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THE RAMSEY NUMBER $N(3,3,3,3;2)$ *

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Abstract. A partition of the nonzero elements of the finite abelian group $\mathbf{Z}/7\mathbf{Z} \times \mathbf{Z}/7\mathbf{Z}$ into four sum-free sets shows that $N(3,3,3,3;2) > 49$. Based on a matrix technique for analyzing the structure of the two nonisomorphic 16-vertex edge-colorings nondegenerate with respect to $N(3,3,3;2)$, an involved argument proves that *no* 65-vertex coloring nondegenerate with respect to $N(3,3,3,3;2)$ exists. Thus $49 < N(3,3,3,3;2) \leq 65$.

0. Introduction

The Ramsey number $N(3,3,3,3;2)$ is the smallest integer such that any four-color edge-coloring of the complete graph on $N(3,3,3,3;2)$ vertices has at least one monochromatic triangle. A monochromatic triangle is a triangle whose three edges are colored with the same color. Greenwood and Gleason [2] prove that $41 < N(3,3,3,3;2) \leq 66$. Erdös [1] wrote that Szalai, a Hungarian sociologist, constructed a monochromatic triangle-free four-color edge-coloring of the complete graph on 65 vertices. The upper bound result presented in this paper proves the nonexistence of the Szalai construction. In this paper we improve both the lower and upper bounds on this Ramsey number.

Definition 0.1. Let G be a group with operation $+$. A set S properly contained in G is said to be *symmetric sum-free* if for all $x, y \in S$ (x, y not necessarily distinct) $x+y \notin S$ and $x^{-1}, y^{-1} \in S$.

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1. Lower bound

We shall achieve a new lower bound for $N(3,3,3,3;2)$ by partitioning the nonzero elements of the finite abelian group $\mathbf{Z}/7\mathbf{Z} \times \mathbf{Z}/7\mathbf{Z}$ into four symmetric sum-free sets S_1, S_2, S_3 and S_4. The vertices of K_{49} (the complete graph on 49 vertices) are labeled with the elements of $\mathbf{Z}/7\mathbf{Z} \times \mathbf{Z}/7\mathbf{Z}$. To each symmetric sum-free set of our partition we associate a distinct color. The edge joining vertices v_1 and v_2 is colored with the color associated with the symmetric sum-free set which contains $v_1 - v_2$. In order for this coloring to be well-defined we must have symmetric sets. The following theorem shows that such a coloring is monochromatic triangle-free.

Theorem 1.1. *Given a symmetric sum-free set coloring of the complete graph on $|G|$ vertices, where G is a group whose nonzero elements have been partitioned into k disjoint symmetric sum-free subsets $S_1, S_2, \ldots, S_k$. The vertices of this graph are labeled with the elements of G. The edge joining vertices a and b is colored with color i, where $a \neq b$ and $a-b$, $b-a \in S_i$. This coloring is monochromatic triangle-free.*

Proof. Suppose $\{a, b, c\}$ is a monochromatic triangle of color i. Then $a-b$, $b-c$, $a-c \in S_i$. But $(a-b) + (b-c) = a-c$ which contradicts the assumption that S_i is sum-free.

Theorem 1.2. $N(3,3,3,3;2) > 49$.

Proof. The following partition of the nonzero elements of $\mathbf{Z}/7\mathbf{Z} \times \mathbf{Z}/7\mathbf{Z}$ into four symmetric sum-free sets yields a four-color monochromatic triangle-free coloring of K_{49}. Herbert Taylor (Jet Propulsion Laboratory, Pasadena, California) showed the author the following partition:

$$S_1 = \{(0,1),(1,0),(1,2),(2,1),(2,6),(6,2),\\ (0,6),(6,0),(6,5),(5,6),(5,1),(1,5)\}$$

$$S_2 = \{(1,1),(1,6),(0,3),(3,0),(3,2),(2,4),\\ (6,6),(6,1),(0,4),(4,0),(4,5),(5,3)\}$$

$$S_3 = \{(2,0),(2,5),(3,1),(3,3),(3,5),(3,6),\\ (5,0),(5,2),(4,6),(4,4),(4,2),(4,1)\}$$

$$S_4 = \{(0,2),(1,3),(1,4),(2,2),(2,3),(3,4),\\ (0,5),(6,4),(6,3),(5,5),(5,4),(4,3)\}\ .$$

2. Upper bound

Only the major steps of the proof that $N(3,3,3,3;2) \leq 65$ are presented here. The complete proof is found in [5]; part of this proof is due to Folkman[1]. Our proof also uses the result of Kalbfleisch and Stanton [3] that there are exactly two non-isomorphic trichromatic colorings of K_{16} which are monochromatic triangle-free. We shall call these two colorings $R(3,3,3;2)$ colorings. Greenwood and Gleason [2] show that one of these colorings can be based on a partition of the nonzero elements of $\mathrm{GF}(2^4)$ into three symmetric sum-free sets. Whitehead [4] shows that the other coloring can be based on a partition of the nonzero elements of $\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$ into three symmetric sum-free sets.

We shall denote the *product* of two $n \times n$ matrices M and N as MN:

$$(MN)_{xy} = \sum_{k=1}^{n} (M)_{xk}(N)_{ky}\ .$$

Denote the Hadamard product of two $n \times n$ matrices M and N as $M \cdot N$:

$$(M \cdot N)_{xy} = (M)_{xy}(N)_{xy}\ .$$

Let the *absolute value* $|M|$ of a $n \times n$ matrix M be defined as follows:

$$|M| = \sum_{j=1}^{n} \sum_{k=1}^{n} |(M)_{jk}|\ .$$

Definition 2.1. Given an m-chromatic $n \times n$ incidence matrix M, where

[1] See Jon Folkman's posthumous notes entitled "Notes on the Ramsey number $N(3,3,3,3)$". The whole upper bound section was motivated by these notes.

$m \geq 1$, the M_i *chromatic adjacency matrix*, for $i = 1, 2, ..., m$, is defined as follows:

$$(M_i)_{xy} = \begin{cases} 1 & \text{if } (M)_{xy} = i\,, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.2. *Let M be an m-chromatic $n \times n$ incidence matrix and $i, j, k \in \{1, 2, ..., m\}$, where $i \neq j \neq k \neq i$. In the graph represented by M the following three statements hold:*

(1) *The number of (i, i, i)-colored monochromatic triangles is $\frac{1}{6}|(M_i^2) \cdot M_i|$.*

(2) *The number of (i, i, j)-colored bichromatic triangles (having 2 edges of color i and 1 edge of color j) is $\frac{1}{2}|(M_i^2) \cdot M_j|$.*

(3) *The number of (i, j, k)-colored trichromatic triangles is $|(M_iM_j) \cdot M_k|$.*

Proof. We prove only statement (2). Statements (1) and (3) follow by similar arguments. Suppose the (x, y) edge and the (z, y) edge of the bichromatic triangle $\{x, y, z\}$ are i-colored and the (x, y) edge is j-colored. This triangle contributes 2 to $|(M_i^2) \cdot M_j|$ as follows:

(a) a length 2 i-path and a length 1 j-path from x to y;

(b) a length 2 i-path and a length 1 j-path from y to x.

An IBM 360/Model 65 program analyzed the two $R(3, 3, 3; 2)$ colorings. The following two equations apply to both of these colorings:

(A) $(M_i^2) \cdot M_j = 2M_j$ for all $i, j = 1, 2, 3$ and $i \neq j$;

(B) $(M_iM_j) \cdot M_k = M_k$ for all $i, j, k = 1, 2, 3$ and $i \neq j \neq k \neq i$.

Theorem 2.3. $N(3, 3, 3, 3; 2) \leq 65$.

Proof. Now $|M_i| = 80$ for all $i = 1, 2, 3$ for both K_{16} colorings, therefore by (A) there are exactly 80 bichromatic triangles of each coloring of the each coloring of the six distinct colorings and by (B) there are exactly 80 trichromatic triangles.

Since $N(3, 3; 2) = 6$ (see [2]), then in either $R(3, 3, 3; 2)$ coloring, there are exactly 5 edges of each of the three colors incident upon each vertex. We shall call the $R(3, 3; 2)$ coloring of the subgraph on the 5 vertices which are connected by 5 edges of the same color to a common vertex v, a *located bichromatic pentagon.* We shall show that every $R(3,3;2)$ coloring in an $R(3,3,3;2)$ coloring, is located. The bichromatic pentagons located by each vertex of either $R(3,3,3;2)$ coloring are distinct.

In fact, a study of the two Kalbfleisch and Stanton [3] $R(3,3,3;2)$ incidence matrices yields that distinct located pentagons of the same coloring intersect in either zero or two vertices. Thus each bichromatic triangle is contained in at most one located bichromatic pentagon. A counting argument shows that the 16 located bichromatic pentagons of a given coloring contain all the bichromatic triangles of the two corresponding colorings. Therefore in either $R(3,3,3;2)$ coloring, every bichromatic triangle is contained in exactly one located bichromatic pentagon. Further study of the Kalbfleisch-Stanton $R(3,3,3;2)$ incidence matrices shows that each edge of color 1 is the intersection of exactly one pair of located bichromatic pentagons of coloring $\{1,2\}$ and the intersection of exactly one pair of located bichromatic pentagons of coloring $\{1,3\}$, similarly for edges of colors 2 and 3.

Every vertex v is contained in at least 5 located bichromatic pentagons of each coloring, since v is contained in a different bichromatic pentagon located by the five vertices of the bichromatic pentagon located by v. If any vertex was contained in more than 5 located bichromatic pentagons of a given coloring, then by counting there would not be enough bichromatic pentagons of the given coloring to give each vertex at least 5 pentagons. Using this result that each vertex is contained in exactly 5 located bichromatic pentagons of each coloring and the result that every bichromatic triangle is contained in exactly one located bichromatic pentagon, an argument yields that every $R(3,3;2)$ coloring in either $R(3,3,3;2)$ coloring is a located bichromatic pentagon.

Equation (A) implies that associated with each edge (p,q) of color i, there are exactly two vertices x_1 and x_2 such that edges (x_1,p), (x_1,q), (x_2,p), (x_2,q) are of color j. Equation (A) implies by symmetry $(M_iM_j)\cdot M_i = 2M_i$ which implies that associated with each edge (p,q) of color i, there are exactly two vertices y_1 and y_2 such that edges (y_1,p) and (y_2,p) are of color i and edges (y_1,q) and (y_2,q) are of color j. Equation (B) implies that associated with each edge (p,q) of color i, there is exactly one z such that edge (p,z) is of color j and edge (q,z) is of color k.

Now we sketch the proof given in Jon Folkman's posthumous notes We derive a contradiction by assuming there is a $R(3,3,3,3;2)$ coloring on 65 vertices. By analogy to M_i matrix we define the chromatic adjacency matrix A_i for our assumed $R(3,3,3,3;2)$ coloring of K_{65}:

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } (A)_{xy} = i, \\ 0 & \text{otherwise,} \end{cases}$$

where A is the four-color 65×65 incidence matrix. As a direct consequence of (1) equation (A), (2) the fact that an $R(3,3,3;2)$ coloring has no monochromatic triangles, and (3) the fact that each vertex of an $R(3,3,3;2)$ coloring has exactly 5 edges of each color incident upon it, we have:

$$(M_i^2)_{xy} = \begin{cases} 5 & \text{if } x = y, \\ 0 & \text{if } x \neq y \text{ and } (M_i)_{xy} = 1, \\ 2 & \text{if } x \neq y \text{ and } (M_i)_{xy} = 0. \end{cases}$$

Below, we show the analogous result for our assumed $R(3,3,3,3;2)$ coloring of K_{65}, namely:

$$\text{(C)} \qquad (A_i^2)_{xy} = \begin{cases} 16 & \text{if } x = y, \\ 0 & \text{if } x \neq y \text{ and } (A_i)_{xy} = 1, \\ 5 & \text{if } x \neq y \text{ and } (A_i)_{xy} = 0. \end{cases}$$

For a fixed vertex w we let P be the set of $\{1, 2\}$ bichromatic pentagons such that each vertex of all pentagons in P are joined to w by an edge of color 3. A vertex $p \neq w$, is said to *cover* $A \in P$ if the edge (p, w) and the edges (p, a) for all $a \in A$ are of color 4. Now let $P_n \subseteq P$ be those elements of P which are covered by exactly n distinct vertices, where n is a nonnegative integer. A complicated argument (involving six lemmas) shows that $n \leq 2$. Therefore $P = P_0 \cup P_1 \cup P_2$.

Since there are 16 bichromatic pentagons of each of the 3 possible colors in a $R(3,3,3;2)$ coloring, and there are 4 $R(3,3,3;2)$ colorings located by w in the assumed $R(3,3,3,3;2)$ coloring, thus $|P| = 4 \cdot 3 \cdot 16$. It can be observed that each vertex except w $(65-1 = 64)$ covers exactly 3 elements of P, so there are $3 \cdot 64$ coverings of the elements of P.

These coverings can be counted as follows:

$$0\cdot|P_0|+2\cdot|P_2|+|P-P_0-P_2| = 2\cdot|P_2|$$
$$= 3\cdot 64-|P_0|-|P_2| = 3\cdot 64.$$

Therefore $|P_2| = |P_0|$. By a series of similar counting arguments it can be shown that $|P_0| \geq 2\cdot|P_0|$. The only possibility is that $|P_0| = 0$. Hence $|P_2| = 0$ and $P = P_1$.

Using the result that $P = P_1$ and applying twice the fact that each vertex in a $R(3,3,3;2)$ coloring is contained in exactly 5 bichromatic pentagons of each coloring, we have:

$$(A_i^2)_{xy} = 5 \quad \text{if } x \neq y \text{ and } (A_i)_{xy} = 0.$$

Now assuming that we have an $R(3,3,3,3;2)$ coloring on K_{65}, means we have no monochromatic triangles, that is:

$$(A_i^2)_{xy} = 0 \quad \text{if } x \neq y \text{ and } (A_i)_{xy} = 1.$$

Finally, $N(3,3,3;2) = 17$ implies that in any $R(3,3,3,3;2)$ coloring on K_{65}, there are exactly 16 edges of each of the four colors incident upon each vertex:

$$(A_i^2)_{xy} = 16 \quad \text{if } x = y.$$

Now equation (C) yields that $A_i^2 + 5A - 11I = 5J$, where I is the 65×65 identity matrix and J is the 65×65 matrix of ones. Now, $(A_i^2)_{xy} = 16$ for $x = y$ implies that symmetric A_i has constant row sum and column sum equal to 16:

$$A_iJ = JA_i = 16J.$$

By simultaneously diagonalizing A_i and J and using standard eigenvalue arguments we find that A_i has one eigenvector equal to 16, m_1 eigenvectors equal to $\frac{1}{2}(-5+\sqrt{69})$, and m_2 eigenvectors equal to $\frac{1}{2}(-5-\sqrt{69})$, where $m_1+m_2 = 65-1 = 64$. Since a complete graph contains no loops, we have that trace $(A_i) = 0$ which implies that $m_1 - m_2 = 288 \times 69^{-\frac{1}{2}}$.

This is a contradiction, since the eigenvalue multiplicities m_1 and m_2 are integers.

Thus Folkman's Theorem that $N(3,3,3,3;2) \leq 65$, is proven. As an immediate corollary, it follows that $N(3,3,3,3,3;2) \leq 322$.

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