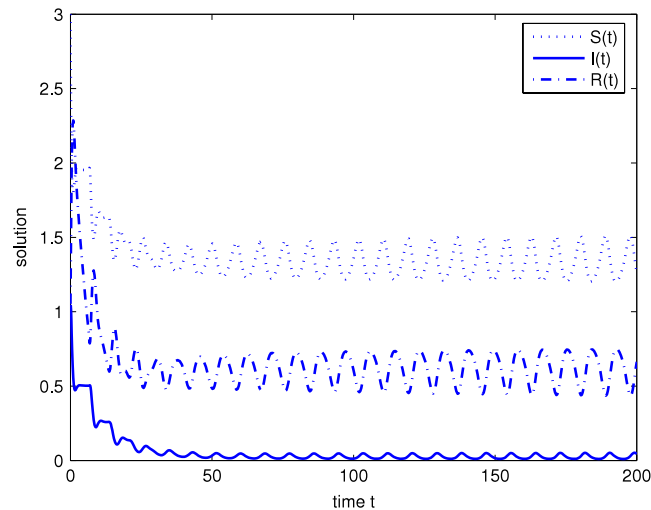


Fig. 3. The temporal solution found by numerical integration of system (1.2) with $\Lambda = 0.2$, $\alpha = 1$, $\beta = 4$, $\gamma = 5$, $\mu_1 = 0.1$, $\mu_2 = 0.2$, $\mu_3 = 0.1$, $\tau = 6.5$, $(\phi_1, \phi_2, \phi_3) \equiv (3, 1, 1)$.

while the endemic equilibrium is not feasible. If $\mathcal{R}_0 > 1$, we have shown in Theorem 3.1 that the endemic equilibrium E^* of system (1.2) is globally asymptotically stable provided that $(\alpha\mu_1 - \beta)(\mu_2 + \gamma) > \alpha\mu_1\gamma e^{-\mu_3\tau}$. We would like to point out here that Theorem 3.1 has room for improvement. Clearly, the assumption (H1) implies that $\alpha > 0$ is sufficiently large. Hence, the result in Theorem 3.1 is not applicable to system (1.1). Recalling the work of Brauer et al. in [13], when $\alpha = 0$, it was shown that there are ranges of parameter values in which Hopf bifurcations and sustained oscillations in system (1.2) are possible, but they have not been able to give necessary and sufficient conditions for the existence of a Hopf bifurcation in these ranges. To further study system (1.2) when $\alpha > 0$, we resorted to numerical simulations. It was shown that for $\mathcal{R}_0 > 1$ and a certain immunity time τ , the solutions of system (1.2) are represented by small amplitude oscillations near the endemic equilibrium E^* . As the immunity period τ increases, the amplitude of these oscillations increases correspondingly. Further increase of τ returns the oscillatory dynamics to the stable steady-state form. This showed the dependence of a long term dynamics of solutions of system (1.2) on the immunity period τ . It is interesting to theoretically study the existence of a Hopf bifurcation at the endemic equilibrium of system (1.2) when $\alpha > 0$. We leave this for further work.

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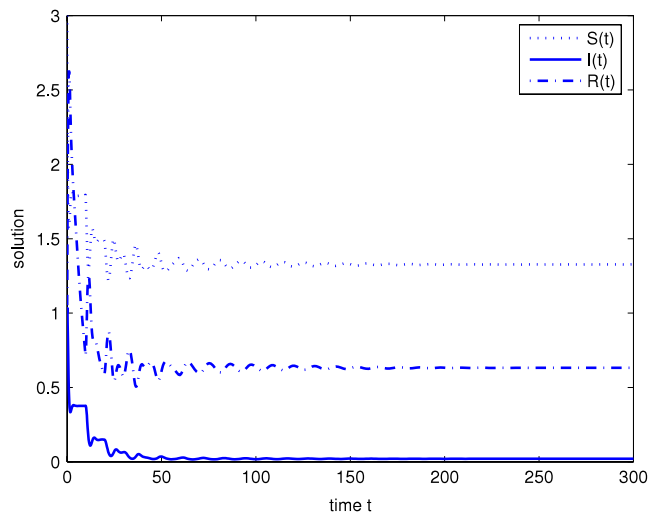


Fig. 4. The temporal solution found by numerical integration of system (1.2) with $\Lambda = 0.2$, $\alpha = 1$, $\beta = 4$, $\gamma = 5$, $\mu_1 = 0.1$, $\mu_2 = 0.2$, $\mu_3 = 0.1$, $\tau = 9.5$, $(\phi_1, \phi_2, \phi_3) \equiv (3, 1, 1)$.

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