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NORMED-CONVERGENCE THEORY FOR SUPERCRITICAL BRANCHING PROCESSES

E. SENETA

Australian National University, Canberra, A.C.T., Australia

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A proof is given of the basic normed-convergence theorem for the ordinary supercritical Bienaymé–Galton–Watson process with finite mean. Part of it is adapted to obtain an analogous result for inhomogeneous supercritical processes (i.e. branching processes in varying environment). This is used in part to give a detailed discussion on the normed-convergence behaviour of the ordinary process in the 'explosive' case (i.e with infinite mean); and rather pathological limit behaviour is found to obtain.

supercritical branching process	varying environment	iteration
inhomogeneous process	norming constants	concavity
explosive homogeneous process	degenerate limit laws	immigration

1. Introduction

We shall denote by $\{Z_n\}$, $n \geq 0$ $(Z_0 = 1)$ the ordinary simple branching process (i.e. Bienaymé–Galton–Watson process). The offspring probability generating function (p.g.f.) will be denoted by $F(s) = \sum_{j=0}^{\infty} f_j s^j$; we shall always suppose that $f_j \not\equiv 1$ for any j, and that $m = F'(1-)$ satisfies $1 < m \leq \infty$. The probability of extinction of $\{Z_n\}$ is denoted by q ($q > 0$ iff $F(0) > 0$). The theorem on which our interest centres is the following.

Theorem A. *Suppose $m < \infty$. Then there exists a sequence of positive constants $\{c_n; n \geq 1\}$, with $c_n \to 0$ as $n \to \infty$, such that the random variables $c_n Z_n$ converge almost surely to a proper non-degenerate random variable W for which* $\mathrm{P}[W = 0] = q$. *If s_0 is any fixed number in* $(0, -\log q)$, *then c_n can be taken as $h_n(s_0)$, where $h_n(s)$ is the inverse func-*

tion of $k_n(s) = -\log \mathrm{E}\{\exp[-sZ_n]\}$. The transform $K(s) = -\log \mathrm{E}\{e^{-sW}\}$ satisfies the Poincaré functional equation (with $k = k_1$)

$$K(ms) = k(K(s)), \qquad s \geqslant 0.$$

This result (actually a somewhat more complete theorem), with convergence in distribution, was proved by the author [7]; and the extension to almost sure convergence was made subsequently by Heyde [4].

The present author's proof of it, with weak convergence, rested heavily on a result on functional iteration of Kuczma; while Heyde noticed that $\{\exp[-h_n(s_0) Z_n]\}$ is a martingale, obviously with bounded mean, so that this approaches a random variable I with proper distribution almost surely; for which, however, it is possible that $\mathrm{P}[I = 0] > 0$. Thus $h_n(s_0) Z_n \to W$ almost surely, where, however, $p \equiv \mathrm{P}[W < \infty] = 1 - \mathrm{P}[I=0]$ may be less than unity; that this may not occur follows by the weak convergence result.

It has been remarked since its publication, to the present author several times that, for didactic purposes, in probabilistic exposition the dependence of the above argument on the iteration–theoretic result is inconvenient, and that a complete deduction of the strong convergence result (which, it should be noted, depends also on a deep (probabilistic) theorem) may be made by beginning with Heyde's insight, and developing a subsequent synthesis. One such proof finally occurs in print in a very recent text-book treatment [1]. In the sequel we give a proof, in two parts, which may be a little shorter overall, and is akin to that of [10] for the process with immigration.

However, this is only an adjunct to our main purpose. The method of the initial steps of the proof (labelled Part I) enables us to obtain some information on normed supercritical processes (to be defined) where the offspring distribution changes with time in deterministic manner ("varying environments"). The result generalizes [5, Theorem 4] in part. There is substantial information on the asymptotic behaviour of the unnormed distribution of the number of descendants in generation n, in the work of Church [2, 3] and Lindvall [6]. See also [9] for earlier references.

Further, this theory can be applied to obtain some information for the homogeneous case with $m = \infty$, not covered by Theorem A. It is already known [8, Theorem 4.4] that in this case there exists no sequence $\{c_n\}$ of positive constants such that $\{c_n Z_n\}$ converges in law to a proper non-degenerate random variable. However, we are able to show, under certain regularity conditions, that with the norming of the process as in

Theorem 1 the limit distribution is concentrated at zero and at infinity, the mass at the origin for a given process depending continuously on s_0 and taking values in $(q, 1)$ as s_0 is varied in $(0, -\log q)$.

2. Proof of Theorem A

Part 1. We begin by using Heyde's martingale argument. As $n \to \infty$,

$$-\log \mathrm{E}\{\exp[-sh_n(s_0)\, Z_n]\} \equiv k_n(h_n(s_0)\, s)$$

$$\to -\log \mathrm{E}\{\exp[-sW]\} \equiv K(s)$$

for $s > 0$, where the last expectation is taken over finite values of I (of total probability measure p), and $0 \leqslant K(s) < \infty$ for all $s > 0$ if $p > 0$ and $K(s) = \infty$ for all $s > 0$ if $p = 0$. But

$$s_0 = k_n(h_n(s_0)) \to K(1),$$

so that $p > 0$.

Part 2. Consider now for $s > 0$ the asymptotic behaviour as $n \to \infty$ of $k_{n+1}(h_n(s_0)s)$:

$$k(k_n(h_n(s_0)s)) \to k(K(s)).$$

On the other hand,

$$k_{n+1}(h_n(s_0)\, s) = k_n(k(h_n(s_0)\, s))$$

$$= k_n(\{k(h_n(s_0)s)/(h_n(s_0)\, s)\}\, h_n(s_0)s).$$

Now $h_n(s_0) \downarrow 0$ as $n \to \infty$, $k_n(s)$ is monotone increasing with s, and $k(s)/s \uparrow m$ as $s \downarrow 0$ since $k(s)$ is strictly concave on $s \geqslant 0$, so it follows that for $n \geqslant n_0 \equiv n_0(\epsilon, s)$ for arbitrary small $\epsilon > 0$, $k_n((m - \epsilon)\, h_n(s_0)\, s) \leqslant k_{n+1}(h_n(s_0)\, s) \leqslant k_n(m\, h_n(s_0)\, s)$. Letting $n \to \infty$,

$$K((m - \epsilon)\, s) \leqslant k(K(s)) \leqslant K(ms).$$

Using the continuity of K on $(0, \infty)$ and letting $\epsilon \to 0+$,

$$K(ms) = k(K(s)), \quad s > 0. \tag{2.1}$$

Letting $s \to 0+$, $s \to \infty$, respectively, using the monotonicity of $K(s)$ on $(0, \infty)$,

$$K(0+) = k(K(0+)), \qquad K(\infty) = k(K(\infty)),$$

where

$$0 \leqslant -\log p \equiv K(0+) < K(\infty) \leqslant \infty,$$

the strict inequality occurring since $K(s)$ is strictly increasing. But the only fix-points of k are 0 and $-\log q \leqslant \infty$, so $p = 1$, and

$$K(\infty) \equiv -\log \mathrm{P}[W = 0] = -\log q.$$

The possibility of degeneracy, $K(s) = cs$, is ruled out by the assumption of non-degeneracy of Z_1 and (2.1).

3. Varying environment

For convenience we shall still denote our process by $\{Z_n\}$ $(Z_0 = 1)$; and denote by $F_n(s)$ the p.g.f. of Z_n.

Let $f_{(1)}(s), f_{(2)}(s), ..., f_{(r)}(s)$ ... be the sequence of successive p.g.f.'s of offspring number. Let

$$k_{(r)}(s) = -\log f_{(r)}(\mathrm{e}^{-s}), \quad s \geqslant 0,$$

denote the corresponding c.g.f.'s, and put, similarly,

$$-\log F_n(\mathrm{e}^{-s}) = k_n(s)$$

$$= k_{(1)}(k_{(2)}(...(k_{(n)}(s))...)) \; (= -\log \mathrm{E}\{\mathrm{e}^{-sZ_n}\}).$$

Put $h_n(s)$ for the inverse function of $k_n(s)$ in a right neighbourhood of the origin, and $h_{(r)}(s)$ for the inverse function of $k_{(r)}(s)$; then

$$h_n(s) = h_{(n)}(h_{(n-1)}(...(h_{(1)}(s))...))$$

in a right neighbourhood of the origin, in fact $[0, r_n)$, where $r_n = k_n(\infty) = -\log F_n(0)$. Clearly $r_n \downarrow r$, $0 \leqslant r \leqslant \infty$. *We shall call the process supercritical if* $r > 0$, and deal only with this case.

Let $s_0 \in (0, r)$. Then

$$\exp[-h_n(s_0) Z_n] \text{ is a martingale with bounded mean, so}$$

$$h_n(s_0) Z_n \to W \text{ a.s., where, however, } p \equiv \mathrm{P}[W < \infty] \tag{3.1}$$

may be less than unity.

Now Section 2, Part 1 above *applies as it stands*, so $p > 0$; and we must expect $p \equiv p(s_0)$.

We may make some further elementary deductions. First, since $\{Z_n = 0\} \Rightarrow \{W = 0\}$ and

$$\mathbf{P}[Z_n = 0] = F_n(0) \uparrow q \equiv \exp[-r]\,(<1),$$

it follows that $\mathbf{P}[W = 0] \geq q$. Further, since $K(1) = s_0 > 0$, it follows that $\mathbf{P}[W = 0] < 1$ (for otherwise $K(s) = 0$ for all s). We may expect also that in general the value of $\mathbf{P}[W = 0]$ will depend on the choice of s_0.

We shall in fact show in the next section, that there exist supercritical processes such that

$$\mathbf{P}[W = 0] = \exp[-s_0], \qquad \mathbf{P}[W = \infty] = 1 - \exp[-s_0], \tag{3.2}$$

where, recall, $0 < s_0 < r$. Such processes may have $r = \infty$ (i.e. $q = 0$) or $r < \infty$ (i.e. $q > 0$). *Thus the concentration of probability at the origin for such a process may take on any value in the interval* $(q, 1)$, *depending on the value of* s_0 *used in the norming.*

A final simple consequence of the above general deliberations: if $p = 1$, since $\mathbf{P}[W = 0] < 1$, it follows that W must have positive probability on $(0, \infty)$.

4. The explosive homogeneous process

We return now to the ordinary (homogeneous) process, in the situation where $m = \infty$, and shall show that the limit random variable W, as defined by (3.1) of the previous section, has the distribution specified by (3.2), under certain additional regularity constraints on the offspring distribution.

Suppose we can prove that $\mathbf{P}[0 < W < \infty] = 0$; then we are done for

$$K(s) = \text{const.} = -\log \mathbf{P}[W = 0] = -\log p,$$

while we know that $K(1) = s_0$.

Assume then, to the contrary, that $\mathbf{P}[0 < W < \infty] > 0$. Then $K(s)$ is on $(0, \infty)$ a continuous strictly monotone increasing function (with $K(0+) = -\log p$), and as $n \to \infty$,

$$k_n(h_n(s_0)\, s) \to K(s), \quad s > 0.$$

On taking inverses,

$$h_n(s)/h_n(s_0) \to H(s) > 0, \qquad -\log p < s < -\log \mathbf{P}[W = 0], \tag{4.1}$$

where $H(s)$ in the specified region (with $0 < H(s) < \infty$) is the inverse function of $K(s)$, $s > 0$. On the other hand, also for

$$0 \leqslant s \leqslant -\log p, \ -\log \mathbf{P}[W=0] \leqslant s,$$

we have

$$h_n(s)/h_n(s_0) \to 0 \text{ or} \to \infty, \tag{4.2}$$

respectively. Thus

$$-\log p < s_0 < -\log \mathbf{P}[W=0].$$

(Obviously $H(s_0) = 1$, so $H(s) > 1$ for $s > s_0$, $H(s) < 1$ for $s < s_0$.) Now it was deduced in [8, §4.5] from a theorem of Szekeres that, provided

$$h'(s) = (1+\beta)\, a s^{\beta} + O\,(s^{\beta+\delta}), \quad s \to 0+ \tag{4.3}$$

for strictly positive β, a, δ, then

$$\chi(s) = \lim_{n\to\infty} (1+\beta)^{-n} \log(1/h_n(s)) \tag{4.4}$$

exists and is continuous and strictly decreasing for $0 < s < r$. On the other hand, from (4.1) we have

$$-\log h_n(s) + \log h_n(s_0) \to -\log H(s),$$

so using (4.4) we obtain for $s \neq s_0$, $-\log p < s < -\log \mathbf{P}[W=0]$, that as $n \to \infty$,

$$(\chi(s) - \chi(s_0))\,(1+\beta)^n \sim -\log H(s),$$

which is nonsensical. Hence, under condition (4.3), our proposition is proved.

To show that (4.3) obtains for examples where $q > 0$ and $q = 0$, respectively, we display

$$F(s) = 1 - b(1-s)^c, \qquad 0 < b \leqslant 1, 0 < c < 1, \tag{4.5}$$

which gives (4.3) with $\beta = c^{-1} - 1$, $a = b^{-1/c}$, $\delta = 1$. When $b = 1$, $q = 0$; otherwise $q > 0$.

5. Related topics

5.1. Other conditions for the explosive case

The reader acquainted with the author's previous work [11] on limit laws for the homogeneous explosive case may well enquire whether it is not possible to replace the condition (4.3) above by a condition of apparent considerable generality in a manner analogous to a similar situation in [11]. This approach is less successful here, and we only sketch it briefly. Focus attention on the function $f(x) = -1/\log k(e^{-1/x})$, $x \geqslant 0$, where $f(0) = 0$ by continuity, and suppose that $f(x)$ is convex or concave on $[0, \infty)$. This implies that $\lim\{-\log k(e^{-y})/y\}$ exists as $y \to \infty$, and, if we denote it by c, satisfies $0 \leqslant c \leqslant 1$ (any c.g.f. $k(s)$ with $k'(0+) = E\{Z_1\} = m < \infty$ satisfies this limiting relation with $c = 1$). Assume henceforth $0 < c < 1$, then $f'(0+) = c^{-1} (> 1)$. Notice also that $\lim f(x) = -1/\log k(1)$ as $x \to \infty$, so that $f(x)$ approaches a finite horizontal asymptote. Taking the last two statements together with the assumed convexity or concavity of f (together with its differentiability any number of times for $x > 0$), we see that $f(x)$ *cannot be convex, and if concave must be increasing to the asymptote, which must be positive.* Hence our assumptions amount to:

(a) $f(x)$ is concave on $[0, \infty)$;

(b) $c^{-1} = \lim_{x \to 0+} f(x)/x$ is finite and exceeds unity.

These assumptions imply that $k(1) < 1$, so that $q > e^{-1}$ (or equivalently $0 < -1/\log r < \infty$), which testifies to their restrictiveness, since the bound on q already excludes, for example, the case $b = 1$ in (4.5).

However, in (a) and (b) we now have conditions totally analogous to that of the ordinary (non-explosive) supercritical process (with c^{-1}, $f(s)$ and $-1/\log r$ playing the roles of m, $k(s)$, r, respectively). The purely analytical deliberations of [7] or [8] can then be applied, if we denote by $\tilde{f}$ the inverse function of f in an appropriate right neighbourhood of the origin to yield the result that for $x \in [0, -1/\log r)$ and x_0 fixed in $(0, -1/\log r)$, as $n \to \infty$,

$$\tilde{f}_n(x) / \tilde{f}_n(x_0) \tag{5.1}$$

approaches a finite limit $\tilde{F}(x)$ positive for $x > 0$, continuous and strictly monotone increasing on $[0, -1/\log r)$, where, note, $\tilde{f}_n(x_0) \to 0$.

However, it is readily checked that

$$\tilde{f}_n(x) = -1/\log h_n(c^{-1/x}), \quad n \geqslant 1,$$

(where a subscript n again denotes the n^{th} iterate), so we can deduce that

$$\chi(s) = \lim_{n\to\infty} \{\mathcal{T}_n(x_0) \log(1/h_n(s))\}, \quad 0 < s < r,$$

when $\chi(s)$ has the properties of the previous section.

Hence clearly we may replace (4.3) by assumptions (a) and (b) of this section to obtain (3.2).

Needless to say, condition (a) is not easy to check. However, we note that the compound Poisson p.g.f.

$$F(s) = \exp[\lambda\{g(s) - 1\}], \quad s \in [0, 1],$$

where $g(s)$ is itself a p.g.f. with $g'(1) = \infty$, certainly satisfies $q > e^{-1}$ if $0 < \lambda \leqslant 1$ (it satisfies (4.3) irrespective of the size of λ). If we put $\lambda = 1$, $g(s) = 1-(1-s)^c$, $0 < c < 1$, we find that (b) is satisfied with this c; and rather tedious manipulation reveals that f is concave, as required. The conditions (a) and (b) are thus not vacuous.

5.2. *Immigration*

If the process described in Section 3 is augmented at each generation by an independent immigration component, where the immigration distribution is also permitted to vary from generation to generation and we denote the resulting process by $\{X_n\}$, it follows precisely as for the homogeneous process in [10] with the notation of Section 3 that $h_n(s_0)\, X_n \to V$ a.s., where $p = \mathrm{P}[V < \infty]$ may be less than unity. It is shown in [10] that here it is possible that $p = 0$. We leave this topic with these few remarks.

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