Available online at www.sciencedirect.com
ScienceDirect
International Journal of Approximate Reasoning
49 (2008) 301–315
INTERNATIONAL JOURNAL OF APPROXIMATE REASONING
www.elsevier.com/locate/ijar

Fuzzy rough approximations of process data

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Received 2 April 2006; received in revised form 9 February 2007; accepted 11 March 2007
Available online 22 October 2007

Abstract

This paper concerns the variable precision fuzzy rough set (VPFRS) model with asymmetric bounds. The discussion of the presented approach is preceded by a comparison of the original crisp rough set paradigm to the variable precision crisp rough set model. As a new aspect, a unified form of expressing the lower and upper crisp approximations is considered. It can be applied to defining new fuzzy rough set models. Crucial notions of the VPFRS model are redefined and explained. A new way of determining the upper variable precision fuzzy rough approximation is proposed. The VPFRS model is used for describing and analyzing the control actions which are accomplished by a human operator, who controls a complex dynamic system. The decision model is expressed by means of a decision table with fuzzy attributes. Decision tables are generated by the fuzzification of crisp data based on a set of fuzzy linguistic values of the attributes. A T-similarity relation is chosen for comparing elements of the universe. In an illustrative example, the task of stabilization of the aircraft's bank angle during a turn maneuver is analyzed.

Keywords: Control theory; Decision model; Fuzzy attributes; Fuzzy rough sets; Variable precision rough sets

1. Introduction

Data obtained from decision processes constitute a valuable source of information, which can be used in knowledge engineering and for design of control systems. Modelling the human operator's controlling behavior is an important issue. Based on process data, the classical control theory tries to create a mathematical model of the human operator. This approach assumes that the human operator can be treated like an additional controller in a closed loop system. In contrast to the classical approach of control theory, a new paradigm in the form of fuzzy set theory was elaborated in the recent decades. This approach turned out to be suitable for modelling the expert's controlling behavior.

Information obtained from the controlled system is typically represented by real numbers. People prefer to deal with concepts or linguistic variables. Therefore, they usually transform crisp data to linguistic values in a process called the fuzzification. The expert's knowledge of proper control actions can be conveniently

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0888-613X/$ - see front matter
doi:10.1016/j.ijar.2007.03.016

formulated as a system of fuzzy decision rules. It is necessary to define input and output variables and the membership functions of the linguistic values which are used in the rules. However, experts cannot always formulate the system of rules explicitly. Hence, the decision system of the human operator has to be discovered, basing on the recorded process data. In such a case the rough set theory can be successfully applied.

The use of the rough set paradigm for modelling the human operator's control in industrial processes was initiated by Mrózek [21,23,24]. He utilized basically decision tables with crisp attributes. The intervals of the attributes values were coded as integers. Only static or slow processes were taken into account.

Modelling dynamic processes by using the crisp rough set description was investigated in our previous work [18]. It concerned the issue of generating and analyzing decision tables, which represented the control actions of a skilled military pilot, performing various flight tasks on a flight simulator. The obtained information systems were relatively large. In such a case the original rough set approach is very sensitive to small changes in data. Therefore, we could effectively adopt the variable precision rough set model (VPRS) introduced by Ziarko [15,33]. A human operator who controls a complex dynamic system must make adequate decisions repeatedly based on his observations. It is possible that the controlled system retains its current state for a longer period of time or returns to that state after a time. In consequence, many control actions of the human operator can be recorded during the process, some of them under similar conditions, and the generated information system will be large. This is why, one should make use of the VPRS model in analysis of such information systems.

The VPRS model is an important contribution toward a successful application of the rough set theory. It admits some level of misclassification while determining the lower and upper approximations of a set. This is especially helpful when large indiscernibility classes are generated with respect to a subset of attributes. Without relaxation of strong inclusion requirement we can get too "pessimistic" lower approximations of sets. This is due to rejection of the entire approximating class, even when only one single element in that (maybe very large) class is not included in the approximated set. Conversely, the obtained upper approximations of sets can be too "optimistic", because even only one common element of a large approximating class and the approximated set is sufficient for counting the class to the upper approximation.

There exist the same reasons for applying the idea of VPRS to fuzzy rough sets, as it was in the case of crisp rough sets. First of all, we should be aware that crisp rough sets are special case of fuzzy rough sets. However, it may be argued that the "true" fuzzy rough sets may not be so sensitive to the phenomena discussed above. This is correct to some extent, the obtained results have fuzzy character. But still, the calculated values of membership in the fuzzy rough lower approximations might be too small and the values of membership in the fuzzy rough upper approximations might be too large. Let us consider, for example, the fuzzy rough lower approximation, which is defined as the greatest lower bound of a set of fuzzy implication values. Only one case, for which the implication result is equal to zero, means that a similarity class cannot be included in the fuzzy rough lower approximation. It is particularly important when dealing with large universes. Nevertheless, we can observe this effect even for relatively small information systems.

In order to utilize decision tables with fuzzy attributes, we need an extension of the original rough set theory. The most important and widely used concept of a fuzzy rough set was originated by Dubois and Prade [7], and developed by many researchers (see, e.g. [10,14,16]).

In [19], we initiated research which aimed at extending the VPRS model on fuzzy rough sets. Our concept of variable precision fuzzy rough sets with asymmetric bounds l and u was then developed and presented in [20].

In an independent work [8], Fernández Salido and Murakami proposed a concept of the variable precision fuzzy rough approximations, which is an extension of the VPRS model with symmetric bounds. They introduced β-precision quasi T-norms and β-precision quasi T-conorms that admit of some misclassification level while aggregating fuzzy data. Another proposal of applying the relaxation idea to fuzzy rough sets was presented in [32]. Furthermore, relaxation of principles was also utilized in the dominance-based rough set approach [11], and studied in the framework of rough mereology [25]. A generalization of VPRS in the form of a parameterized rough set model was proposed in [13].

In the following paper, we want to give more insight into issues connected with application of the variable precision fuzzy rough set model (VPFRS) to modelling the human operator's decision system. This is a new contribution to our previous work, in which mainly theoretical aspects of the VPFRS model were considered.

In particular, we discuss how to construct from process data decision tables with fuzzy attributes, choose an adequate fuzzy similarity relation, and analyze decision tables with the help of the VPFRS model. The basic notions of our approach are redefined: the implication-based and T-norm-based inclusion sets, and the measures of lower and upper α-inclusion errors are proposed. Furthermore, we give a new definition of the upper variable precision fuzzy rough set approximation.

In our model of VPFRS, we generalize the notions which were introduced by Ziarko in his crisp VPRS concept. Before we go on with the details of our approach, we want to discuss some aspects of the standard crisp rough sets and the crisp VPRS model. We propose a unified way of expressing rough approximations which is important for fuzzy generalizations of the rough set theory.

2. Rough sets versus variable precision rough sets

We start our discussion with the crisp rough set concept proposed by Pawlak [22]. Any crisp subset of an universe U can be approximated by means of an indiscernibility relation R, representing our lack of knowledge about elements of U.

The lower approximation $\underline{R}(A)$ and upper approximation $\overline{R}(A)$ of a crisp set A are defined as follows:

$$\underline{R}(A) = \{x \in U : [x]_R \subseteq A\}, \tag{1}$$
$$\overline{R}(A) = \{x \in U : [x]_R \cap A \neq \emptyset\}, \tag{2}$$

where $[x]_R$ denotes an indiscernibility (equivalence) class containing an element $x \in U$.

Properties of rough sets have been thoroughly studied. Many generalizations of the basic definition have been proposed, e.g. [9,12,14]. An interesting extension of the rough set theory, elaborated by Polkowski and Skowron [25], bases on the mereology of Leśniewski. An important notion in the framework of rough mereology is the rough inclusion.

Let us reconsider the rough approximations given by (1) and (2) in the context of set inclusion. We can define the lower and upper approximations in an alternative way, utilizing solely the notion of set inclusion.

Definition 1. Given an indiscernibility relation R, the lower approximation $\underline{R}(A)$ and upper approximation $\overline{R}(A)$ of a crisp set A are defined as follows:

$$\underline{R}(A) = \{x \in U : \forall S \subseteq [x]_R \wedge S \neq \emptyset, S \subseteq A\}, \tag{3}$$
$$\overline{R}(A) = \{x \in U : \exists S \subseteq [x]_R \wedge S \neq \emptyset, S \subseteq A\}. \tag{4}$$

The only difference between (3) and (4) is the quantifier used, emphasizing two extreme (ideal) cases of approximation obtained by applying the indiscernibility relation R. It is obvious that an indiscernibility class can only be included in the lower approximation of the set A, when all subsets of that class are included in A. In contrast to that, an indiscernibility class can be included in the upper approximation of the set A, when some non-empty subset of that class is included in the set A. Moreover, we can define the lower and upper approximations in a similar way, using the notion of membership in a set.

Definition 2. Given an indiscernibility relation R, the lower approximation $\underline{R}(A)$ and upper approximation $\overline{R}(A)$ of a crisp set A are defined as follows:

$$\underline{R}(A) = \{x \in U : \forall y \in [x]_R,\ y \in A\}, \tag{5}$$
$$\overline{R}(A) = \{x \in U : \exists y \in [x]_R,\ y \in A\}. \tag{6}$$

We can see again the contrast between all needed elements and some sufficient element in the case of the lower and upper approximations, respectively.

A unified form of the lower and upper approximation given by (3) and (4) may do not seem so important, when we consider crisp sets. However, there is no single method of performing basic operations on fuzzy sets. In consequence, we can get many fuzzy rough set generalizations, and they depend on the form of the rough set definition which we try to generalize.

In applications of the rough set theory to real data, one has to cope with inconsistencies in information systems caused by noise and errors. As explained in the previous section, the need for admitting some level of misclassification is especially urgent in the case of large information systems.

The idea of relaxation of strict inclusion requirements was introduced by Ziarko [33] in the form of the variable precision rough set model, by means of a modified relation of set inclusion. It can be explained using the notion of inclusion error, $e(A,B)$, of a non-empty (crisp) set A in a (crisp) set B, defined as follows:

$$e(A,B) = 1 - \frac{\mathrm{card}(A \cap B)}{\mathrm{card}(A)}. \tag{7}$$

To limit the inclusion error, we apply a lower limit l and an upper limit u, introduced in the extended version of VPRS [15], which satisfy the requirement

$$0 \leqslant l < u \leqslant 1. \tag{8}$$

The crisp VPRS model was generalized more recently to a probabilistic rough set approach [27,34]. In this version of VPRS the lower limit l reflects the highest acceptable degree of the conditional probability $P(A|E)$ to include an equivalence class E in the negative region of an approximated set A. The upper limit u represents the least acceptable degree of the conditional probability $P(A|E)$ to include an equivalence class E in the positive region of the set A. The limits l and u satisfy the following constraint:

$$0 \leqslant l < P(A) < u \leqslant 1, \tag{9}$$

where $P(A)$ denotes the prior probability of the subset $A \in U$.

We retain in our further consideration a non-probabilistic interpretation of VPRS. Basing on the limits l and u which satisfy the constraint (8), one can define the u-lower and the l-upper approximation of any subset A of the universe U by an indiscernibility relation R.

The u-lower approximation of A by R is a set

$$\underline{R}_u A = \{x \in U : \ e([x]_R, A) \leqslant 1 - u\}, \tag{10}$$

where $[x]_R$ denotes an indiscernibility class of R containing the element x.

The l-upper approximation of A by R is a set

$$\overline{R}_l A = \{x \in U : \ e([x]_R, A) < 1 - l\}, \tag{11}$$

or alternatively

$$\overline{R}_l A = \{x \in U : \ e([x]_R, \overline{A}) > l\}. \tag{12}$$

The definitions (10) and (11) use the same notion of inclusion error and can be perceived as a weakened form of (3) and (4). Not all subsets of an indiscernibility class need to be included in the approximated set, and no subset of the indiscernibility class included in the set is sufficient for the acceptance of the class in the lower and upper approximations, respectively. In this way, we abandon the ideals of approximation and admit of some level of misclassification.

In order to get more realistic and intuitively acceptable results in applications of the VPRS model, we modified the standard definition of the positive region of a set [18]. This idea can also be used to encompass the variable precision fuzzy rough set model. In our definition of the positive region, we use only those elements of the lower approximation, for which there is no contradiction between the set A and the indiscernibility classes. This is an intermediate way between the very restrictive approach of the original rough set theory and the excessively tolerant VPRS model. The modified definition of the u-positive region of a crisp set A consists in rejecting all elements of the approximating classes, which are not included in the set A

$$\mathrm{Pos}_{R_u}(A) = A \cap \underline{R}_u A. \tag{13}$$

The idea of using intersection with the approximated set was also used by Inuiguchi [14] in his definition of the classification-oriented rough sets. For the classification-oriented counterpart of the upper approximation he takes the union with the approximated set, which produces the possible region of the set. The generalizations of rough sets given in [14] do not concern the VPRS model.

By applying the idea of the possible region of a set to the VPRS model, we obtain

$$\text{Psb}_{R_l}(A) = A \cup \overline{R}_l A. \tag{14}$$

Let us observe that the VPRS model does not retain all properties of the original rough sets, e.g. the basic property [22] of the lower and upper approximations of a set A, which is stated by

$$\underline{R}(A) \subseteq A \subseteq \overline{R}(A). \tag{15}$$

The property (15) does not hold in general in the VPRS model. This is due to relaxing the strong inclusion requirement. Only the u-positive and l-possible regions of a set A satisfy this property

$$\text{Pos}_{R_u}(A) \subseteq A \subseteq \text{Psb}_{R_l}(A). \tag{16}$$

Prior to extending the crisp VPRS model to a fuzzy one, we need to discuss the problem of a suitable representation of fuzzy information systems.

3. Decision tables with fuzzy attributes

Let us introduce the necessary description for construction of fuzzy decision tables, needed for representing the human operator's controlling behavior. To this end, we adopt an extension of Bodjanova's idea of fuzzy concepts [2,8].

We have a finite universe U with N elements: $U = \{x_1, x_2, \ldots, x_N\}$. Each element x of the universe U is described with the help of fuzzy attributes, which are divided into a subset of n condition attributes $C = \{c_1, c_2, \ldots, c_n\}$, and a subset of m decision attributes $D = \{d_1, d_2, \ldots, d_m\}$.

For each fuzzy attribute a set of linguistic values can be given. We denote by $V_{i1}, V_{i2}, \ldots, V_{in_i}$ the linguistic values of the condition attribute c_i, and by $W_{j1}, W_{j2}, \ldots, W_{jm_j}$ the linguistic values of the decision attribute d_j, where n_i and m_j is the number of the linguistic values of the ith condition and the jth decision attribute, respectively, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

For any element $x \in U$, its membership degrees in all linguistic values of the condition attribute c_i (or decision attribute d_j) have to be determined. It is done during the fuzzification stage, by utilizing the recorded crisp value of a particular attribute of x. When the linguistic values of an attribute have the form of singletons or disjoint intervals, with membership degree equal to 1 in the original domain of the attribute, then only one linguistic value can be assigned to that attribute. In that case we get a classical crisp decision table. In general, we obtain a non-zero membership of x to more than one linguistic value of an attribute. Moreover, we may say that the value of an attribute for a given element x is a fuzzy set in the domain of all linguistic values of that attribute. So, we denote by $V_i(x)$ the fuzzy value of the condition attribute c_i for any x, as a fuzzy set in the domain of the linguistic values of c_i

$$V_i(x) = \{\mu_{V_{i1}}(x)/V_{i1}, \mu_{V_{i2}}(x)/V_{i2}, \ldots, \mu_{V_{in_i}}(x)/V_{in_i}\}.$$

$W_j(x)$ denotes the fuzzy value of the decision attribute d_j for any x, as a fuzzy set in the domain of the linguistic values of d_j

$$W_j(x) = \{\mu_{W_{j1}}(x)/W_{j1}, \mu_{W_{j2}}(x)/W_{j2}, \ldots, \mu_{W_{jm_j}}(x)/W_{jm_j}\}.$$

The problem of comparing objects described by fuzzy sets has been widely studied in the literature (see, e.g. [4,6,8]). Many different forms of similarity relation have been invented and investigated, e.g. Greco et al. [10] proposed approximation of fuzzy sets by means of fuzzy relations which are only reflexive. In our considerations, when we focus on the analysis of the recorded process data, the symmetry and some kind of transitivity of the fuzzy similarity relation should be assumed.

After fuzzification of real crisp numbers obtained from the control process, each row of the decision table (in a vector representation) contains the membership degrees of a particular element x in all possible linguistic values of the condition and decision attributes. We use further a symmetric, reflexive and T-transitive fuzzy similarity relation [4,8], which is defined by means of the distance between the compared elements. For the sake of brevity the following formulas will only be given for condition attributes.

If we want to compare any two elements x and y of the universe U with respect to the condition attribute c_i, then the similarity between x and y could be expressed by

$$S_{c_i}(x,y) = 1 - \max_{k=1,n_i} \left|\mu_{V_{ik}}(x) - \mu_{V_{ik}}(y)\right|. \tag{17}$$

The above definition of $S_{c_i}(x,y)$ is one of many possible measures of similarity between the fuzzy sets $V_i(x)$ and $V_i(y)$. This is the case of T-similarity relation based on the Łukasiewicz T-norm [8].

In order to evaluate the similarity $S_C(x,y)$, with respect to condition attributes C, we have to aggregate the results obtained for all attributes $c_i, i = 1,2,\ldots,n$. This can be done by using the T-norm operator `min` as follows

$$S_C(x,y) = \min_{i=1,n} S_{c_i}(x,y) = \min_{i=1,n} \left(1 - \max_{k=1,n_i} \left|\mu_{V_{ik}}(x) - \mu_{V_{ik}}(y)\right|\right). \tag{18}$$

By the calculation of similarity for all pairs of elements of the universe U, we obtain a symmetric similarity matrix. Every row of the similarity matrix forms a fuzzy set in the domain of U. If the value of similarity between the elements x and y is equal to 1, they do belong to the same similarity class. It means that two rows of the similarity matrix must be merged into one fuzzy set with the membership degrees equal to 1 for x and y. This way, we obtain a family of fuzzy similarity classes $\widetilde{C} = \{C_1, C_2, \ldots, C_{\tilde{n}}\}$, for condition attributes C and a family of fuzzy similarity classes $\widetilde{D} = \{D_1, D_2, \ldots, D_{\tilde{m}}\}$, for decision attributes D.

The generated partitions $\widetilde{C}$ and $\widetilde{D}$ satisfy the property of covering U sufficiently and the property of disjointness [7]. For the partition $\widetilde{C}$ with $\tilde{n}$ elements, the properties of covering and disjointness are expressed as follows

$$\inf_{x\in U} \max_{i=1,\tilde{n}} \mu_{C_i}(x) > 0, \tag{19}$$

$$\forall i,j \in \{1,2,\ldots,\tilde{n}\} \wedge i \neq j, \quad \sup_{x\in U} \min(\mu_{C_i}(x), \mu_{C_j}(x)) < 1. \tag{20}$$

Now, we are able to calculate the approximations of $\widetilde{D}$ by $\widetilde{C}$. This will be done by using the VPFRS model.

4. Variable precision fuzzy rough approximations

Here, we redefine and discuss our approach [20] to variable precision fuzzy rough sets with asymmetric bounds l and u. Our VPFRS model bases on the use of fuzzy R-implication operators and extends the basic idea of inclusion error discussed in Section 2. We apply a notion of inclusion degree of any fuzzy set A in a fuzzy set B, with respect to particular elements of A.

Because every similarity class is a fuzzy set in the domain of U, calculating the approximations of particular members of the family $\widetilde{D}$ by the family $\widetilde{C}$ entails the problem of determining the degree of inclusion of one fuzzy set into another. This is an important issue of the fuzzy set theory. There is no unique solution. Different measures of fuzzy sets inclusion were considered in the literature (see, e.g. [2,17]). The use of implication operators in the inclusion indicator was proposed by Bandler and Kohout [1,3]. A most general (axiomatic) approach, given by Sinha–Dougherty [5], can also be implemented by applying the generalized Łukasiewicz implicators.

A crucial notion of our VPFRS model is the inclusion degree of a fuzzy set A in a fuzzy set B obtained with respect to particular elements (or singletons) of a set A. This way, we construct a fuzzy set which will be called the fuzzy inclusion set of A in B, and denoted by $\mathrm{Incl}(A,B)$. There are many possibilities to define such an inclusion set. According to the above remarks, we apply to this end an implication operator, denoted by $\rightarrow$.

Definition 3. The implication-based inclusion set of a non-empty fuzzy set A in a fuzzy set B, denoted by $\mathrm{Incl}(A,B)$, is defined as follows

$$\mu_{\mathrm{Incl}(A,B)}(x) = \begin{cases} \mu_A(x) \rightarrow \mu_B(x) & \text{if } \mu_A(x) > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

We set $\mu_{\mathrm{Incl}(A,B)}(x) = 0$, for $\mu_A(x) = 0$, in order to focus on the support of the set A. It would be possible to use a unique definition for determining the inclusion for all elements of the universe, but the form of the definition (21) helps to simplify the computational algorithm.

Let us apply a T-norm operator (e.g. `min`), denoted by $*$.

Definition 4. The T-norm-based inclusion set of a non-empty fuzzy set A in a fuzzy set B, denoted by $\mathrm{Incl}'(A,B)$, is defined as follows

$$\mu_{\mathrm{Incl}'(A,B)}(x) = \mu_A(x) * \mu_B(x). \tag{22}$$

This is equivalent to assuming a special form of implication in (21), namely the quasi implicator of Mamdani, which is often applied in fuzzy inference systems. We need two different definitions of fuzzy inclusion set, in order to maintain the compatibility between the fuzzy rough set model of Dubois and Prade [7] and our VPFRS model. Fuzzy implication will be used in the case of the lower variable precision fuzzy rough approximation, and T-norm (fuzzy intersection) for the upper variable precision fuzzy rough approximation, respectively.

When using fuzzy implication, we require that the degree of inclusion with respect to x should be equal to 1, if the inequality $\mu_A(x) \leqslant \mu_B(x)$ for that x is satisfied

$$\mu_A(x) \to \mu_B(x) = 1 \quad \text{if } \mu_A(x) \leqslant \mu_B(x). \tag{23}$$

We can easy show that the requirement (23) is always satisfied by residual implicators (R-implicators) [26] defined, using a T-norm operator $*$, as follows

$$x \to y = \sup\{\lambda \in [0,1] : x * \lambda \leqslant y\}. \tag{24}$$

The determination of the lower approximation of a set in the (crisp or fuzzy) VPFRS model can be interpreted as counting the indiscernibility classes into the lower approximation, basing on ‘‘better’’ elements (concerning their membership in the set $\mathrm{Incl}(A,B)$) and disregarding ‘‘the worst’’ elements of the indiscernibility classes, provided that an admissible error is not exceeded. So, we must determine the error that would be made, when ‘‘the worst’’ elements of the approximating fuzzy set, in the sense of their membership in the fuzzy inclusion set $\mathrm{Incl}(A,B)$, were omitted.

The rejection of ‘‘bad’’ elements can be done by utilizing the notion of α-cut, defined for any fuzzy set A and $\alpha \in [0,1]$ by

$$A_\alpha = \{x \in U : \ \mu_A(x) \geqslant \alpha\}. \tag{25}$$

Definition 5. The lower α-inclusion error, $\underline{e}_\alpha(A,B)$ of any non-empty fuzzy set A in a fuzzy set B is defined as follows

$$\underline{e}_\alpha(A,B) = 1 - \frac{\mathrm{power}(A \cap \mathrm{Incl}(A,B)_\alpha)}{\mathrm{power}(A)}, \tag{26}$$

where $\mathrm{power}(A)$ denotes the cardinality of a fuzzy set A. For any finite fuzzy set A with n elements, defined on U

$$\mathrm{power}(A) = \sum_{i=1}^{n} \mu_A(x_i). \tag{27}$$

An α value will be needed to express how many ‘‘bad’’ elements may be disregarded without violating the admissible error.

We interpret the determination of the upper VPFRS approximation of a set, as counting the indiscernibility classes into the upper approximation, basing on ‘‘worse’’ elements (concerning their membership in the set $\mathrm{Incl}'(A,B)$) and disregarding ‘‘the best’’ elements of the indiscernibility classes, provided that an admissible error is not exceeded. So, we must determine the error that would be made, when ‘‘the best’’ elements of the approximating fuzzy set, in the sense of their membership in the fuzzy inclusion set $\mathrm{Incl}'(A,B)$, were omitted.

Definition 6. The upper α-inclusion error, $\overline{e}_\alpha(A,B)$ of any non-empty fuzzy set A in a fuzzy set B is defined as follows

$$\overline{e}_\alpha(A,B) = 1 - \frac{\mathrm{power}(A \cap (\overline{\mathrm{Incl}'(A,B)})_\alpha)}{\mathrm{power}(A)}. \tag{28}$$

An α value will be needed to express how many "good" elements may be disregarded without violating the admissible error.

Now, we show that the measures of inclusion error used in (10)–(12) are special cases of the proposed measures (26) and (28).

Proposition 1. *For any non-empty crisp set A and any crisp set B, for any $\alpha \in (0,1]$, it holds that $\underline{e}_\alpha(A,B) = e(A,B)$, and $\overline{e}_\alpha(A,B) = e(A,\overline{B})$.*

Proof. First, we show that for any crisp sets A and B, the inclusion set $\mathrm{Incl}(A,B)$ is equal to the crisp intersection $A \cap B$. For any crisp set C

$$\mu_C(x) = \begin{cases} 1 & \text{for } x \in C, \\ 0 & \text{for } x \notin C. \end{cases} \tag{29}$$

Every implicator $\rightarrow$ satisfies the conditions: $1 \rightarrow 0 = 0$, and $1 \rightarrow 1 = 1, 0 \rightarrow 1 = 1, 0 \rightarrow 0 = 1$.

Thus, applying the definition (21), we get

$$\mu_{\mathrm{Incl}(A,B)}(x) = \mu_{A\cap B}(x) = \begin{cases} 1 & \text{if } x \in A \text{ and } x \in B, \\ 0 & \text{otherwise.} \end{cases} \tag{30}$$

Taking into account (27) and (29), we get for any finite crisp set C

$$\mathrm{power}(C) = \mathrm{card}(C). \tag{31}$$

Furthermore, applying (25) for any $\alpha \in (0,1]$, we obtain

$$C_\alpha = C. \tag{32}$$

By Eqs. (30)–(32), we finally have

$$\frac{\mathrm{power}(A \cap \mathrm{Incl}(A,B)_\alpha)}{\mathrm{power}(A)} = \frac{\mathrm{power}(A \cap (A\cap B)_\alpha)}{\mathrm{power}(A)} = \frac{\mathrm{card}(A\cap B)}{\mathrm{card}(A)}.$$

Hence, we obtain $\underline{e}_\alpha(A,B) = e(A,B)$, for any $\alpha \in (0,1]$.

For any crisp sets A and B, we get $\mathrm{Incl}'(A,B) = A \cap B$ by definition.

By repeating the above steps, we can show that

$$\frac{\mathrm{power}(A \cap (\overline{\mathrm{Incl}'(A,B)})_\alpha)}{\mathrm{power}(A)} = \frac{\mathrm{card}(A \cap \overline{A\cap B})}{\mathrm{card}(A)} = \frac{\mathrm{card}(A\cap \overline{B})}{\mathrm{card}(A)}.$$

Thus, $\overline{e}_\alpha(A,B) = e(A,\overline{B})$, for any $\alpha \in (0,1]$. □

We will apply the introduced lower and upper α-inclusion error, with the aim of implementing the idea of relaxation of strong inclusion requirements in the framework of fuzzy rough sets. The notion of fuzzy rough set was proposed by Dubois and Prade [7] and generalized by Radzikowska and Kerre [26].

For a given fuzzy set F and a fuzzy partition $\Phi = \{F_1, F_2, \ldots, F_n\}$ on the universe U, the membership functions of the lower and upper approximation of F by Φ are defined by

$$\mu_{\underline{\Phi}F}(F_i) = \inf_{x\in U} \mu_{F_i}(x) \rightarrow \mu_F(x), \tag{33}$$

$$\mu_{\overline{\Phi}F}(F_i) = \sup_{x\in U} \mu_{F_i}(x) * \mu_F(x). \tag{34}$$

The pair of sets $(\underline{\Phi}F, \overline{\Phi}F)$ is called a fuzzy rough set.

Furthermore, we assume that the implicator $\rightarrow$ does satisfy the boundary condition $0 \rightarrow a = 1$. This is true for all R-implicators and most other popular implication operators. Since any T-norm is a monotonic operator, satisfying the boundary condition $0*1 = 0$, it also fulfills the requirement $0*a = 0$. Thus, we can determine the lower and upper limit in (33) and (34), respectively, by taking into account only the support of the approximating class F_i, denoted by $\mathrm{supp}(F_i)$

$$\mu_{\underline{\Phi}F}(F_i) = \inf_{x \in \text{supp}(F_i)} \mu_{F_i}(x) \rightarrow \mu_F(x), \tag{35}$$

$$\mu_{\overline{\Phi}F}(F_i) = \sup_{x \in \text{supp}(F_i)} \mu_{F_i}(x) * \mu_F(x), \tag{36}$$

where

$$\text{supp}(F_i) = \{x \in U : \mu_{F_i}(x) > 0\}.$$

In the analysis of process data, we apply the proposals given in the previous section. Decision tables will be given in the form of Table 1. The admissible inclusion error will be expressed by using a lower limit l and an upper limit u.

For a given decision table we approximate particular fuzzy similarity classes $D_j \in \widetilde{D}$, $j = 1, 2, \ldots, \tilde{m}$, generated with respect to the decision attributes D, by all elements of the fuzzy partition $\widetilde{C}$, generated with respect to the condition attributes C. Thus, to apply the definitions (35) and (36), we take the fuzzy partition $\widetilde{C}$, instead of Φ.

According to the discussion given above, we admit of some level of tolerance, by taking into account only "better" elements of the approximating class in the case of the lower approximation, and "worse" elements of the approximating class in the case of the upper approximation, respectively.

Definition 7. The u-lower approximation of a fuzzy set D_j by a fuzzy partition $\widetilde{C}$ is a fuzzy set on the domain $\widetilde{C}$ with membership function expressed by

$$\mu_{\underline{\widetilde{C}}_u D_j}(C_i) = \inf_{x \in S_{i_u}} \mu_{\text{Incl}(C_i, D_j)}(x), \tag{37}$$

where

$$S_{i_u} = \text{supp}(C_i \cap \text{Incl}(C_i, D_j)_{\alpha_u}),$$
$$\alpha_u = \sup\{\alpha \in [0, 1] : \underline{e}_\alpha(C_i, D_j) \leqslant 1 - u\}.$$

The set S_{i_u} contains those elements of the approximating class C_i that are included in D_j at least to the degree α_u. The membership $\mu_{\underline{\widetilde{C}}_u D_j(C_i)}$ is determined by using "better" elements from S_{i_u}, instead of the whole class C_i. This helps to prevent the situation, when a few "bad" elements of a large class C_i significantly reduce the lower approximation of the set D_j.

The same idea of tolerance can be applied to defining the upper approximation. We take into account only "the best" elements of the complement of the intersection of the approximating class C_i and the approximated set D_j.

Definition 8. The l-upper approximation of a fuzzy set D_j by a fuzzy partition $\widetilde{C}$ is a fuzzy set on the domain $\widetilde{C}$ with membership function expressed by

$$\mu_{\overline{\widetilde{C}}_l D_j}(C_i) = \sup_{x \in S_{i_l}} \mu_{\text{Incl}'(C_i, D_j)}(x), \tag{38}$$

where

$$S_{i_l} = \text{supp}(C_i \cap (\overline{\text{Incl}'(C_i, D_j)})_{\alpha_l}),$$
$$\alpha_l = \sup\{\alpha \in [0, 1] : \overline{e}_\alpha(C_i, D_j) \leqslant l\}.$$

Table 1
Decision table with fuzzy attributes

	c_1	c_2	$\cdots$	c_n	d_1	d_2	$\cdots$	d_m
x_1	$V_1(x_1)$	$V_2(x_1)$	$\cdots$	$V_n(x_1)$	$W_1(x_1)$	$W_2(x_1)$	$\cdots$	$W_m(x_1)$
x_2	$V_1(x_2)$	$V_2(x_2)$	$\cdots$	$V_n(x_2)$	$W_1(x_2)$	$W_2(x_2)$	$\cdots$	$W_m(x_2)$
	$\cdots$							
x_N	$V_1(x_N)$	$V_2(x_N)$	$\cdots$	$V_n(x_N)$	$W_1(x_N)$	$W_2(x_N)$	$\cdots$	$W_m(x_N)$

The set S_{i_l} contains those elements of the approximating class C_i that are included in D_j at most to the degree α_l. The membership $\mu_{\overline{\widetilde{C}_l}D_j}(C_i)$ is determined by using "worse" elements from S_{i_l}, instead of the whole class C_i. This helps to prevent the situation, when a few "good" elements of a large class C_i significantly increase the lower approximation of the set D_j.

The sets S_{i_u} and S_{i_l} are used to give more insight into the presented VPFRS model. In practice, the algorithm for determining the approximations (37) and (38) can be quite straightforwardly implemented. Since we use limit values of membership function of the inclusion sets $\mathrm{Incl}(C_i, D_j)$ and $\mathrm{Incl}'(C_i, D_j)$, we get $\mu_{\underline{\widetilde{C}}_u D_j}(C_i) = \alpha_u$ and $\mu_{\overline{\widetilde{C}}_l D_j}(C_i) = 1 - \alpha_l$, respectively. Nevertheless, the given form of (37) and (38) is suitable for applying different, i.e. non-limit-based classes of fuzzy rough sets.

Now, we show that the approximations (37) and (38) are equivalent to fuzzy rough approximations of Dubois and Prade, when $u = 1$ and $l = 0$.

Proposition 2. *For any elements $C_i \in \widetilde{C}$ and $D_j \in \widetilde{D}$ of fuzzy partitions $\widetilde{C}$ and $\widetilde{D}$, if $u = 1$ and $l = 0$, then it holds $\mu_{\underline{\widetilde{C}}_u D_j}(C_i) = \mu_{\underline{\widetilde{C}} D_j}(C_i)$, and $\mu_{\overline{\widetilde{C}}_l D_j}(C_i) = \mu_{\overline{\widetilde{C}} D_j}(C_i)$.*

Proof. For $u = 1$, it is required that $\underline{e}_{\alpha_u}(C_i, D_j) = 0$. This means that no elements of the approximating similarity class C_i can be discarded:

$$\underline{e}_{\alpha_u}(C_i, D_j) = 1 - \frac{\mathrm{power}(C_i \cap \mathrm{Incl}(C_i, D_j)_{\alpha_u})}{\mathrm{power}(C_i)} = 0, \quad C_i \cap \mathrm{Incl}(C_i, D_j)_{\alpha_u} = C_i.$$

According to (21), in the case, when $\mathrm{supp}(\mathrm{Incl}(C_i, D_j)) \subset \mathrm{supp}(C_i)$, we get $\alpha_u = 0$, and $\mathrm{Incl}(C_i, D_j)_0 = U$.

In the other case, when $\mathrm{supp}(\mathrm{Incl}(C_i, D_j)) = \mathrm{supp}(C_i)$, we have $\alpha_u > 0$.

Thus, we obtain

$$\mathrm{supp}(C_i) \subseteq \mathrm{Incl}(C_i, D_j)_{\alpha_u}, \quad S_{i_u} = \mathrm{supp}(C_i \cap \mathrm{Incl}(C_i, D_j)_{\alpha_u}) = \mathrm{supp}(C_i). \tag{39}$$

By taking into account (21) and (39), we finally have, for $u = 1$

$$\mu_{\underline{\widetilde{C}}_u D_j}(C_i) = \inf_{x \in S_{i_u}} \mu_{\mathrm{Incl}(C_i, D_j)}(x) = \inf_{x \in \mathrm{supp}(C_i)} \mu_{C_i}(x) \rightarrow \mu_{D_j}(x).$$

A similar proof can be given for the l-upper approximation, when $l = 0$. □

Furthermore, we need a generalized measure of u-approximation quality in order to deal with fuzzy sets and fuzzy relations.

For the family $\widetilde{D} = \{D_1, D_2, \ldots, D_{\tilde{m}}\}$ and the family $\widetilde{C} = \{C_1, C_2, \ldots, C_{\tilde{n}}\}$ the u-approximation quality of $\widetilde{D}$ by $\widetilde{C}$ is defined as follows

$$\gamma_{\widetilde{C}_u}(\widetilde{D}) = \frac{\mathrm{power}(\mathrm{Pos}_{\widetilde{C}_u}(\widetilde{D}))}{\mathrm{card}(U)}, \tag{40}$$

where

$$\mathrm{Pos}_{\widetilde{C}_u}(\widetilde{D}) = \bigcup_{D_j \in \widetilde{D}} \omega(\underline{\widetilde{C}}_u D_j) \cap D_j.$$

The fuzzy extension ω denotes a mapping from the domain $\widetilde{C}$ into the domain of the universe U, which is expressed for any fuzzy set F on the domain $\widetilde{C}$ by

$$\mu_{\omega(F)}(x) = \mu_F(C_i), \quad \text{if } \mu_{C_i}(x) = 1. \tag{41}$$

In the definition (40), we generalize the notion of positive region, discussed in Section 2. For any fuzzy set A on U and a similarity relation S, the positive region of A is defined as follows

$$\mathrm{Pos}_{S_u}(A) = A \cap \omega(\underline{S}_u A). \tag{42}$$

The u-approximation quality of $\widetilde{D}$ by $\widetilde{C}$ will be used as a measure of consistency of the human operator's decision model.

5. Example

Let us consider the task of stabilization of the aircraft's bank angle during a right turn maneuver, performed by a pilot. Three condition attributes c_1, c_2, c_3 were taken into account:

c_1: bank angle deviation from the required value
(values: V_{11} – "Large Negative", V_{12} – "Small Negative", V_{13} – "Zero",
V_{14} – "Small Positive", V_{15} – "Large Positive");
c_2: change of the bank angle
(values: V_{21} – "Negative", V_{22} – "Zero", V_{23} – "Positive";
c_3: change of the aileron deflection angle in the previous moment
(values: V_{31} – "Negative (Decrease)", V_{32} – "Zero (No Change)",
V_{33} – "Positive (Increase)").

One decision attribute d_1 was used:

d_1 change of the aileron deflection angle,
(values: W_{11} – "Negative (Decrease)", W_{12} – "Zero (No Change)",
W_{13} – "Positive (Increase)").

The membership functions selected for all linguistic values of the attributes have a typical "trapezoidal" shape.

In reality, the process of right turn stabilization is more complicated, but a simplified description is sufficient for our considerations. The decision table with fuzzy attributes was generated by the fuzzification stage. It is convenient to represent fuzzy attributes as vectors (Table 2).

In order to analyze the obtained decision table with the help of VPFRS the following steps were executed:

(1) Determining the similarity matrix on the domain $U \times U$ with respect to all condition attributes and the similarity matrix with respect to all decision attributes, according to (18).

Table 2
Decision table with fuzzy attributes in vector form

	c_1	c_2	c_3	d_1
x_1	(0.9, 0.1, 0.0, 0.0, 0.0)	(0.8, 0.2, 0.0)	(0.0, 1.0, 0.0)	(0.0, 0.0, 1.0)
x_2	(0.1, 0.9, 0.0, 0.0, 0.0)	(0.9, 0.1, 0.0)	(0.0, 0.0, 1.0)	(0.0, 0.0, 1.0)
x_3	(0.0, 1.0, 0.0, 0.0, 0.0)	(0.0, 1.0, 0.0)	(0.0, 0.0, 1.0)	(0.0, 0.0, 1.0)
x_4	(0.0, 0.0, 0.9, 0.1, 0.0)	(0.0, 0.0, 0.1)	(0.0, 0.0, 1.0)	(0.8, 0.2, 0.0)
x_5	(0.0, 0.1, 0.9, 0.0, 0.0)	(1.0, 0.0, 0.0)	(0.8, 0.2, 0.0)	(0.0, 0.1, 0.9)
x_6	(0.0, 0.0, 1.0, 0.0, 0.0)	(0.0, 1.0, 0.0)	(0.0, 0.1, 0.9)	(0.0, 1.0, 0.0)
x_7	(0.0, 0.0, 1.0, 0.0, 0.0)	(0.0, 1.0, 0.0)	(0.0, 1.0, 0.0)	(0.0, 1.0, 0.0)
x_8	(0.0, 0.0, 1.0, 0.0, 0.0)	(0.0, 1.0, 0.0)	(0.0, 1.0, 0.0)	(0.0, 1.0, 0.0)
x_9	(0.0, 0.0, 1.0, 0.0, 0.0)	(0.0, 1.0, 0.0)	(0.0, 1.0, 0.0)	(0.0, 1.0, 0.0)
x_{10}	(0.0, 0.0, 1.0, 0.0, 0.0)	(0.0, 1.0, 0.0)	(0.0, 1.0, 0.0)	(0.0, 1.0, 0.0)
x_{11}	(0.0, 0.0, 1.0, 0.0, 0.0)	(0.0, 1.0, 0.0)	(0.0, 1.0, 0.0)	(1.0, 0.0, 0.0)
x_{12}	(0.7, 0.3, 0.0, 0.0, 0.0)	(1.0, 0.0, 0.0)	(1.0, 0.0, 0.0)	(0.0, 0.0, 1.0)
x_{13}	(0.0, 1.0, 0.0, 0.0, 0.0)	(0.9, 0.1, 0.0)	(0.0, 0.0, 1.0)	(0.0, 0.0, 1.0)
x_{14}	(0.0, 1.0, 0.0, 0.0, 0.0)	(0.0, 1.0, 0.0)	(0.0, 0.0, 1.0)	(0.0, 0.0, 1.0)
x_{15}	(0.0, 0.0, 0.9, 0.1, 0.0)	(0.0, 0.0, 1.0)	(0.0, 0.0, 1.0)	(0.9, 0.1, 0.0)
x_{16}	(0.0, 0.1, 0.9, 0.0, 0.0)	(1.0, 0.0, 0.1)	(0.9, 0.1, 0.0)	(0.0, 0.1, 0.9)
x_{17}	(0.0, 0.0, 1.0, 0.0, 0.0)	(0.0, 1.0, 0.0)	(0.0, 0.1, 0.9)	(0.0, 1.0, 0.0)
x_{18}	(0.0, 0.0, 1.0, 0.0, 0.0)	(0.0, 1.0, 0.0)	(0.0, 1.0, 0.0)	(0.0, 1.0, 0.0)
x_{19}	(0.0, 0.0, 1.0, 0.0, 0.0)	(0.0, 1.0, 0.0)	(0.0, 1.0, 0.0)	(0.0, 1.0, 0.0)
x_{20}	(0.0, 0.0, 0.0, 0.6, 0.4)	(0.0, 0.0, 1.0)	(0.0, 1.0, 0.0)	(1.0, 0.0, 0.0)

(2) Determining the family of similarity classes $\widetilde{C}$ and $\widetilde{D}$. We obtained 11 similarity classes with respect to condition and 6 similarity classes with respect to decision attribute.
(3) Calculating the u-lower approximation of particular decision similarity classes by the family of condition similarity classes, in the domain of $\widetilde{C}$, according to (37).
(4) Determining the u-lower approximation of $\widetilde{D}$ by $\widetilde{C}$ in the domain of U, and calculating the u-approximation quality of $\widetilde{D}$ by $\widetilde{C}$, according to (40).
(5) Evaluating the importance of each condition attribute for the human operator's decision model.

Let us take a closer look at approximating the decision similarity class D_4 with respect to the condition similarity classes C_6 and C_7. Table 3 contains the fuzzy similarity classes D_4, C_6, C_7 and the inclusion sets $\mathrm{Incl}(C_6, D_4)$, $\mathrm{Incl}(C_7, D_4)$. The Łukasiewicz implication operator was used in the calculation.

Next, we determine the membership degree of C_6 in the lower approximation of D_4. We start with the limit case, assuming $u = 1$. This means that no inclusion error is allowed. Hence, we seek for the biggest $\alpha \in [0, 1]$, denoted by α_u, for which $\underline{e}_\alpha(C_6, D_4) = 1 - \frac{\mathrm{power}(C_6 \cap \mathrm{Incl}(C_6, D_4)_\alpha)}{\mathrm{power}(C_6)} = 0$.

We find, for $u = 1$, that $\alpha_u = 0.9$,
$\mathrm{Incl}(C_6, D_4)_{0.9} = \{x_6, x_7, x_8, x_{9,10}, x_{11}, x_{17}, x_{18}, x_{19}\}$,
and finally, we obtain the membership degree $\mu_{\underline{\widetilde{C}}_{0.9} D_4}(C_6) = \alpha_u = 0.9$.
Assuming $u = 0.9$, we find that $\alpha_u = 1$,
$\mathrm{Incl}(C_6, D_4)_1 = \{x_6, x_7, x_8, x_9, x_{10}, x_{17}, x_{18}, x_{19}\}$,
$\underline{e}_{\alpha_u}(C_6, D_4) = 1 - 2.6/2.7 = 0.037 \leqslant 1 - u = 0.1$.

Observe, that omitting "the weakest" element x_{11} from $\mathrm{Incl}(C_6, D_4)$ leads to an increase (from 0.9 to 1) of the membership degree of C_6 in the lower approximation of D_4. The error made is equal to 0.037.

When using the similarity classes D_4 and C_7 and assuming $u = 1$, we get $\alpha_u = 0$. We cannot omit any element from the inclusion set $\mathrm{Incl}(C_7, D_4)$. Therefore, $\mu_{\underline{\widetilde{C}}_u D_4}(C_7) = \alpha_u = 0$.

For $u = 0.85$, we obtain $\alpha_u = 1$,
$\mathrm{Incl}(C_7, D_4)_1 = \{x_6, x_7, x_8, x_9, x_{10}, x_{17}, x_{18}, x_{19}\}$,
$\underline{e}_{\alpha_u}(C_7, D_4) = 1 - 6.2/7.2 = 0.139 \leqslant 1 - u = 0.15$,
and $\mu_{\underline{\widetilde{C}}_u D_4}(C_7) = 1$.

Table 3
Inclusion sets

	D_4	C_6	C_7	$\mathrm{Incl}(C_6, D_4)$	$\mathrm{Incl}(C_7, D_4)$
x_1	0.0	0.0	0.0	0.0	0.0
x_2	0.0	0.0	0.0	0.0	0.0
x_3	0.0	0.0	0.0	0.0	0.0
x_4	0.2	0.0	0.0	0.0	0.0
x_5	0.1	0.0	0.0	0.0	0.0
x_6	1.0	1.0	0.1	1.0	1.0
x_7	1.0	0.1	1.0	1.0	1.0
x_8	1.0	0.1	1.0	1.0	1.0
x_9	1.0	0.1	1.0	1.0	1.0
x_{10}	1.0	0.1	1.0	1.0	1.0
x_{11}	0.0	0.1	1.0	0.9	0.0
x_{12}	0.0	0.0	0.0	0.0	0.0
x_{13}	0.0	0.0	0.0	0.0	0.0
x_{14}	0.0	0.0	0.0	0.0	0.0
x_{15}	0.1	0.0	0.0	0.0	0.0
x_{16}	0.1	0.0	0.0	0.0	0.0
x_{17}	1.0	1.0	0.1	1.0	1.0
x_{18}	1.0	0.1	1.0	1.0	1.0
x_{19}	1.0	0.1	1.0	1.0	1.0
x_{20}	0.0	0.0	0.0	0.0	0.0

The use of $u = 0.85$ results in a maximal increase (from 0 to 1) of the membership degree of C_7 in the lower approximation of D_4. Owing to the relaxation of strong inclusion requirements in the VPFRS model, we could prevent the exclusion of the class C_7 from the lower approximation. The error made is equal to 0.139.

Finally, we determine the u-lower approximation of D_4 by the family $\widetilde{C}$.

For $u = 1$, we get

$$\underline{\widetilde{C}}_u D_4 = \{0/C_1, 0/C_2, 0/C_3, 0.1/C_4, 0.1/C_5, 0.9/C_6, 0/C_7, 0/C_8, 0/C_9, 0.1/C_{10}, 0/C_{11}\}$$

and for $u = 0.85$

$$\underline{\widetilde{C}}_u D_4 = \{0/C_1, 0/C_2, 0/C_3, 0.1/C_4, 0.1/C_5, 1/C_6, 1/C_7, 0/C_8, 0/C_9, 0.1/C_{10}, 0/C_{11}\}.$$

We determine the u-lower approximation in the domain of U. Using again the similarity class D_4, we determine the positive region of D_4.

For $u = 1$, we obtain

$$\mathrm{Pos}_{\widetilde{C}_u}(D_4) = \{0/x_1, 0/x_2, 0/x_3, 0.1/x_4, 0.1/x_5, 0.9/x_6, 0/x_7, 0/x_8, 0/x_9, 0/x_{10}, 0/x_{11}, 0/x_{12}, 0/x_{13}, 0/x_{14}, 0.1/x_{15}, 0.1/x_{16}, 0.9/x_{17}, 0/x_{18}, 0/x_{19}, 0/x_{20}\}$$

and for $u = 0.85$, we have

$$\mathrm{Pos}_{\widetilde{C}_u}(D_4) = \{0/x_1, 0/x_2, 0/x_3, 0.1/x_4, 0.1/x_5, 1/x_6, 1/x_7, 1/x_8, 1/x_9, 1/x_{10}, 0/x_{11}, 0/x_{12}, 0/x_{13}, 0/x_{14}, 0.1/x_{15}, 0.1/x_{16}, 1/x_{17}, 1/x_{18}, 1/x_{19}, 0/x_{20}\}.$$

The results of u-approximation quality of $\widetilde{D}$ by $\widetilde{C}$ before and after removing each condition attribute are given in Table 4. We denote by L – Łukasiewicz, G – Gaines, KD – Kleene-Dienes, Gd – Gödel implications operators. We see that even for a small universe, the value of u-approximation quality increases, when we use the VPFRS model with $u < 1$. The u-approximation quality is a good measure of consistency of the human operator's decision model. The analyzed pilot's decision system has a relatively high consistency. Calculations after discarding particular condition attributes lead to a conclusion that the condition attributes c_1 and c_2 are

Table 4
u-Approximation quality for different values of required inclusion degree

Method	Removed attribute	$\gamma_{\widetilde{C}_u}(\widetilde{D})$			
		$u = 1$	$u = 0.9$	$u = 0.85$	$u = 0.8$
Ł-inf	None	0.700	0.710	0.950	0.950
	c_1	0.520	0.535	0.715	0.775
	c_2	0.540	0.560	0.770	0.810
	c_3	0.630	0.630	0.950	0.950
G-inf	None	0.630	0.710	0.950	0.950
	c_1	0.391	0.506	0.506	0.746
	c_2	0.345	0.523	0.622	0.772
	c_3	0.630	0.630	0.950	0.950
KD-inf	None	0.680	0.700	0.940	0.950
	c_1	0.495	0.505	0.685	0.745
	c_2	0.525	0.530	0.710	0.775
	c_3	0.595	0.615	0.935	0.950
Gd-inf	None	0.630	0.710	0.950	0.950
	c_1	0.390	0.485	0.485	0.725
	c_2	0.310	0.510	0.510	0.750
	c_3	0.630	0.630	0.950	0.950

important in the decision model. The condition attribute c_3 could be omitted from the decision table, when $u = 0.8$ is assumed.

6. Conclusions

The presented variable precision fuzzy rough set (VPFRS) model with asymmetric bounds is a suitable tool for analyzing information systems with crisp or fuzzy attributes. We proposed to express the human operator's decision model in the form of decision table with fuzzy attributes. The fuzzy character of attributes corresponds with the human ability to inference, using linguistic concepts rather than numbers. Particular steps of the VPFRS approach were illustrated by simple examples. It was shown that relaxation of strong inclusion requirements of one fuzzy set in another fuzzy set (admitting of a certain misclassification level in the human operator's control) leads to an increase of the approximation quality of the decision system. In this paper, we discussed an extended and improved version of our VPFRS model. It is compatible with the fuzzy rough set concept of Dubois and Prade. However, there is more than one possibility to define fuzzy rough sets. This is not only due to different forms of fuzzy operators (intersection, union, implication) that can be used in definitions of fuzzy rough approximations. We proposed a unified form of the crisp lower and upper approximations that could also be applied to defining new models of fuzzy rough set. Thus, our aim of future research is to introduce and investigate new classes of fuzzy rough sets and VPFRS models, which go beyond the widely used concept of Dubois and Prade.

References

[1] W. Bandler, L. Kohout, Fuzzy power sets and fuzzy implication operators, Fuzzy Sets and Systems 4 (1980) 13–30.
[2] S. Bodjanova, Approximation of fuzzy concepts in decision making, Fuzzy Sets and Systems 85 (1997) 23–29.
[3] P. Burillo, N. Frago, R. Fuentes, Inclusion grade and fuzzy implication operators, Fuzzy Sets and Systems 114 (2000) 417–429.
[4] S.-M. Chen, M.-S. Yeh, P.-Y. Hsiao, A comparison of similarity measures of fuzzy values, Fuzzy Sets and Systems 72 (1995) 79–89.
[5] C. Cornelis, C. Van der Donck, E. Kerre, Sinha–Dougherty approach to the fuzzification of set inclusion revisited, Fuzzy Sets and Systems 134 (2003) 283–295.
[6] V.V. Cross, T.A. Sudkamp, Similarity and Compatibility in Fuzzy Set Theory, Physica-Verlag, Heidelberg, New York, 2002.
[7] D. Dubois, H. Prade, Putting rough sets and fuzzy sets together, in: Słowiński [29], pp. 203–232.
[8] J.M. Fernández Salido, S. Murakami, Rough set analysis of a general type of fuzzy data using transitive aggregations of fuzzy similarity relations, Fuzzy Sets and Systems 139 (2003) 635–660.
[9] S. Greco, B. Matarazzo, R. Słowiński, The use of rough sets and fuzzy sets in MCDM, in: T. Gal, T.J. Stewart, T. Hanne (Eds.), Advances in Multiple Criteria Decision Making, Kluwer Academic Publishers, Boston Dordrecht London, 1999, pp. 14.1–14.59.
[10] S. Greco, B. Matarazzo, R. Słowiński, Rough set processing of vague information using fuzzy similarity relations, in: C.S. Calude, G. Paun (Eds.), Finite Versus Infinite—Contributions to an Eternal Dilemma, Springer-Verlag, Berlin, Heidelberg, New York, 2000, pp. 149–173.
[11] S. Greco, B. Matarazzo, R. Słowiński, J. Stefanowski, Variable consistency model of dominance-based rough set approach, in: W. Ziarko, Y. Yao (Eds.), Rough Sets and Current Trends in Computing, Lecture Notes in Artificial Intelligence, vol. 2005, Springer-Verlag, Berlin, Heidelberg, New York, 2001, pp. 170–181.
[12] S. Greco, M. Inuiguchi, R. Słowiński, Rough sets and gradual decision rules, in: Wang et al. [31], pp. 156–164.
[13] S. Greco, B. Matarazzo, R. Słowiński, Rough membership and bayesian confirmation measures for parametrized rough sets, in: Ślęzak et al. [28], pp. 314–324.
[14] M. Inuiguchi, Generalizations of rough sets: From crisp to fuzzy cases, in: Tsumoto et al. [30], pp. 26–37.
[15] J.D. Katzberg, W. Ziarko, Variable precision extension of rough sets, Fundamenta Informaticae 27 (1996) 155–168.
[16] T.Y. Lin, Topological and fuzzy rough sets, in: Słowiński [29], pp. 287–304.
[17] T.Y. Lin, Coping with Imprecision Information—Fuzzy Logic, Downsizing Expo, Santa Clara Convention Center, 1993.
[18] A. Mieszkowicz-Rolka, L. Rolka, Variable precision rough sets in analysis of inconsistent decision tables, in: L. Rutkowski, J. Kacprzyk (Eds.), Advances in Soft Computing, Physica-Verlag, Heidelberg, 2003, pp. 304–309.
[19] A. Mieszkowicz-Rolka, L. Rolka, Fuzziness in information systems, in: A. Skowron, M.S. Szczuka (Eds.), Electronic Notes in Theoretical Computer Science, vol. 82, Elsevier, 2003. http://www.elsevier.nl/locate/entcs/volume82.html.
[20] A. Mieszkowicz-Rolka, L. Rolka, Variable precision fuzzy rough sets, in: J.F. Peters et al. (Eds.), Transactions on Rough Sets I, Lecture Notes in Computer Science (Journal Subline), vol. 3100, Springer-Verlag, Berlin, Heidelberg, New York, 2004, pp. 144–160.
[21] A. Mrózek, Rough sets in computer implementation of rule-based control of industrial processes, in: Słowiński [29], pp. 19–31.
[22] Z. Pawlak, Rough Sets: Theoretical Aspects of Reasoning about Data, Kluwer Academic Publishers, Boston, Dordrecht, London, 1991.

[23] Z. Pawlak, AI and intelligent industrial applications: The rough set perspective, Cybernetics and Systems: An International Journal 31 (2000) 227–252.
[24] J.F. Peters, A. Skowron, Z. Suraj, An application of rough sets methods in control design, Fundamenta Informaticae 43 (2000) 269–290.
[25] L. Polkowski, Toward rough set foundations. Mereological approach, in: Tsumoto et al. [30], pp. 8–25.
[26] A.M. Radzikowska, E.E. Kerre, A comparative study of fuzzy rough sets, Fuzzy Sets and Systems 126 (2002) 137–155.
[27] D. Ślęzak, W. Ziarko, Variable precision bayesian rough set model, in: Wang et al. [31], pp. 312–315.
[28] D. Ślęzak et al. (Eds.), Rough Sets and Current Trends in Computing, Lecture Notes in Artificial Intelligence, vol. 3641, Springer-Verlag, Berlin, Heidelberg, New York, 2005.
[29] R. Słowiński (Ed.), Intelligent Decision Support: Handbook of Applications and Advances of the Rough Sets Theory, Kluwer Academic Publishers, Boston, Dordrecht, London, 1992.
[30] S. Tsumoto et al. (Eds.), Rough Sets and Current Trends in Computing, Lecture Notes in Artificial Intelligence, vol. 3066, Springer-Verlag, Berlin, Heidelberg, New York, 2004.
[31] G. Wang, Q. Liu, Y. Yao, A. Skowron (Eds.), Rough Sets, Fuzzy Sets, Data Mining, and Granular Computing, Lecture Notes in Computer Science, vol. 2639, Springer-Verlag, Berlin, Heidelberg, New York, 2003.
[32] W.N. Liu, J. Yao, Y. Yao, Rough approximations under level fuzzy sets, in: Tsumoto et al. [30], pp. 78–83.
[33] W. Ziarko, Variable precision rough sets model, Journal of Computer and System Sciences 46 (1) (1993) 39–59.
[34] W. Ziarko, Probabilistic rough sets, in: Ślęzak et al. [28], pp. 283–293.