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Inversion of Stochastic Partial Differential Operators—The Linear Case

G. ADOMIAN

Center for Applied Mathematics, University of Georgia, Athens, Georgia 30602

AND

K. MALAKIAN

McDonnell Douglas Technical Services, Astronautics Division, Houston, Texas 77058

The operator-theoretic (or inverse) method for stochastic differential equations is generalized to stochastic partial differential operators. This paper treats the linear case.

INTRODUCTION

The iterative method of Adomian [1–3] for linear or nonlinear stochastic differential equations, recently extended by Adomian and Malakian [4] and Adomian and Sibul [5] is further generalized in this paper to the case of linear, deterministic, or stochastic, partial differential equations. We begin with the deterministic case.

LINEAR DETERMINISTIC PARTIAL DIFFERENTIAL EQUATIONS

Let L_x and L_t be linear partial differential operators (e.g., $\partial^2/\partial x^2$, $\partial/\partial t$, etc.) and consider an equation of the form

$$L_t u + L_x u = g(t, x); \tag{1}$$

g, the forcing term (or system input), is allowed to be *stochastic* but the operators are deterministic. $u = u(t, x)$. We require that the inverses L_t^{-1} and L_x^{-1} exist. Let us rewrite (1) in the form

$$L_{t,x} u = g. \tag{2}$$

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If the inverse $L_{t,x}^{-1}$ exists, the solution of (1) or (2) is

$$u = L_{t,x}^{-1} g. \tag{3}$$

Hence our objective is the determination of $L_{t,x}^{-1} = [L_t + L_x]^{-1}$. To do this, write (1) in the two forms:

$$\begin{aligned} L_t u &= g - L_x u, \\ L_x u &= g - L_t u. \end{aligned} \tag{4}$$

Assuming the individual inverses L_t^{-1} and L_x^{-1} are known or determinable, we have from (4) the integral equations

$$\begin{aligned} u &= L_t^{-1} g - L_t^{-1} L_x u, \\ u &= L_x^{-1} g - L_x^{-1} L_t u. \end{aligned} \tag{5}$$

If we take the specific operators $L_t = \partial/\partial t$ and $L_x = \partial^2/\partial x^2$ as a typical example, the first equation in (5) implies that $g \in C^2$ with respect to x. The second implies $g \in C^1$ with respect to t. Equations (2) and (5) yield the operator identities

$$\begin{aligned} L_{t,x}^{-1} &= L_t^{-1} - L_t^{-1} L_x L_{t,x}^{-1}, \\ L_{t,x}^{-1} &= L_x^{-1} - L_x^{-1} L_t L_{t,x}^{-1} \end{aligned} \tag{6}$$

We can add the two equations in (5), and divide by 2, to obtain a single equation for u, or, do the same with (6) to write the operator equation

$$L_{t,x}^{-1} = (1/2)[(L_t^{-1} + L_x^{-1}) - (L_t^{-1} L_x + L_x^{-1} L_t) L_{t,x}^{-1}]. \tag{7}$$

If we define $L^{-1} = (1/2)[L_t^{-1} + L_x^{-1}]$ and $L^{-1}R = (1/2)[L_t^{-1} L_x + L_x^{-1} L_t]$, we can write

$$L_{t,x}^{-1} g = u = L^{-1} g - L^{-1} R u \tag{8}$$

as in the *ordinary* linear stochastic differential equation of the form $\mathscr{L} y = x$ where $\mathscr{L} = L + R$, L being a deterministic operator and R is stochastic, whose solution is known [1–3] to be

$$u = u_0 + u_1 + u_2 + \cdots,$$

where each $u_i = -L^{-1} R u_{i-1}$ for $i \geqslant 1$ and $u_0 = L^{-1} g$, an approach which yields statistical separability without the need for closure approximations. This is viewed as a decomposition of u into $\sum_{i=0}^{\infty} \lambda^i u_i$ or, equivalently of the inverse operator, $L_{t,x}^{-1} = \sum_{i=0}^{\infty} \lambda^i H_i$ where λ is later set equal to unity. Since the problem has been reduced to one whose solution is known, we can write immediately

$$
\begin{aligned}
u_0 &= L^{-1}g = (1/2)[L_t^{-1} + L_x^{-1}]\,g,\\
u_1 &= -L^{-1}Ru_0 = -(1/2)[L_t^{-1}L_x + L_x^{-1}L_t]\,u_0,\\
u_2 &= -L^{-1}Ru_1 = -(1/2)[L_t^{-1}L_x + L_x^{-1}L_t]\,u_1,\\
&\text{etc.}
\end{aligned}
\tag{9}
$$

from which u is known.

A solution can also be obtained directly from Eq. (7). Parametrizing with λ, we write

$$L_{t,x}^{-1} = (1/2)[(L_t^{-1} + L_x^{-1}) - \lambda(L_t^{-1}L_x + L_x^{-1}L_t)\,L_{t,x}^{-1}]. \tag{10}$$

Substituting $L_{t,x}^{-1} = \sum_{i=0}^{\infty} \lambda^i H_i$,

$$\sum \lambda^i H_i = (1/2)\left[(L_t^{-1} + L_x^{-1}) - \lambda(L_t^{-1}L_x + L_x^{-1}L_t)\sum \lambda^i H_i\right].$$

Equating comparable powers of λ,

$$
\begin{aligned}
H_0 &= (1/2)(L_t^{-1} + L_x^{-1}),\\
H_1 &= -(1/2)(L_t^{-1}L_x + L_x^{-1}L_t)H_0\\
&= -(1/2)^2(L_t^{-1}L_x + L_x^{-1}L_t)(L_t^{-1} + L_x^{-1}),\\
&\vdots\\
H_n &= (-1)^n(1/2)^{n+1}(L_t^{-1}L_x + L_x^{-1}L_t)^n(L_t^{-1} + L_x^{-1}),\\
&\vdots
\end{aligned}
\tag{11}
$$

λ is simply a device for grouping terms and (9) and (11) are equivalent. We set $\lambda = 1$ and have the desired inverse $L_{t,x}^{-1} = H_0 + H_1 + \cdots$ or

$$L_{t,x}^{-1} = \sum_{n=0}^{\infty} (-1)^n(1/2)^{n+1}(L_t^{-1}L_x + L_x^{-1}L_t)^n(L_t^{-1} + L_x^{-1}) \tag{12}$$

and

$$u = \sum_{n=0}^{\infty} (-1)^n(1/2)^{n+1}(L_t^{-1}L_x + L_x^{-1}L_t)^n(L_t^{-1} + L_x^{-1})\,g \tag{13}$$

is the solution.

Convergence

We do not of course calculate an infinite sum but an approximation $\phi_n = \sum_{k=0}^{n-1} u_k$, i.e., for $n \geqslant 1$

$$\phi_n = \sum_{k=0}^{n-1} (-1)^k (1/2)^{k+1} [L_t^{-1} L_x + L_x^{-1} L_t]^k \cdot [L_t^{-1} + L_x^{-1}]\, g(t, x). \quad (14)$$

Now consider

$$L_{t,x}\phi_n = (L_t + L_x) \sum_{k=0}^{n-1} (-1)^k (1/2)^{k+1} [L_t^{-1} L_x + L_x^{-1} L_t]^k \cdot [L_t^{-1} + L_x^{-1}]\, g. \quad (15)$$

The first term is

$$\begin{aligned} L_{t,x}\phi_1 &= (L_t + L_x)(1/2)[L_t^{-1} + L_x^{-1}]\, g \\ &= g + (1/2)[L_t L_x^{-1} + L_x L_t^{-1}]\, g. \end{aligned}$$

The second term is

$$\begin{aligned} L_{t,x}\phi_2 &= L_{t,x}\phi_1 + L_{t,x} u_1 \\ &= g + (1/2)[L_t L_x^{-1} + L_x L_t^{-1}]\, g \\ &\quad - (L_t + L_x)(1/2)[L_t^{-1} L_x + L_x^{-1} L_t](1/2)[L_t^{-1} + L_x^{-1}]\, g \\ &= g + (1/2)[L_t L_x^{-1} + L_x L_t^{-1}]\, g - (1/2)[L_t L_x^{-1} + L_x L_t^{-1}]\, g \\ &\quad - (1/2)\, g - (1/2)^2 [L_t L_x^{-1} L_t L_x^{-1} + L_x L_t^{-1} L_x L_t^{-1}]\, g. \end{aligned}$$

We note the second and third terms vanish. The next calculation removes the fourth and fifth terms and adds $+(1/2)^2 g + (1/2)^3 [L_t L_x^{-1} L_t L_x^{-1} L_t L_x^{-1} + L_x L_t^{-1} L_x L_t^{-1} L_x L_t^{-1}]\, g$, etc.

$$L_{t,x}\phi_n = g + (-1)^{n-1}(1/2)^{n-1} g + (-1)^{n-1}(1/2)^n [(L_t L_x^{-1})^n - (L_x L_t^{-1})^n]\, g. \quad (16)$$

In the limit as $n \to \infty$, the left side is $L_{t,x} u$ if $\lim_{n\to\infty} \phi_n = u$. We assume $\| g \| < \infty$ a.s. and the operators L_t, L_x, and L_t^{-1} are bounded in norm. Then we can state:

Theorem 1. *ϕ_n converges to u, if and only if $\| L_t^{-1} L_x + L_x^{-1} L_t \| < 1$.*

Theorem 2. *$\lim_{n\to\infty} L_{t,x}\phi_n = L_{t,x} \lim_{n\to\infty} \phi_n = g$ and $\lim_{n\to\infty} \phi_n$ satisfies the equation $\mathscr{L} u = g$ as $n \to \infty$. The approximate solution satisfies the partial differential equation if and only if $\| (L_t L_x^{-1})^n - (L_x L_t^{-1})^n \| < 2^n$.*

Alternatively, the requirement with the ordinary *differential equation* [1] *is* $\|L^{-1}R\| < 1$. *With our earlier definitions*

$$\|(L_t^{-1}L_x + L_x^{-1}L_t)\| < 1 \tag{17}$$

where the choice of the norm depends on the specific statistical measure of interest.

Linear Stochastic Partial Differential Equations

$$\mathscr{L}_{t,x}u = g; \tag{18}$$

g may be stochastic. $\mathscr{L}_{t,x} = \mathscr{L}_t + \mathscr{L}_x$ where $\mathscr{L}_t = L_t + R_t$ and $\mathscr{L}_x = L_x + R_x$, i.e., $\mathscr{L}_t$ and $\mathscr{L}_x$ decompose into deterministic parts L_t and L_x and (zero-mean) random parts given by R_t and R_x. Assume $\mathscr{L}_{t,x}^{-1}$, L_t^{-1}, and L_x^{-1} exist. We have

$$\begin{aligned} L_t u &= g - R_t u - L_x u - R_x u, \\ L_x u &= g - R_x u - L_t u - R_t u, \end{aligned} \tag{19}$$

where the initial conditions, whether deterministic or random, are accounted for in taking the inverses L_t^{-1} and L_x^{-1} as shown in an example at the end of this paper. Equivalently,

$$\begin{aligned} u &= L_t^{-1}g - L_t^{-1}R_t u - L_t^{-1}L_x u - L_t^{-1}R_x u, \\ u &= L_x^{-1}g - L_x^{-1}R_x u - L_x^{-1}L_t u - L_x^{-1}R_t u. \end{aligned} \tag{20}$$

Since $u = \mathscr{L}_{t,x}^{-1} g$,

$$\begin{aligned} \mathscr{L}_{t,x}^{-1} g &= L_t^{-1}g - L_t^{-1}R_t\mathscr{L}_{t,x}^{-1} g - L_t^{-1}L_x\mathscr{L}_{t,x}^{-1} g - L_t^{-1}R_x\mathscr{L}_{t,x}^{-1} g, \\ \mathscr{L}_{t,x}^{-1} g &= L_x^{-1}g - L_x^{-1}R_x\mathscr{L}_{t,x}^{-1} g - L_x^{-1}L_t\mathscr{L}_{t,x}^{-1} g - L_x^{-1}R_t\mathscr{L}_{t,x}^{-1} g, \end{aligned} \tag{21}$$

yielding the operator equations

$$\begin{aligned} \mathscr{L}_{t,x}^{-1} &= L_t^{-1} - L_t^{-1}R_t\mathscr{L}_{t,x}^{-1} - L_t^{-1}L_x\mathscr{L}_{t,x}^{-1} - L_t^{-1}R_x\mathscr{L}_{t,x}^{-1}, \\ \mathscr{L}_{t,x}^{-1} &= L_x^{-1} - L_x^{-1}R_x\mathscr{L}_{t,x}^{-1} - L_x^{-1}L_t\mathscr{L}_{t,x}^{-1} - L_x^{-1}R_t\mathscr{L}_{t,x}^{-1}. \end{aligned} \tag{22}$$

Adding as before

$$\begin{aligned} \mathscr{L}_{t,x}^{-1} = (1/2)[(L_t^{-1} + L_x^{-1}) &- L_t^{-1}(R_t + L_x + R_x)\,\mathscr{L}_{t,x}^{-1} \\ &- L_x^{-1}(R_x + L_t + R_t)\,\mathscr{L}_{t,x}^{-1}]. \end{aligned} \tag{23}$$

Parametrizing and representing $\mathscr{L}_{t,x}^{-1}$ by $\sum \lambda^n H_n$ we have

$$\sum \lambda^n H_n = (1/2)\Big[(L_t^{-1} + L_x^{-1}) - \lambda L_t^{-1}(R_t + L_x + R_x)\sum \lambda^n H_n - \lambda L_x^{-1}(R_x + L_t + R_t)\sum \lambda^n H_n\Big]. \tag{24}$$

Then

$$H_0 = (1/2)(L_t^{-1} + L_x^{-1}),$$
$$H_1 = -(1/2)^2[L_t^{-1}(R_t + L_x + R_x) + L_x^{-1}(R_x + L_t + R_t)](L_t^{-1} + L_x^{-1}),$$
$$\vdots$$
$$H_n = (-1)^n(1/2)^{n+1}[L_t^{-1}(R_t + L_x + R_x) + L_x^{-1}(R_x + L_t + R_t)]^n(L_t^{-1} + L_x^{-1}).$$

The inverse operator is therefore given by

$$\mathscr{L}_{t,x}^{-1} = \sum_{n=0}^{\infty} (-1)^n(1/2)^{n+1}[L_t^{-1}(R_t + L_x + R_x) + L_x^{-1}(R_x + L_t + R_t)]^n(L_t^{-1} + L_x^{-1}). \tag{25}$$

Special Cases

Case 1: $R_t, R_x = 0$.

$$\mathscr{L}_{t,x}^{-1} = \sum_{n=0}^{\infty} (-1)^n(1/2)^{n+1}[L_t^{-1}L_x + L_x^{-1}L_t]^n(L_t^{-1} + L_x^{-1}) \tag{26}$$

which is the same as (12).

Case 2: $L_x, R_x = 0$.
The equation is $\mathscr{L}_t u = g$, which has been previously solved [1].

Conditions

The operators L_t, L_x, L_t^{-1}, and L_x^{-1} must all be bounded in norm; R_t and R_x must be bounded a.s. in norm. The necessary and sufficient condition is given by

$$\|(L_t^{-1}(R_x + L_t + R_t) + L_x^{-1}(R_x + L_t + R_t)\| < 1 \qquad \text{a.s.} \tag{27}$$

Alternate Approach: $\mathscr{L}_{t,x}u = g$

$$\mathscr{L}_{t,x} = L_{t,x} + R_{t,x}, \tag{28}$$

i.e., instead of decomposition of the stochastic partial differential operator into ordinary differential operators, we decompose it into a deterministic partial differential operator and a zero-mean random partial differential operator. If $\mathscr{L}_{t,x}^{-1}$ and $L_{t,x}^{-1}$ exist,

$$L_{t,x}u = g - R_{t,x}u,$$
$$u = L_{t,x}^{-1} g - L_{t,x}^{-1} R_{t,x} u;$$

hence

$$\mathscr{L}_{t,x}^{-1} = L_{t,x}^{-1} - L_{t,x}^{-1} R_{t,x} \mathscr{L}_{t,x}^{-1} \tag{29}$$

is the operator equation. Applying the previous procedure,

$$\mathscr{L}_{t,x}^{-1} = L_{t,x}^{-1} - \lambda L_{t,x}^{-1} R_{t,x} \mathscr{L}_{t,x}^{-1}. \tag{30}$$

Decompose $\mathscr{L}_{t,x}^{-1}$ into partial differential operators by $\sum \lambda^n H_n$;

$$\sum \lambda^n H_n = L_{t,x}^{-1} - \lambda L_{t,x}^{-1} R_{t,x} \sum \lambda^n H_n, \tag{31}$$

$$H_0 = L_{t,x}^{-1},$$
$$H_1 = -L_{t,x}^{-1} R_{t,x} L_{t,x}^{-1},$$
$$H_2 = L_{t,x}^{-1} R_{t,x} L_{t,x}^{-1} R_{t,x} L_{t,x}^{-1},$$
$$\vdots$$
$$\mathscr{L}_{t,x}^{-1} = \sum_{n=0}^{\infty} (-1)^n (L_{t,x}^{-1} R_{t,x})^n L_{t,x}^{-1}, \tag{32}$$

$$u = \sum_{n=0}^{\infty} (-1)^n (L_{t,x}^{-1} R_{t,x})^n L_{t,x}^{-1} g. \tag{33}$$

Statistical Measures for Stochastic Partial Differential Equations

Let ϕ_n be the approximate solution for u. Statistical measures such as the mean, correlation, etc., can be obtained in the same manner as in earlier work [1–6].

Example

Let us illustrate the procedure with a single problem:

$$\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = g(t, x) = xe^{tx} + t^2 e^{tx}.$$

Thus $\partial u/\partial t = L_t u$ and $\partial^2 u/\partial x^2 = L_x u$. Then

$$u_0 = (1/2)[L_t^{-1} + L_x^{-1}]\, g, \tag{34}$$

where $L_t^{-1} g$ and $L_t^{-1} x$ are evaluated using the initial conditions $u(x, 0) = u(0, t) = 1$ (we can approximate the exponential with the leading terms of the expansions) and substitute into (34) to yield the first term of the series for $u = u_0 + u_1 + \cdots$. The second term is given by

$$u_1 = -(1/2)[L_t^{-1} L_x + L_x^{-1} L_t]\, u_0,$$

where L_x and L_t are known and u_0, L_t^{-1}, and L_x^{-1} have been found above. Then

$$u_2 = -(1/2)[L_t^{-1} L_x + L_x^{-1} L_t]\, u_1$$

is determined and similarly for higher terms, i.e.,

$$u_i = -(1/2)[L_t^{-1} L_x + L_x^{-1} L_t]\, u_{i-1}, \qquad i \geqslant 2.$$

The solution is the series for e^{tx}.

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