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On a Spectral Problem in Vibration Mechanics: Computation of Elastic Tanks Partially Filled with Liquids*

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1. INTRODUCTION

In this study we proceed on a theoretical and numerical analysis of the vibrations of a fluid confined in a deformable shell. The so-called hydroelastic problem has seldom been met in a global way; one can find in [1] a study considered by us as the origin of our work ([1] is exclusively devoted to the case of thin shells; the solutions' properties are obtained from the equations in a computational way).

In the present analysis, we state within the linearized theory the general equations of the coupled system fluid–elastic shell (considered as a three-dimensional body), which corresponds to the study of small vibrations, and we introduce a variational formulation of the problem. By an appropriate choice of the function spaces it is possible to associate a spectral problem for which we have defined the operators' properties. In particular, we demonstrate compactness properties resulting from a previous work [2]. We then show directly the existence of an eigenfrequencies and eigenfunctions spectrum which defines the vibration modes. The knowledge of the vibration modes characterizes the system from a mechanical point of view.

The theoretical study is achieved by a numerical analysis by the finite element method. In Section 2, we present the matrix formulation of the approximate problem. We then study the different finite element types used in the computation: fluid elements of isoparametric type, shell elements, coupling elements.

After describing the resolution of the approximate problem, numerical results are shown in the case of the first stage of the civil applications launch vehicle Diamant B. The results allow us to identify the mode of vibration characterizing the Pogo effect. We recall that this phenomenon consists of

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an instability due to coupling between the main structure (tanks and shells), the so-called secondary structure (pipes, pumps), and the thrust.

2. Description of the Problem

The liquid is supposed to be inviscid, incompressible, irrotational, and contained in a deformable shell.

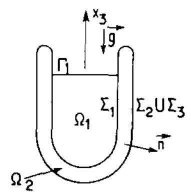

Fig. 1. Notations.

Notation (Fig. 1). Ω_1 and Ω_2 are open bounded sets of $\mathbf{R}^3$.

The *liquid* occupies the initial volume Ω_1 with the boundary (supposed smooth) $\partial\Omega_1 = \Gamma_1 \cup \Sigma_1$, where Γ_1 is the free surface at rest and Σ_1 the wetted surface of the shell. $n(n_j)$ represents the external normal to Ω_1, ρ_f the density of the fluid, $\Phi(x, t)$ the velocity potential of the fluid ($x = (x_1, x_2, x_3)$), $Y(x', t)$ the normal displacement at the free surface ($x' = (x_1, x_2)$), and g the gravity vector.

The *shell*, supposed thick, is represented by a domain Ω_2 with the boundary $\partial\Omega_2 = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$. On Σ_2 and Σ_3 (meas Σ_2, meas $\Sigma_3 > 0$) act, respectively, a force and displacement field. $n'(n_j')$ represents the external normal to Ω_2, $n' = -n$ on Σ_1; ρ_c is the density of the shell, $u = (u_1, u_2, u_3)$ is the displacement field vector referred to the equilibrium (static) configuration, $\boldsymbol{\sigma}$ is the tensor whose components σ_{ij} represent the stress variations between the actual and the initial states; finally, $f(f_i)$ represents the given body forces per unit volume. We shall set here $f_i = 0$.

2.1. *Equations of the Coupled Problem*

The Bernoulli theorem permits to calculate the pressure perturbation in any point M of the shell or of the free surface. After linearization, one obtains

$$p = -\rho_f \left(\frac{\partial \Phi}{\partial t} - g \cdot u(M) \right).$$

In the linear approximation corresponding to the study of the small vibrations, the model presently used supposes that the initial state is quasinatural and the normal variations are negligible during the deformation [3]. The dynamic equations are the following:

(i) for the fluid

$$\Delta\Phi = 0 \qquad \text{in} \qquad \Omega_1 \times]0, T[$$

with the boundary conditions

$$\frac{\partial \Phi}{\partial n} = \sum_{j=1}^{3} \frac{\partial u_j}{\partial t} n_j \qquad \text{on} \qquad \Sigma_1 \times]0, T[\quad \text{(nonpenetration condition)}$$

and

$$\left.\begin{aligned} \frac{\partial \Phi}{\partial n} &= \frac{\partial Y}{\partial t} \\ \frac{\partial \Phi}{\partial t} + gY &= 0 \end{aligned}\right\} \qquad \text{on} \qquad \Gamma_1 \times]0, T[\quad \text{(free surface condition);}$$

(ii) for the shell

$$\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} = \rho_c \frac{\partial^2 u_i}{\partial t^2} \qquad \text{in} \qquad \Omega_2 \times]0, T[$$

with the boundary conditions

$$u_i = 0 \qquad \text{on} \qquad \Sigma_3 \times]0, T[,$$

$$\sum_j \sigma_{ij} n_j' = 0 \qquad \text{on} \qquad \Sigma_2 \times]0, T[,$$

$$\sum_j \sigma_{ij} n_j' = \rho_f \left(\frac{\partial \Phi}{\partial t} + g u_3 \right) n_i' \qquad \text{on} \qquad \Sigma_1 \times]0, T[.$$

We recall here that the constitutive equations for linear elasticity are of the form

$$\sigma_{ij}(u) = \sum_{h,k} \lambda_{ijhk} \epsilon_{hk}(u),$$

where λ_{ijhk} are the elasticity coefficients and

$$\epsilon_{hk}(u) = \frac{1}{2} \left(\frac{\partial u_h}{\partial x_k} + \frac{\partial u_k}{\partial x_h} \right).$$

In the following, we assume that the Tong [1] hypothesis holds: the inner product $u \cdot g$ is replaced by $(u \cdot n)(g \cdot n)$.

Consequently, the boundary condition on $\Sigma_1 \times]0, T[$ becomes

$$\sum_j \sigma_{ij} n_j' = \rho_f \left(\frac{\partial \Phi}{\partial t} - (u \cdot n)(g \cdot n) \right) n_i'$$

and later on leads to the introduction of symmetric operators.

2.2. *Variational Formulation*

Let $H^1(\Omega_2)$ be the Sobolev space of real functions with the second power absolutely integrable for the Lebesgue measure with derivatives of order one in $L^2(\Omega_2)$ [4], and let V be the space

$$V = \{v \in (H^1(\Omega_2))^3;\ v\,|_{\Sigma_3} = 0\}.$$

We can show, using the Green formula, that the problem in Section 2.1 is equivalent to the following variational problem.

Find the functions $(\Phi(\cdot), Y(\cdot), U(\cdot))$ defined on $]0, T[$ and taking values in $H^1(\Omega_1) \times L^2(\Gamma_1) \times V$, satisfying

$$\int_{\Omega_1} \rho_f \nabla\Phi\,\nabla\psi\,dx - \int_{\Gamma_1} \rho_f \frac{\partial Y}{\partial t}\psi\,d\sigma - \int_{\Sigma_1} \rho_f \left(\frac{\partial u}{\partial t}\cdot n\right)\psi\,d\sigma = 0 \qquad \forall\psi \in H^1(\Omega_1), \tag{1}$$

$$\int_{\Gamma_1} \rho_f \frac{\partial \Phi}{\partial t}\zeta\,d\sigma + g\int_{\Gamma_1} \rho_f Y\zeta\,d\sigma = 0 \qquad \forall\zeta \in L^2(\Gamma_1), \tag{2}$$

$$\int_{\Omega_2} \sum_{i,j} \sigma_{ij}(u)\,\epsilon_{ij}(v)\,dx + \int_{\Omega_2} \rho_c \left(\frac{\partial^2 u}{\partial t^2}\cdot v\right) dx + \int_{\Sigma_1} \rho_f \frac{\partial\Phi}{\partial t}(n\cdot v)\,d\sigma - \int_{\Sigma_1} \rho_f (g\cdot n)(u\cdot n)(v\cdot n)\,d\sigma = 0 \qquad \forall v \in V \tag{3}$$

with the initial conditions

$$\begin{aligned} u(x,0) &= u_0(x), & \frac{\partial u}{\partial t}(x,0) &= u_1(x) \quad \text{in} \quad \Omega_2, \\ \Phi(x',0) &= \Phi_0(x'), & \frac{\partial \Phi}{\partial t}(x',0) &= \Phi_1(x') \quad \text{on} \quad \Gamma_1. \end{aligned} \tag{4}$$

3. Preliminary Studies

Before defining the function spaces for the whole problem, it is necessary to establish some lemmas concerning the equations of the shell alone and of the fluid alone.

3.1. *The Elastic Problem*

We assume that the elastic coefficients λ_{ijhk} have the usual symmetric $(\lambda_{ijhk} = \lambda_{ijkh} = \lambda_{khij})$ and ellipticity properties, and we define

$$b(u, v) = \sum_{i,j} \int_{\Omega_2} \sigma_{ij}(u)\,\epsilon_{ij}(v)\,dx. \tag{5}$$

As a result from the Korn inequality and its consequences [5], we have the following.

LEMMA 1. *The bilinear form $b(u, v)$ equipped V with a norm equivalent to the usual norm of $(H^1(\Omega_2))^3$.*

Let V' be the dual space of V. We define the linear operator $B \in \mathscr{L}(V, V')$ by means of

$$b(u, v) = \langle Bu, v \rangle \qquad \forall u, v \in V, \tag{6}$$

where $\langle , \rangle$ represent the inner product in the duality between V and V'.

Let $D(B)$ be the domain of the unbounded operator B in $(L^2(\Omega_2))^3$. Because of the coercivity and symmetric properties of $b(u, v)$, the operator B is positive self-adjoint in $(L^2(\Omega_2))^3$ with the domain $D(B)$.

On the other hand, it follows from the Sobolev theorem that the mapping from V into $(L^2(\Omega_2))^3$ is compact, and we have the next lemma.

LEMMA 2. *The operator B defined by (6) is inversible positive self-adjoint, and B^{-1} is compact in $(L^2(\Omega_2))^3$.*

3.2. *Study of the Fluid*

From the notions developed in [2] for vibrations of a liquid contained in a rigid tank we shall establish compactness results.

This problem related to the $u \equiv 0$ case is written from (1)–(4): Find $\Phi(\cdot)$, $Y(\cdot)$ defined on $]0, T[$ taking their values in $H^1(\Omega_1) \times L^2(\Gamma_1)$ and satisfying

$$\int_{\Omega_1} \nabla\Phi \, \nabla\psi \, dx - \int_{\Gamma_1} \frac{\partial Y}{\partial t} \psi \, d\sigma = 0 \qquad \forall \psi \in H^1(\Omega_1), \tag{7}$$

$$\int_{\Gamma_1} \frac{\partial \Phi}{\partial t} \zeta \, d\sigma + g \int_{\Gamma_1} Y\zeta \, d\sigma = 0 \qquad \forall \zeta \in L^2(\Gamma_1), \tag{8}$$

with the initial conditions

$$\Phi(x', 0) = \Phi_0(x'), \qquad \frac{\partial \Phi}{\partial t}(x', 0) = \Phi_1(x') \qquad \text{on} \qquad \Gamma_1 . \tag{9}$$

3.2.1. *Recalls*

We set

$$\frac{\partial \Phi}{\partial n} = \frac{\partial Y}{\partial t} = -g^{-1} \frac{\partial^2 \Phi}{\partial t^2},$$

and we studied the problem (7), (8): Find $\Phi(t) \in H^1(\Omega_1)$ such that

$$\int_{\Omega_1} \nabla\Phi(t)\, \nabla\psi\, dx + \frac{1}{g} \int_{\Gamma_1} \frac{\partial^2\Phi}{\partial t^2}\, \psi\, d\sigma = 0 \qquad \forall\psi \in H^1(\Omega_1)$$

with the initial conditions (9).

For the resolution, we introduce the subspace of $H^1(\Omega_1)$ defined by

$$\Delta\Phi = 0 \quad \text{in} \quad \Omega_1\,, \qquad \frac{\partial\Phi}{\partial n} = 0 \quad \text{on} \quad \Sigma_1\,, \qquad \Phi = \varphi \quad \text{on} \quad \Gamma_1\,, \quad (10)$$

where $\varphi \in H^{1/2}(\Gamma_1)$.

We define [6] the operator $A \in \mathscr{L}(H^{1/2}(\Gamma_1), H^{-1/2}(\Gamma_1))$ by setting $A\varphi = (\partial\Phi/\partial n)\,|_{\Gamma_1}$, which leads to the following evolution problem on the manifold Γ_1:

$$\int_{\Gamma_1} A\varphi(t)\, \psi\, d\sigma = \frac{1}{g} \int_{\Gamma_1} \frac{\partial^2\varphi}{\partial t^2}\, \psi\, d\sigma \qquad \forall\psi \in H^{1/2}(\Gamma_1).$$

We show that the problem has a unique solution which is a linear combination of stationary solutions such as $\Phi(x, t) = \phi(x)\, e^{i\omega t}$. The operator A^{-1} is compact in $L^2(\Gamma_1)$, so the circular eigenfrequencies ω and the eigenfunctions $\varphi(x') = \phi(x)\,|_{\Gamma_1}$ are given by the spectral problem

$$\int_{\Gamma_1} A\varphi\psi\, d\sigma = \frac{\omega^2}{g} \int_{\Gamma_1} \varphi\psi\, d\sigma \qquad \forall\psi \in H^{1/2}(\Gamma_1). \qquad (11)$$

We obtain $\phi(x)$ by solving Eqs. (10). Let us recall the following lemma.

LEMMA 3. *There exists for* (11) *an infinite set of eigenvalues* ($\omega_0 = 0$, $< \omega_1 \leqslant \cdots \leqslant \omega_n \to \infty$)), *and the corresponding eigenfunctions* $\varphi_n(x')$ *form an orthogonal and total sequence in* $H^{1/2}(\Gamma_1)$.

We recall, too, that the first eigenfunction $\varphi_0(x')$ corresponding to $\omega_0 = 0$ is a constant [2].

3.2.2. *Consequences*

Let W (resp. S_1) be the subspace of $H^1(\Omega_1)$ (resp. $L^2(\Gamma_1)$) orthogonal with respect to the constant functions; $d(\Phi) = \int_{\Omega_1} |\, \nabla\Phi\, |^2\, dx$ defines a norm on W, and by a lemma due to Deny and Lions [7] we have the following.

LEMMA 4. $d(\Phi) = \int_{\Omega_1} |\, \nabla\Phi\, |^2\, dx$ *defines on* W *an equivalent norm to the usual norm of* $H^1(\Omega_1)$.

Let W_1 be

$$W_1 = \left\{ \Phi \in W \mid \Delta\Phi = 0 \text{ in } \Omega_1 , \frac{\partial \Phi}{\partial n} = 0 \text{ on } \Sigma_1 \right\} .$$

Following (10), we have (Green's formula)

$$\int_{\Omega_1} | \nabla \Phi |^2 \, dx = \int_{\Gamma_1} A\varphi \cdot \varphi \, d\sigma \qquad \forall \Phi \in W_1 .$$

As, on the one hand, $A \in \mathscr{L}(H^{1/2}(\Gamma_1), H^{-1/2}(\Gamma_1))$ and, on the other hand, the trace operator is continuous from $H^1(\Omega_1)$ into $H^{1/2}(\Gamma_1)$, the topology induced by $d(\Phi)$ on the subspace W_1 is equivalent to the topology of $H^{1/2}(\Gamma_1)$, and we have the following.

LEMMA 5. *The sequence* $\{\phi_n(x)\}_{n=1}^{\infty}$, *obtained by solving* (10) *with the boundary condition involving the eigenfunctions* $\{\varphi_n(x')\}_{n=1}^{\infty}$ *of* A, *is an orthogonal total sequence in* W_1.

3.2.3. *Back to the Problem* (7), (8)

Let T_1 be

$$T_1 = \left\{ \Phi \in W_1 , \frac{\partial \Phi}{\partial n} \bigg|_{\Gamma_1} = Y \in S_1 \right\} . \tag{12}$$

T_1 corresponds, by means of (10), to the domain $D(A)$ of the unbounded operator A in $L^2(\Gamma_1)$.

The results mentioned in Section 3.2.1 lead us to look for the stationary solutions of (7), (8), such as

$$\Phi(x, t) = i\nu\phi(x) \, e^{i\nu t}, \qquad Y(x, t) = y(x) \, e^{i\nu t}$$

with

$$(\phi, y) \in W \times S_1 .$$

So we can avoid the trivial solution $\phi = \text{const}$, which corresponds to $\nu = 0$.

After a substitution in (7), (8) we obtain the following spectral problem: Find $(\phi, y) \in W \times S_1$ such that

$$\int_{\Omega_1} \nabla\phi \, \nabla\psi \, dx - \int_{\Gamma_1} y\psi \, d\sigma = 0 \qquad \forall \psi \in H^1(\Omega_1),$$

$$-\frac{\nu^2}{g} \int_{\Gamma_1} \phi\zeta \, d\sigma + \int_{\Gamma_1} y\zeta \, d\sigma = 0 \qquad \forall \zeta \in S_1 .$$

We notice that we have

$$\left.\frac{\partial \phi}{\partial n}\right|_{\Gamma_1} = y, \qquad \left.\frac{\partial \psi}{\partial n}\right|_{\Gamma_1} = \zeta \quad \text{in} \quad T_1$$

and, when using Green's formula,

$$\int_{\Omega_1} \nabla\phi \, \nabla\psi \, dx = \int_{\Gamma_1} y\psi \, d\sigma = \int_{\Gamma_1} \zeta\phi \, d\sigma.$$

The above spectral problem is therefore equivalent to

$$\int_{\Gamma_1} y\zeta \, d\sigma - \frac{\nu^2}{g} \int_{\Omega_1} \nabla\phi \, \nabla\psi \, dx = 0 \qquad \forall(\psi, \zeta) \in T_1 \times S_1 . \tag{13}$$

Recalling the notations of [2] stated in Section 3.2.1, we can set

$$y = A\varphi \qquad \text{with} \qquad \varphi = \Phi \,|_{\Gamma_1} ,$$

$$\zeta = A\psi \qquad \text{with} \qquad \psi = \Psi \,|_{\Gamma_1} .$$

Another form of (13) is

$$\int_{\Gamma_1} A\varphi A\psi \, d\sigma - \frac{\nu^2}{g} \int_{\Gamma_1} \varphi A\psi \, d\sigma = 0 \qquad \forall\psi \in D(A),$$

where $D(A)$ represents the domain of A.

Problem (7), (8) consists now, as is quite natural, of the spectral decomposition of A. We can formulate the previous results (Lemmas 3 and 5) as follows.

LEMMA 6. *There exists, for the problem* (7), (8), *an infinite positive sequence of circular eigenfrequencies* ω_n . *The corresponding eigenfunctions constitute the sequences* $\{\phi_n\}$, $\{\varphi_n\}$, *where* $\{\phi_n\}_{n=1,\ldots,\infty}$ *is orthogonal and total in* W_2 *and,* $\{\varphi_n\}_{n=1,\ldots,\infty}$ *is orthogonal and total in* S_1 *and such that*

$$\left.\begin{aligned} \int_{\Omega_1} \nabla\phi_n(x) \, \nabla\phi_m(x) \, dx &= \frac{\omega_n^{\,2}}{g} \int_{\Gamma_1} \varphi_n(x') \, \varphi_m(x') \, d\sigma \\ \int_{\Gamma_1} y_n(x') \, y_m(x') \, d\sigma &= \frac{\omega_n^{\,2}}{g} \int_{\Omega_1} \nabla\phi_n(x) \, \nabla\phi_m(x) \, dx \end{aligned}\right\} \forall m, y_n = \left.\frac{\partial \phi_n}{\partial n}\right|_{\Gamma_1} . \tag{14}$$

We define the symmetric unbounded operator C_1: $T_1 \to S_1$ by setting

$$C_1\phi = y \left(= \left.\frac{\partial \phi}{\partial n}\right|_{\Gamma_1} \right) .$$

It verifies that

$$C_1\phi = A\varphi \qquad (\varphi = \Phi|_{\Gamma_1}).$$

$$(C_1\phi, \psi) = \int_{\Gamma_1} y\psi\, d\sigma = \int_{\Omega_1} \nabla\phi\, \nabla\psi\, dx \qquad \forall\psi \in T_1 .$$

Lemma 6 shows the existence of a discrete spectrum for C_1; hence, by a theorem from [8, p. 462], we have the following.

LEMMA 7. *$C_1^{-1}: S_1 \to T_1$ is a compact operator (for the topology induced on T_1 by W).*

The operator C_1^{-1} is generally known as the Neumann operator of the problem [9].

3.2.5. *Extension of the Results*

The preceding methods can be extended without difficulties to the following boundary value problem:

$$\begin{aligned} &\Delta\Phi = 0 && \text{in} \quad \Omega_1 \times]0, T[, \\ &\left.\begin{aligned} \frac{\partial\Phi}{\partial n} &= \frac{\partial Y}{\partial t} \\ \frac{\partial\Phi}{\partial t} + gY &= 0 \end{aligned}\right\} && \text{on} \quad \partial\Omega_1 \times]0, T[. \end{aligned} \tag{15}$$

Let us set

$$\varphi = \Phi\Big|_{\partial\Omega_1} \qquad \text{and} \qquad A\varphi = \frac{\partial\Phi}{\partial n}\Big|_{\partial\Omega_1},$$

where $A \in \mathscr{L}(H^{1/2}(\partial\Omega_1), H^{-1/2}(\partial\Omega_1))$. From the compactness of the mapping from $H^{1/2}(\partial\Omega_1)$ into $L^2(\partial\Omega_1)$ we get the compactness of A^{-1} into $L^2(\partial\Omega_1)$.

Let $\mathscr{S}$ be the subspace of $L^2(\partial\Omega_1)$ orthogonal with respect to the constant functions, and let T be

$$T = \left\{\Phi \in W; \Delta\Phi = 0 \text{ in } \Omega_1, \frac{\partial\Phi}{\partial n}\Big|_{\partial\Omega_1} \in \mathscr{S}\right\}. \tag{16}$$

The solution of (15) leads to the spectral problem

$$\int_{\partial\Omega_1} y\zeta\, d\sigma - \frac{\nu^2}{g}\int_{\Omega_1} \nabla\phi\, \nabla\psi\, dx = 0 \qquad \forall\psi \in T,$$

which is similar to (13).

We define $C: T \to \mathscr{S}$ by setting $(\partial\phi/\partial n)|_{\partial\Omega_1} = C\phi$.

By similar methods we can prove the existence of a discrete spectrum and we have the following lemma.

LEMMA 8. *The operator defined by* $C^{-1}: \mathscr{S} \to T$ *is compact* (T *is equipped with the topology of* W).

4. Spectral Problem Associated with the Initial Problem

4.1. *Description of the Problem*

Let $\mathfrak{H}$ be the Cartesian product of the three sets

$$\mathfrak{H} = W \times L^2(\Gamma_1) \times V.$$

$\mathfrak{H}$ is equipped with the topology of the product and $\mathfrak{h} = (\psi, \zeta, v)$ represents a member of $\mathfrak{H}$.

We have already seen the interest of stationary solutions which are related to the notion of a spectral problem (Section 3.2.1). We therefore search the stationary solutions of the problem under the form of

$$\begin{gathered} \Phi(x, t) = i\nu\phi(x)\, e^{i\nu t}, \\ u(x, t) = u(x)\, e^{i\nu t}, \qquad Y(x', t) = y(x')\, e^{i\nu t}, \end{gathered} \tag{17}$$

where the triplet $P = (\phi, y, u)$ belongs to $\mathfrak{H}$ in order to eliminate the trivial solution ($\phi = \text{const}$, $u = 0$, $y = 0$) corresponding to $\nu = 0$.

The problem (1)–(4) leads then to

$$i\nu \left[\int_{\Omega_1} \rho_f \nabla\phi \nabla\psi \, dx - \int_{\Gamma_1} \rho_f y\psi \, d\sigma - \int_{\Sigma_1} \rho_f(un)\, \psi \, d\sigma\right] = 0 \qquad \forall\psi \in H^1(\Omega_1), \tag{18}$$

$$-\nu^2 \int_{\Gamma_1} \rho_f \phi\zeta \, d\sigma + g \int_{\Gamma_1} \rho_f y\zeta \, d\sigma = 0 \qquad \forall\zeta \in L^2(\Gamma_1), \tag{19}$$

$$\begin{aligned} \sum_{i,j} \int_{\Omega_2} \sigma_{ij}(u)\, \epsilon_{ij}(v)\, dx - \nu^2 \int_{\Omega_2} \rho_c(uv)\, dx - \nu^2 \int_{\Sigma_1} \rho_f\phi(vn)\, d\sigma & \\ - \int_{\Sigma_1} \rho_f(gn)\,(un)\,(vn)\, d\sigma = 0 \qquad \forall v \in V. & \end{aligned} \tag{20}$$

4.2. *The Reduced Problem*

Let $\mathfrak{R}$ be the subspace of $W \times L^2(\Gamma_1) \times V$ defined by

$$\mathfrak{R} = \left\{ \mathfrak{h} \in \mathfrak{R} \mid \Delta\psi = 0 \text{ in } \Omega_1; \frac{\partial\psi}{\partial n} = (vn) \text{ on } \Sigma_1; \frac{\partial\psi}{\partial n} = \zeta \text{ on } \Gamma_1 \right\}. \tag{21}$$

The fact that $\mathfrak{h}$ belongs to $\mathfrak{R}$ leads in particular to

$$\int_{\Sigma_1} (vn)\, d\sigma + \int_{\Gamma_1} \zeta\, d\sigma = 0.$$

LEMMA 9. *$\mathfrak{R}$ is a closed subspace of $\mathfrak{H}$.*

For the purpose of demonstration, we consider a Cauchy sequence in $\mathfrak{R}$, that is $\mathfrak{h}_n = (\phi_n, y_n, u_n)$. We show that $\lim_{n\to\infty} \mathfrak{h}_n = \mathfrak{h} = (\phi, y, u)$ belongs to $\mathfrak{R}$.

This follows, on the one hand, from $\Delta\phi = 0$ in the sense of distributions in Ω_1 so

$$\left.\frac{\partial\phi_n}{\partial n}\right|_{\partial\Omega_1} \to \left.\frac{\partial\phi}{\partial n}\right|_{\partial\Omega_1} \quad \text{in} \quad H^{-1/2}(\partial\Omega_1).$$

On the other hand, we consider the limit of the sequences in the equation

$$\int_{\Omega_1} \nabla\phi_n \nabla\psi\, dx = \int_{\Gamma_1} y_n \psi\, d\sigma + \int_{\Sigma_1} (u_n n)\, \psi\, d\sigma \qquad \forall\psi \in H^1(\Omega_1).$$

Fundamental Remark. Equation (18) is verified if we search P in the space $\mathfrak{R}$. The application of Green's formula in a symmetrical way leads to

$$\begin{aligned} \int_{\Omega_1} \nabla\phi \nabla\psi\, dx &= \int_{\Gamma_1} y\psi\, d\sigma + \int_{\Sigma_1} (un)\, \psi\, d\sigma \\ &= \int_{\Gamma_1} \phi\zeta\, d\sigma + \int_{\Sigma_1} (vn)\, \phi\, d\sigma \qquad \forall\mathfrak{h} \in \mathfrak{R}. \end{aligned} \tag{21'}$$

This fundamental remark implies the reduction of Eqs. (19) and (20).

LEMMA 10. *The problem* (18)–(20) *becomes the following: Find the scalar ν and $P = (\phi, y, u)$ such that*

$$\begin{aligned} &\sum_{i,j} \int_{\Omega_2} \sigma_{ij}(u)\, \epsilon_{ij}(v)\, dx - \int_{\Sigma_1} \rho_f(gn)\,(un)\,(vn)\, d\sigma + g \int_{\Gamma_1} \rho_f\, y\zeta\, d\sigma \\ &\quad - \nu^2 \left[\int_{\Omega_2} \rho_c(uv)\, dx + \int_{\Omega_1} \rho_f \nabla\phi \nabla\psi\, dx \right] = 0 \qquad \forall\mathfrak{h} = (\psi, \zeta, v) \in \mathfrak{R}. \end{aligned} \tag{22}$$

We are now able to state Property 1, which correlates the problem of vibration to the analysis of a spectral problem.

Property 1. The following spectral problem is associated with the problem (1)–(4): Find ν^2 and $P \in \mathfrak{K}$ such that

$$\mathscr{M}(P, \mathfrak{h}) = \nu^2 \mathscr{N}(P, \mathfrak{h}) \qquad \forall \mathfrak{h} \in \mathfrak{K} \tag{23}$$

with

$$\mathscr{M}(P, \mathfrak{h}) = \sum_{i,j} \int_{\Omega_2} \sigma_{ij}(u)\, \epsilon_{ij}(v)\, dx + g \int_{\Gamma_1} \rho_f\, y\zeta\, d\sigma - \int_{\Sigma_1} \rho_f (gn)\,(un)\,(vn)\, d\sigma,$$

$$\mathscr{N}(P, \mathfrak{h}) = \int_{\Omega_1} \rho_f\, \nabla\phi\, \nabla\psi\, dx + \int_{\Omega_2} \rho_c (uv)\, dx. \tag{24}$$

Remark. The problem (18)–(20) is equivalent to the problem (23), (24) on the subspace of $L^2(\Gamma_1) \times V$, for which we have

$$\int_{\Gamma_1} \zeta\, d\sigma + \int_{\Sigma_1} (vn)\, d\sigma = 0.$$

5. Resolution of the Preceding Spectral Problem

5.1.

$\mathscr{M}(P, \mathfrak{h})$ and $\mathscr{N}(P, \mathfrak{h})$ are two continuous symmetric bilinear forms on $\mathfrak{K}$ equipped with the topology of $\mathfrak{H}$. Let $\mathfrak{H}'$ be the strong dual of $\mathfrak{H}$. There exists two self-adjoint operators $\mathfrak{M}$ and $\mathfrak{N} \in \mathscr{L}(\mathfrak{H}, \mathfrak{H}')$ such that

$$\langle \mathfrak{M}P, \mathfrak{h} \rangle = \mathscr{M}(P, \mathfrak{h}), \qquad \langle \mathfrak{N}P, \mathfrak{h} \rangle = \mathscr{N}(P, \mathfrak{h}). \tag{25}$$

The norm induced on $\mathfrak{K}$ by $\langle \mathfrak{N}P, \mathfrak{h} \rangle$ is the norm induced by $W \times (L^2(\Omega_2))^3$.

5.2

For the spectral analysis (23), a first difficulty comes through the fact that $\mathfrak{M}$ is not necessary a positive operator. Hence, we consider *the hypothesis of V-coercivity with regard to* $(L^2(\Omega_2))^3$: There exist two constants $\lambda_0 \geqslant 0$ and $\alpha_0 > 0$ such that

$$\sum_{i,j} \int_{\Omega_2} \sigma_{ij}(u)\, \epsilon_{ij}(u)\, dx - \rho_f \int_{\Sigma_1} (gn)\,(un)^2\, d\sigma + \lambda_0 \int_{\Omega_2} \rho_c \mid u \mid^2 dx \geqslant \alpha_0 \| u \|_V^2 . \tag{26}$$

This hypothesis is connected with the geometry of the system and appears to be justified because the coefficient $\rho_f(gn)$ is generally small; on the other hand, it is obviously verified in the case of thin shells of thickness h, where the integration in Ω_2 is replaced by an integration on the mean surface γ_2 (it is therefore sufficient to choose $\lambda_0 \geqslant (gn)\, \rho_f/\rho_c h$).

Under the hypothesis (26), let the operator $\mathfrak{M}'$ be defined by

$$\langle \mathfrak{M}'P, \mathfrak{h} \rangle = \langle \mathfrak{M}P, \mathfrak{h} \rangle + \lambda_0 \langle \mathfrak{N}P, \mathfrak{h} \rangle. \tag{27}$$

(24), (26), and Lemma 1 lead to the coercivity of the self-adjoint operator $\mathfrak{M}'$.

5.3

It remains to prove that "$\mathfrak{M}'$ is compact in regard to $\mathfrak{N}$." This property is equivalent to the following one: From every sequence $\{P_n\} \in \mathfrak{K}$ such that $\langle \mathfrak{M}'P_n , P_n \rangle <$ const, we can extract a subsequence which converges strongly for the norm $\langle \mathfrak{N}P, P \rangle$. $\mathfrak{M}'$ is the sum of the two operators $\mathfrak{M}_1$ and $\mathfrak{M}_2$, defined by

$$\langle \mathfrak{M}_1 P, \mathfrak{h} \rangle = \sum_{i,j} \int_{\Omega_2} \sigma_{ij}(u)\, \epsilon_{ij}(v)\, dx - \rho_f \int_{\Sigma_1} (gn)\,(un)\,(vn)\, d\sigma + \lambda_0 \int_{\Omega_2} \rho_c (uv)\, dx,$$

$$\langle \mathfrak{M}_2 P, \mathfrak{h} \rangle = g \int_{\Gamma_1} y\zeta\, d\sigma + \lambda_0 \int_{\Omega_1} \nabla\phi\, \nabla\psi\, dx.$$

We recall here that P (resp. $\mathfrak{h}$) represents the triplet (ϕ, y, u) (resp. (ψ, ζ, v)) as an element of $\mathfrak{K}$.

It follows from Lemma 2 and the hypothesis (26) that $\mathfrak{M}_1^{-1}$ is compact in $(L^2(\Omega_2))^3$. On the other hand, we have

$$\int_{\Gamma_1} y_n{}^2\, d\sigma < \text{const} \qquad \text{and} \qquad \int_{\Sigma_1} (u_n n)^2\, d\sigma < \text{const},$$

owing to the continuity property of the trace operator from $H^1(\Omega_2)$ into $L^2(\partial\Omega_2)$. The sequence P_n is such that $\partial\phi_n/\partial n \in \mathscr{S}$ and $|\, \partial\phi_n/\partial n\, |^2 <$ const.

Using Lemma 8, we can show the compactness property of $\mathfrak{M}_2^{-1}$. $\mathfrak{M}'$, being the sum of two compact operators (in regard to $(L^2(\Omega_2))^3$ and W) is therefore "compact in regard to $\mathfrak{N}$."

These results are gathered in the following theorem.

THEOREM. *$\mathfrak{M}'$ is a coercive, self-adjoint operator on $\mathfrak{K}$, and $\mathfrak{M}'^{-1}$ is compact in $\mathfrak{K}$ (for the topology induced by $\langle \mathfrak{N}P, \mathfrak{h} \rangle$).*

We can therefore apply a classical theorem of spectral analysis [8, p. 237] in order to obtain the following property.

Property 2. The spectral problem

$$\langle \mathfrak{M}'P, \mathfrak{h}\rangle - \alpha\langle \mathfrak{N}P, \mathfrak{h}\rangle = 0 \qquad \forall \mathfrak{h} \in \mathfrak{K} \tag{28}$$

has an infinite sequence of real eigenvalues:

$$0 < \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_n \leqslant \cdots, \qquad \lim_{n\to\infty} \alpha_n = +\infty.$$

The corresponding eigenfunctions $P_n = (\phi_n, y_n, u_n)$ form on $\mathfrak{K}$ an orthogonal and total sequence, such that

$$\langle \mathfrak{M}'P_n, P_m\rangle = \alpha_n\langle \mathfrak{N}P_n, P_m\rangle = \alpha_n\delta_{nm} \qquad \forall m.$$

6. Consequences: Resolution of the Initial Problem

Referring to the problem (1)–(4) with which we have connected the spectral problem (23), $\mathscr{M}(P, \mathfrak{h}) = \nu^2\mathscr{N}(P, \mathfrak{h})$, the circular eigenfrequencies are given by

$$\nu_n^2 = \alpha_n - \lambda_0, \tag{29}$$

and the corresponding eigenfunctions P_n form an orthogonal and total sequence in $W \times (L^2(\Omega_2))^3$ such that

$$\begin{aligned}
&\Delta\phi_n = 0 && \text{in} \quad \Omega_1,\\
&\frac{\partial\phi_n}{\partial n} = (u_n n) && \text{on} \quad \Sigma_1,\\
&\left.\begin{aligned}\frac{\partial\phi_n}{\partial n} &= y_n\\ \nu_n^2\phi_n &= y_n\end{aligned}\right\} && \text{on} \quad \Gamma_1,\\
&\sum_{i,j} \frac{\partial\sigma_{ij}(u_n)}{\partial x_j} = \nu_n^2\rho_c u_{n_i} && \text{in} \quad \Omega_2, \qquad i = 1, 2, 3,\\
&u_n = 0 && \text{on} \quad \Sigma_3,\\
&\sum_j \sigma_{ij}(u_n)\, n_j = 0 && \text{on} \quad \Sigma_2,\\
&\sum_{i,j} n_i\sigma_{ij}(u_n)\, n_j = \rho_f\nu_n^2\phi_n - \rho_f(u_n \cdot n)\,(g \cdot n) && \text{on} \quad \Sigma_1.
\end{aligned} \tag{30}$$

It must be observed that the system is unstable (in the usual sense in linear vibration mechanic) if $\alpha_n - \lambda_0$ is negative for the first values of n.

Conclusion. For the initial hydroelastic problem, we have proved the existence of an infinite sequence of eigenfrequencies such as $\lim_{n\to\infty} \nu_n = \infty$. The eigenfunction given by the triplets (ϕ_n, y_n, u_n) are the corresponding vibration shapes. The system is completely characterized from the mechanical point of view by the knowledge of the vibration modes.

Remark. If meas $\Sigma_3 = 0$, we can introduce [5, p. 115] the space $\mathring{V} = (H^1(\Omega_2))^3/\mathscr{R}$, where $\mathscr{R}$ is the set of rigid solid displacements. The results of the theoretical analysis remain the same if we take $W \times L^2(\Gamma_1) \times \mathring{V}$ as a new definition of the space $\mathfrak{H}$. For the following numerical analysis, we shall use Eqs. (18)–(20) with $\mathfrak{H} = H^1(\Omega_1) \times L^2(\Gamma_1) \times (H^1(\Omega_2))^3$. The solutions are then defined to within a rigid body displacement.

7. Numerical Analysis by the Finite Element Method

According to the chosen application we shall deal with the problem of axisymmetric vibrations of an axisymmetric shell partially filled with liquid, within the thin shell theory.

7.1. *Matrix Equations of the Discretized Problem*

In this analysis v is not necessary zero on Σ_3. Taking into account the preceding Remark in Section 6, the function space here considered is

$$\mathfrak{H} = H^1(\Omega_1) \times L^2(\Gamma_1) \times (H^1(\Omega_2))^3.$$

Equations (18)–(20) of the weak problem recalled here,

$$i\nu \left[\int_{\Omega_1} \rho_f \nabla\phi \nabla\psi \, dx - \int_{\Gamma_1} \rho_f y\psi \, d\sigma - \int_{\Sigma_1} \rho_f(un) \psi \, d\sigma\right] = 0 \qquad \forall \psi \in H^1(\Omega_1), \tag{18}$$

$$-\nu^2 \int_{\Gamma_1} \rho_f \phi\zeta \, d\sigma + g \int_{\Gamma_1} \rho_f y\zeta \, d\sigma = 0 \qquad \forall \zeta \in L^2(\Gamma_1), \tag{19}$$

$$\sum_{i,j} \int_{\Omega_2} \sigma_{ij}(u) \epsilon_{ij}(v) \, dx - \nu^2 \int_{\Omega_2} \rho_c(uv) \, dx - \nu^2 \int_{\Sigma_1} \rho_f \phi(vn) \, d\sigma - \int_{\Sigma_1} \rho_f(gn)(un)(vn) \, d\sigma = 0 \qquad \forall v \in (H^1(\Omega_2))^3, \tag{20}$$

are discretized by the finite element method. The domain is covered by a finite number of simplex models such as the open simplexes are disjoint, and two adjacent simplexes have a common side. The trial functions used are piecewise interpolation polynomials [11]. The unknowns are then the nodal values of the functions. $\mathscr{Y}^*$, $\mathscr{U}^*$, Φ^* are, after assembly procedure, the column matrices constituted by the nodal values of the functions y, u, ϕ. The matrix formulation of the discretized problem is

$$\left\{\begin{bmatrix} S & 0 & 0 \\ 0 & K+k & 0 \\ 0 & 0 & 0 \end{bmatrix} - \nu^2 \begin{bmatrix} 0 & 0 & A_1 \\ 0 & M & A_2 \\ \bar{A}_1 & \bar{A}_2 & -F \end{bmatrix}\right\} \begin{Bmatrix} \mathscr{Y}^* \\ \mathscr{U}^* \\ \Phi^* \end{Bmatrix} = 0 \tag{31}$$

(in $\bar{A}_1$ the overbar represents the transpose quantity). The different submatrices of the system (31) ($(S, K, k) \in \mathfrak{K}$; $(M, A_1, A_2, F) \in \mathscr{M}$) are associated with the bilinear forms of the functional equations (18)–(20) and constitute the matrices resulting from the application of the finite element method. Their detailed definitions will be found in Section 7.3.

$\mathfrak{K}$ and $\mathscr{M}$ are, respectively, the stiffness and mass matrices of the mechanical system.

7.2. *Description of the Elements*

The function spaces $H^1(\Omega_1)$, $L^2(\Gamma_1)$, $(H^1(\Omega_2))^3$ of the variational coupled problem are approximated. Because of the symmetry of the physical problem, the analysis becomes twodimensional. As a result, the simplexes on which the interpolations are to be constructed are torus (axis $0X$). The cross section will be represented in a meridian plane (X, Y).

Let M_i be the node of coordinates (X_i, Y_i) and (y_i, ϕ_i, u_i^*) (resp. ζ_i, ψ_i, v_i^*), the values of the unknown functions at the node M_i before assemblage.

We also put

$$X_{ij} = X_j - X_i, \qquad Y_{ij} = Y_j - Y_i.$$

7.2.1. *Fluid Finite Elements*

Three types of finite elements are considered.

(a) *Rectangular* (Fig. 2). We proceed on a quadratic development of the solution

$$\begin{aligned} \phi(X, Y) &= a_0 + a_1X + a_2Y + a_3XY, \\ \psi(X, Y) &= b_0 + b_1X + b_2Y + b_3XY. \end{aligned} \tag{32}$$

This corresponds to a Lagrangian-type interpolation formula in the element domain (nodal unknowns: $\phi_1, \phi_2, \phi_3, \phi_4$).

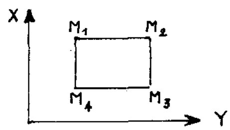

FIG. 2. Rectangular element.

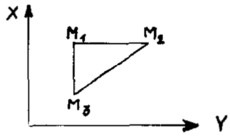

FIG. 3. Triangular element.

(b) *Triangular* (Fig. 3). The chosen interpolation is linear and results from the truncated developments:

$$\begin{aligned} \phi(X, Y) &= c_0 + c_1 X + c_2 Y, \\ \psi(X, Y) &= d_0 + d_1 X + d_2 Y \end{aligned} \tag{33}$$

(nodal unknowns: ϕ_1, ϕ_2, ϕ_3).

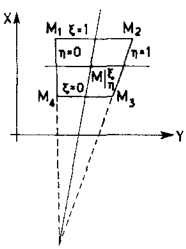

FIG. 4. Isoparametric element.

(c) *Isoparametric trapezoidal* (Fig. 4). A point M of the cross section is represented by the coordinate lines: $\xi =$ const, $\eta =$ const. The interpolation functions chosen for ϕ and ψ are defined from

$$\begin{aligned} \phi(\xi, \eta) &= e_0 + e_1\xi + e_2\eta + e_3\xi\eta, \\ \psi(\xi, \eta) &= f_0 + f_1\xi + f_2\eta + f_3\xi\eta. \end{aligned} \tag{34}$$

It is clear that the continuity condition for ϕ and ψ is satisfied along the element boundaries.

7.2.2. *Element of the Free Surface* (Fig. 5)

This element has the shape of a circular ring generated by a linear segment M_1M_2 of the meridian straight line of the free surface. The functions y, ζ

FIG. 5. Free surface ring.

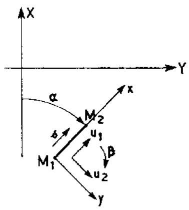

FIG. 6. Conical frustrum element.

are linearly interpolated with respect to Y (nodal unknowns: (y_1, y_2); (ζ_1, ζ_2) for the test function).

7.2.3. *Elastic Shell Element* (Fig. 6)

The shell, supposed to be thin, is represented schematically by conical frustrum elements of revolution around the axis $0X$ generated by the segment M_1M_2.

We consider here a local system of coordinates (x, y). Let us put

$$s = x/L.$$

L is the length of the segment M_1M_2.

The components $[u_1(s), u_2(s)]$ of the displacement vector $\mathbf{u}$ of the shell are respectively interpolated by Lagrangian and Hermitian polynomials [12]:

$$\begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix} = \begin{bmatrix} L_1(s) \\ L_2(s) \end{bmatrix} u^*, \qquad \begin{bmatrix} v_1(s) \\ v_2(s) \end{bmatrix} = \begin{bmatrix} L_1(s) \\ L_2(s) \end{bmatrix} v^* \tag{35}$$

with

$$L_1(s) = [1 - s \quad 0 \quad 0 \quad s \quad 0 \quad 0],$$
$$L_2(s) = [0 \quad 1 - 3s^2 + 2s^3 \quad L(s - 2s^2 + s^3) \quad 0 \quad 3s^2 - 2s^3 \quad L(-s^2 + s^3)],$$

and

$$u^* = \begin{bmatrix} u_{11} \\ u_{21} \\ \beta_1 \\ \hline u_{12} \\ u_{22} \\ \beta_2 \end{bmatrix}, \qquad v^* = \begin{bmatrix} v_{11} \\ v_{21} \\ \gamma_1 \\ \hline v_{12} \\ v_{22} \\ \gamma_2 \end{bmatrix}, \qquad \beta = \frac{du_2}{dx} \quad \left(\text{resp. } \gamma = \frac{dv_2}{dx}\right).$$

u_{ij}, v_{ij} ($i = 1, 2; j = 1, 2$) are the values of the ith components of the displacement vector $\mathbf{u}$ (resp. $\mathbf{v}$) at the node M_j. Let us recall that $\mathbf{v}$ is the test function associated with $\mathbf{u}$.

β_i (resp. γ_i) are the values at the node M_i of the angle of rotation of the normal to the middle surface of the shell.

Let us recall here that the thin shell theory [13] is a surface theory. Then, the particular geometry of the body allows the introduciton of kinematic simplifying assumptions.

The deformation of the shell can then be expressed in terms of the middle surface displacement (u_1, u_2) and of the rotations of the associated normals. Therefore the tensor ϵ (ϵ_{ij}) takes into account the curvatures of the middle surface. Due to the chosen interpolation functions, the continuity condition for u_i (resp. v_i) is satisfied along the element boundaries.

7.3. *Description of the Discretized Problem*

The above-described finite elements will allow us to state the matrices corresponding to the approximation of the bilinear symmetric forms of Eqs. (18)–(20).

In the following paragraphs, we describe in detail the matrix equation put down in Section 7.1.

7.3.1. *Approximation of* $\int_{\Omega_1} \rho_f \nabla\phi \nabla\psi \, dx$

(a) *Rectangular element.* Let ϕ^* and ψ^* be

$$\phi^* = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix}, \qquad \psi^* = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix}.$$

The matrix F_R corresponding to the form $\overline{\phi^* F_R \psi^*}$ is

$$F_R = 2\pi\rho_f \bar{B} \left[\int_{X_3}^{X_1} \int_{Y_1}^{Y_2} \bar{L} L Y \, dX \, dY \right] B$$

with

$$L = \begin{bmatrix} 0 & 1 & X \\ 1 & 0 & Y \end{bmatrix}, \qquad B = \frac{1}{\text{area}} \begin{bmatrix} Y_2 & -Y_1 & Y_1 & -Y_2 \\ X_3 & -X_3 & X_1 & -X_1 \\ -1 & 1 & -1 & 1 \end{bmatrix}.$$

(b) *Triangular element.* We obtain the matrix F_T corresponding to $\overline{\phi^* F_T \psi^*}$ through the relation

$$F_T = 2\pi\rho_f \bar{B} \left[\int_{\text{triangle}} \bar{L} L Y \, dX \, dY \right] B,$$

where

$$L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \frac{1}{2 \times \text{area}} \begin{bmatrix} Y_{12} & 0 & -Y_{12} \\ X_{13} & -X_{13} & 0 \end{bmatrix}.$$

(c) *Isoparametric element.* The calculus of the discretized bilinear form $\overline{\phi^*} F_I \psi^*$ is done after the following change of variables:

$$X = X_2 + \xi X_{21},$$

$$Y = Y_1 + \eta Y_{13} + \xi\eta Y_{32}.$$

We get

$$\varphi(\xi, \eta) = [(1-\eta)\,\xi \quad \eta\xi \quad \eta(1-\xi) \quad (1-\eta)(1-\xi)] \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix}$$

and

$$F_I = 2\pi\rho_f \int_0^1 \int_0^1 \bar{B}B(\xi, \eta)\, Y(\xi, \eta) \cdot \left| \frac{D(X, Y)}{D(\xi, \eta)} \right| d\xi\, d\eta,$$

where B is the matrix which represents $\nabla\boldsymbol{\varphi}$ as a function of the nodal unknowns.

The assemblage of the elementary matrices F_I, F_T, F_R gives the submatrix F of the set (31).

7.3.2. *Approximation of* $\rho_f \int_{\Gamma_1} y\psi\, d\sigma$ *and* $\rho_f \int_{\Gamma_1} \phi\zeta\, d\sigma$

The matrices associated with these two bilinear forms are transposed one by the other. Using the notations of Section 7.2.2, let us put

$$y^* = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \psi^* = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \quad \zeta^* = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}, \quad \phi^* = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix},$$

setting, furthermore,

$$c_1 = \begin{bmatrix} \zeta^* \\ \psi^* \end{bmatrix}, \quad C_1 = \begin{bmatrix} y^* \\ \phi^* \end{bmatrix}.$$

The elementary matrices associated with $\rho_f \int_{\Gamma_1} y\psi\, d\sigma$ and $\rho_f \int_{\Gamma_1} \phi\zeta\, d\sigma$ are, respectively,

$$\bar{C}_1 \begin{bmatrix} 0 & a_1 \\ 0 & 0 \end{bmatrix} c_1 \quad \text{and} \quad \bar{C}_1 \begin{bmatrix} 0 & 0 \\ \bar{a}_1 & 0 \end{bmatrix} c_1$$

with

$$a_1 = 2\pi\rho_f \bar{N} \left[\int_{Y_1}^{Y_2} \bar{T}TY\, dY \right] N \qquad (\bar{a}_1 = a_1),$$

where

$$N = \frac{1}{Y_{12}} \begin{bmatrix} Y_2 & -Y_1 \\ -1 & 1 \end{bmatrix}, \qquad T = [1 \quad Y].$$

The assemblage of these elementary matrices forms the submatrix

$$\begin{bmatrix} 0 & A_1 \\ \overline{A_1} & 0 \end{bmatrix}$$

of the set (31).

7.3.3. *Approximation of* $g\rho_f \int_{\Gamma_1} y\zeta \, d\sigma$.

Using the interpolations defined in Section 7.2.2, the elementary matrix associated with this form is ga_1. a_1 has been defined in Section 7.3.2.

The assemblage of these elementary matrices on the whole free surface provides the submatrix S of the set (31).

7.3.4. (a) *Approximation of* $\sum_{i,j} \int_{\Omega_2} \sigma_{ij}(u)\, \epsilon_{ij}(v)\, dx$.

The geometry of the element has been described in Section 7.2.3. The calculus of the associated elementary matrix is carried out in the local system of coordinates (x, y). After a change of basis, the assemblage procedure is done in the global system of coordinates (X, Y).

The relations between displacements and strains are the following [13]:

$$\begin{aligned} \epsilon_x &= \frac{du_1}{dx}, \qquad & \epsilon_z &= \frac{u_1}{Y}\frac{dY}{dx} + \frac{u_2 \cos\alpha}{Y}, \\ \kappa_x &= \frac{d^2u_2}{dx^2}, \qquad & \kappa_z &= \frac{1}{Y}\frac{dY}{dx}\frac{du_2}{dx} \end{aligned} \tag{36}$$

κ_x and κ_z are the variations of curvature of the middle surface, respectively, in the directions (x, z).

The same relations (36) are also utilized for the test function $v(v_1, v_2)$.

The relations between the generalized stresses (resulting from an integration of the three-dimensional stresses with respect to the thickness of the shell) and the generalized strains, are the following:

$$\sigma = D\epsilon$$

with

$$\sigma = \begin{bmatrix} N_x \\ N_z \\ M_x \\ M_z \end{bmatrix}, \qquad \epsilon = \begin{bmatrix} \epsilon_x \\ \epsilon_z \\ \kappa_x \\ \kappa_z \end{bmatrix},$$

$$D = \begin{bmatrix} C_L & C_{LC} & 0 & 0 \\ C_{LC} & C_C & 0 & 0 \\ 0 & 0 & D_L & D_{LC} \\ 0 & 0 & D_{LC} & D_C \end{bmatrix},$$

where D is the classical matrix of the characteristic coefficients of the material in linear orthotropic elasticity (the indexes L, C indicate longitudinal or circumferential stiffness).

Relations (35) and (36) permit the calculation of ϵ as a function of u^* and v^* in the form

$$\epsilon(s, u^*) = A(s)\, u^*,$$
$$\epsilon(s, v^*) = A(s)\, v^*.$$

The matrix associated with the bilinear symmetric form

$$\sum_{i,j} \int_{\Omega_2} \sigma_{ij}(u)\, \epsilon_{ij}(v)\, dx$$

on the shell element is $\overline{u^*K_C'v^*}$, where

$$K_C' = 2\pi L \int_0^1 \overline{A(s)}\, DA(s)\, (Y_1 + sY_{12})\, ds.$$

K_C' is the elementary matrix which represents the bilinear form in the local basis.

The submatrix K of the set (31) is obtained afterwards by assembling the matrices

$$K_C = \bar{\mathcal{R}} K_C' \mathcal{R},$$

where

$$\mathcal{R} = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}$$

with

$$R = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) *Approximation of* $\int_{\Omega_2} \rho_c(uv)\, dx$. Let m_C' be the matrix which represents this form for a shell element.

We obtain in the local basis (x, y)

$$m_C' = 2\pi\rho_c L \int_0^1 P(s)\, (Y_1 + sY_{12})\, ds.$$

Using the polynomial interpolation and the notations of Section 7.2.3, we have

$$P(s) = \overline{L_1(s)}\, L_1(s) + \overline{L_2(s)}\, L_2(s).$$

In the (X, Y) coordinate system, the elementary matrix m_C' becomes

$$m_C = \bar{\mathscr{R}} m_C' \mathscr{R}.$$

The matrix M of the set (31) is then obtained by assembling the matrices m_C.

(c) *Approximation of* $-\int_{\Sigma_1} \rho_f(gn)\,(un)\,(vn)\, d\sigma$. We obtain the matrix relation $-\overline{u^*} k_C' v^*$. Using the same notation as in Section 7.2.3, the corresponding k_C' matrix is

$$k_C' = 2\pi\rho_f L g \sin\alpha \int_0^1 \overline{L_2(s)}\, L_2(s)\,(Y_1 + Y_{12}s)\, ds.$$

$k_C = \bar{\mathscr{R}} k_C' \mathscr{R}$ leads to the submatrix k of the set (31).

7.3.5. *Approximation of* $\int_{\Sigma_1} \rho_f \phi(vn)\, d\sigma$ *and* $\int_{\Sigma_1} \rho_f(un)\, \psi\, d\sigma$

After discretization on a finite element (on the wetted surface of the shell), the bilinear reciprocally symmetric forms are

$$\overline{u^*} a_2 \psi^* \qquad \text{and} \qquad \overline{\phi^*}\, \bar{a}_2 v^*.$$

Let us recall that these forms represent coupling terms between the fluid and the shell. Using the results of Section 7.2.3 as well as the linear interpolation of φ (resp. ψ) along Σ_1 (Sections 7.2.1(b) and 7.2.1(c)), we have

$$a_2 = 2\pi\rho_f L \int_0^1 \overline{L_2(s)}\, [1 - s \quad s]\,(Y_1 + sY_{12})\, ds.$$

The elementary matrices associated with the discretized bilinear forms are

$$\bar{C}_2 \begin{bmatrix} 0 & a_2 \\ 0 & 0 \end{bmatrix} c_2 \qquad \text{and} \qquad \bar{C}_2 \begin{bmatrix} 0 & 0 \\ \bar{a}_2 & 0 \end{bmatrix} c_2,$$

where

$$c_2 = \begin{bmatrix} v^* \\ \psi^* \end{bmatrix}, \qquad C_2 = \begin{bmatrix} u^* \\ \phi^* \end{bmatrix}.$$

After a change of basis, the assemblage is made on the following elementary matrices:

$$\begin{bmatrix} 0 & \bar{\mathscr{R}} a_2 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 0 & 0 \\ \overline{\bar{\mathscr{R}} a_2} & 0 \end{bmatrix}.$$

We then obtain the submatrix $\left[\begin{smallmatrix} 0 & \bar{A}_2 \\ A_2 & 0 \end{smallmatrix}\right]$ of the set (31).

7.4. *Resolution of the Discretized Problem—Numerical Results*

7.4.1. *Resolution*

Let us recall here that, in the case of axisymmetric vibrations, the velocity potential is determined to within an additive constant (function of time only). Consequently, we show that if F is the $n \times n$ submatrix associated with the bilinear form $\int_{\Omega_2} \rho_f \nabla\phi\nabla\psi \, dx$, then the order of F is $n - 1$.

The set (31) then becomes

$$\left\{\left[\begin{array}{c|c} \kappa_{11} & 0 \\ \hline 0 & 0 \end{array}\right] - \nu^2 \left[\begin{array}{c|c} \overset{R}{m}_{11} & \overset{R}{m}_{12} \\ \hline \overset{R}{m}_{12} & 0 \end{array}\right]\right\}\left\{\begin{array}{c} \mathscr{Y}^* \\ \mathscr{U}^* \\ \hline \Phi_1^* \end{array}\right\} = 0 \qquad (37)$$

with

$$\kappa_{11} = \left[\begin{array}{c|c} S & 0 \\ \hline 0 & K + k \end{array}\right],$$

and we therefore have a relation between $\mathscr{Y}^*$, $\mathscr{U}^*$, and Φ^*:

$$\mathscr{L}(\mathscr{Y}^*, \mathscr{U}^*, \Phi^*) = 0.$$

The reduction of the set (31) can be done by considering the following equation:

$$\begin{bmatrix} m_{11} & m_{12} \\ \overline{m}_{12} & m_{22} \end{bmatrix} \begin{bmatrix} \delta \\ \Phi_R^* \end{bmatrix} = \begin{bmatrix} F^* \\ 0 \end{bmatrix}, \qquad (38)$$

where the first quantity in square brackets is $\mathscr{M}$, the mass matrix of the set (31), and where

$$\delta = \begin{bmatrix} \mathscr{Y}^* \\ \mathscr{U}^* \\ \Phi_1^* \end{bmatrix}$$

and F^* is an arbitrary vector. (Φ_R^* represents the column matrix constituted by all the unknowns Φ^* except one.)

Once (37) solved, the set (38) gives Φ_R^*. The method avoids the use of the constraint relation $\mathscr{L}$. For the resolution, we use the algorithms of a general existing code ordinarily used for the static and dynamic analysis of structures by the finite element method (displacement method) [14].

As a matter of fact, the procedure for recovering the $\Phi_R{}^*$ corresponds to a regular, mixed problem of linear elasticity where the boundary conditions hold (at the same point) simultaneously on the displacements and the forces components [15].

7.4.2. *Numerical Results*

The computation is carried out on a structure composed of two tanks with an intermediate bulkhead separating the two liquids. This structure constitutes the first stage of the civil launch vehicle Diamant B (Fig. 7).

The tanks considered here are cylindrical shells with ellipsoidal bottoms partially filled with liquid (the chosen configuration leads to a level of liquids corresponding to 10% of burned propellant).

The modelization used for each tank, through the different finite elements previously described, is shown in Fig. 8. Figures 9 and 10 represent the shapes of the free surface and of the shell and the velocity potential along the axis of revolution.

Figure 11 shows a comparison between the first five numerical and experimental eigenfrequencies.

The good agreement obtained shows the precision of the presented method.

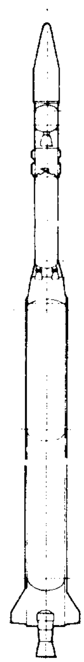

FIG. 7. Launcher configuration.

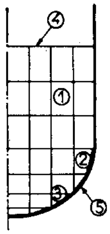

FIG. 8. Axisymmetrical elements. (i) Fluid elements: (1) rectangular, (2) isoparametric, (3) triangular, (4) free surface. (ii) Shell elements: (5) stiffness and mass. (iii) Coupling elements: fluid–shell, fluid–free surface.

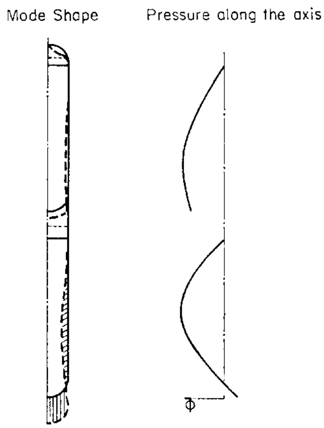

FIG. 9. Second structural mode: mode shape, —— initial configuration, ----- deformed configuration.

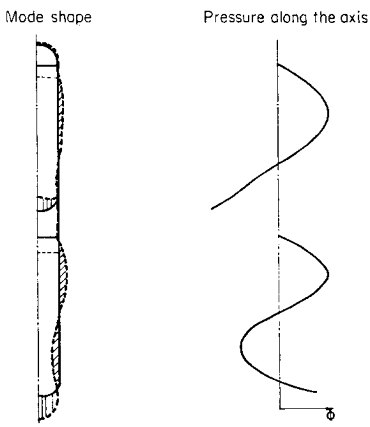

FIG. 10. Fourth structural mode: mode shape, —— initial configuration, ----- deformed configuration.

Modes	1	2	3	4	5
Test	28.6	48.1 - 48.4	79.5	91.5/92.8	119/120
Computation	27.8	48.3	75.9	95.0	124.2

FIG. 11. Frequencies: Comparison with experimental results.

From the values of the velocity potential φ at the bottom of the tanks, we get the values of the pressure at these points. These values are used as known quantities in a numerical model of the Pogo loop stability.

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