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On an inhomogeneous slip-inflow boundary value problem for a steady flow of a viscous compressible fluid in a cylindrical domain

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ABSTRACT

We investigate a steady flow of a viscous compressible fluid with inflow boundary condition on the density and inhomogeneous slip boundary conditions on the velocity in a cylindrical domain $\Omega = \Omega_0 \times (0, L) \in \mathbb{R}^3$. We show existence of a solution $(v, \rho) \in W_p^2(\Omega) \times W_p^1(\Omega)$, $p > 3$, where v is the velocity of the fluid and ρ is the density, that is a small perturbation of a constant flow $(\bar{v} \equiv [1, 0, 0],\ \bar{\rho} \equiv 1)$. We also show that this solution is unique in a class of small perturbations of $(\bar{v}, \bar{\rho})$. The term $u \cdot \nabla w$ in the continuity equation makes it impossible to show the existence applying directly a fixed point method. Thus in order to show existence of the solution we construct a sequence (v^n, ρ^n) that is bounded in $W_p^2(\Omega) \times W_p^1(\Omega)$ and satisfies the Cauchy condition in a larger space $L_\infty(0, L; L_2(\Omega_0))$ what enables us to deduce that the weak limit of a subsequence of (v^n, ρ^n) is in fact a strong solution to our problem.

1. Introduction

The mathematical description of a flow of a viscous, compressible fluid usually lead to problems of mixed character as the momentum equation is elliptic (in stationary case) or parabolic (in case of time-dependent flow) in the velocity, while the continuity equation is hyperbolic in the density. Therefore, the application of standard methods usually applied to elliptic or hyperbolic problems fails in the mathematical analysis of the compressible flows and a combination of such techniques, as well as development of new mathematical tools is required. As a result a consistent theory of weak solutions to the Navier–Stokes equations for compressible fluids has been developed quite recently in

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the 90's, mainly due to the work of Lions [12] and Feireisl [7]. An overview of these results is given in the monograph [18]. A modification of this approach in case of steady flows with slip boundary conditions has been developed by Mucha and Pokorný in a two dimensional case in [15] and in 3D in [24].

The issue of regular solutions is less investigated and the problems are considered mainly with Dirichlet boundary conditions. If we assume that the velocity does not vanish on the boundary, the hyperbolicity of the continuity equation makes it necessary to prescribe the density on the part of the boundary where the flow enters the domain. In [26] Valli and Zajaczkowski investigate a time-dependent system with inflow boundary condition, obtaining also a result on existence of a solution to stationary problem. The existence of regular solutions to stationary problems with an inflow condition on the density has been investigated by Kellogg and Kweon [9] and Kweon and Song [11]. Their results require some smallness assumptions on the data, and the regularity of solutions is a subject to some constraints on the geometry of the boundary near the points where the inflow and outlow parts of the boundary meet. In [10] Kellogg and Kweon consider a domain where the inflow and outflow parts of the boundary are separated, obtaining regular solutions.

The lack of general existence results inhibits the development of qualitative analysis of compressible flows. Therefore it is worth to mention here the papers by Plotnikov and Sokolowski who has investigated shape optimization problems with inflow boundary condition in 2D [22] and 3D [23] dealing with weak solutions. More recently Plotnikov, Ruban and Sokolowski have investigated shape optimization problems working with strong solutions in [20] and [21].

It seems interesting both from the mathematical point of view and in the eye of applications to investigate problems with inflow boundary condition on the density combined with slip boundary conditions on the velocity, that enables to describe precisely the action between the fluid and the boundary. Such problem is investigated in this paper. The domain is a three dimensional cylinder and we assume that the fluid slips along the boundary with a given friction coefficient and there is no flow across the wall of the cylinder. We show existence of a regular solution that can be considered a small perturbation of a constant solution. The method of the proof is outlined in the next part of the introduction and now we are in a position to formulate our problem more precisely.

The flow is described by the Navier–Stokes system supplied with the slip boundary conditions on the velocity. The complete system reads

$$\begin{aligned}
&\rho v\cdot\nabla v-\mu\Delta v-(\mu+\nu)\nabla\operatorname{div} v+\nabla\pi(\rho)=0 && \text{in } \Omega,\\
&\operatorname{div}(\rho v)=0 && \text{in } \Omega,\\
&n\cdot\mathbf{T}\big(v,\pi(\rho)\big)\cdot\tau_k+f v\cdot\tau_k=b_k,\quad k=1,2 && \text{on } \Gamma,\\
&n\cdot v=d && \text{on } \Gamma,\\
&\rho=\rho_{\mathrm{in}} && \text{on } \Gamma_{\mathrm{in}},
\end{aligned}\tag{1.1}$$

where $v:\mathbb{R}^3\to\mathbb{R}^3$ is the unknown velocity field of the fluid and $\rho:\mathbb{R}^3\to\mathbb{R}$ is the unknown density. We assume that the pressure is a function of the density of a class C^3. Further, μ and ν are viscosity coefficients satisfying $\mu>0$, $\nu+2\mu>0$ and $f>0$ is a friction coefficient. The domain Ω is a cylinder in $\mathbb{R}^3$ of a form $\Omega=\Omega_0\times(0,L)$ where $\Omega_0\in\mathbb{R}^2$ is a set with a boundary regular enough and L is a positive constant (see Fig. 1). We want to show existence of a solution that can be considered a small perturbation of a constant flow $(\bar v,\bar\rho)\equiv([1,0,0],1)$. Thus we denote the subsets of the boundary $\Gamma=\partial\Omega$ as $\Gamma=\Gamma_{\mathrm{in}}\cup\Gamma_{\mathrm{out}}\cup\Gamma_0$, where $\Gamma_{\mathrm{in}}=\{x\in\Gamma\colon\ \bar v\cdot n<0\}$, $\Gamma_{\mathrm{out}}=\{x\in\Gamma\colon\ \bar v\cdot n>0\}$ and $\Gamma_0=\{x\in\Gamma\colon\ \bar v\cdot n=0\}$.

By n we denote the outward unit normal to Γ and τ_1,τ_2 are the unit tangent vectors to Γ. Since the boundary has singularities at the junctions of Γ_{in} and Γ_{out} with Γ_0, for the boundary traces we will consider functional spaces that are algebraic sums of spaces defined on the boundary. More precisely for $s,q\in\mathbb{R}$ we shall denote $W^q_s(\Gamma):=W^q_s(\Gamma_{\mathrm{in}})+W^q_s(\Gamma_{\mathrm{out}})+W^q_s(\Gamma_0)$. We assume that $b\in W^{1-1/p}_p(\Gamma)$, $\rho_{\mathrm{in}}\in W^1_p(\Gamma_{\mathrm{in}})$ and $d\in W^{2-1/p}_p(\Gamma)$ are given functions and $d=0$ on Γ_0 what means that Γ_0 is an impermeable wall.

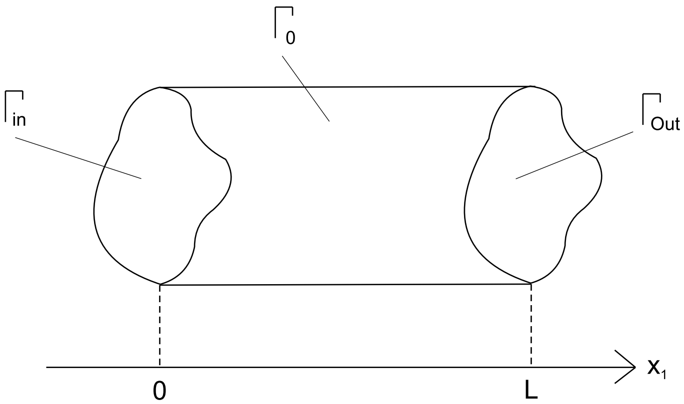

Fig. 1. The domain.

For simplicity we consider the momentum equation with zero r.h.s., but our proofs work without any modification for the r.h.s. ρF where F is small enough in L_p.

We shall make here some remarks concerning notation. Since we will usually use the spaces of functions defined on Ω, we will skip Ω in notation of the spaces, for example we will write L_2 instead of $L_2(\Omega)$. For the density we will use estimates in the space $L_\infty(0,L;L_2(\Omega_0))$. For simplicity we will denote this space by $L_\infty(L_2)$. A constant dependent on the data that can be controlled, but not necessarily small, will be denoted by C, and E shall denote a constant that can be arbitrarily small provided that the data is small enough.

In order to formulate our main result let us define a quantity D_0 that measures how the boundary data b, d and ρ_{in} differ from the values of, respectively, $f\bar{v}\cdot\tau_i$, $n\cdot\bar{v}$ and $\bar{\rho}$ in appropriate norms. We have $\bar{v}\cdot\tau_i=\tau_i^{(1)}$ and $\bar{v}\cdot n=n^{(1)}$, thus we define

$$D_0=\left\|b_i-f\tau_i^{(1)}\right\|_{W_p^{1-1/p}(\Gamma)}+\left\|d-n^{(1)}\right\|_{W_p^{2-1/p}(\Gamma)}+\|\rho_{\text{in}}-1\|_{W_p^1(\Gamma_{\text{in}})}. \tag{1.2}$$

Our main result is:

Theorem 1. *Assume that D_0 defined in* (1.2) *is small enough, f is large enough and $p>3$. Then there exists a solution $(v,\rho)\in W_p^2(\Omega)\times W_p^1(\Omega)$ to the system* (1.1) *and*

$$\|v-\bar{v}\|_{W_p^2}+\|\rho-\bar{\rho}\|_{W_p^1}\leqslant E(D_0), \tag{1.3}$$

where $E(D_0)$ can be arbitrarily small provided that D_0 is small enough. This solution in unique in the class of solutions satisfying the estimate (1.3)*.*

The major difficulty in the proof of Theorem 1 is in the term $u\cdot\nabla w$ in the continuity equation, that yields impossible a direct application of a fixed point argument. To overcome this problem one can apply the method of elliptic regularization, known rather from the theory of weak solutions (see [18]). This method has been applied to a similar problem in a two dimensional case in [19]. However, it complicates considerably the computations since we have to find the bound on the artificial diffusive term. Here we apply a method of successive approximations, that leads to a more direct proof. In order to prove Theorem 1 we will construct a sequence $(u^n,w^n)\in W_p^2\times W_p^1$ that converges to the solution of (1.1). Due to the presence of the term $u\cdot\nabla w$ we cannot show directly the convergence in $W_p^2\times W_p^1$, but we can show that (u^n,w^n) is a Cauchy sequence in a larger space $H^1\times L_\infty(L_2)$ and thus converges in this space to the weak solution of (1.1). On the other hand, the sequence is bounded in $W_p^2\times W_p^1$, what enables us to show that the weak solution is in fact strong. A similar approach has been applied in [4] to an evolutionary Navier–Stokes system in a framework of Besov spaces. Another method based on the same idea is to construct an operator that maps certain ball to itself in stronger topology and is a contraction in weaker topology, and apply a generalization of the Banach theorem that gives a unique fixed point for such operator. Such approach has been applied, among others, by Dutto and Novotný in [6] to show existence of a solution to steady compressible Navier–Stokes equations in an exterior domain in 2D, and by Novotný and Pokorný in [17] to prove existence for a system describing steady flow of viscoelastic fluid.

We start with removing the inhomogeneity from the boundary condition $(1.1)_4$. To this end let us construct $u_0 \in W^2_p(\Omega)$ such that

$$n \cdot u_0|_\Gamma = d - n^{(1)}. \tag{1.4}$$

Due to the assumption of smallness of $d - n^{(1)}$ in $W^{2-1/p}_p(\Gamma)$ we can assume that

$$\|u_0\|_{W^2_p} \ll 1. \tag{1.5}$$

From now on we assume (1.5) in all our results. Now we consider

$$u = v - \bar{v} - u_0 \quad \text{and} \quad w = \rho - \bar{\rho}.$$

One can easily verify that (u, w) satisfies the following system:

$$\begin{aligned}
&\partial_{x_1} u - \mu\Delta u - (\nu+\mu)\nabla \operatorname{div} u + \pi'(1)\nabla w = F(u, w) && \text{in } \Omega,\\
&\operatorname{div} u + \partial_{x_1} w + (u + u_0)\cdot\nabla w = G(u, w) && \text{in } \Omega,\\
&n \cdot 2\mu\mathbf{D}(u)\cdot\tau_i + f\, u\cdot\tau_i = B_i, \quad i = 1, 2 && \text{on } \Gamma,\\
&n \cdot u = 0 && \text{on } \Gamma,\\
&w = w_{\text{in}} && \text{on } \Gamma_{\text{in}},
\end{aligned} \tag{1.6}$$

where

$$\begin{aligned}
F(u, w) &= -w(u + \bar{v} + u_0)\cdot\nabla(u + u_0) - (u_0\cdot\nabla u) - u\cdot\nabla u_0\\
&\qquad + \mu\Delta u_0 + (\nu+\mu)\nabla \operatorname{div} u_0 - u_0\cdot\nabla u_0 - \big[\pi'(w+1) - \pi'(1)\big]\nabla w,\\
G(u, w) &= -(w+1)\operatorname{div} u_0 - w \operatorname{div} u
\end{aligned} \tag{1.7}$$

and

$$B_i = b_i - 2\mu n\cdot\mathbf{D}(u_0)\cdot\tau_i - f\tau_i^{(1)}.$$

From now on we will denote $\pi'(1) =: \gamma$. We see that F and G also depend on $\nabla u, u_0, \nabla u_0$, but for simplicity we will write $F(u, w)$ and $G(u, w)$. In order to prove Theorem 1 it is enough to show the existence of a solution (u, w) to the system (1.6) provided that $\|B\|_{W^{1-1/p}_p(\Gamma)}$ and $\|u_0\|_{W^2_p(\Omega)}$ are small enough. As we already mentioned, we will construct a sequence that converges to the solution. The sequence will be defined as

$$
\begin{aligned}
&\partial_{x_1} u^{n+1} - \mu \Delta u^{n+1} - (\nu+\mu)\nabla \operatorname{div} u^{n+1} + \gamma \nabla w^{n+1} = F\big(u^n, w^n\big) && \text{in } \Omega,\\
&\operatorname{div} u^{n+1} + \partial_{x_1} w^{n+1} + \big(u^n + u_0\big)\cdot \nabla w^{n+1} = G\big(u^n, w^n\big) && \text{in } \Omega,\\
&n \cdot 2\mu \mathbf{D}\big(u^{n+1}\big)\cdot \tau_i + f\, u^{n+1}\cdot \tau_i = B_i, \quad i=1,2 && \text{on } \Gamma,\\
&n\cdot u^{n+1} = 0 && \text{on } \Gamma,\\
&w^{n+1} = w_{\text{in}} && \text{on } \Gamma_{\text{in}}.
\end{aligned}
\tag{1.8}
$$

As we will see in the sequel, our method does not require any particular starting point for the sequence (u^n, w^n), but only some smallness assumptions on the starting point (u^0, w^0), hence without loss of generality we can set $(u^0, w^0) = (0, 0)$. In order to show the existence of the sequence defined in (1.8) we have to solve a linear system:

$$
\begin{aligned}
&\partial_{x_1} u - \mu \Delta u - (\nu+\mu)\nabla \operatorname{div} u + \gamma \nabla w = F && \text{in } \Omega,\\
&\operatorname{div} u + \partial_{x_1} w + (\bar{u} + u_0)\cdot \nabla w = G && \text{in } \Omega,\\
&n \cdot 2\mu \mathbf{D}(u)\cdot \tau_i + f\, u\cdot \tau_i = B_i, \quad i=1,2 && \text{on } \Gamma,\\
&n\cdot u = 0 && \text{on } \Gamma,\\
&w = w_{\text{in}} && \text{on } \Gamma_{\text{in}},
\end{aligned}
\tag{1.9}
$$

where $(F, G, \bar{u}, u_0) \in L_p \times W_p^1 \times W_p^2 \times W_p^2$ are given functions and $\bar{u}\cdot n = 0$ on Γ.

Let us now outline the strategy of the proof, and thus the structure of the paper. In Section 2 we show the *a priori* estimate (2.35) on a solution to the linear system (1.9). We start with an energy estimate in $H^1 \times L_\infty(L_2)$. Next the properties of the slip boundary conditions enables us to show that the vorticity of the velocity on the boundary has the same regularity as the velocity, and this fact makes it possible to find a bound on $\|w\|_{W_p^1}$. Then the estimate (2.35) results directly from the elliptic regularity of the Lamé system.

The linear system (1.9) is solved in Section 3. First we show the existence of a weak solution using the Galerkin method modified to deal with the continuity equation. Next we can show that this solution is in fact strong using *a priori* estimate and symmetry of the slip boundary conditions.

In Section 4 we show the estimate in $W_p^2 \times W_p^1$ on the sequence (u^n, w^n) and, as a result, the Cauchy condition satisfied by this sequence in the space $H^1 \times L_\infty(L_2)$. These results are derived by application of the estimates for the linear system.

In Section 5 we apply the results of Section 4 passing to the limit with (u^n, w^n) and then showing that the limit is a solution to (1.6). Finally we show that this solution is unique in a class of solutions satisfying the estimate (1.3).

2. A priori bounds

The main result of this section is the estimate (2.35) in $W_p^2 \times W_p^1$. In order to show it we start with an energy estimate in $H^1 \times L_\infty(L_2)$. Next we consider the equation on the vorticity of the velocity and apply the Helmholtz decomposition to derive the bound on $\|w\|_{W_p^1}$ and finally using the classical elliptic theory we conclude (2.35).

In our proofs we shall not need explicit formulas on the functions $F(u, w)$ and $G(u, w)$, what will be important is that they depend quadratically on u and w. More precisely, we will show a following estimate

Lemma 2. *Let $(u, w) \in W_p^2 \times W_p^1$ and let $F(u, w)$ and $G(u, w)$ be defined in (1.7). Then*

$$
\big\|F(u, w)\big\|_{L_p} + \big\|G(u, w)\big\|_{W_p^1} \leqslant C\big[\big(\|u\|_{W_p^2} + \|w\|_{W_p^1}\big)^2 + \|u_0\|_{W_p^2}\big]. \tag{2.1}
$$

Proof. Since by the imbedding theorem $W^1_p(\Omega) \subset L_\infty(\Omega)$, the estimate on $\|G\|_{W^1_p}$ is straightforward, and the only part of F that deserves attention is $\delta\pi'(w)\nabla w$, where

$$\delta\pi'(w) := \pi'(w+1) - \pi'(1). \tag{2.2}$$

We will apply a fact that for a C^1-function f we have

$$f(x) - f(y) = (x-y)\int_0^1 f'\big[tx + (1-t)y\big]\,dt. \tag{2.3}$$

Thus we have

$$\delta\pi'(w) = w\int_0^1 \pi''(tw+1)\,dt.$$

Since π is a C^3-function, the above implies

$$\big\|\delta\pi'(w)\nabla w\big\|_{L_p} \leqslant C(\pi)\|w\|_\infty\|\nabla w\|_{L_p} \leqslant C\|w\|^2_{W^1_p}.$$

The other parts of F can be estimated directly giving (2.1). □

Next, we derive the 'energy' estimate in $H^1 \times L_\infty(L_2)$. It is stated in the following lemma

Lemma 3. *Let (u, w) be a solution to the system (1.9) with $(F, G, B, w_{\mathrm{in}}, \bar{u}) \in V^* \times L_2 \times L_2(\Gamma) \times L_2(\Gamma_{\mathrm{in}}) \times W^2_p$, with $\|\bar{u}\|_{W^2_p}$ small enough and f large enough. Then*

$$\|u\|_{H^1} + \|w\|_{L_\infty(L_2)} \leqslant C\big[\|F\|_{V^*} + \|G\|_{L_2} + \|B\|_{L_2(\Gamma)} + \|w_{\mathrm{in}}\|_{L_2(\Gamma_{\mathrm{in}})}\big], \tag{2.4}$$

where

$$V = \big\{v \in H^1(\Omega)\colon\ v\cdot n|_\Gamma = 0\big\} \tag{2.5}$$

and V^* is the dual space of V.

Proof. We apply a general identity

$$\int_\Omega \big(-\mu\Delta u - (\nu+\mu)\nabla \operatorname{div} u\big) v\,dx = \int_\Omega \big\{2\mu\mathbf{D}(u) : \nabla v + \nu \operatorname{div} u \operatorname{div} v\big\}\,dx - \int_\Gamma n\cdot\big[2\mu\mathbf{D}(u) + \nu\operatorname{div} u\,\mathbf{Id}\big]\cdot v\,d\sigma. \tag{2.6}$$

For u, v satisfying the boundary conditions $(1.9)_{3,4}$ the boundary term in (2.6) equals

$$\int_\Gamma \left\{\sum_{i=1}^2 \big[B_i - f(u\cdot\tau_i)\big](v\cdot\tau_i)\right\} d\sigma.$$

Thus multiplying $(1.9)_1$ by u and integrating over Ω we get

$$\int_\Omega \big\{2\mu \mathbf{D}^2(u) + \nu \operatorname{div}^2 u\big\}\,dx + \int_\Gamma \left(f + \frac{n^{(1)}}{2}\right)|u|^2\,d\sigma - \gamma \int_\Omega w \operatorname{div} u\,dx$$
$$= \int_\Omega F \cdot u\,dx + \int_\Gamma \big\{B_1(u\cdot\tau_1) + B_2(u\cdot\tau_2)\big\}\,d\sigma. \tag{2.7}$$

From now on (not only in this proof but also later) we will use the summation convention when taking the sum over the tangential components. Applying $(1.9)_2$ and the boundary conditions we get

$$\int_\Omega w \operatorname{div} u\,dx = \int_\Omega G w\,dx + \frac{1}{2}\int_\Omega w^2 \operatorname{div}(\bar u + u_0)\,dx$$
$$- \frac{1}{2}\int_{\Gamma_{\mathrm{out}}} w^2\big(1 + u_0^{(1)}\big)\,d\sigma + \frac{1}{2}\int_{\Gamma_{\mathrm{in}}} w_{\mathrm{in}}^2\big(1 + u_0^{(1)}\big)\,d\sigma.$$

For $\|u_0\|_{W_p^2}$ small enough we have by the imbedding theorem $1 + u_0^{(1)} > 0$ a.e. on Γ_{out} what yields $\int_{\Gamma_{\mathrm{out}}} w^2(1 + u_0^{(1)})\,d\sigma > 0$. Moreover, for the friction f large enough on Γ_{in} the boundary term in (2.7) will be positive. To derive the bound on $\|u\|_{H^1}$ from (2.7) we apply a well-known Korn inequality:

$$\int_\Omega \big[2\mu \mathbf{D}^2(u) + \nu \operatorname{div}^2 u\big]\,dx + \int_\Gamma f(u\cdot\tau)^2\,d\sigma \geqslant C\|u\|_{H^1}^2. \tag{2.8}$$

As this is a standard result we skip the proof, let us only notice that we can modify the proof of Lemma 2.1 in [14] and actually simplify it considerably using the fact that the friction is large enough. Combining (2.8) with (2.7) we derive the following inequality

$$C\|u\|_{H^1}^2 \leqslant \int_\Omega F\cdot u\,dx + \int_\Gamma B_i(u\cdot\tau_i)\,d\sigma + \frac{1}{2}\int_\Omega w^2 \operatorname{div}(\bar u + u_0)\,dx$$
$$- \frac{1}{2}\int_{\Gamma_{\mathrm{in}}} w_{\mathrm{in}}^2\big(1 + u_0^{(1)}\big)\,d\sigma. \tag{2.9}$$

In order to derive (2.4) from (2.9) we have to estimate $\|w\|_{L_\infty(L_2)}$ in terms of $\|u\|_{H^1}$ and the data. To show this estimate we refer to Section 3 where the linear system (1.9) is solved. Namely, we have $w = S(G - \operatorname{div} u)$ where the operator S is defined in (3.7) and thus the estimate (3.8) implies

$$\|w\|_{L_\infty(L_2)} \leqslant C\big(\|G\|_{L_2} + \|u\|_{H_1} + \|w_{\mathrm{in}}\|_{L_2(\Gamma_{\mathrm{in}})}\big). \tag{2.10}$$

The above inequality combined with (2.9) yields (2.4). □

Now we consider the vorticity of the velocity $\alpha = \operatorname{rot} u$. The properties of the slip boundary conditions enables us to express the tangential components of α on the boundary in terms of the velocity. We arrive at the following system

$$\begin{aligned}
&\partial_{x_1}\alpha - \mu\Delta\alpha = \operatorname{rot} F && \text{in } \Omega,\\
&\alpha\cdot\tau_2 = \left(2\chi_1 - \frac{f}{\nu}\right)u\cdot\tau_1 + \frac{B_1}{\nu} && \text{on } \Gamma,\\
&\alpha\cdot\tau_1 = \left(\frac{f}{\nu} - 2\chi_2\right)u\cdot\tau_2 - \frac{B_2}{\nu} && \text{on } \Gamma,\\
&\operatorname{div}\alpha = 0 && \text{on } \Gamma,
\end{aligned} \tag{2.11}$$

where χ_i denote the curvatures of the curves generated by tangent vectors τ_i. In order to show the boundary relations $(2.11)_{2,3}$ it is enough to differentiate $(1.9)_4$ with respect to the tangential directions and apply $(1.9)_3$. A rigorous proof, modifying the proof in the two dimensional case from [16], is given in Appendix A. The condition $\operatorname{div}\alpha = 0$ in Ω results simply from the fact that $\alpha = \operatorname{rot} u$. We introduce this relation as a boundary condition $(2.11)_4$, that completes the conditions on the tangential parts of the vorticity. What is remarkable in the boundary conditions $(2.11)_{2,3}$ is that the tangential parts of the vorticity on the boundary has the same regularity as the velocity itself and the data. This feature of slip boundary conditions makes it possible to show the higher estimate on the vorticity (see [13, 14,24]).

In order to derive the bound on the vorticity we can follow [24, Lemma 4], and construct α_0, a divergence-free extension of the boundary data $(2.11)_{2,3}$, for example as a solution to the Stokes problem with zero r.h.s. and the boundary conditions $(2.11)_{2,3}$ supplied with $\alpha_0\cdot n = 0$. The theory of the Stokes system then yields

$$\|\alpha_0\|_{W_p^1} \leqslant C\big[\|u\|_{W_p^{1-1/p}(\Gamma)} + \|B\|_{W_p^{1-1/p}(\Gamma)}\big]. \tag{2.12}$$

Then the function $\alpha - \alpha_0$ satisfies the system

$$\begin{aligned}
&-\mu\Delta(\alpha-\alpha_0) = \operatorname{rot}[F - \partial_{x_1}u] + \mu\Delta\alpha_0 && \text{in } \Omega,\\
&(\alpha-\alpha_0)\cdot\tau_1 = 0 && \text{on } \Gamma,\\
&(\alpha-\alpha_0)\cdot\tau_2 = 0 && \text{on } \Gamma,\\
&\operatorname{div}(\alpha-\alpha_0) = 0 && \text{on } \Gamma.
\end{aligned} \tag{2.13}$$

Here we have used the fact that $\partial_{x_1}\alpha = \operatorname{rot}\partial_{x_1}u$ to preserve the rotational structure of the r.h.s. For the above system we have the following estimate (see [27])

$$\|\alpha\|_{W_p^1} \leqslant C\big[\|F\|_{L_p} + \|\partial_{x_1}u\|_{L_p} + \|\alpha_0\|_{W_p^1}\big]. \tag{2.14}$$

The term with α_0 can be bounded by (2.12) and to deal with $\partial_{x_1}u$ we apply the interpolation inequality (A.3). We obtain the term $\|u\|_{H^1}$ that we bound using (2.4) and finally arrive at

$$\begin{aligned}
\|\alpha\|_{W_p^1} \leqslant{}& C(\epsilon)\big[\|F\|_{L_p} + \|G\|_{W_p^1} + \|w_{\text{in}}\|_{L_2(\Gamma_{\text{in}})} + \|u\|_{W_p^{1-1/p}(\Gamma)} + \|B\|_{W_p^{1-1/p}(\Gamma)}\big]\\
&+ \epsilon\|u\|_{W_p^2}.
\end{aligned} \tag{2.15}$$

With the bound on the vorticity at hand the next step is to consider the Helmholtz decomposition of the velocity (the proof can be found in [8]):

$$u = \nabla\phi + A, \tag{2.16}$$

where $\phi|_\Gamma = 0$ and $\operatorname{div} A = 0$. We see that the field A satisfies the following system

$$\begin{aligned} \operatorname{rot} A &= \alpha \quad \text{in } \Omega, \\ \operatorname{div} A &= 0 \quad \text{in } \Omega, \\ A \cdot n &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{2.17}$$

This is the standard rot–div system and we have $\|A\|_{W_p^2} \leqslant C\|\alpha\|_{W_p^1}$, what by (2.15) can be rewritten as

$$\|A\|_{W_p^2} \leqslant C(\epsilon)\big[\|F\|_{L_p} + \|G\|_{W_p^1} + \|u\|_{W_p^{1-1/p}(\Gamma)} + \|B\|_{W_p^{1-1/p}(\Gamma)} + \|w_{\text{in}}\|_{W_p^1(\Gamma_{\text{in}})}\big] + \epsilon\|u\|_{W_p^2} \tag{2.18}$$

for any $\epsilon > 0$. Now we substitute the Helmholtz decomposition to $(1.9)_1$. We get

$$\nabla\big[-(\nu + 2\mu)\Delta\phi + \gamma w\big] = F - \partial_{x_1} A + \mu\Delta A - \partial_{x_1}\phi, \tag{2.19}$$

but $\Delta\phi = \operatorname{div} u$ and denoting the l.h.s. of the above equation by $\bar{F}$ we obtain

$$-(\nu + 2\mu)\operatorname{div} u + \gamma w = \bar{H}, \tag{2.20}$$

where $\nabla\bar{H} = \bar{F}$. Combining the last equation with $(1.9)_2$ we arrive at

$$\bar{\gamma} w + w_{x_1} + (\bar{u} + u_0)\nabla w = H, \tag{2.21}$$

where $\bar{\gamma} = \frac{\gamma}{\nu+2\mu}$ and

$$H = \frac{\bar{H}}{\nu + 2\mu} + G. \tag{2.22}$$

Eq. (2.21) makes it possible to estimate the W_p^1-norm of the density in terms of W_p^1-norm of H. The latter will be controlled since (2.19) enables us to bound $\|\nabla H\|_{L_p}$ and $\|H\|_{L_p}$ using interpolation and the energy estimate (2.4). The details are presented in the proof of Lemma 5, but first we estimate $\|w\|_{W_p^1}$ in terms of H. The result is stated in the following lemma:

Lemma 4. *Assume that w satisfies Eq.* (2.21) *with $H \in W_p^1$. Then*

$$\|w\|_{W_p^1} \leqslant C\big[\|H\|_{W_p^1} + \|w_{\text{in}}\|_{W_p^1(\Gamma_{\text{in}})}\big]. \tag{2.23}$$

Proof. In order to find a bound on $\|w\|_{L_p}$ we multiply (2.21) by $|w|^{p-2}w$ and integrate over Ω. Integrating by parts and next using the boundary conditions we get

$$\int_\Omega |w|^{p-2} w w_{x_1}\,dx = \frac{1}{p}\int_\Omega \partial_{x_1}|w|^p\,dx = \frac{1}{p}\int_{\Gamma_{\text{out}}} |w|^p\,d\sigma - \frac{1}{p}\int_{\Gamma_{\text{in}}} |w|^p\,d\sigma,$$

since $n^{(1)} \equiv 0$ on Γ_0, $n^{(1)} \equiv -1$ on Γ_{in} and $n^{(1)} \equiv 1$ on Γ_{out}. Similarly, applying the boundary conditions we get

$$\int_\Omega (\bar{u}+u_0)\cdot\big(|w|^{p-2}w\nabla w\big)dx = \frac{1}{p}\int_\Omega (\bar{u}+u_0)\cdot\nabla|w|^p\,dx$$
$$= -\frac{1}{p}\int_\Omega \operatorname{div}(\bar{u}+u_0)|w|^p\,dx + \frac{1}{p}\int_{\Gamma_{\mathrm{out}}} u_0^{(1)}|w|^p\,d\sigma$$
$$-\frac{1}{p}\int_{\Gamma_{\mathrm{in}}} u_0^{(1)}|w|^p\,d\sigma.$$

Thus multiplying (2.21) by $|w|^{p-2}w$ we get

$$\bar{\gamma}\|w\|_{L_p}^p - \frac{1}{p}\int_\Omega \operatorname{div}(\bar{u}+u_0)|w|^p\,dx + \frac{1}{p}\int_{\Gamma_{\mathrm{out}}} |w|^p\big(1+u_0^{(1)}\big)\,d\sigma$$
$$\leqslant \|H\|_{L_p}\|w\|_{L_p}^{p-1} + \frac{1}{p}\int_{\Gamma_{\mathrm{in}}} |w_{\mathrm{in}}|^p\big(1+u_0^{(1)}\big)\,d\sigma. \tag{2.24}$$

By the imbedding theorem the smallness of $\|\bar{u}+u_0\|_{W_p^2}$ implies $1+u_0^{(1)} > 0$ a.e. in Ω and $\bar{\gamma} - \|\operatorname{div}(\bar{u}+u_0)\|_\infty > 0$. Thus the boundary term on the l.h.s. is positive and the term with $\operatorname{div}(\bar{u}+u_0)$ can be combined with the first term of the l.h.s., what yields

$$C\|w\|_{L_p}^p \leqslant \|H\|_{L_p}\|w\|_{L_p}^{p-1} + C\|w_{\mathrm{in}}\|_{L_p(\Gamma_{\mathrm{in}})}^p,$$

and so

$$\|w\|_{L_p} \leqslant C\big[\|H\|_{L_p} + \|w_{\mathrm{in}}\|_{L_p(\Gamma_{\mathrm{in}})}\big]. \tag{2.25}$$

The derivatives of the density are estimated in a similar way. In order to find a bound on w_{x_i} we differentiate (2.21) with respect to x_i. If we assume that $w \in W_p^1$ then (2.21) implies $\tilde{u}\cdot\nabla w \in W_p^1$, where

$$\tilde{u} := \big[1+(\bar{u}+u_0)^{(1)}, (\bar{u}+u_0)^{(2)}, (\bar{u}+u_0)^{(3)}\big]. \tag{2.26}$$

Thus $\tilde{u}\cdot\nabla w_{x_i} := (\tilde{u}\cdot\nabla w)_{x_i} - \tilde{u}_{x_i}\cdot\nabla w \in L_p$. Hence we can differentiate (2.21) with respect to x_i, multiply by $|w_{x_i}|^{p-2}w_{x_i}$ and integrate. Since $\tilde{u}_{x_i} = (\bar{u}+u_0)_{x_i}$, we have

$$\int_\Omega \tilde{u}_{x_i}\cdot\big(|w_{x_i}|^{p-2}w_{x_i}\nabla w\big)\,dx \leqslant \big\|\nabla(\bar{u}+u_0)\big\|_{L_\infty}\|\nabla w\|_{L_p}^p \leqslant C\|\bar{u}+u_0\|_{W_p^2}\|\nabla w\|_{L_p}.$$

Next, since $\tilde{u}\cdot\nabla w_{x_i}\in L_p$, we can write

$$\begin{aligned}
&\int_\Omega \tilde{u}\cdot|w_{x_i}|^{p-2}w_{x_i}\nabla w_{x_i}\,dx\\
&\quad=\frac{1}{p}\int_\Omega \tilde{u}\cdot\nabla|w_{x_i}|^p\,dx=-\frac{1}{p}\int_\Omega |w_{x_i}|^p\operatorname{div}\tilde{u}\,dx+\frac{1}{p}\int_\Gamma |w_{x_i}|^p\tilde{u}\cdot n\,d\sigma\\
&\quad=-\frac{1}{p}\int_\Omega |w_{x_i}|^p\operatorname{div}\tilde{u}\,dx-\frac{1}{p}\int_{\Gamma_{\text{in}}}|w_{in,x_i}|^p\big(1+u_0^{(1)}\big)\,d\sigma+\frac{1}{p}\int_{\Gamma_{\text{out}}}|w_{x_i}|^p\big(1+u_0^{(1)}\big)\,d\sigma.
\end{aligned}$$

For $i=2,3$ we have $w_{in,x_i}\in L_p(\Gamma_{\text{in}})$ and hence the above defines the trace of $|w_{x_i}|^p$ on Γ_{out}. We arrive at

$$\begin{aligned}
&\bar{\gamma}\|w_{x_i}\|_{L_p}^p-\frac{1}{p}\int_\Omega \operatorname{div}(\bar{u}+u_0)|w_{x_i}|^p\,dx+\frac{1}{p}\int_{\Gamma_{\text{out}}}|w_{x_i}|^p\big(1+u_0^{(1)}\big)\,d\sigma\\
&\quad\leqslant\|H_{x_i}\|_{L_p}\|w_{x_i}\|_{L_p}^{p-1}+\frac{1}{p}\int_{\Gamma_{\text{in}}}|w_{in,x_i}|^p\big(1+u_0^{(1)}\big)\,d\sigma+C\|\bar{u}+u_0\|_{W_p^2}\|\nabla w\|_{L_p}^p.
\end{aligned}\tag{2.27}$$

For $i=2,3$ it gives directly the bound on $\|w_{x_i}\|_{L_p}$. In order to estimate w_{x_1} we start the same way differentiating (2.21) with respect to x_1 and multiplying by $|w_{x_1}|^{p-2}w_{x_1}$. The difference in comparison to w_{x_2} and w_{x_3} is that w_{x_1} is not given on Γ_{in}. In order to overcome this difficulty we can observe that on Γ_{in} Eq. (2.21) reduces to

$$\bar{\gamma}w_{\text{in}}+(\bar{u}+u_0)^{(2)}w_{in,x_2}+(\bar{u}+u_0)^{(3)}w_{in,x_3}+\big[1+(\bar{u}+u_0)^{(1)}\big]w_{x_1}=H,$$

what can be rewritten as

$$w_{x_1}=\frac{1}{1+(\bar{u}+u_0)^{(1)}}\big[H-\bar{\gamma}w_{\text{in}}-(\bar{u}+u_0)_\tau\cdot\nabla_\tau w_{\text{in}}\big].$$

Thus we have

$$\|w_{x_1}\|_{L_p(\Gamma_{\text{in}})}\leqslant C\big[\|H|_{\Gamma_{\text{in}}}\|_{L_p(\Gamma_{\text{in}})}+\|w_{\text{in}}\|_{W_p^1(\Gamma_{\text{in}})}\big].$$

Using this bound in (2.27), $i=1$, we arrive at the estimate

$$\|w_{x_1}\|_{L_p}^p\leqslant C\big[\|H_{x_1}\|_{L_p}\|w_{x_1}\|_{L_p}^{p-1}+\|\bar{u}+u_0\|_{W_p^2}\|\nabla w\|_{L_p}^p+\|H\|_{L_p(\Gamma_{\text{in}})}^p+\|w_{\text{in}}\|_{W_p^1(\Gamma_{\text{in}})}^p\big].\tag{2.28}$$

The boundary term $\|H\|_{L_p(\Gamma_{\text{in}})}$ can by replaced by $\|H\|_{W_p^1}$ due to the trace theorem. Thus combining (2.27) (for x_2 and x_3) with (2.28) we get

$$\|\nabla w\|_{L_p}^p\leqslant C\big[\|\nabla H\|_{L_p}\|\nabla w\|_{L_p}^{p-1}+\|\bar{u}+u_0\|_{W_p^2}\|\nabla w\|_{L_p}^p+\|H\|_{W_p^1}^p+\|w_{\text{in}}\|_{W_p^1(\Gamma_{\text{in}})}^p\big].\tag{2.29}$$

The term $\|\bar{u}+u_0\|_{W_p^2}\|\nabla w\|_{L_p}^p$ can be put on the l.h.s. due to the smallness assumption and thus we get

$$\|\nabla w\|_{L_p} \leqslant C\big[\|H\|_{W^1_p} + \|w_{\text{in}}\|_{W^1_p(\Gamma_{\text{in}})}\big], \tag{2.30}$$

what combined with (2.25) yields

$$\|w\|_{W^1_p} \leqslant C\big[\|H\|_{W^1_p} + \|H\|_{L_p(\Gamma_{\text{in}})} + \|w_{\text{in}}\|_{W^1_p(\Gamma_{\text{in}})}\big]. \tag{2.31}$$

Applying again the trace theorem to the term $\|H\|_{L_p(\Gamma_{\text{in}})}$ we arrive at (2.23). □

The next step is to estimate H in terms of the data. The result is in the following:

Lemma 5. *Let H be defined in* (2.22)*. Then $\forall \delta > 0$ we have*

$$\|H\|_{W^1_p} \leqslant \delta\|u\|_{W^2_p} + C(\delta)\big[\|F\|_{L_p} + \|G\|_{W^1_p} + \|B\|_{W^{1-1/p}_p(\Gamma)} + \|w_{\text{in}}\|_{W^1_p(\Gamma_{\text{in}})}\big]. \tag{2.32}$$

Proof. Applying first the interpolation inequality (A.3) and then the estimate (2.4) we get

$$\|H\|_{L_p} \leqslant \delta_1\|\nabla H\|_{L_p} + C(\delta_1)\big[\|F\|_{L_2} + \|G\|_{L_2} + \|B\|_{L_2(\Gamma)} + \|w_{\text{in}}\|_{L_2(\Gamma_{\text{in}})}\big]. \tag{2.33}$$

Next, by (2.19) we have

$$\|\nabla H\|_{L_p} \leqslant C\big[\|F\|_{L_p} + \|G\|_{W^1_p} + \|A\|_{W^2_p} + \|\partial_{x_1}\phi\|_{L_p}\big],$$

where $u = \nabla\phi + A$ is the Helmholtz decomposition. Now we use the bound (2.18) on $\|A\|_{W^2_p}$. We obtain a term $\|u\|_{W^{1-1/p}_p(\Gamma)}$, that we estimate using the trace theorem and the interpolation inequality (A.3). The same inequality is applied to estimate $\|\partial_{x_1}\phi\|_{L_p}$. We arrive at

$$\begin{aligned}\|\nabla H\|_{L_p} \leqslant{}& C\big[\|F\|_{L_p} + \|G\|_{W^1_p} + \|B\|_{W^{1-1/p}_p(\Gamma)} + \|w_{\text{in}}\|_{W^1_p(\Gamma_{\text{in}})}\big]\\ &+ \delta_1\|u\|_{W^2_p} + C(\delta_1)\big[\|F\|_{L_2} + \|G\|_{L_2} + \|B\|_{L_p(\Gamma)}\big].\end{aligned} \tag{2.34}$$

Combining (2.33) and (2.34) we get (2.32). □

Now we are ready to show the *a priori* estimate in $W^2_p \times W^1_p$ on the solution of the linear problem.

Lemma 6. *Let (u, w) be a solution to* (1.9) *with $(F, G, B, w_{\text{in}}, \bar{u}) \in L_p \times W^1_p \times W^{1-1/p}_p(\Gamma) \times W^1_p(\Gamma_{\text{in}}) \times W^2_p$, with $\|\bar{u}\|_{W^2_p}$ small enough and f large enough. Then*

$$\|u\|_{W^2_p} + \|w\|_{W^1_p} \leqslant C\big[\|F\|_{L_p} + \|G\|_{W^1_p} + \|B\|_{W^{1-1/p}_p(\Gamma)} + \|w_{\text{in}}\|_{W^1_p(\Gamma_{\text{in}})}\big]. \tag{2.35}$$

Proof. If (u, w) is a solution to (1.9), then in particular the velocity satisfies the Lamé system

$$\begin{aligned}&\partial_{x_1}u - \mu\Delta u - (\nu+\mu)\nabla\operatorname{div}u = F - \gamma\nabla w && \text{in } \Omega,\\ &n\cdot 2\mu\mathbf{D}(u)\cdot\tau_i + f\,u\cdot\tau_i = B_i, \quad i = 1, 2 && \text{on } \Gamma,\\ &n\cdot u = 0 && \text{on } \Gamma.\end{aligned} \tag{2.36}$$

The classical theory of elliptic equations (Agmon, Douglis, Nirenberg [2,3]) yields

$$\|u\|_{W^2_p} \leqslant C\big[\|F\|_{L_p} + \|w\|_{W^1_p} + \|B\|_{W^{1-1/p}_p(\Gamma)} + \|u\|_{W^1_p}\big].$$

Applying the interpolation inequality (A.3) to the term $\|u\|_{W^1_p}$ and then the energy estimate (2.4) we get

$$\|u\|_{W^2_p} \leqslant C\big[\|F\|_{L_p} + \|G\|_{W^1_p} + \|w\|_{W^1_p} + \|B\|_{W^{1-1/p}_p(\Gamma)} + \|w_{\text{in}}\|_{L_2(\Gamma_{\text{in}})}\big]. \tag{2.37}$$

In order to complete the proof we combine (2.23) and (2.32) obtaining

$$\|w\|_{W^1_p} \leqslant \delta\|u\|_{W^2_p} + C(\delta)\big[\|F\|_{L_p} + \|G\|_{W^1_p} + \|B\|_{W^{1-1/p}_p(\Gamma)} + \|w_{\text{in}}\|_{W^1_p(\Gamma_{\text{in}})}\big], \tag{2.38}$$

and choosing for example $\delta = \frac{1}{2C}$ where C is the constant from (2.37) we arrive at (2.35). □

3. Solution of the linear system

In this section we show the existence of the sequence (u^n, w^n) defined in (1.8). To this end we have to solve the linear system (1.9) where $(F, G, \bar{u}, u_0) \in L_p \times W^1_p \times W^2_p \times W^2_p$ are given functions such that $\bar{u} \cdot n = 0$ on Γ. First we apply the Galerkin method to prove the existence of a weak solution and next we show that this solution is strong. For simplicity we will denote $\bar{u} + u_0$ by $\bar{u}$.

3.1. Weak solution

Let us recall the definition of the space V (2.5). A natural definition of a weak solution to the system (1.9) is a couple $(u, w) \in V \times L_\infty(L_2)$ such that

$$\int_\Omega \big\{v \cdot \partial_{x_1} u + 2\mu\mathbf{D}(u) : \nabla v + \nu \operatorname{div} u \operatorname{div} v - \gamma w \operatorname{div} v\big\}\, dx + \int_\Gamma f(u \cdot \tau_i)(v \cdot \tau_i)\, d\sigma$$
$$= \int_\Omega F \cdot v\, dx + \int_\Gamma B_i(v \cdot \tau_i)\, d\sigma \tag{3.1}$$

is satisfied $\forall v \in V$ and $(1.9)_2$ is satisfied in $\mathcal{D}'(\Omega)$, i.e. $\forall \phi \in \bar{C}^\infty(\Omega)$:

$$-\int_\Omega w\tilde{u} \cdot \nabla\phi\, dx - \int_\Omega w\phi \operatorname{div}\tilde{u}\, dx + \int_{\Gamma_{\text{out}}} w\phi\, d\sigma = \int_\Omega \phi(G - \operatorname{div} u)\, dx + \int_{\Gamma_{\text{in}}} w_{\text{in}}\phi\, d\sigma, \tag{3.2}$$

where $\tilde{u}$ is defined in (2.26). Let us introduce an orthonormal basis of V: $\{\omega_i\}_{i=1}^\infty$. We consider finite dimensional spaces: $V^N = \{\sum_{i=1}^N \alpha_i\omega_i\colon\ \alpha_i \in \mathbf{R}\} \subset V$. The sequence of approximations to the velocity will be searched for in a standard way as $u^N = \sum_{i=1}^N c_i^N \omega_i$. Due to Eq. $(1.9)_2$ we have to define the approximations to the density in an appropriate way. Namely, we set $w^N = S(G^N - \operatorname{div} u^N)$, where $S : L_2(\Omega) \to L_\infty(L_2)$ is defined as

$$w = S(v) \iff \begin{cases} \partial_{x_1} w + \bar{u} \cdot \nabla w = v & \text{in } \mathcal{D}'(\Omega), \\ w = w_{\text{in}} & \text{on } \Gamma_{\text{in}}. \end{cases} \tag{3.3}$$

We want the image of S to be in the space $L_\infty(L_2)$ so that we can apply the theory of transport equation treating x_1 as a 'time' variable to show that S is well defined. In order to solve the system on the r.h.s. of (3.3) we can search for a change of variables $x = \psi(z)$ satisfying the identity

$$\partial_{z_1} = \partial_{x_1} + \bar{u} \cdot \nabla_x. \tag{3.4}$$

We construct the mapping ψ in the following:

Lemma 7. *Let $\|\bar{u}\|_{W^2_p}$ be small enough. Then there exists a set $U \subset \mathbb{R}^3$ and a diffeomorphism $x = \psi(z)$ defined on U such that $\Omega = \psi(U)$ and (3.4) holds. Moreover, if $z_n \to z$ and $\psi(z_n) \to \Gamma_0$ then $n^1(z) = 0$, where n is the outward normal to U.*

Remark 1. The last condition states that the first component of the normal to $\psi^{-1}(\Gamma_0)$ vanishes, but since ψ is defined only on U we formulate this condition using the limits. It means simply that the image $U = \psi^{-1}(\Omega)$ is also a cylinder with a flat wall. It will be important in the construction of the operator S.

Proof of Lemma 7. The identity (3.4) means that ψ must satisfy

$$\frac{\partial \psi^1}{\partial z_1} = 1 + \bar{u}^1(\psi), \qquad \frac{\partial \psi^2}{\partial z_1} = \bar{u}^2(\psi), \qquad \frac{\partial \psi^3}{\partial z_1} = \bar{u}^3(\psi). \tag{3.5}$$

A natural condition is that $\psi(\Gamma_{in}) = \Gamma_{in}$. Thus we can search for $\psi(z_1, z_2, z_3) = \psi_{z_2,z_3}(z_1)$, where for all (z_2, z_3) such that $(z_2, z_3, 0) \in \Gamma_{in}$ the function $\psi_{z_2,z_3}(\cdot)$ is a solution to a system of ODE:

$$\begin{cases} \partial_s \psi^1_{z_2,z_3} = 1 + \bar{u}^1(\psi_{z_2,z_3}), \qquad \partial_s \psi^2_{z_2,z_3} = \bar{u}^2(\psi_{z_2,z_3}), \qquad \partial_s \psi^3_{z_2,z_3} = \bar{u}^3(\psi_{z_2,z_3}), \\ \psi_{z_2,z_3}(0) = (0, z_2, z_3). \end{cases} \tag{3.6}$$

The r.h.s. of the system (3.6) is a Lipschitz function with a constant $K = \|\nabla\bar{u}\|_\infty$ and thus provided that $\|\bar{u}\|_{W^2_p}$ is small enough the system (3.6) has a unique solution defined on some interval $(0, b_{z_1,z_2})$, where b_{z_1,z_2} depends on z_2, z_3 and $\|\nabla\bar{u}\|_\infty$. Provided that the latter is small enough the function $\psi(z) = \psi_{z_2,z_3}(z_1)$ will be defined on U such that $\Omega = \psi(U)$.

Now we show that $\psi(z) = \psi_{z_2,z_3}(z_1)$ is a diffeomorphism. The derivatives with respect to z_1 are given by (3.5) and the remaining derivatives can be expressed in terms of $\bar{u}$ so we can see that $J\psi = 1 + E(\bar{u})$, where $E(\bar{u})$ is small (and thus $J\psi > 0$) provided that $\|\bar{u}\|_{W^2_p}$ is small.

To see that ψ is $1-1$ we can write it in a form $\psi(z) = z + \epsilon(z)$, where $\|\nabla\epsilon\|_{L_\infty}$ is small. Assume that $\psi(z^1) = \psi(z^2)$ and $z^1 \neq z^2$. Then there exists i such that $|z^1_i - z^2_i| \geqslant \frac{1}{3}|z^1 - z^2|$ (the lowercase denotes the coordinate). On the other hand, we have $|z^1_i - z^2_i| = |\epsilon_i(z^1) - \epsilon_i(z^2)| \leqslant \|\nabla\epsilon\|_{L_\infty}|z^1 - z^2|$, what contradicts the smallness of $\|\nabla\epsilon\|_{L_\infty}$.

We have shown that the mapping ψ given by (3.6) is a diffeomorphism defined on U such that $\psi(U) = \Omega$. Let us denote $\phi = \psi^{-1}$. Now it is natural to define the subsets of ∂_U as $\partial_U = U_{in} \cup U_{out} \cup U_0$ where $U_{in} = \Gamma_{in}$, $U_{out} = \{z\colon\ z = \lim \phi(x_n),\ x_n \to \Gamma_{out}\}$ and $U_0 = \{z\colon\ z = \lim \phi(x_n),\ x_n \to \Gamma_0\}$.

In order to complete the proof we have to show that $n^1(z) = 0$ for $z \in U_0$. But to this end it is enough to observe that

$$D\psi(z)\big([1, 0, 0]\big) = \big[1 + \bar{u}^1(x), \bar{u}^2(x), \bar{u}^3(x)\big],$$

where $x = \psi(z)$. But for $x \in \Gamma_0$ the vector on the r.h.s. is tangent to Γ_0 since $\bar{u} \cdot n = 0$. We can conclude that on U_0 the image in ψ of a straight line $\{(s, z_2, z_3)\colon\ s \in (0, b)\}$ is a curve tangent to Γ_0, and thus U_0 is a sum of such lines and so we have $n^1(z) = 0$. The proof of Lemma 7 is completed. □

Now we can define $S(v)$ for a continuous function v as

$$S(v)(x) = w_{\mathrm{in}}\big(0, \phi_2(x), \phi_3(x)\big) + \int_0^{\phi_1(x)} v\big(\psi\big(s, \phi_2(x), \phi_3(x)\big)\big)\, ds. \tag{3.7}$$

The condition $n^1 = 0$ on $\phi(\Gamma_0)$ guarantees that a straight line (s, z_1, z_2): $s \in (0, b)$ has a picture in Ω and thus we integrate along a curve contained in Ω. It means that S is well defined for continuous functions defined on Ω and the construction of ψ clearly ensures that S satisfies (3.3). Next we have to extend S on $L_2(\Omega)$. To this end we need an estimate in $L_\infty(L_2)$. It is given by the following

Lemma 8. *Let S be defined in* (3.7). *Then*

$$\big\| S(v) \big\|_{L_\infty(L_2)} \leqslant C\big[\|w_{\mathrm{in}}\|_{L_2(\Gamma_{\mathrm{in}})} + \|v\|_{L_2(\Omega)}\big]. \tag{3.8}$$

Proof. Let Ω_{x_1} be denoted an x_1-cut of Ω and let $\bar{x} := (x_2, x_3)$. Then by (3.7) we have

$$\begin{aligned}
\big\| S(v) \big\|^2_{L_2(\Omega_{x_1})} &= \int_{\Omega_{x_1}} \left[w_{\mathrm{in}}\big(0, \phi_2(x), \phi_3(x)\big) + \int_0^{\phi_1(x)} v\big(\psi\big(s, \phi_2(x), \phi_3(x)\big)\big)\, ds \right]^2 d\bar{x} \\
&\leqslant 2\|w_{\mathrm{in}}\|^2_{L_2(\Gamma_{\mathrm{in}})} + C \int_{\Omega_{x_1}} \int_0^{\phi_1(x)} v^2\big(\psi\big(s, \phi_2(x), \phi_3(x)\big)\big)\, ds\, d\bar{x} \\
&\leqslant C\big[\|w_{\mathrm{in}}\|^2_{L_2(\Gamma_{\mathrm{in}})} + \|v\|^2_{L_2(\Omega)}\big].
\end{aligned}$$

The above holds for every $x_1 \in (0, L)$ what implies (3.8). □

Now we can define $S(v)$ for $v \in L_2(\Omega)$ using a standard density argument. Let us take a sequence of smooth functions $v_n \to v$ in $L_2(\Omega)$. By (3.8) the sequence $S(v_n)$ satisfies

$$\big\| S(v_n) \big\|_{L_\infty(L_2)} \leqslant C\big[\|w_{\mathrm{in}}\|_{L_2(\Gamma_{\mathrm{in}})} + \sup_n \|v_n\|_{L_2}\big]. \tag{3.9}$$

The bound on the r.h.s. is uniform in n and thus $S(v_n) \rightharpoonup^* \eta$ in $L_\infty(L_2)$, and η satisfies the estimate (3.8). In particular for $\phi \in \bar{C}^\infty(\Omega)$ we have

$$\int_\Omega S(v_n)\tilde{u} \cdot \nabla\phi\, dx \to \int_\Omega \eta\tilde{u} \cdot \nabla\phi\, dx \quad \text{and} \quad \int_\Omega S(v_n)\phi \operatorname{div}\tilde{u}\, dx \to \int_\Omega \eta\phi \operatorname{div}\tilde{u}\, dx.$$

In order to show that $\eta = S(v)$, i.e. η solves the system on the r.h.s. of (3.3) we have to show that $\int_{\Gamma_{\mathrm{out}}} S(v_n)\phi\, d\sigma \to \int_{\Gamma_{\mathrm{out}}} \eta\phi\, d\sigma$. To this end notice that the proof of Lemma 8 implies in particular that $\|S(v_n)\|_{L_2(\Gamma_{\mathrm{out}})}$ satisfies the estimate (3.9). Thus $S(v_n) \rightharpoonup \zeta$ in $L_2(\Gamma_{\mathrm{out}})$ for some $\zeta \in L_2(\Gamma_{\mathrm{out}})$, and in particular $\int_{\Gamma_{\mathrm{out}}} S(v_n)\phi\, d\sigma \to \int_{\Gamma_{\mathrm{out}}} \zeta\phi\, d\sigma$. We have to verify that $\eta|_{\Gamma_{\mathrm{out}}} = \zeta$. This would not be obvious if we only had $S(v_n) \in L_\infty(L_2)$, but indeed the proof of lemma 8 implies a stronger condition that supremum (not only the essential supremum) of $\|S(v_n)\|_{L_2(\Omega_{x_1})}$ is bounded, thus we must have $\zeta = \eta|_{\Gamma_{\mathrm{out}}}$. We have shown that $\tilde{u} \cdot \nabla\eta = v$ in $\mathcal{D}'(\Omega)$, thus indeed $\eta = S(v)$.

Having the operator S well defined we are ready to proceed with the Galerkin method. Taking $F = F^N$, $u = u^N = \sum_i c_i^N \omega_i$, $v = \omega_k$, $k = 1 \ldots N$, and $w = w^N = S(G^N - \operatorname{div} u^N)$ in (3.1), where F^N and G^N are orthogonal projections of F and G on V^N, we arrive at a system of N equations

$$B^N(u^N, \omega_k) = 0, \quad k = 1 \dots N, \tag{3.10}$$

where $B^N : V^N \to V^N$ is defined as

$$\begin{aligned} B^N(\xi^N, v^N) = {} & \int_\Omega \{\xi^N \partial_{x_1} v^N + 2\mu \mathbf{D}(\xi^N) : \nabla v^N + \operatorname{div} \xi^N \operatorname{div} v^N\} \, dx \\ & - \gamma \int_\Omega S(G^N - \operatorname{div} \xi^N) \operatorname{div} v^N \, dx + \int_\Gamma [f(\xi^N \cdot \tau_j) - B_i](v^N \cdot \tau_j) \, d\sigma \\ & - \int_\Omega F^N \cdot v^N \, dx. \end{aligned} \tag{3.11}$$

Now, if u^N satisfies (3.10) for $k = 1 \dots N$ and $w^N = S(G^N - \operatorname{div} u^N)$, then a pair (u^N, w^N) satisfies (3.1)–(3.2) for $(v, \phi) \in (V^N \times \bar{C}^\infty(\Omega))$. We will call such a pair an approximate solution to (3.1)–(3.2).

The following lemma gives existence of solution to the system (3.10):

Lemma 9. *Let $F, G \in L^2(\Omega)$, $w_{\text{in}} \in L_2(\Gamma_{\text{in}})$, $B \in L_2(\Gamma)$ and assume that f is large enough and $\|\bar{u}\|_{W^2_p}$ is small enough. Then there exists $u^N \in V^N$ satisfying (3.10) for $k = 1 \dots N$. Moreover,*

$$\|u^N\|_{H^1} \leqslant C(DATA). \tag{3.12}$$

Proof. In order to solve the system (3.10) we will apply a well-known result in finite dimensional Hilbert spaces, Lemma 15 in Appendix A. Thus we define the operator $P^N : V^N \to V^N$ as

$$P^N(\xi^N) = \sum_k B^N(\xi^N, \omega_k)\omega_k \quad \text{for } \xi^N \in V^N. \tag{3.13}$$

In order to apply Lemma 15 we have to show that $(P(\xi^N), \xi^N) > 0$ on some sphere in V^N. Since $B^N(\cdot,\cdot)$ is linear with respect to the second variable, we clearly have

$$\begin{aligned} (P(\xi^N), \xi^N) = B^N(\xi^N, \xi^N) = {} & \underbrace{2\mu \int_\Omega D^2(\xi^N) \, dx + \nu \int_\Omega \operatorname{div}^2 \xi^N \, dx}_{I_1} \\ & + \underbrace{\int_\Omega \xi^N \partial_{x_1} \xi^N \, dx + \int_\Gamma f(\xi^N \cdot \tau_i)^2 \, d\sigma}_{I_2} \underbrace{- \gamma \int_\Omega S(G^N - \operatorname{div} \xi^N) \operatorname{div} \xi^N \, dx}_{I_3} \\ & - \int_\Omega F \cdot \xi^N \, dx - \int_\Gamma B_i(\xi^N \cdot \tau_i) \, d\sigma. \end{aligned} \tag{3.14}$$

Using the Korn inequality similarly as in the proof of the energy estimate (2.4) we get

$$I_1 + I_2 \geqslant C\|\xi^N\|^2_{H^1} \tag{3.15}$$

for f large enough. We have to find a bound on I_3. Denoting $\eta^N = S(G^N - \operatorname{div}\xi^N)$ we have

$$-\int_\Omega \eta^N \operatorname{div}\xi^N\,dx = \int_\Omega \eta^N\big(\partial_{x_1}\eta^N + \bar{u}\cdot\nabla\eta^N\big)\,dx - \int_\Omega \eta^N G^N\,dx. \tag{3.16}$$

Using (3.8) we get

$$-\int_\Omega \eta^N G^N\,dx \geqslant -\|\eta^N\|_{L^2}\|G^N\|_{L^2} \geqslant -C\|G^N\|_{L_2}\big(\|G^N\|_{L_2} + \|\xi^N\|_{H^1} + \|w_{\mathrm{in}}\|_{L_2(\Gamma_{\mathrm{in}})}\big). \tag{3.17}$$

With the first integral on the r.h.s. of (3.16) we have

$$\begin{aligned}\int_\Omega \eta^N\big(\partial_{x_1}\eta^N + \bar{u}\cdot\nabla\eta^N\big)\,dx &= \int_U \eta^N(z)\partial_{z_1}\eta^N(z)J\psi(z)\,dz \\ &= \int_U \eta^N(z)\partial_{z_1}\eta^N(z)\,dz + \int_U \eta^N(z)\partial_{z_1}\eta^N(z)\big[J\psi(z)-1\big]\,dz.\end{aligned} \tag{3.18}$$

The first integral can be rewritten as a boundary integral and since $n^1(z)=0$ on $\phi(\Gamma_0)$, it reduces to

$$\frac{1}{2}\int_{\partial U}\big[\eta^N(z)\big]^2 n^1(z)\,d\sigma(z) = -\frac{1}{2}\int_{U_{\mathrm{in}}}\big[\eta^N(z)\big]^2\,d\sigma(z) + \frac{1}{2}\int_{U_{\mathrm{out}}}\big[\eta^N(z)\big]^2\,d\sigma(z) \geqslant -\int_{\Gamma_{\mathrm{in}}} w_{\mathrm{in}}^2\,d\sigma(x).$$

In the last passage we used the fact that $\phi|_{\Gamma_{\mathrm{in}}}$ is the identity and that $n^1(z) > 0$ on U_{out}, what is true provided that ϕ does not differ too much from the identity on Γ_{out}, what in turn holds under the smallness assumptions on $\bar{u}$.

With the second integral on the r.h.s. of (3.18) we have

$$\begin{aligned}\int_U \eta^N(z)\partial_{z_1}\eta^N(z)\big[J\psi(z)-1\big]\,dz &\geqslant -\sup_U|J\psi-1|\int_U \eta^N(z)\big(G^N - \operatorname{div}_x\xi^N\big)(z)\,dz \\ &\geqslant -E\|\eta^N\|_{L_2(U)}\big[\|G^N\|_{L_2(U)} + \|\operatorname{div}_x\xi^N\|_{L_2(U)}\big] \\ &\geqslant -E\big[\|G^N\|^2_{L_2(\Omega)} + \|\xi^N\|^2_{H^1(\Omega)} + \|w_{\mathrm{in}}\|^2_{L_2(\Gamma_{\mathrm{in}})}\big].\end{aligned}$$

Combining this estimate with (3.15) we get

$$\big(P^N(\xi^N),\xi^N\big) \geqslant C\big[\|\xi^N\|^2_{H^1(\Omega)} - D\|\xi^N\|_{H^1(\Omega)} - D^2\big], \tag{3.19}$$

where $D = \|F\|_{L^2(\Omega)} + \|G\|_{L^2(\Omega)} + \|w_{\mathrm{in}}\|_{L_2(\Gamma_{\mathrm{in}})} + \|B\|_{L_2(\Gamma)}$. Thus there exists $\tilde{C} = \tilde{C}(\mu,\Omega,D)$ such that $(P^N(\xi^N),\xi^N) > 0$ for $\|\xi\| = \tilde{C}$, and applying Lemma 15 we conclude that $\exists\xi^*$: $P^N(\xi^*) = 0$ and $\|\xi^*\| \leqslant \tilde{C}$. Moreover, since $\{\omega_k\}_{k=1}^N$ is the basis of V^N, we have $P^N(\xi^*) = 0 \Leftrightarrow B^N(\xi^*,\omega_k) = 0$, $k = 1\dots N$. Thus ξ^* is a solution to (3.10). $\square$

Now showing the existence of the weak solution is straightforward. The result is in the following:

Lemma 10. *Assume that* $F, G \in L_2(\Omega)$, $w_{in} \in L_2(\Gamma_{in})$, $B \in L_2(\Gamma)$. *Assume further that* f *is large enough and* $\|\bar{u}\|_{W^2_p}$ *is small enough. Then there exists* $(u, w) \in V \times W$ *that is a weak solution to the system* (1.9). *Moreover, the weak solution satisfies the estimate* (2.4).

Proof. The estimates (3.8) and (3.12) imply that $\|u^N\|_{H^1} + \|w^N\|_{L_\infty(L_2)} \leqslant C(DATA)$. Thus

$$u^N \rightharpoonup u \quad \text{in } H^1 \quad \text{and} \quad w^N \rightharpoonup^* w \quad \text{in } L_\infty(L_2)$$

for some $(u, w) \in H^1 \times L_\infty(L_2)$. It is very easy to verify that (u, w) is a weak solution. First, passing to the limit in (3.1) for (u^N, w^N) we see that u satisfies (3.1) with w. On the other hand, taking the limit in (3.2) we verify that $w = S(G - \operatorname{div} u)$. We conclude that (u, w) satisfies (3.1)–(3.2), thus we have the weak solution. To show the boundary condition on the density we can rewrite the r.h.s. of (3.3) as

$$\begin{cases} w_{x_1} + \dfrac{\bar{u}^{(2)}}{1+\bar{u}^{(1)}} w_{x_2} + \dfrac{\bar{u}^{(3)}}{1+\bar{u}^{(1)}} w_{x_3} = \dfrac{v}{1+\bar{u}^{(1)}} & \text{in } \mathcal{D}'(\Omega), \\ w = w_{in} & \text{on } \Gamma_{in}, \end{cases} \tag{3.20}$$

and, treating x_1 as a 'time' variable, adapt Di Perna–Lions theory of transport equation (see [5]) that implies the uniqueness of solution to (3.20) in the class $L_\infty(L_2)$. The proof is thus complete. □

3.2. Strong solution

Having the weak solution of the linear system (1.9) we can show quite easily that this solution is strong if the data has the appropriate regularity. The following lemma gives existence of a strong solution to (1.9).

Lemma 11. *Let* $F \in L_p$, $G \in W^1_p$, $w_{in} \in W^1_p(\Gamma_{in})$, $B \in W^{1-1/p}_p(\Gamma)$ *and assume that* f *is large enough and* $\|\bar{u}\|_{W^2_p}$ *is small enough. Then there exists* $(u, w) \in W^2_p \times W^1_p$ *that is a strong solution to* (1.9) *and satisfies the estimate* (2.35).

Proof. Since (1.9) is a linear system, the *a priori* estimate (2.35) will imply the regularity of the weak solution once we can deal with the singularity of the boundary at the junctions of Γ_0 with Γ_{in} and Γ_{out}. This however can be done easily since Ω is symmetric w.r.t. the plane $\{x_1 = 0\}$ and the slip boundary conditions preserve this symmetry. More precisely, for $\{\tilde{x} = (-x_1, x_2, x_3)$: $x = (x_1, x_2, x_3) \in \Omega\}$ we can consider a vector field

$$\tilde{u}(\tilde{x}) = \left[-u^1(x), u^2(x), u^3(x)\right]. \tag{3.21}$$

Then on Γ_{in} we have $\tilde{u} \cdot n = u \cdot n$ and $n \cdot \mathbf{D}(\tilde{u}) \cdot \tau_i + \tilde{u} \cdot \tau_i = n \cdot \mathbf{D}(u) \cdot \tau_i + u \cdot \tau_i$. Hence we can extend the weak solution on the negative values of x_1 using (3.21) and, applying the estimate (2.35), show that the extended solution is in $W^2_p \times W^1_p$. An identical argument can be applied on Γ_{out} and we conclude that (u, w) is a strong solution to (1.9). □

4. Bounds on the approximating sequence

In this section we will show the bounds on the sequence $\{(u^n, w^n)\}$ of solutions to (1.8). The term $u \cdot \nabla w$ in the continuity equation makes it impossible to show directly the convergence in $W^2_p \times W^1_p$ to the strong solution of (1.6). We can show however that the sequence of iterated solutions is bounded in $W^2_p \times W^1_p$, and using this bound we can conclude it is a Cauchy sequence in $H^1 \times L_\infty(L_2)$, and thus converges in this space to some couple (u, w). On the other hand, the boundedness implies weak convergence in $W_p^2 \times W_p^1$, and the limit must be (u, w). The following lemma gives the boundedness of (u^n, w^n) in $W_p^2 \times W_p^1$.

Lemma 12. *Let $\{(u^n, w^n)\}$ be a sequence of solutions to* (1.8) *starting from $(u^0, w^0) = (0, 0)$. Then*

$$\|u^n\|_{W_p^2} + \|w^n\|_{W_p^1} \leqslant M, \tag{4.1}$$

where M can be arbitrarily small provided that $\|u_0\|_{W_p^2}$ (extension of the boundary data (1.4)*, not to be confused with u^0 from (u^0, w^0), the starting point of the sequence (u^n, w^n)), $\|B\|_{W_p^{1-1/p}(\Gamma)}$, $\|w_{\mathrm{in}}\|_{W_p^1(\Gamma_{\mathrm{in}})}$ and $\|\bar{u}\|_{W_p^2}$ are small enough and f is large enough.*

Proof. The estimate (2.35) for the iterated system reads

$$\begin{aligned} &\|u^{n+1}\|_{W_p^2} + \|w^{n+1}\|_{W_p^1} \\ &\quad \leqslant C\big[\big\|F\big(u^n, w^n\big)\big\|_{L_p} + \big\|G\big(u^n, w^n\big)\big\|_{W_p^1} + \|B\|_{W_p^{1-1/p}(\Gamma)} + \|w_{\mathrm{in}}\|_{W_p^1(\Gamma_{\mathrm{in}})}\big]. \end{aligned} \tag{4.2}$$

Denoting $A_n = \|u^n\|_{W_p^2} + \|w^n\|_{W_p^1}$ and $b = \|u_0\|_{W_p^2} + \|B\|_{W_p^{1-1/p}(\Gamma)} + \|w_{\mathrm{in}}\|_{W_p^1(\Gamma_{\mathrm{in}})}$, from (2.1) and (4.2) we get

$$A_{n+1} \leqslant C A_n^2 + b, \tag{4.3}$$

thus A_n is bounded by a constant that can be arbitrarily small provided that A_0 and b are small enough. Indeed let us fix $0 < \delta < \frac{1}{4C}$ and assume that $b < \delta$. Then (4.3) entails an implication $A_n \leqslant 2b \Rightarrow A_{n+1} \leqslant 2b$ and we can conclude that

$$\left.\begin{array}{l} \delta < \dfrac{1}{4}, \\ b < \delta, \\ A_0 < 2b \end{array}\right\} \quad \Longrightarrow \quad A_n < 2\delta \quad \forall n \in \mathbb{N}. \tag{4.4}$$

Hence if we fix $0 < \epsilon < \frac{1}{4}$ and assume that $\|u_0\|_{W_p^2} + \|B\|_{W_p^{1-1/p}(\Gamma)} + \|w_{\mathrm{in}}\|_{W_p^1(\Gamma_{\mathrm{in}})} < \epsilon$ then starting the iteration from $(u^0, w^0) = (0, 0)$ we have

$$\|u^n\|_{W_p^2} + \|w^n\|_{W_p^1} \leqslant 2\delta \quad \forall n \in \mathbb{N}. \qquad \square \tag{4.5}$$

The next lemma almost completes the proof of the Cauchy condition in $H^1 \times L_\infty(L_2)$ for the sequence of iterated solutions.

Lemma 13. *Let the assumptions of Lemma* 12 *hold. Then we have*

$$\|u^{n+1} - u^{m+1}\|_{H^1} + \|w^{n+1} - w^{m+1}\|_{L_\infty(L_2)} \leqslant E(M)\big(\|u^n - u^m\|_{H^1} + \|w^n - w^m\|_{L_\infty(L_2)}\big), \tag{4.6}$$

where M is the constant from (4.1)*.*

Proof. Subtracting $(1.8)_m$ from $(1.8)_n$ we arrive at

$$\begin{aligned}&\partial_{x_1}\big(u^{n+1}-u^{m+1}\big)-\mu\Delta\big(u^{n+1}-u^{m+1}\big)-(\nu+\mu)\nabla\operatorname{div}\big(u^{n+1}-u^{m+1}\big)\\&\quad+\gamma\nabla\big(w^{n+1}-w^{m+1}\big)=F\big(u^n,w^n\big)-F\big(u^m,w^m\big),\end{aligned}$$

$$\begin{aligned}&\operatorname{div}\big(u^{n+1}-u^{m+1}\big)+\partial_{x_1}\big(w^{n+1}-w^{m+1}\big)+\big(u^n+u_0\big)\cdot\nabla\big(w^{n+1}-w^{m+1}\big)\\&\quad=G\big(u^n,w^n\big)-G\big(u^m,w^m\big)+\big(u^n-u^m\big)\cdot\nabla w^m,\end{aligned}$$

$$\begin{gathered}n\cdot 2\mu\mathbf{D}\big(u^{n+1}-u^{m+1}\big)\cdot\tau_i+f\big(u^{n+1}-u^{m+1}\big)\cdot\tau_i|_\Gamma=0,\\ n\cdot\big(u^{n+1}-u^{m+1}\big)\big|_\Gamma=0,\\ w^{n+1}-w^{m+1}|_{\Gamma_{\text{in}}}=0.\end{gathered}$$

The estimate (2.4) applied to this system yields

$$\begin{aligned}&\big\|u^{n+1}-u^{m+1}\big\|_{H^1}+\big\|w^{n+1}-w^{m+1}\big\|_{L_\infty(L_2)}\\&\quad\leqslant\big\|F\big(u^n,w^n\big)-F\big(u^m,w^m\big)\big\|_{V^*}+\big\|G\big(u^n,w^n\big)-G\big(u^m,w^m\big)\big\|_{L_2}+\big\|\big(u^n-u^m\big)\cdot\nabla w^m\big\|_{L_2}.\end{aligned}$$

In order to derive (4.6) from the above inequality we have to examine the l.h.s. The part with G is the most straightforward and we have

$$\big\|G\big(u^n,w^n\big)-G\big(u^m,w^m\big)\big\|_{L_2}\leqslant E(M)\big(\big\|u^n-u^m\big\|_{H^1}+\big\|w^n-w^m\big\|_{L_\infty(L_2)}\big).\tag{4.7}$$

The function F is more complicated and we have to look at the difference more carefully. A direct calculation yields $F(u^n,w^n)-F(u^m,w^m)=F_1^{n,m}+F_2^{n,m}$, where

$$\big\|F_1^{n,m}\big\|_{V^*}\leqslant E(M)\big(\big\|u^n-u^m\big\|_{H^1}+\big\|w^n-w^m\big\|_{L_\infty(L_2)}\big)\tag{4.8}$$

and

$$F_2^{n,m}=-\big[\delta\pi'\big(w^n\big)-\delta\pi'\big(w^m\big)\big]\nabla w^n+\delta\pi'\big(w^m\big)\nabla\big(w^n-w^m\big)=:F_{2,1}^{n,m}+F_{2,2}^{n,m},\tag{4.9}$$

where $\delta\pi'(\cdot)$ is defined in (4.11). Since we are interested in the V^*-norm of $F_2^{n,m}$, we have to multiply $F_{2,1}^{n,m}$ and $F_{2,2}^{n,m}$ by $v\in V$ and integrate. With $F_{2,2}^{n,m}$ we get

$$\begin{aligned}&\int_\Omega\delta\pi'\big(w^m\big)\nabla\big(w^n-w^m\big)\cdot v\,dx\\&\quad=-\int_\Omega\delta\pi'\big(w^m\big)\big(w^n-w^m\big)\operatorname{div}v\,dx-\int_\Omega\big(w^n-w^m\big)\nabla\big[\delta\pi'\big(w^m\big)\big]\cdot v\,dx,\end{aligned}$$

and thus we have to estimate $\delta\pi'(w^m)$ in terms of w^m. Using (2.3) we can write

$$\delta\pi'\big(w^m\big)=w^m\int_0^1\pi''\big(tw^m+1\big)\,dt,\tag{4.10}$$

what yields

$$\|\delta\pi'(w^m)\|_{L_q} \leqslant C(\pi)\|w^m\|_{L_q}, \quad 1 \leqslant q \leqslant \infty. \tag{4.11}$$

Now we have to estimate $\|\nabla\delta\pi'(w^m)\|_{L_p}$. Since π is a C^3-function, we can take the gradient of (4.10) and verify that

$$\|\nabla\delta\pi'(w^m)\|_{L_p} \leqslant C(\pi)\|\nabla w^m\|_{L_p}. \tag{4.12}$$

Thus we have

$$\begin{aligned}\left|\int_\Omega \delta\pi'(w^m)(w^n - w^m)\operatorname{div} v\,dx\right| &\leqslant \|\delta\pi'(w^m)\|_{L_\infty}\|w^n - w^m\|_{L_2}\|\operatorname{div} v\|_{L_2} \\ &\leqslant C\|w^m\|_{W_p^1}\|w^n - w^m\|_{L_\infty(L_2)}\|v\|_V. \end{aligned}\tag{4.13}$$

Next, since $p > 3$, by the Sobolev imbedding theorem we have

$$\begin{aligned}\left|\int_\Omega (w^n - w^m)\nabla[\delta\pi'(w^m)]\cdot v\,dx\right| &\leqslant \|w^n - w^m\|_{L_2}\|\nabla\delta\pi'(w^m)\|_{L_p}\|v\|_{L_6} \\ &\leqslant C\|w^m\|_{W_p^1}\|w^n - w^m\|_{L_\infty(L_2)}\|v\|_V. \end{aligned}\tag{4.14}$$

Combining (4.13) and (4.14) we get

$$\|F_{2,2}^{n,m}\|_{V^*} \leqslant E(M)\|w^n - w^m\|_{L_\infty(L_2)}. \tag{4.15}$$

In order to estimate $F_{2,1}^{n,m}$ we will use again (2.3) to write

$$\delta\pi'(w^n) - \delta\pi'(w^m) = (w^n - w^m)\int_0^1 p''[tw^n + (1-t)w^m + 1]\,dt, \tag{4.16}$$

what yields $\|\delta\pi'(w^n) - \delta\pi'(w^m)\|_{L_2} \leqslant C\|w^n - w^m\|_{L_2}$. With this observation we can estimate

$$\begin{aligned}\left|\int_\Omega [\delta\pi'(w^n) - \delta\pi'(w^m)]\nabla w^n\cdot v\,dx\right| &\leqslant \|\delta\pi'(w^n) - \delta\pi'(w^m)\|_{L_2}\|\nabla w^n\|_{L_p}\|v\|_{L_6} \\ &\leqslant E(\|w^n\|_{W_p^1})\|w^n - w^m\|_{L_\infty(L_2)}\|v\|_V, \end{aligned}$$

what yields

$$\|F_{2,1}^{n,m}\|_{V^*} \leqslant E(M)\|w^n - w^m\|_{L_\infty(L_2)}. \tag{4.17}$$

Combining the estimates on $F_1^{n,m}$, $F_{2,1}^{n,m}$ and $F_{2,2}^{n,m}$ we get

$$\|F(u^n, w^n) - F(u^m, w^m)\|_{V^*} \leqslant E(M)[\|u^n - w^n\|_{H^1} + \|w^n - w^m\|_{L_\infty(L_2)}]. \tag{4.18}$$

The part that remains to estimate is $(u^n - u^m) \cdot \nabla w^m$. We shall notice here that this is the term which makes it impossible to show the convergence in $W^2_p \times W^1_p$ directly. Namely, if we would like to apply the estimate (2.35) to the system for the difference then we would have to estimate $\|(u^n - u^m) \cdot \nabla w^m\|_{W^1_p}$ what cannot be done as we do not have any knowledge about $\|w\|_{W^2_p}$.

Fortunately we only need to estimate the L_2-norm of this awkward term, what is straightforward. Namely, we have

$$\big\|\big(u^n - u^m\big) \cdot \nabla w^m\big\|_{L_2} \leqslant \big\|u^n - u^m\big\|_{L_q}\big\|\nabla w^m\big\|_{L_p} \leqslant C\big\|w^m\big\|_{W^1_p}\big\|u^n - u^m\big\|_{H^1}, \tag{4.19}$$

since $q = \frac{2p}{p-2} < 6$ for $p > 3$. We have thus completed the proof of (4.6). $\square$

Now, Lemma 12 implies that the constant $E(M) < 1$ provided that the data is small enough and the starting point $(u^0, w^0) = (0, 0)$. It completes the proof of the Cauchy condition in $H^1 \times L_\infty(L_2)$ for the sequence $\{(u^n, w^n)\}$.

Remark 2. Lemmas 12 and 13 hold for any starting point (u^0, w^0) small enough in $W^2_p \times W^1_p$, not necessarily $(0, 0)$, but we can start the iteration from $(0, 0)$ without loss of generality.

5. Proof of Theorem 1

In this section we prove our main result, Theorem 1. First we show existence of the solution passing to the limit with the sequence (u^n, w^n) and next we show that this solution is unique in the class of solutions satisfying (1.3).

Existence of the solution. Since we have the Cauchy condition on the sequence (u^n, w^n) only in the space $H^1(\Omega) \times L_\infty(L_2)$, first we have to show the convergence in the weak formulation of the problem (1.6), transferring the derivatives of the density on the test function. The sequence (u^n, w^n) satisfies in particular the following weak formulation of (1.8)

$$\begin{aligned}&\int_\Omega \big\{v \cdot \partial_{x_1} u^{n+1} + 2\mu \mathbf{D}\big(u^{n+1}\big) : \nabla v + \nu \operatorname{div} u^{n+1} \operatorname{div} v - \gamma w^{n+1} \operatorname{div} v\big\}\, dx \\ &\quad + \int_\Gamma f\big(u^{n+1} \cdot \tau_i\big)(v \cdot \tau_i)\, d\sigma = \int_\Omega F\big(u^n, w^n\big) \cdot v\, dx + \int_\Gamma B_i(v \cdot \tau_i)\, d\sigma\end{aligned} \tag{5.1}$$

and

$$\begin{aligned}&-\int_\Omega w^{n+1}\big[\tilde{u}^n \cdot \nabla \phi + \operatorname{div} \tilde{u}^n \phi\big]\, dx + \int_{\Gamma_{\text{out}}} w^{n+1} \phi\, d\sigma \\ &\quad = \int_\Omega \phi\big(G\big(u^n, w^n\big) - \operatorname{div} u^{n+1}\big)\, dx + \int_{\Gamma_{\text{in}}} w_{\text{in}} \phi\, d\sigma\end{aligned} \tag{5.2}$$

$\forall (v, \phi) \in V \times \bar{C}^\infty(\Omega)$, where $\tilde{u}^n = [1 + (u^n + u_0)^{(1)}, (u^n + u_0)^{(2)}, (u^n + u_0)^{(3)}]$.

Now using the convergence in $H^1 \times L_\infty(L_2)$ combined with the bound (4.1) in $W^2_p \times W^1_p$ we can pass to the limit in (5.1)–(5.2). The convergence in all the terms on the r.h.s. of (5.1) is obvious and the only nontrivial step to show the convergence of $F(u^n, w^n)$ is to show that

$$\int_\Omega \delta \pi'\big(w^n\big) \nabla w^n \cdot v\, dx \to \int_\Omega \delta \pi'(w) \nabla w \cdot v\, dx,$$

what is equivalent to

$$-\int_\Omega \delta\pi'(w)\big(w^n - w\big)\operatorname{div} v\,dx - \int_\Omega \big(w^n - w\big)\nabla\delta\pi'(w)\cdot v\,dx$$
$$+\int_\Omega \big[\delta\pi'\big(w^n\big) - \delta\pi'(w)\big]\nabla w^n \cdot v\,dx \to 0. \tag{5.3}$$

The first and second integral on the l.h.s. converges by (4.11) and (4.12) respectively (both with w instead of w^m). Finally, we easily verify that $\|\delta\pi'(w^n) - \delta\pi'(w)\|_{L_p} \leqslant C(\pi)\|w^n - w\|_{L_p}$ what entails convergence of the third integral. Hence (5.3) holds.

We conclude that (u, w) satisfies

$$\int_\Omega \big\{v\cdot\partial_{x_1}u + 2\mu\mathbf{D}(u) : \nabla v + \nu\operatorname{div} u\operatorname{div} v - \gamma w\operatorname{div} v\big\}\,dx + \int_\Gamma f(u\cdot\tau_i)(v\cdot\tau_i)\,d\sigma$$
$$= \int_\Omega F(u,w)\cdot v\,dx + \int_\Gamma B_i(v\cdot\tau_i)\,d\sigma \tag{5.4}$$

$\forall v \in V$. In (5.2) we have to check the convergence in the boundary term. We can use the same argument as in the proof of the existence of solution to the linear system when we have passed to the limit with finite dimensional approximations. Namely, in fact w^n satisfies the Cauchy condition not only in $L_\infty(L_2)$. A stronger fact holds that w^n is a Cauchy sequence in $L_2(\Omega_{x_1})$ for every $x_1 \in [0, L]$, where Ω_{x_1} denotes the x_1-cut of Ω. In particular $w^n \to \zeta$ in $L_2(\Gamma_{\text{out}})$ for some $\zeta \in L_2(\Gamma_{\text{out}})$ and since $\sup_{x_1\in[0,L]}\|w\|_{L_2(\Omega_{x_1})} < \infty$ we conclude that $\zeta = w|_{\Gamma_{\text{out}}}$. This result combined with the obvious convergence of other terms in (5.2) implies

$$-\int_\Omega w[\tilde u\cdot\nabla\phi + \operatorname{div}\tilde u\phi]\,dx + \int_{\Gamma_{\text{out}}} w\phi\,d\sigma = \int_\Omega \phi\big(G(u,w) - \operatorname{div} u\big)\,dx + \int_{\Gamma_{\text{in}}} w_{\text{in}}\phi\,d\sigma \tag{5.5}$$

$\forall\phi \in \bar C^\infty(\Omega)$, where $\tilde u = [1 + (u+u_0)^{(1)}, (u+u_0)^{(2)}, (u+u_0)^{(3)}]$.

Hence we have shown that (u, w) satisfies (5.4)–(5.5), the weak formulation of (1.6). Now we want to show that the strong formulation also holds.

The bound in $W^2_p \times W^1_p$ implies $(u^{n_k}, w^{n_k}) \rightharpoonup (\bar u, \bar w)$ in $W^2_p \times W^1_p$ for some $(\bar u, \bar w) \in W^2_p \times W^1_p$. On the other hand, we have $(u^{n_k}, w^{n_k}) \to (u, w)$ in $H^1 \times L_\infty(L_2)$, thus we conclude that $(\bar u, \bar w) = (u, w)$.

Hence we can integrate by parts in (5.4)–(5.5) to obtain

$$\int_\Omega \big[F(u,w) - \mu\Delta u - (\mu+\nu)\nabla\operatorname{div} u + \gamma\nabla w\big]\cdot v\,dx$$
$$= \int_\Gamma \big[B_i(v\cdot\tau_i) - n\cdot\big[2\mu\mathbf{D}(u) + \nu\operatorname{div} u\,\mathbf{Id}\big]\cdot v - f(u\cdot\tau_i)(v\cdot\tau_i)\big]\,d\sigma \tag{5.6}$$

and

$$\int_\Omega \big[w_{x_1} + (u+u_0)\cdot\nabla w\big]\phi\,dx = \int_\Omega \big[G(u,w) - \operatorname{div} u\big]\phi\,dx. \tag{5.7}$$

From these equations we conclude that $(1.6)_{1,2}$ are satisfied a.e. in Ω and $(1.6)_3$ is satisfied a.e. on Γ. It remains to verify that $(1.6)_4$ is satisfied a.e. on Γ and $(1.6)_5$ holds a.e. on Γ_{in}. The condition $(1.6)_4$ results from the convergence $u^n \to u$ in H^1.

Finally, $w^n \rightharpoonup w$ in W^1_p implies that $w^n|_{\Gamma_{\mathrm{in}}} \rightharpoonup tr\, w|_{\Gamma_{\mathrm{in}}}$ in $L_p(\Gamma_{\mathrm{in}})$. On the other hand $w^n|_{\Gamma_{\mathrm{in}}} \to w_{\mathrm{in}}$ in $W^1_p(\Gamma_{\mathrm{in}})$ since it is a constant sequence. We conclude that $w|_{\Gamma_{\mathrm{in}}} = w_{\mathrm{in}}$.

Uniqueness. In order to prove the uniqueness of the solution consider (v_1, ρ_1) and (v_2, ρ_2) being two solutions to (1.1) satisfying (1.3). We will prove that

$$\|v_1 - v_2\|^2_{H^1} + \|\rho_1 - \rho_2\|^2_{L_2} = 0. \tag{5.8}$$

For simplicity let us denote $u := v_1 - v_2$ and $w := \rho_1 - \rho_2$. We will show that

$$\|u\|_{H^1} \leqslant E\|w\|_{L_2} \tag{5.9}$$

and

$$\|w\|_{L_2} \leqslant C\|u\|_{H^1}, \tag{5.10}$$

what obviously implies (5.8). Subtracting Eqs. (1.1) for (v_1, ρ_1) and (v_2, ρ_2) we get

$$\begin{aligned}
& w v_2 \cdot \nabla v_2 + \rho_1 u \cdot \nabla v_2 + \rho_1 v_1 \cdot \nabla u - \mu \Delta u - (\mu + \nu)\nabla \operatorname{div} u + I_\pi \nabla w + w \nabla I_\pi = 0, \\
& \rho_1 \operatorname{div} u + w \operatorname{div} v_2 + u \cdot \nabla \rho_2 + v_1 \cdot \nabla w = 0, \\
& n \cdot 2\mu \mathbf{D}(u) \cdot \tau|_\Gamma = 0, \\
& n \cdot u|_\Gamma = 0, \\
& w|_{\Gamma_{\mathrm{in}}} = 0,
\end{aligned} \tag{5.11}$$

where

$$I_\pi = \int_0^1 \pi'\big((t\rho_1) + (1-t)\rho_2\big)\,dt. \tag{5.12}$$

Notice that $I_\pi \in W^1_p$ since $\rho_i \in W^1_p$ and $\pi \in C^3$. In order to show (5.9) we follow the proof of (2.4) multiplying $(5.11)_1$ by $\rho_1 u$ (it will be clarified soon why take the test function $\rho_1 u$ instead of u). Using (2.6) we get

$$\begin{aligned}
& \int_\Omega \big(2\mu \mathbf{D}^2(u) + \nu\rho_1 \operatorname{div}^2 u\big)\,dx \\
& \quad + \underbrace{\int_\Omega \big\{2\mu\big[(\rho_1 - 1)\mathbf{D}(u) : \nabla u + \mathbf{D}(u) : (u \otimes \nabla\rho_1)\big] + \nu(\operatorname{div} u) u \cdot \nabla\rho_1\big\}\,dx}_{I_1} \\
& \quad \underbrace{- \int_\Omega \big\{w u \nabla\rho_1 + \rho_1^2 u^2 \cdot \nabla v_2 + u w \rho_1 v_2 \cdot \nabla v_2\big\}\,dx}_{I_2} + \underbrace{\int_\Omega \rho_1^2 (v_1 \cdot \nabla u) \cdot u\,dx}_{I_3}
\end{aligned}$$

$$+\underbrace{\int_{\Omega} \rho_1 wu \cdot \nabla I_\pi \, dx}_{I_4} - \underbrace{\int_{\Omega} wu \cdot \nabla(I_\pi \rho_1) dx}_{I_5} - \int_{\Omega} I_\pi w \rho_1 \operatorname{div} u \, dx + \int_{\Gamma} \rho_1 f u^2 \, d\sigma = 0.$$

We have $|I_1| + |I_2| \leqslant E(\|u\|_{H^1}^2 + \|w\|_{L_2}^2)$ and in order to deal with I_3 let us split it into two parts:

$$2I_3 = \underbrace{\int_{\Omega} \left\{ \left(\rho_1^2 v_1^{(1)} - 1\right) \partial_{x_1} |u|^2 + \rho_1^2 v_1^{(2)} \partial_{x_2} |u|^2 + \rho_1^2 v_1^{(3)} \partial_{x_3} |u|^2 \right\} dx}_{I_3^1} + \underbrace{\int_{\Omega} \partial_{x_1} |u|^2 \, dx}_{I_3^2}.$$

We have $|I_3^1| \leqslant E\|u\|_{H^1}^2$ and $I_3^2 = \int_\Gamma |u|^2 n^{(1)} \, d\sigma = -\int_{\Gamma_{\text{in}}} |u|^2 \, d\sigma + \int_{\Gamma_{\text{out}}} |u|^2 \, d\sigma$. In order to examine I_4 and I_5 we have to differentiate (5.12) what yields

$$\nabla I_\pi = I_\pi^1 \nabla \rho_1 + I_\pi^2 \nabla \rho_2, \tag{5.13}$$

where

$$I_\pi^1 = \int_0^1 \pi''\big(t\rho_1 + (1-t)\rho_2\big) t \, dt \quad \text{and} \quad I_\pi^2 = \int_0^1 \pi''\big(t\rho_1 + (1-t)\rho_2\big)(1-t) \, dt.$$

We have

$$\left| \int_{\Omega} \rho_1 I_\pi^1 u w \nabla \rho_1 \, dx \right| \leqslant \left\| \rho_1 I_\pi^1 \right\|_{L_\infty} \|\nabla \rho_1\|_{L_p} \|u\|_{L_6} \|w\|_{L_2} \leqslant E\big(\|u\|_{H^1}^2 + \|w\|_{L_2}^2\big),$$

and the same for $\int_\Omega \rho_1 I_\pi^2 u w \nabla \rho_2 \, dx$. Thus the application of (5.13) to I_4 yields $|I_4| \leqslant E(\|u\|_{H^1}^2 + \|w\|_{L_2}^2)$. To estimate $|I_5|$ it is enough to use (5.13) to compute $\nabla(I_\pi \rho_1)$ and then with the same arguments as in case of I_4 we get $|I_5| \leqslant E(\|u\|_{H^1}^2 + \|w\|_{L_2}^2)$. Summarizing our estimates we can write

$$\begin{aligned} &\|u\|_{H^1}^2 + \int_{\Gamma_{\text{in}}} \left(\rho_1 f - \frac{1}{2}\right) |u|^2 \, d\sigma + \int_{\Gamma_0} \rho_1 f |u|^2 \, d\sigma + \int_{\Gamma_{\text{out}}} \left(\rho_1 f + \frac{1}{2}\right) |u|^2 \, d\sigma \\ &\quad \leqslant \int_{\Omega} I_\pi w \rho_1 \operatorname{div} u \, dx + E\|w\|_{L_2}^2. \end{aligned} \tag{5.14}$$

The boundary integrals over Γ_0 and Γ_{out} will be nonnegative for any $f \geqslant 0$ and the integral over Γ_{in} will be nonnegative for f large enough on Γ_{in}. Now in order to obtain (5.9) from (5.14) we can express $\rho_1 \operatorname{div} u$ in terms of w using Eq. $(5.11)_2$ (this is why we have tested $(5.11)_1$ with $\rho_1 u$ instead of u) and rewrite (5.14) as

$$\|u\|_{H^1}^2 \leqslant -\underbrace{\int_{\Omega} I_\pi w^2 \operatorname{div} v_2 \, dx}_{I_6} - \underbrace{\int_{\Omega} I_\pi w u \cdot \nabla \rho_2 \, dx}_{I_7} - \underbrace{\int_{\Omega} I_\pi v_1 w \cdot \nabla w \, dx}_{I_8} + E\|w\|_{L_2}^2. \tag{5.15}$$

We verify easily that $|I_6| + |I_7| \leqslant E(\|u\|^2_{H_1} + \|w\|^2_{L_2})$. We have to put a little more effort to find a bound on I_8. Let us integrate by parts:

$$2I_8 = \int_\Omega I_\pi v_1 \nabla w^2\,dx = -\int_\Omega w^2 \operatorname{div}(I_\pi v_1)\,dx + \int_\Gamma w^2 I_\pi v_1 \cdot n\,d\sigma.$$

The boundary term reduces to $\int_{\Gamma_{\text{out}}} I_\pi w^2 v_1^{(1)}\,d\sigma > 0$ and in order to deal with the first term on the l.h.s. notice that

$$\operatorname{div}(I_\pi v_1) = \operatorname{div} v_1 I_\pi + I_\pi^1 v_1 \cdot \nabla\rho_1 + I_\pi^2 v_1 \cdot \nabla\rho_2,$$

hence

$$2I_8 \leqslant \underbrace{-\int_\Omega w^2 \operatorname{div} v_1 I_\pi\,dx}_{I_8^1} \underbrace{-\int_\Omega w^2 v_1 \cdot \nabla\rho_1 I_\pi^1\,dx}_{I_8^2} \underbrace{-\int_\Omega w^2 v_1 \cdot \nabla\rho_2 I_\pi^2\,dx}_{I_8^3}.$$

Obviously we have $|I_8^1| \leqslant E\|w\|^2_{L_2}$. In order to bound I_8^2 we can apply the continuity equation that yields $v_i \cdot \nabla\rho_i = -\rho_i \operatorname{div} v_i$, what implies $|I_8^2| = |\int_\Omega w^2\rho_1 \operatorname{div} v_i I_\pi^1\,dx| \leqslant E\|w\|^2_{L_2}$. In the term I_8^3 we can rewrite the mixed component as $v_1 \cdot \nabla\rho_2 = u \cdot \nabla\rho_2 + v_2 \cdot \nabla\rho_2$ and conclude that $|I_8^3| \leqslant E(\|u\|^2_{H^1} + \|w\|^2_{L_2})$. Combining the above results with (5.14) we get (5.9).

In order to show (5.9) we express the pointwise value of w using $(5.11)_2$:

$$\begin{aligned} w^2(x_1, x_2) &= \int_0^{x_1} w w_s(s, x_2)\,ds = -\int_0^{x_1} \frac{\rho_1}{v_1^{(1)}} w \operatorname{div} u(s, x_2)\,ds \\ &\quad - \int_0^{x_1} \frac{1}{v_1^{(1)}}\big(w^2 \operatorname{div} v_2 + wu \cdot \nabla\rho_2\big)(s, x_2)\,ds \\ &\quad - \frac{1}{2}\int_0^{x_1} \frac{1}{v_1^{(1)}}\big[v_1^{(2)}\partial_{x_2} w^2 + v_1^{(3)}\partial_{x_3} w^2\big](s, x_2)\,ds \\ &=: w_1^2 + w_2^2 + w_3^2. \end{aligned}$$

We estimate directly the first two components of the l.h.s. obtaining

$$\int_\Omega w_1^2\,dx \leqslant \epsilon\|w\|^2_{L_2} + C(\epsilon)\|u\|^2_{H_1} \quad \forall\epsilon > 0$$

and $\int_\Omega w_2^2\,dx \leqslant E(\|w\|^2_{L_2} + \|u\|^2_{H^1})$. To complete the proof we have to find a bound on w_3^2. To this end notice that

$$\int_\Omega w_3^2\,dx = \frac{1}{2}\int_0^L \int_{P_{x_1}} \frac{1}{v_1^{(1)}}\big[v_1^{(2)}\partial_{x_2} w^2 + v_1^{(3)}\partial_{x_3} w^2\big]\,dx\,dx_1,$$

where $P_{x_1} = \Omega_0 \times (0, x_1)$. Integrating by parts in the inner integral we get

$$\int_{\Omega} w_3^2\,dx = \frac{1}{2}\int_0^L \left\{ -\int_{P_{x_1}} w^2 \left[\partial_{x_2} \frac{v_1^{(2)}}{v_1^{(1)}} + \partial_{x_3} \frac{v_1^{(3)}}{v_1^{(1)}} \right] dx + \int_{\partial P_{x_1}} \frac{w^2}{v_1^{(1)}} \left[v_1^{(2)} n^{(2)} + v_1^{(3)} n^{(3)} \right] d\sigma \right\} dx_1.$$

The boundary integral reduces to $\int_{\Gamma_0 \cap \partial P_{x_1}} w^2 v \cdot n\,d\sigma = 0$, what implies $\int_\Omega w_3^3\,dx \leqslant E\|w\|_{L_2}^2$ and (5.10) easily follows completing the proof of the uniqueness, and hence the proof of the theorem. □

Appendix A

Vorticity on the boundary. In order to show the boundary relation $(2.11)_{3,4}$ we have to differentiate $(1.9)_4$ with respect to tangential directions at a given point $x_0 \in \Gamma$. Without loss of generality we can assume that $n(x_0) = (1, 0, 0)$, $\tau_1(x_0) = (0, 1, 0)$ and $\tau_2(x_0) = (0, 0, 1)$. Then we can rewrite $(1.9)_3$ as (all the quantities are taken at x_0):

$$\begin{cases} \mu\big(u^1,_2 + u^2,_1\big) + f u^2 = B_1, \\ \mu\big(u^1,_3 + u^3,_1\big) + f u^3 = B_2. \end{cases} \tag{A.1}$$

Differentiating $(1.9)_4$ with respect to the tangential direction τ_1 we get

$$\left(\frac{d}{d\tau_1} n\right) \cdot u + u^1,_2 = 0. \tag{A.2}$$

If we denote by χ_1 the curvature of the curve generated by τ_1 then we have $\frac{d}{d\tau_1} n = \chi_1 \tau_1$ and (A.2) can be rewritten as $\chi_1(\tau_1 \cdot u) + u^1,_2 = 0$. Combining this equation with $(A.1)_1$ we get

$$u^2,_1 - u^1,_2 = \left(2\chi_1 - \frac{f}{\mu}\right)(u \cdot \tau_1) + \frac{B_1}{\mu},$$

what is exactly $(2.11)_3$. $(2.11)_4$ can be shown in the same way differentiating $(1.9)_4$ with respect to the tangential direction τ_2.

Lemma 14 *(Interpolation inequality).* $\forall \epsilon > 0$, $\exists C(\epsilon, p, \Omega)$ *such that* $\forall f \in W_p^1(Q)$:

$$\|f\|_{L_p} \leqslant \epsilon \|\nabla f\|_{L_p} + C\|f\|_{L_2}. \tag{A.3}$$

Proof. Inequality (A.3) results from the inequality $\|f\|_{L_p} \leqslant C(p, \Omega)\|f\|_{W_2^1}^{\theta} \|f\|_{L_2}^{1-\theta}$ for $2 \leqslant p < \infty$, where $\theta = \frac{n(p-2)}{2p}$ (see [1, Theorem 5.8]). Using Cauchy inequality with ϵ we get A.3. □

The last auxiliary result we use is a following fact on finitely dimensional Hilbert spaces (the proof can be found in [25]):

Lemma 15. *Let X be a finite dimensional Hilbert space and let $P : X \to X$ be a continuous operator satisfying*

$$\exists M > 0\colon \quad \big(P(\xi), \xi\big) > 0 \quad \textit{for } \|\xi\| = M. \tag{A.4}$$

Then $\exists \xi^*\colon \|\xi^*\| \leqslant M$ and $P(\xi^*) = 0$.

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