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Balanced pairs in partial orders

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Abstract

An α-balanced pair in a partially ordered set $P = (X, <)$ is a pair (x, y) of elements of X such that the proportion of linear extensions of P with x below y lies between α and $1 - \alpha$. The 1/3–2/3 Conjecture states that, in every finite partial order P, not a chain, there is a $1/3$-balanced pair. This was first conjectured in a 1968 paper of Kislitsyn, and remains unsolved. We survey progress towards a resolution of the conjecture, and discuss some of the many related problems.

1. Introduction

Let us start with a very well-known problem.

There are n objects, in some underlying linear (total) order $\prec$, with the order completely unknown to you. Your task is to discover the linear order $\prec$, by asking questions of the form "Is it true that $x \prec y$?" (this is a *comparison* between x and y). You will receive the result of each comparison immediately, before you have to make the next comparison (i.e., we are considering sequential rather than parallel algorithms). How many comparisons will you need, in the worst case?

This is the problem of *comparison sorting*. The answer is that $(1 + o(1))n \log n$ comparisons are necessary and sufficient. Here and throughout the paper, logarithms are taken base 2. One version of the proof that this number of comparisons is necessary goes as follows: there are $n!$ possible linear orders, but if k comparisons are sufficient then there are only 2^k possible outcomes of the comparison process, therefore $2^k \geqslant n!$, or $k \geqslant \log n! = (1 + o(1))n \log n$. This is sometimes called the *information theoretic lower bound*. Merge sort (or binary insertion sort) produces a matching upper bound.

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[1] Part of this survey was written while the author was at the University of Memphis.

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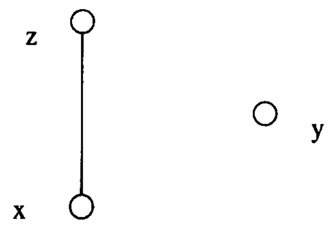

Fig. 1. The 3-element partial order T.

Now suppose that an unskilled person has kindly started the sorting process for you, so that you start with some information about $\prec$. This information will be in the form of a *partial order* $<$ on the set X of objects. How many comparisons do you need to determine $\prec$, starting from this partial information? The number of linear orders consistent with the information $<$ is simply the number $e(P)$ of *linear extensions* of the partially ordered set $P = (X, <)$.

There are two ways to think about linear extensions, and we shall freely use both. If $P = (X, <)$ is a partial order, a linear extension of P can be thought of as a linear order $\prec$ on X *extending* $<$, i.e., such that $x \prec y$ whenever $x < y$. (We shall adopt the convention that $\prec$ denotes a linear order, while $<$ is reserved for the partial order we are studying.) Alternatively, we can think of a linear extension as a bijection λ from X to $[n] \equiv \{1,\dots,n\}$, such that $\lambda x \prec \lambda y$ whenever $x < y$. Here, $n = |X|$, and $\prec$ denotes the standard linear order on $[n]$. If, as is sometimes convenient, we assume that $X = [n]$, then a linear extension is formally a permutation of $[n]$.

Let $C(P)$ denote the number of comparisons required to find the unknown linear extension $\prec$, in the worst case, starting from partial information given by the partial order P. The information theoretic argument above then gives the lower bound $C(P) \geqslant \log e(P)$. Is this at all a good bound? Let us see a couple of examples showing that it is not sharp, at any rate.

Example 1. Our first example is the three-element partial order $T = (X, <)$ on $X = \{x, y, z\}$, with $x < z$ and y incomparable to both. There are three linear extensions $\prec$ of T, namely $y \prec x \prec z$, $x \prec y \prec z$, and $x \prec z \prec y$. See Fig. 1.

In the worst case, we clearly need 2 comparisons to sort, starting from T. We can generalise this example by stacking k copies of T on top of one another, with each element of one copy below each element of the one above — this is a *linear sum* of the k copies of T. This new partial order T_k will have 3^k linear extensions, and require $2k$ comparisons in the worst case. Thus $C(T_k) = (2/\log 3)\log e(T_k) \simeq 1.2619 \log e(T_k)$.

Example 2. Our next example is initially infinite. We define a partial order L on the set $\{x_i : i \in \mathbb{Z}\}$ by setting $x_i < x_j$ if $i \preceq j - 2$, where $\prec$ denotes the usual linear order on $\mathbb{Z}$. The partial order L is shown in Fig. 2.

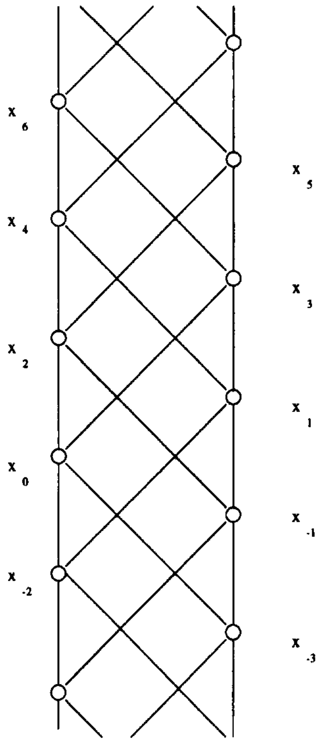

Fig. 2. The infinite ladder L.

For $n \geqslant 1$, let L_n denote the restriction of L to the set $\{x_i : 1 \leqslant i \leqslant n\}$. This example was first considered in this context by Linial [24]; as we shall see, it has a relatively large value of $C(P)$ compared with $e(P)$.

The number of linear extensions of L_n is the Fibonacci number F_n (with the convention $F_0 = F_1 = 1$) — to see this, note that the linear extensions of L_n break into two classes:

- those with x_n top, which are in 1–1 correspondence with linear extensions of L_{n-1},
- those with x_{n-1} top, and necessarily x_n second top, which are thus in 1–1 correspondence with linear extensions of L_{n-2},

and so we have $e(L_n) = e(L_{n-1}) + e(L_{n-2})$. If we are asked to sort, starting from L_n, in the worst case — for instance if the linear order we seek is in fact $x_1 \prec x_2 \prec \cdots \prec x_n$ — we might be required to compare all the $n-1$ incomparable pairs (x_i, x_{i+1}). Thus

$$C(L_n) = (1 + o(1))(\log(1+\sqrt{5}) - 1)^{-1} \log e(L_n) \simeq 1.4404 \log e(L_n)$$

as $n \longrightarrow \infty$.

These examples suggest that the right question is: is there a constant R such that $C(P) \leqslant R \log e(P)$ for all partially ordered sets P? And, if so, what is the least such constant — say R_0?

The answer to the first question is 'yes'. In 1976, Fredman [14] came close to resolving this by showing that $C(P) \leqslant 2n + \log e(P)$, and in 1984 Kahn and Saks [19] showed that $C(P) \leqslant \log e(P)/\log(11/8) \simeq 2.1766 \log e(P)$.

The second question is still unresolved. Currently, the best known bounds appear to be

$$(\log(1+\sqrt{5}) - 1)^{-1} \simeq 1.4404 \leqslant R_0 \leqslant 2.1226.$$

Here the example of L_n shows the lower bound, while the upper bound (only a small numerical improvement on the one given by Kahn and Saks) will be established as Theorem 4.2 below. A better upper bound is claimed in Brightwell, Felsner and Trotter [6], but the proof given there is incorrect. It seems probable that the lower bound above is the true value of R_0. We shall discuss these results in detail in Section 4.3, but our main concern is with another problem that arises naturally in connection with this one.

One approach to the sorting problem is to try to show that, for every partially ordered set $P = (X, <)$ (other than a linear order) there is at least one 'good comparison' to make, which will advance the sorting process whatever the outcome of the comparison. In this context, a good comparison will be one between a pair of elements (x, y) such that x is below y in about half of the linear extensions of P. The point is that, whatever the result of the comparison query, the number of linear extensions will be almost halved. Indeed, if we could always find a pair (x, y) such that $x \prec y$ in *exactly* half of the linear extensions $\prec$, then we would be able to find $\prec$ using just $\log e(P)$ comparisons. This is of course vastly over-optimistic, as our examples show. Indeed, in Example 1, every incomparable pair breaks the set of linear extensions into two parts, one of which is twice as large as the other. This leads to the following definitions and conjecture.

For a partial order $P = (X, <)$, and elements $x, y \in X$, we define $\mathbb{P}(x \prec y)$ to be the proportion of linear extensions of P in which x is below y. For $0 < \alpha \leqslant \frac{1}{2}$, an *$\alpha$-balanced pair* is a pair (x, y) of elements of X with

$$\alpha \leqslant \mathbb{P}r(x \prec y) \leqslant 1 - \alpha.$$

The 1/3–2/3 Conjecture. In every finite partial order that is not a chain (linear order), there is some $\frac{1}{3}$-balanced pair.

If true, the three-element partial order T of Example 1 (or indeed a stack T_k of k copies of T) would show that the result is best possible.

It is not hard to see that the conjecture would imply $C(P) \leqslant \log e(P)/\log \frac{3}{2} \simeq 1.7095 \log e(P)$. We shall give more details, and return to this issue, in Section 4.3.

The 1/3–2/3 Conjecture apparently originated in a 1968 paper of Kislitsyn [22], in a Russian journal. It was also formulated independently by Fredman in about 1975, and again by Linial [24] in 1984. On each occasion, the motivation for the problem was the connection with sorting discussed above, but it also stands by itself as a very natural and appealing problem in pure combinatorics.

The 1/3–2/3 Conjecture has so far defied all attempts to solve it, and remains one of the major open problems in the combinatorial theory of partial orders. Saks [26] wrote a short survey article on the conjecture in 1985, and there is also an account of the major results on the topic in Trotter's chapter [29] in the *Handbook of Combinatorics*. The purpose of the present survey is to discuss the various partial results that have been obtained in recent years, and to mention some of the many variations and related problems.

The conjecture is widely believed to be true. The evidence for this is not just that no-one has been able to find a counterexample, but also that all the apparently most promising areas to search for counterexamples have been found to contain none. We shall discuss this in detail in the next section.

Let us introduce some more notation. For a partial order $P=(X,<)$, let the *balance constant* $b(P)$ be the maximum, over all pairs $x,y\in X$, of $\min\{\mathbb{P}(x\prec y),\mathbb{P}(y\prec x)\}$. So $b(P)\geqslant\alpha$ if and only if P contains an α-balanced pair. For $\mathscr{A}$ a class of partial orders, let $b(\mathscr{A})$ be the infimum, over all partial orders $P\in\mathscr{A}$, of $b(P)$. Let $\mathscr{P}$ be the class of all partial orders that are not chains. Then the 1/3–2/3 Conjecture states that $b(\mathscr{P})$ is equal to 1/3.

We next explain, following Brightwell [3], how to extend the above definitions to a certain class of countably infinite partial orders. For a fixed natural number k, we say that a partial order $P=(X,<)$ (finite or infinite) is *k-thin* if every element of X is incomparable with at most k others. We say that P is *thin* if it is k-thin for some k. Suppose now that $P=(X,<)$ is an infinite partial order that is thin and also has connected incomparability graph. Then P is necessarily countable and *locally finite*, i.e., every interval $[x,y]\equiv\{z : x\leqslant z\leqslant y\}$ is finite. Also, we can find an increasing sequence $X_1,X_2,\ldots$ of subsets of X that are convex (i.e., if x and y are in some X_n, then so is the entire interval $[x,y]$) and whose union is X. We thus get a sequence of partial orders $P_1,P_2,\ldots$ obtained by restricting the order $<$ to each X_n in turn. Now, if x and y are incomparable elements of some X_n, then we can consider the sequence of probabilities $\mathbb{P}(x\prec y)$ in the partial orders $P_n,P_{n+1},\ldots$. Brightwell [3] shows that this sequence is convergent, and independent of the sequence (X_n) chosen, provided that P is thin. It is then natural to define the limit as $\mathbb{P}(x\prec y)$ in the infinite partial order P.

If the incomparability graph of P falls into several connected components, then P has the structure of the linear sum of partial orders on each of these components separately. So, if x and y are incomparable elements of any infinite thin partial order, we define $\mathbb{P}(x\prec y)$ to be the limit probability in the appropriate component of the linear sum. For any thin partial order P, we can then define $b(P)$, as before, as the supremum, over all incomparable pairs $x,y\in X$, of the smaller of $\mathbb{P}(x\prec y)$ and $\mathbb{P}(y\prec x)$.

For example, consider the infinite ladder L of Example 2. A natural choice is to define $X_n=\{x_i : -n\leqslant i\leqslant n\}$ for each n. Now consider $\mathbb{P}(x_0\prec x_1)$ in each of the P_n in turn. The number of linear extensions of P_n is F_{2n+1}, as before, while the total number with $x_0\prec x_1$ is $F_{n+1}F_n$. Thus in P we have

$$\mathbb{P}(x_0\prec x_1)=\lim_{n\to\infty}\frac{F_{n+1}F_n}{F_{2n+1}}=\frac{5+\sqrt{5}}{10}\simeq 0.7236.$$

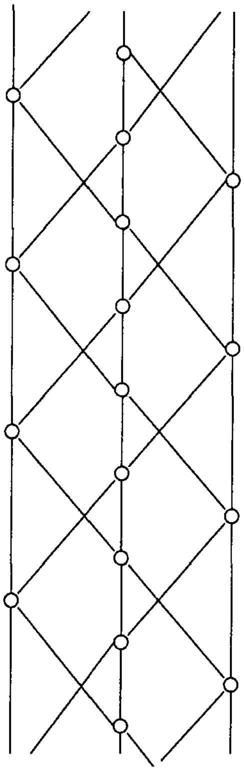

Fig. 3. The ladder M.

By symmetry (or by generalising the calculation), we have $\mathbb{P}(x_i \prec x_{i+1}) = (5 + \sqrt{5})/10$ for every i; so every incomparable pair is very unbalanced, and we have $b(L) = (5 - \sqrt{5})/10 \simeq 0.2764$. In other words, the 1/3–2/3 Conjecture fails for the infinite partial order L. This example is independently due to Brightwell [3] and Trotter.

As we shall see in Section 3, this is in fact the worst example: for any infinite thin partial order P, not a chain, there is a pair (x, y) of elements such that

$$\frac{5-\sqrt{5}}{10} \leqslant \mathbb{P}(x \prec y) \leqslant \frac{5+\sqrt{5}}{10}.$$

Thus we are in the perhaps curious position that the 1/3–2/3 Conjecture itself is still wide open, while its infinite analogue is solved.

We conclude this section with another example in the same vein as L, which we shall make reference to on a few occasions later.

Example 3. Consider the infinite ladder M shown in Fig. 3. This is defined on the set $\{x_i\colon i \in \mathbb{Z}\}$ by setting $x_i < x_j$ if $i \leqslant j-2$ and i is even, or if $i \leqslant j-3$ and

i is odd. In the figure, the even numbered elements are on the 'spine' of M. It is possible to show (as was done in the author's PhD thesis) that $b(M) = (17 + \sqrt{17})/68 \simeq 0.3106$, so M is also a counterexample to the infinite version of the 1/3–2/3 Conjecture.

Let M_n be the restriction of M to $\{x_i : 0 \leqslant i \leqslant n-1\}$. The number E_k of linear extensions of M_{2k+1} satisfies the recurrence $E_k = 3E_{k-1} + 2E_{k-2}$, and is given by

$$E_k = \frac{1}{\sqrt{17}}\left[\left(\frac{3+\sqrt{17}}{2}\right)^{k+1} - \left(\frac{3-\sqrt{17}}{2}\right)^{k+1}\right].$$

The finite pieces M_{4k+3} (especially) are very close to being counterexamples to the 1/3–2/3 Conjecture. The only $\frac{1}{3}$-balanced pairs in this partial order are (x_{4k}, x_{4k+1}), the second and third highest elements, and (x_1, x_2), the second and third lowest. As $k \to \infty$, $b(M_{4k+3}) = \mathbb{P}(x_2 \prec x_1)$ tends to $(7 - \sqrt{17})/8 \simeq 0.3596$.

In the next section, we look at some special types of partial order for which the 1/3–2/3 Conjecture has been proved. Then in Section 3 we look at the various weaker bounds that have been obtained on $b(\mathscr{P})$. Section 4 deals with variations on the problem, and finally we look at algorithmic aspects in Section 5. Terminology is for the most part standard: see for instance Trotter's book [28].

2. Special cases

The general 1/3–2/3 Conjecture has proved hard to resolve. In this section, we look at some special cases of classes of partial orders where the conjecture has been proved.

We start with some preliminaries, with the aim of providing some intuition, as well as setting up some notation and terminology.

Suppose that $(X, <)$ is a counterexample to the 1/3–2/3 Conjecture, so that, for each incomparable pair (x, y), either $\mathbb{P}(x \prec y) > 2/3$ or $\mathbb{P}(y \prec x) > 2/3$. Now define an auxiliary order $\prec_*$ on X by setting $x \prec_* y$ if $\mathbb{P}(x \prec y) > 2/3$. Note that $\prec_*$ is a linear order on X: if $x \prec_* y$, and $y \prec_* z$, then we have $\mathbb{P}(x \prec y \prec z) > 1/3$, so $\mathbb{P}(x \prec z) > 2/3$ and $x \prec_* z$. Without loss of generality, we can label X as $x_1, \ldots, x_n$, with $x_i \prec_* x_j$ whenever $i < j$.

Now it seems intuitively clear that, if we do indeed have a counterexample, then each x_i will be forced to appear close to position i in all linear extensions. So we would expect each element in a counterexample to be incomparable with few others. Roughly speaking, a counterexample ought to be 'tall and thin'.

What features of the partial order $P = (X, <)$ make x likely to be lower than an incomparable element y in a randomly chosen linear extension? For this to happen, there will need to be elements u 'pushing x below y', i.e., with $x < u$ but y incomparable with u, and/or elements v 'pushing y above x', i.e., with $y > v$ but x incomparable

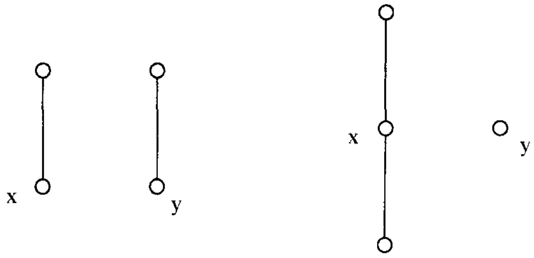

Fig. 4. Forbidden suborders for semiorders.

with v. If such elements exist, we can choose ones so that the relations in question are covering relations. Accordingly, we call an element z *good* for the pair (x, y) if either:

- z covers x and is incomparable with y, or
- z is covered by y and is incomparable with x.

So, in a counterexample to the 1/3–2/3 Conjecture, we expect to see, for each i, several elements of X that are good for (x_i, x_{i+1}), and few or none that are good for (x_{i+1}, x_i).

A *semiorder* is a partial order $(X, <)$ such that there is a linear order $\prec_0$ of X with the property that, if $x \prec_0 y$, then there is no z good for (y, x). Equivalently, a semiorder is a partial order containing no induced copy of either of the two partial orders shown in Fig. 4.

Thus, at first sight, semiorders are good candidates for counterexamples to the 1/3–2/3 Conjecture.

As support for our intuitive picture of what a counterexample 'should' look like, note that the infinite ladders L and M in the previous section are semiorders, have low width, and have all elements incomparable with at most four others.

By contrast, the 1/3–2/3 Conjecture has been proved for the following special cases: partial orders of width 2, partial orders in which each element is incomparable with at most 5 others, and semiorders. We shall look at these results in detail in Section 2.1 below; they seem to provide strong evidence that the 1/3–2/3 Conjecture is true.

At the other end of the spectrum, it proved surprisingly hard to rule out 'short and fat' partial orders as counterexamples to the 1/3–2/3 Conjecture. However, there is now a variety of results, which we shall look at in Section 2.2, ruling out partial orders with sufficiently many minimal (or maximal) elements. In particular, the conjecture has been proved for height 2 partial orders.

2.1. Ruling out 'likely' counterexamples

We begin with what turns out to be a useful lemma, justifying the intuition about the need, in a counterexample, for good elements for each incomparable pair.

Lemma 2.1. *Suppose that (x, y) is an incomparable pair of elements in a partial order $P = (X, <)$ with $\mathbb{P}(x \prec y) > 2/3$. Suppose further that, for every $z \in X \setminus \{x, y\}$,* *either* $\mathbb{P}(z \prec x) \geqslant 2/3$ *or* $\mathbb{P}(y \prec z) \geqslant 2/3$. *Then there are at least two elements of* X *that are good for* (x, y).

Proof. We partition the set $L(P)$ of linear extensions of P into three classes. The class L_1 consists of those linear extensions with y below x, and so $|L_1| < e(P)/3$. The class L_2 consists of those linear extensions where $x \prec z \prec y$, for some element z good for (x, y), and the class L_3 consists of the remaining linear extensions.

We claim that $|L_3| \leqslant |L_1|$. To see this, consider the map f from L_3 to L_1 defined as follows. Given a linear extension $\prec$ in L_3, we obtain $\prec' = f(\prec)$ by exchanging x and y in $\prec$. Note that $\prec'$ is a linear extension of P: if not, then there is some element z between x and y in $\prec$ with either $x < z$ or $z < y$, and we can choose z to either cover x or be covered by y. But then z is good for (x, y), so $\prec$ is in L_2. Now it is clear that f is an injection from L_3 to L_1, so we have $|L_3| \leqslant |L_1|$, as claimed.

Since $|L_3| \leqslant |L_1| < e(P)/3$, we have $|L_2| > e(P)/3$. By hypothesis, there is no single element z with $x \prec z \prec y$ in more than $e(P)/3$ linear extensions, so there are at least two elements good for (x, y), as desired. □

Given a partial order $P = (X, <)$ and a linear extension $\prec_0$, we say that P is *2-separated in the order* $\prec_0$ if, for every incomparable $\prec_0$-consecutive pair (x, y) of elements of X, there are two elements of X that are good for (x, y). Lemma 2.1 then implies that, if $P = (X, <)$ is a counterexample to the 1/3–2/3 Conjecture with auxiliary linear order $\prec_*$, then P is 2-separated in $\prec_*$.

Lemma 2.1 first appeared in [4]. That paper also contains the next two applications. The first is a proof of one of the first results in the area, Linial's 1984 result [24] that the 1/3–2/3 Conjecture holds for partial orders of width 2.

Theorem 2.2. *Let* $P = (X, <)$ *be a partial order of width exactly* 2. *Then* $b(P) \geqslant 1/3$.

Proof. We may assume that P has two minimal elements since, if there is just one minimal element z, then a balanced pair in $P - z$ will also be balanced in P. Label the two minimals x and y, so that $\mathbb{P}(x \prec y) \geqslant 1/2$.

Take a partition of X into two chains, and label the one containing x as $x = x_1 < x_2 < \dots$. Now take r maximal so that $\mathbb{P}(x_r \prec y) \geqslant 1/2$. We claim that one of the pairs (x_r, y) or (y, x_{r+1}) is balanced. Indeed, suppose not, so that $\mathbb{P}(x_r \prec y) > 2/3$ and $\mathbb{P}(y \prec x_{r+1}) > 2/3$. Then Lemma 2.1 can be applied to the pair (x_r, y), since all other elements z of X are either below x_r (so $\mathbb{P}(z \prec x_r) = 1$), or above either x_{r+1} or y (so $\mathbb{P}(y \prec z) > 2/3$). Thus there are two elements good for (x_r, y), but the only possible good element is x_{r+1}, a contradiction. □

This proof is in fact essentially equivalent to Linial's original (which is if anything even shorter than the one given here); certainly the idea of finding a $\frac{1}{3}$-balanced pair including the higher of the two minimal elements is common to the two proofs.

Aigner [1] showed that, if $P = (X, <)$ is a width 2 partial order, then either P is a linear sum of singletons and copies of the three-element partial order T, or there is a pair (x, y) with $\mathbb{P}(x \prec y)$ *strictly* between 1/3 and 2/3.

Theorem 2.3. *If $P = (X, <)$ is a semiorder, not a chain, then $b(P) \geqslant 1/3$.*

Proof. We suppose that P is a counterexample, and argue to a contradiction. Since P is a semiorder, there is a linear order $\prec_0$ on its ground-set X such that, whenever $x \prec_0 y$, every element below x is also below y, and every element above y is also above x. Label the elements of X so that $x_1 \prec_0 x_2 \prec_0 \dots \prec_0 x_n$.

If we have $x_i < x_{i+1}$ for any i, then P breaks up as the linear sum of $\{x_1, \dots, x_i\}$ and $\{x_{i+1}, \dots, x_n\}$, and we can treat each part separately. Therefore we may assume that each pair (x_i, x_{i+1}) is incomparable.

Note that, if $x \prec_0 y$, then the map exchanging the positions of x and y is an injection from the set of linear extensions with $y \prec x$ to the set with $x \prec y$, and so $\mathbb{P}(x \prec y) \geqslant 1/2$. Suppose now that there is no $\frac{1}{3}$-balanced pair (x, y) in P. Then we must have $\prec_* = \prec_0$, i.e., $\mathbb{P}(x \prec y) > 2/3$ whenever $x \prec_0 y$. Thus, by Lemma 2.1, P is 2-separated in the order $\prec_0$.

Consider the $n-1$ incomparable pairs (x_i, x_{i+1}). For each of these pairs, there are at least two elements of X good for the pair. So there are at least $2n-2$ instances of an element z good for a pair (x_i, x_{i+1}).

Now consider any element $z \in X$. The semiorder structure implies that there are indices i and j with $0 \leqslant i < j \leqslant n$, such that z is above all of $\{x_1, \dots, x_i\}$, incomparable with all of $\{x_{i+1}, \dots, x_j\} \setminus \{z\}$, and below all of $\{x_{j+1}, \dots, x_n\}$. Thus z is good for the pairs (x_i, x_{i+1}) and (x_j, x_{j+1}), if these are indeed pairs of elements of X, and no others. Moreover, we have $i = 0$, so that the pair (x_i, x_{i+1}) is *not* a pair of elements of X, exactly when z is minimal. Similarly, the pair (x_j, x_{j+1}) is not a pair of elements in X when $j = n$, which is when z is maximal.

Thus the number of instances of an element z good for a pair (x_i, x_{i+1}) is equal to $2n - |\mathrm{Min}(P)| - |\mathrm{Max}(P)|$. Since P has at least two minimals and two maximals, this number is at most $2n-4$, which contradicts our earlier conclusion that there are at least $2n-2$ instances. □

One might begin to ask whether there are any examples of partial orders that are 2-separated in any linear extension. In fact, there are very many: one way to 'construct' an example is to take two large antichains A and B, and put in each relation of the form $a < b$, with $a \in A$ and $b \in B$, with probability 1/2. It is easy to check that, with high probability, each pair of elements of A has two good elements of B, and vice versa. The explicit small example in Fig. 5 was found by Brightwell and Wright [8].

Brightwell and Wright [8] were able to combine the ideas above with other techniques to prove the following result. Recall that a partial order is *k-thin* if every element is incomparable with at most k others.

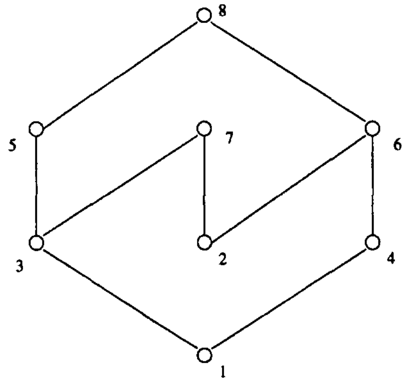

Fig. 5. A 2-separated partial order.

Theorem 2.4. *Let $P = (X, <)$ be a 5-thin partial order, not a chain. Then $b(P) \geqslant 1/3$.*

The proof of Theorem 2.4 involves constructing a list of 38372 'initial segments' $(X, <, \prec, F)$, consisting of a partial order $(X, <)$, a linear extension $\prec$ and a down-set F, with the following properties.

- If $P = (Y, <')$ is a 5-thin partial order, not a chain, which is 2-separated in an order $\prec'$, then there is some $(X, <, \prec, F)$ on the list such that: $(X, <)$ is (isomorphic to) a down-set of $(Y, <')$, $\prec$ is an initial segment of $\prec'$, and every element of F is below every element of $Y \setminus X$. We say that $(Y, <', \prec')$ is a *continuation* of $(X, <, \prec, F)$.
- For every $(X, <, \prec, F)$ and every continuation $(Y, <', \prec')$, there is a $\frac{1}{3}$-balanced pair of elements of F.

Loosely, every possible candidate partial order has some initial segment on the list, but every item on the list is ruled out as an initial segment of a counterexample.

As might be guessed, the proof of Theorem 2.4 was computer-assisted. Similar techniques and up-to-date computing power could possibly be used to extend this result to the 6-thin case, but there is reason to believe that the number of items on the list will undergo a severe combinatorial explosion, and the 7- or 8-thin case is unlikely to succumb to the methods of [8] without significant new ideas.

2.2. *Ruling out 'unlikely' counterexamples*

As we mentioned above, a 'random' height-2 partial order is unlikely to be 2-separated. However, it is also extremely unlikely to be a counterexample to the 1/3–2/3 Conjecture: one would expect every pair (x, y) of minimal elements to have $\mathbb{P}(x \prec y)$ close to 1/2.

In itself, it is easy enough to show that *almost every* height-2 partial order on a large number of elements does have $\mathbb{P}(x \prec y)$ close to 1/2 for *every* pair of minimals (and every pair of maximals) — for instance, a very strong form of this is immediate from results of Brightwell [5]. However, it is not so straightforward to show that, in *every* (large) height-2 order, there is *some* pair (x, y) with $\mathbb{P}(x \prec y)$ close to 1/2. Obviously we should expect some trade-off between the size of the partial order, and how close to 1/2 we can get.

Results along these lines are contained in papers of Komlós [23], Friedman [15] and Trotter, Gehrlein and Fishburn [30], all written in about 1990. All the proofs are informative, but unfortunately we do not have the space to do them full justice here.

We start by seeing the most that can be achieved if we are willing to assume that our partial order is very large: Komlós [23] proves the following result.

Theorem 2.5. *For every $\varepsilon > 0$, there is a function $M(n) = o(n)$ such that, if $P = (X, <)$ is an n-element partial order with at least $M(n)$ minimal elements, then $b(P) \geqslant \frac{1}{2} - \varepsilon$.*

In particular, for every ε there is an n_0 such that every height-2 partial order P with at least n_0 elements has a $(\frac{1}{2} - \varepsilon)$-balanced pair.

In [23], Komlós derives Theorem 2.5 from a general Ramsey-style result, of which an immediate consequence is as follows.

Theorem 2.6. *For any fixed $\varepsilon > 0$ and $k \in \mathbb{N}$, there is some number $N = N(k, \varepsilon)$ such that, in any collection $(Z_i)_{i=1}^N$ of N random variables taking values in $[k]$, some pair (Z_m, Z_n) of them satisfies*

$$|\mathrm{Prob}(Z_m < Z_n) - \mathrm{Prob}(Z_n < Z_m)| < \varepsilon.$$

Proof of Theorem 2.5. Given any positive ε and δ, choose k to be an integer at least $2/\varepsilon$ such that $t = (1-\varepsilon)^k$ is at most $\delta\varepsilon/5$. Now consider an n-element poset $P = (X, <)$ with $M \geqslant \max(4tn/\varepsilon, 2N(2k, \varepsilon/2))$ minimal elements, where $N(\cdot,\cdot)$ is the function in Theorem 2.6. We aim to prove that P contains a pair of minimal elements u and v with $\mathbb{P}(u \prec v)$ within ε of $\frac{1}{2}$. Note that, for sufficiently large n, we may take $M < \delta n$, so this will suffice to prove the result.

Consider a random order-preserving map f from $P = (X, <)$ to $[0, 1]$, defined by assigning to each element x of X a uniform $[0, 1]$ random variable $f(x)$, and conditioning on the function f being order-preserving. For $x, y \in X$, the probability that $f(x)$ is less than $f(y)$ is just $\mathbb{P}(x \prec y)$. Now, for a minimal element x, let $R(x)$ be the lowest value of $f(y)$ over all elements y above x in P. Conditioned on the value of $R(x)$, $f(x)$ is a uniform random variable on $[0, R(x)]$, independent of any other $f(y)$.

Since $\sum_{x\in X} \mathrm{Prob}(f(x) < t) = tn$, the number of elements x for which $\mathrm{Prob}(f(x) < t) \leqslant \varepsilon/2$ is at most $2tn/\varepsilon$. Thus, since $M \geqslant 4tn/\varepsilon$, there is a set S of at least $M/2$ minimal elements x with $\mathrm{Prob}(f(x) < t) < \varepsilon/2$, and thus certainly $\mathrm{Prob}(R(x) < t) < \varepsilon/2$.

Now we define a discretised version Z_x of the random variable $f(x)$ for each $x \in S$, with a view to applying Theorem 2.6 to the resulting family. To be more precise, we divide the range $[0, t)$ into k equal pieces $A_1, \ldots, A_k$, so that $A_i = [(i-1)t/k, it/k)$. Also, we divide the range $[t,1)$ into k unequal pieces $B_1,\dots,B_k$, with $B_i = [(1-\varepsilon)^{k-i+1}, (1-\varepsilon)^{k-i})$. These ranges are chosen so that, for any value of $R(x)$ greater than $t = (1-\varepsilon)^k$, the probability that $f(x)$ is in any of the intervals A_i or B_i, conditioned on the value of $R(x)$, is at most ε. Now, for $x \in S$, define the random variable Z_x by setting $Z_x = i$ if $f(x) \in A_i$, and $Z_x = k+i$ if $f(x) \in B_i$.

Since there are at least $M/2 \geqslant N(2k, \varepsilon/2)$ random variables Z_x, some pair of them, say Z_u and Z_v, satisfy

$$|\mathrm{Prob}(Z_u < Z_v) - \mathrm{Prob}(Z_v < Z_u)| < \varepsilon/2.$$

Also, $\mathrm{Prob}(Z_u = Z_v) \leqslant \mathrm{Prob}(R(u) < t) + \mathrm{Prob}(Z_u = Z_v \mid R(u) \geqslant t)$. The first term is at most $\varepsilon/2$ because $u \in S$, and the second is at most ε by the choice of the intervals A_i and B_i, and the independence of $f(u)$ and $f(v)$ given $R(u)$. Hence

$$\begin{aligned}&|\mathrm{Prob}(f(u) < f(v)) - \mathrm{Prob}(f(v) < f(u))| \\ &\quad \leqslant |\mathrm{Prob}(Z_u < Z_v) - \mathrm{Prob}(Z_v < Z_u)| + \mathrm{Prob}(Z_u = Z_v) \leqslant 2\varepsilon,\end{aligned}$$

which implies that

$$|\mathbb{P}(u \prec v) - \tfrac{1}{2}| < \varepsilon,$$

as required. □

For the next result, due to Friedman [15], many fewer minimal elements are required, but we obtain a somewhat weaker conclusion.

Theorem 2.7. *For any $\varepsilon > 0$ there is a C such that, if $P = (X, <)$ is an n-element partial order with at least $C\sqrt{n}$ minimal elements, then $b(P) \geqslant (1/e) - \varepsilon$.*

Friedman obtains the same conclusion if P has height at most $2\log\log n - C(\varepsilon)$. His techniques are geometric, and are based on an idea of Kahn and Linial [18] which we shall discuss in the next section. The required constants $C = C(\varepsilon)$ can be calculated explicitly, but still some moderately large lower bound on n will be necessary to apply these results.

Trotter, Gehrlein and Fishburn [30] removed this restriction in the case of height-2 orders, at the necessary cost of weakening the conclusion still further.

Theorem 2.8. *For every height-2 partial order $P = (X, <)$, not a chain, $b(P) \geqslant \frac{1}{3}$.*

The basic approach in [30] is to apply a method of Kahn and Saks, which we shall discuss in the next section, to deal with all but the small cases, which then require a substantial amount of special treatment.

3. Looser bounds

Recall that the 1/3–2/3 Conjecture states that every partial order, not a chain, has a $\frac{1}{3}$-balanced pair. If one cannot prove this, there is an obvious type of partial result to aim for, namely to prove that every partial order, not a chain, has an α-balanced pair, for some value of α with $0 < \alpha < \frac{1}{3}$, i.e., to show that $b(\mathscr{P}) \geqslant \alpha$.

For several years, it was not even known whether $b(\mathscr{P})$ was positive. This was resolved by Kahn and Saks [19] in 1984, who proved $b(\mathscr{P}) \geqslant \frac{3}{11} \simeq 0.2727$. Elegant geometric arguments were later given by Khachiyan [21] and Kahn and Linial [18], both proving somewhat weaker bounds. The current state of the art is that $b(\mathscr{P}) \geqslant (5-\sqrt{5})/10 \simeq 0.2764$ — this was proved by Brightwell, Felsner and Trotter [6] in 1995, using an extension of the methods of Kahn and Saks, along with some new techniques. The significance of this is less the modest numerical improvement over the Kahn-Saks bound, but more that, in view of Example 2 in the Introduction, this is the best possible constant in the wider class of infinite thin partial orders. Since the proofs in [19] and [6] go through in the infinite case, this represents a natural barrier.

All the proofs we shall discuss involve the concept of *average height* or *average height difference*. Given an n-element partial order $P = (X, <)$, and an element $x \in X$, the *average height* $h(x) = h_P(x)$ of x in P is the average, over all linear extensions $\lambda : X \longrightarrow [n]$, of $\lambda(x)$. Thus the average heights of elements of X are all rational numbers between 1 and n. Since there are n elements, there will then be some pair whose average heights are within 1 of each other. Indeed, unless P is a chain, there is a pair x, y with

$$0 \leqslant h(y) - h(x) < 1.$$

We define the *average height difference* $h(x, y) = h_P(x, y)$ to be $h(y) - h(x)$, which can also be viewed as the average of $\lambda(y) - \lambda(x)$, over linear extensions λ of P.

If $P = (X, <)$ is an infinite, thin, locally finite, partial order, and (X_i) is an increasing sequence of convex subsets containing elements x and y, whose union is X, then we can define $h_P(x, y)$ to be the limit of the average height difference $h_{P_i}(x, y)$ as $i \longrightarrow \infty$, where P_i is the restriction of $<$ to X_i. This definition will not depend on the sequence (X_i) chosen, and almost all of what follows can be translated to this setting.

It seems natural to ask whether a pair x, y with $|h(x, y)| \leqslant 1$ is always balanced. So far, this has been the approach that has proved successful in finding lower bounds for $b(\mathscr{P})$.

3.1. The geometric approach

Let us first look at a geometric approach to this issue. We follow Kahn and Linial [18]. For a partial order $P = (X, <)$ with $X = [n]$, define the *order polytope* $\mathcal{O}(P)$ to be

$$\{\mathbf{a} \in [0,1]^n : a_i \prec a_j \text{ whenever } i < j\}.$$

Here $\prec$ is the standard order on $[0,1]$. It is obvious that $\mathcal{O}(P)$ is a compact convex full-dimensional set — a *convex body.*

For a linear order σ of $[n]$, thought of as a permutation, define

$$\mathscr{A}_\sigma = \{\mathbf{a} \in [0,1]^n\colon\ a_{\sigma^{-1}(1)} \leqslant a_{\sigma^{-1}(2)} \leqslant \cdots \leqslant a_{\sigma^{-1}(n)}\}.$$

Up to the set of measure 0 where two co-ordinates are equal, $[0,1]^n$ is partitioned into the $n!$ sets $\mathscr{A}_\sigma$ of equal volume $1/n!$. The order polytope $\mathcal{O}(P)$ is just the union of those $\mathscr{A}_\sigma$ where σ is a linear extension of P.

An immediate consequence is that the volume of $\mathcal{O}(P)$ is just the number of linear extensions of P, divided by $n!$. Also, $\mathbb{P}(i \prec j)$ is the proportion of the volume of $\mathcal{O}(P)$ lying in the halfspace given by $a_i \leqslant a_j$. The centroid of $\mathcal{O}(P)$ is at the vector $[1/(n+1)](h(1), h(2), \ldots, h(n))$ given by the average heights of the elements. Finally, note that if i, j are incomparable in P, then there is a vector $\mathbf{a} \in \mathcal{O}(P)$ with $a_i = 1$ and $a_j = 0$.

All of this suggests the following question: given a convex body $\mathscr{B}$ in $\mathbb{R}^n$ whose centroid $\mathbf{h}$ has the property that $0 \leqslant h_2 - h_1 \leqslant 1/(n+1)$, and containing points $\mathbf{c}$ and $\mathbf{d}$ with $c_1 - c_2 = 1$ and $d_2 - d_1 = 1$, what is the minimum of the ratio $\mathrm{Vol}(\mathscr{B}^-)/\mathrm{Vol}(\mathscr{B})$, where $\mathscr{B}^- = \{\mathbf{a} \in \mathscr{B} : a_1 \geqslant a_2\}$?

In [18], Kahn and Linial show that an extremal convex body is formed by taking any $(n-1)$-dimensional convex body $\mathscr{A}$ in the hyperplane $a_2 - a_1 = 1/(n-1)$, any points $\mathbf{c}$ and $\mathbf{d}$ as above, and setting $\mathscr{B}$ equal to the union of the cones formed from $(\mathbf{c}, \mathscr{A})$ and $(\mathbf{d}, \mathscr{A})$. It is easy to check that the centroid of such a double cone $\mathscr{B}$ has the required property, while

$$\frac{\mathrm{Vol}(\mathscr{B}^-)}{\mathrm{Vol}(\mathscr{B})} = \frac{1}{2}\left(\frac{n-1}{n}\right)^{n-1} \geqslant \frac{1}{2e}.$$

Thus $b(\mathscr{P}) \geqslant 1/2\mathrm{e} \simeq 0.1839$.

The geometric aspects of Kahn and Linial's proof constitute a neat application of the Brunn–Minkowski Theorem. The way this is used is to consider the $(n-1)$-dimensional slices

$$\mathscr{S}_z \equiv \{\mathbf{a} \in \mathcal{O}(P) : a_2 - a_1 = z\}$$

for $-1 \leqslant z \leqslant 1$. Since the order polytope is convex, the Brunn–Minkowski Theorem implies that the function $f(z) = \mathrm{Vol}(\mathscr{S}_z)^{1/(n-1)}$ is concave. For the rest of the proof, see Kahn and Linial [18].

A similar approach was used slightly earlier by Khachiyan [21] to prove the weaker statement that $b(\mathscr{P}) \geqslant \mathrm{e}^{-2}$. The new idea introduced by Friedman [15] is to apply the same geometric method to a tailored variant of the order polytope, resulting in improved results under the conditions described in the previous section (see Theorem 2.7).

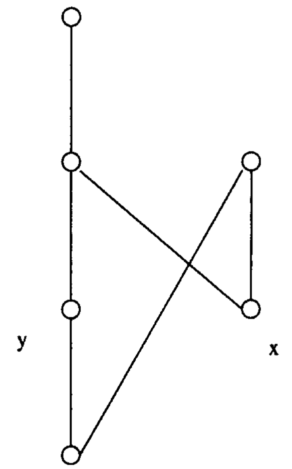

Fig. 6. An pair of elements with $h(y) - h(x) = 1$ and $\mathbb{P}(x \prec y) = \frac{8}{11}$.

3.2. *The combinatorial approach*

Let us turn now to the earlier, but more complicated, proof that $b(\mathscr{P}) \geqslant \frac{3}{11}$, given by Kahn and Saks [19]. Again, the idea is to start with a pair (x, y) of elements of $P = (X, <)$ such that $0 \leqslant h(y) - h(x) < 1$. Given a fixed such pair (x, y), we define two sequences of numbers (a_i) and (b_i) as follows. For $i \geqslant 1$, let a_i be the proportion of linear extensions λ of P where $\lambda(y) - \lambda(x) = i$, and b_i the proportion where $\lambda(x) - \lambda(y) = i$.

The condition on the average heights is then equivalent to $0 \leqslant \sum_{i \geqslant 1} ia_i - \sum_{i \geqslant 1} ib_i < 1$, while $\mathbb{P}(x \prec y)$ is simply the sum of the a_i's. Of course, the pair of sequences $((a_i), (b_i))$ satisfies certain conditions. For example, it is clear that $\sum_{i \geqslant 1} a_i + \sum_{i \geqslant 1} b_i = 1$, and that $a_1 = b_1$. The general approach of Kahn and Saks is to find a collection of conditions on the sequences which together imply that $\sum_{i \geqslant 1} b_i > \frac{3}{11}$. The number $\frac{3}{11}$ arises because of the sequences given by: $b_1 = a_1 = \frac{3}{11}$, $a_2 = \frac{4}{11}$, $a_3 = \frac{1}{11}$, and all other a_i and b_i equal to 0. This pair of sequences satisfies the average height constraint, and indeed does correspond to a pair of elements (x, y) with height difference equal to 1 in a partial order, as shown in Fig. 6. This example is due to Trotter — see for instance [29].

One way to view this line of attack is as a combinatorial version of the Kahn–Linial approach, where the a_i and b_i play roles similar to those of the volumes of the slices $\mathscr{S}_z$. What we have not yet translated is the convexity of the order polytope, or (essentially equivalently) the concavity of the function $f(z)$. Intuitively, this ought to be reflected in various convexity conditions on the sequences (a_i) and (b_i): most of the remaining conditions can be seen in that light.

It is not too hard to see that, at the extremes, if $a_i = 0$, then so are all of $a_{i+1}, a_{i+2}, \ldots$, and similarly for the b_i's. At the centre, one can show, by exhibiting an explicit injection, that $a_2 + b_2 \leqslant a_1 + b_1$. It is substantially harder to show that $a_i^2 \geqslant a_{i-1}a_{i+1}$, and similarly for the b_i's, for each i — Kahn and Saks prove this using the Alexandrov–Fenchel inequalities for mixed volumes.

There is one final constraint to be added, namely that $a_i \leqslant a_{i+1} + a_{i-1}$, and similarly for the b_i's, for each i. This requires a little work to prove: a simpler proof than the original appears in Felsner and Trotter [12]. Again, the proof involves finding an explicit injection from the set of linear extensions counted by a_i to that counted by $a_{i+1} + a_{i-1}$.

Subject to all the various conditions collected above, Kahn and Saks then show that $\sum_{i\geqslant 1} b_i > \frac{3}{11}$, as they require. Let us go into slightly more detail. For a fixed value B with $0 < B \leqslant \frac{1}{3}$, we define the *fully packed* pair of sequences $((a_i),(b_i))$ with parameter B by setting $a_1 = b_1 = B$, $b_2 = b_3 = \ldots = 0$, and then making each successive a_i as large as possible subject to the constraints. This means that, for some j, $a_i = 2^{i-1}B$ for $i \leqslant j$, and either

- $a_{j+1} \leqslant a_j$,
- $a_{j+1} \geqslant a_j$, and $a_{j+2} = a_{j+1} - a_j$,

with all subsequent a_i being 0, and the sum of all a_i and b_i being 1. Set $H(B)$ equal to the 'average height difference' $\sum_{i\geqslant 1} ia_i - b_1$ of the fully packed sequences with parameter B. As B decreases from $\frac{1}{3}$ towards 0, $H(B)$ increases continuously from $\frac{2}{3}$ to infinity.

The following result is essentially from Kahn and Saks [19], and is stated more explicitly in Brightwell, Felsner and Trotter [6].

Lemma 3.1. *For any $h \geqslant 2/3$, let $((a_i),(b_i))$ be any pair of sequences satisfying:*

$$\begin{array}{ll}
a_1 = b_1, & \\
a_i = 0 \Rightarrow a_{i+1} = 0 \ (i \geqslant 1), \ b_i = 0 \Rightarrow b_{i+1} = 0 \ (i \geqslant 1), & \\
\sum_{i\geqslant 1} a_i + \sum_{i\geqslant 1} b_i = 1, & \\
a_2 + b_2 \leqslant a_1 + b_1, & \\
a_{i+1} \leqslant a_i + a_{i+2} \ (i \geqslant 1), & b_{i+1} \leqslant b_i + b_{i+2} \ (i \geqslant 1), \\
a_{i+1}^2 \geqslant a_i a_{i+2} \ (i \geqslant 1), & b_{i+1}^2 \geqslant b_i b_{i+2} \ (i \geqslant 1), \\
\sum_{i\geqslant 1} ia_i - \sum_{i\geqslant 1} ib_i \geqslant h. &
\end{array}$$

Then $\sum_{i\geqslant 1} b_i \geqslant B$, where B is the unique value such that $H(B) = h$.

In particular, for $h = 1$, the fully packed sequence is that given earlier with $B = \frac{3}{11}$ so, combining Lemma 3.1 with results stating that all the given inequalities do hold for the pair $((a_i),(b_i))$ of sequences associated with a pair at average height distance h, we obtain that $b(\mathscr{P}) \geqslant \frac{3}{11}$. Moreover, we have the following.

Theorem 3.2. *Let $P = (X, <)$ be a finite partial order, and let (x, y) be a pair of incomparable elements with $h(y) - h(x) \leqslant 1$. Then $\mathbb{P}(x \prec y) \geqslant \frac{3}{11}$. If $h(y) - h(x) < 1$, then $\mathbb{P}(x \prec y) > \frac{3}{11}$.*

As we have seen, Theorem 3.2 is best possible. To obtain their improved lower bound on $b(\mathscr{P})$, Brightwell, Felsner and Trotter [6] considered *three* elements x, y, z, with $h(x) \leqslant h(y) \leqslant h(z) \leqslant h(x) + 2$. They broke the analysis up into several cases, depending on the relations (if any) among x, y and z.

Theorem 3.3. *Let $P = (X, <)$ be a finite partial order with three elements x, y, z, not forming a chain $x < y < z$, satisfying $h(x) \leqslant h(y) \leqslant h(z) \leqslant h(x)+2$. Then one of the following three situations arises*:
- *one of $\mathbb{P}(y \prec x)$ and $\mathbb{P}(z \prec y)$ is at least $\frac{1}{3}$*,
- *$\{x, y, z\}$ forms an antichain, and one of $\mathbb{P}(y \prec x)$ and $\mathbb{P}(z \prec y)$ is at least* 0.2786,
- *$x < z$, y is incomparable with both x and z, and $\mathbb{P}(y \prec x)+\mathbb{P}(z \prec y) \geqslant (5-\sqrt{5})/5 \simeq$ 0.5528, so one of $\mathbb{P}(y \prec x)$ and $\mathbb{P}(z \prec y)$ is at least $(5-\sqrt{5})/10 \simeq 0.2764$.*

The approach used to prove Theorem 3.3 is heavily based on that used by Kahn and Saks, but there are several other techniques used as well, notably the use of the following non-linear inequality, which is proved using the Ahlswede–Daykin Four Functions Inequality.

Theorem 3.4. *For x, y, z any elements of a finite partial order P, and $i, j \geqslant 1$, let $L(i, j)$ denote the number of linear extensions λ in which $\lambda(y)-\lambda(x) = i$ and $\lambda(z)-\lambda(y) = j$. Then*

$$L(1,1)L(2,2) \leqslant L(1,2)L(2,1).$$

For further details, the interested reader is referred to [6].

For the second case in Theorem 3.3, where $\{x, y, z\}$ forms an antichain, the assertion in Theorem 3.3 above is slightly stronger than that given in [6]; the form presented above requires replacing Lemma 6.3 of [6] with the statement that, if B and B' are both at most 0.2786, and $B(1-\varepsilon) + B'(1-\varepsilon') \geqslant \frac{1}{11}$, then $H(B,\varepsilon) + H(B',\varepsilon') \geqslant 2$: to verify this is routine but tedious, given the formulae for $H(B,\varepsilon)$ in various ranges (the function $H(B,\varepsilon)$ is a generalised version of the function $H(B)$ we introduced earlier).

We will make use of this more explicit result later, when we correct an error occurring in a later section of Brightwell, Felsner and Trotter [6]. There is much room (and, as we shall see later, motivation) for improvement in this case; the constant 0.2786 given here is that which can be obtained without essentially changing the proof from [6], which was not designed to give an especially good constant. The three-element antichain $\{x_{-1}, x_0, x_1\}$ in the infinite ladder M (Example 3) has $h(x_{-1}, x_0) = h(x_0, x_1) = 1$, and $\mathbb{P}(x_0 \prec x_{-1}) = \mathbb{P}(x_1 \prec x_0) = (17+\sqrt{17})/68 \simeq 0.3106$, so it will not be possible to improve the constant 0.2786 of Theorem 3.3 beyond this value.

For our present purposes, it is the third case in Theorem 3.3 that is the crucial one. Indeed, it is easy to show that any partial order not containing three elements x, y, z with $h(x) \leqslant h(y) \leqslant h(z) \leqslant h(x) + 2$ is either a chain or a two-element antichain, so Theorem 3.5 implies the following.

Theorem 3.5. $b(\mathscr{P}) \geqslant (5-\sqrt{5})/10 \simeq 0.2764$.

The proofs of Theorem 3.3 and Theorem 3.5 carry over to the case where P is infinite and thin, replacing average height with average height difference in the obvious way. In that case, the infinite ladder L shows the results to be best possible so that, as we mentioned earlier, the infinite version of the 1/3–2/3 Conjecture is now solved.

Theorem 3.6. *If $\mathscr{Q}$ is the class of infinite thin partial orders, then*

$$b(\mathscr{Q}) = (5-\sqrt{5})/10 \simeq 0.2764.$$

4. Variations on the theme

4.1. Average height difference

We have seen that two elements x, y of a partial order that are close in average height also make up a balanced pair. This begs the question of whether, in every finite partial order not a chain, there is a pair of elements with average heights within h of each other, for some fixed $h<1$. In view of the proof of Theorem 3.5, it would be even more interesting to show that there is a triple of elements x, y, z with $h(x) \leqslant h(y) \leqslant h(z) \leqslant 2-\varepsilon$, for some fixed $\varepsilon > 0$. Any such result would automatically give a better bound for $b(\mathscr{P})$.

For a partial order P, define $\gamma(P)$ to be the minimum, over all pairs (x, y) of distinct elements of P, of $|h(x,y)|$. We are interested in the supremum of $\gamma(P)$ over all finite partial orders P.

Saks [26] gave an example of a family of partial orders P_i with

$$\gamma(P_i) = \frac{1}{2}\prod_{j=2}^{i+1}\frac{2^j}{2^j-1} \to \gamma^* \simeq 0.8657.$$

These partial orders are constructed as follows. The initial partial order P_1 is the three-element partial order T of Example 1. Each subsequent P_i is made up of a linear sum of two copies of P_{i-1}, together with a single isolated element. Thus $|P_i| = 2^{i+1}-1$. Suppose that the average height of an element x in P_i is h. Then, in P_{i+1}, the average height of the corresponding element x' in the lower of the two copies of P_i is $h + h/|P_{i+1}|$, since the second term is the probability that the isolated element z comes below x'. Thus, all gaps in P_{i+1} between elements other than z are of size $\gamma(P_i)(1+1/|P_{i+1}|)$ or larger. But the gaps involving z are even larger, so we have $\gamma(P_{i+1}) = \gamma(P_i)2^{i+2}/(2^{i+2}-1)$, as claimed.

I would venture to suggest that $\gamma(P) \leqslant \gamma^*$ for all finite partial orders P, i.e., that the P_i are asymptotically optimal for this problem. In any case, it seems to me to be a very worthwhile problem just to bound $\gamma(P)$ away from 1 for all finite P; this doesnot seem to have received much attention. The infinite ladders L and M have $\gamma(L) = \gamma(M) = 1$,

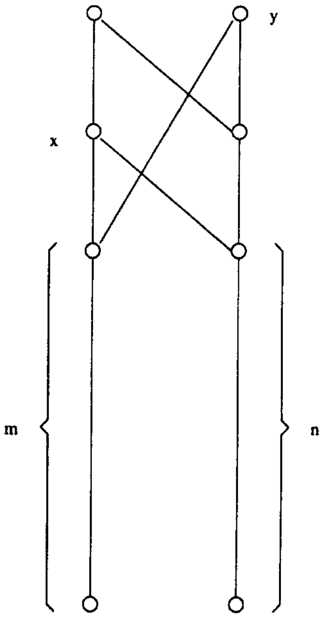

Fig. 7. A partial order realising a fully packed sequence.

so an attack on this problem is likely to have to concentrate on the 'top' elements of the partial order.

4.2. *Probability vs. average height difference*

Another question suggested during the previous section is that of how close the relationship between $\mathbb{P}(x \prec y)$ and $h(x, y)$ is. In particular, for each fixed value of h, let $G(h)$ be the supremum of the set $\{\mathbb{P}(y \prec x) : x, y$ incomparable elements of some partial order with $h(x, y) \geqslant h\}$: what is $G(h)$?

Lemma 3.1 tells us that $G(h) \geqslant H^{-1}(h)$, for $h \geqslant \frac{2}{3}$, and we have already seen (Fig. 6) that this gives the correct value for $h = 1$, namely $G(1) = \frac{3}{11}$.

In the range $\frac{2}{3} \leqslant h \leqslant \frac{7}{5}$, Lemma 3.1 gives $G(h) \geqslant H^{-1}(h) = (5 - 2h)/11$, and in fact we have equality here. To see this, it is enough to show that, for any pair (s, t) of rationals with $0 \leqslant t \leqslant s$, there is a pair (x, y) in a partial order realising the fully packed sequence $b_1 = a_1 = s/(3s+2t)$, $a_2 = (s+t)/(3s+2t)$, $a_3 = t/(3s+2t)$, with all other entries 0. (Notation is as in Section 3.2.) A partial order achieving this is shown in Fig. 7, with m and n non-negative integers such that $s/t = (m+n+1)/(n+1)$.

On the other hand, I suspect that fully packed sequences are not realisable for typical values of h larger than $\frac{7}{5}$.

Lemma 3.1 is not valid as it stands for $h < \frac{2}{3}$, but the results and techniques of Kahn and Saks [19] do allow one to construct the optimal sequence, subject to the

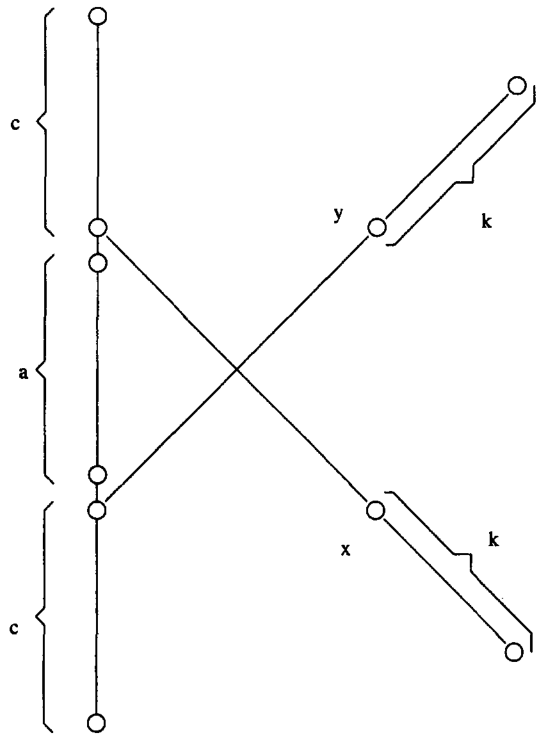

Fig. 8. An upper bound for $G(0)$.

constraints, in that case too. Again, it is not in general clear when the sequences are (approximately) realisable.

One particularly interesting question is that of determining the value of $G(0)$ — how unbalanced can a pair at the *same* average height be? There is a geometric result bearing directly on this problem, namely that, if H is a hyperplane through the centroid of a convex body in $\mathbb{R}^n$, then at least a proportion 1/e of the body lies on each side. This result was discovered independently by Grünbaum [16] and Hammer, and later rediscovered by Mityagin [25]. Applying it to the order polytope of a partial order with elements x and y such that $h(x) = h(y)$ yields $G(0) \geqslant 1/e \simeq 0.3679$. This was observed by Kahn and Linial [18], who also remark that the Kahn–Saks techniques from [19] give the same bound.

I am unaware of any upper bound on $G(0)$ in the literature. Accordingly, I offer the example in Fig. 8, with every expectation that it can be beaten. In this example, take $c \simeq (k-1)a/2$, with $c \gg a \gg k \gg 1$. It is fairly easy to verify that this choice of c suffices to make $h(x)$ and $h(y)$ approximately equal, and a short calculation reveals that $\mathbb{P}(x \prec y) \simeq 3e^{-2} \simeq 0.4060$, which is thus an upper bound on $G(0)$. This example is similar to ones discovered by Fishburn [13] in a related context, namely that of finding how large (or small) $\mathbb{P}(x \prec z)$ can be, given the values of $\mathbb{P}(x \prec y)$ and $\mathbb{P}(y \prec z)$.

4.3. Sorting time

Let us return to the problem asked at the beginning of the paper: find a bound for the worst-case sorting time $C(P)$ in terms of $e(P)$.

Fredman [14] showed that $C(P) \leqslant 2n + \log e(P)$, for all n-element partial orders P. This implies that, for any $\varepsilon > 0$, all 'sufficiently sparse' partial orders, with at most $2^{2n/\varepsilon}$ linear extensions, have $C(P) \leqslant (1+\varepsilon)\log e(P)$. So we are really concerned here with partial orders that are 'almost sorted', having only on the order of A^n linear extensions, for some smallish constant A. As we indicated in Section 1, a lower bound on $b(\mathscr{P})$ translates into an upper bound on $C(P)$, namely we have the following simple result.

Theorem 4.1. *For any finite partial order P,*

$$C(P) \leqslant \frac{\log e(P)}{-\log(1-b(\mathscr{P}))}.$$

Proof. Given P, choose as the first comparison a pair (x, y) with $b(\mathscr{P}) \leqslant \mathbb{P}(x \prec y) \leqslant 1 - b(\mathscr{P})$. Whatever the outcome of this comparison, there are at most $e(P)(1 - b(\mathscr{P}))$ feasible linear extensions left. Similarly, after k comparisons, whatever the outcomes, we can reduce to a set of at most $e(P)(1-b(\mathscr{P}))^k$ possible linear orders. For $k = -\log e(P)/\log(1-b(\mathscr{P}))$, this is at most 1, so k comparisons suffice. $\square$

For a finite partial order P, set $R(P) = C(P)/\log e(P)$. Theorem 4.1 allows us to convert any lower bound on $b(\mathscr{P})$ to an upper bound on $R(P)$ for all finite partial orders P. Indeed, the result of Kahn and Saks [19] that $b(\mathscr{P}) \geqslant \frac{3}{11}$ yielded the first proof that $R(P)$ is bounded above at all. Having cleared this hurdle, one is then led automatically to ask for the value of the supremum R_0 of $R(P)$ over all finite partial orders P.

We look first at upper bounds, and begin by noting that Theorems 3.5 and 4.1 give $R_0 \leqslant 2.1427$. In fact, we can do a little better by using Theorem 3.3 directly. It is claimed in Brightwell et al. [6] (Theorem 8.1) that $R_0 \leqslant 4/\log 5 \simeq 1.7227$. Unfortunately, the proof is incorrect, and all that can be shown using that proof is the following rather more modest improvement.

Theorem 4.2. *For every finite partial order P,*

$$R(P) \leqslant 2.1226.$$

Proof. Set $\beta = 0.2786$, the constant appearing in Theorem 3.3. We claim that $e(P)(1-\beta)^{C(P)} \geqslant 1$ for all finite partial orders P. This will imply that $R(P) \leqslant -1/\log(1-\beta) \leqslant 2.1226$, as required.

We proceed by induction on $e(P)$, the result being trivially true for partial orders with at most 2 linear extensions. In the induction, what we show is that either we can make one comparison and reduce the number of linear extensions by a factor of

$(1-\beta)$ or better, or we can make two comparisons and reduce the number by a factor of $(1-\beta)^2$ or better.

If P is neither a chain nor a two-element antichain, then we can find three elements x, y, z with $h(x) \leqslant h(y) \leqslant h(z) \leqslant h(x)+2$. By Theorem 3.5, we have one of $\mathbb{P}(y \prec x) \geqslant \beta$, $\mathbb{P}(z \prec y) \geqslant \beta$, or: $x < z$, y incomparable with both x and z, and $\mathbb{P}(y \prec x) + \mathbb{P}(z \prec y) \geqslant (5-\sqrt{5})/5$. Note that, since $h(x) \leqslant h(y)$, we always have $\mathbb{P}(x \prec y) \geqslant G(0) \geqslant 0.3679$, and similarly for $\mathbb{P}(y \prec z)$.

In the first case, when $\mathbb{P}(y \prec x) \geqslant \beta$, the single comparison between x and y reduces the number of linear extensions by a factor of at worst $1-\beta$, and we are done by induction. Similarly, the single comparison between y and z is good in the case when $\mathbb{P}(z \prec y) \geqslant \beta$.

If $\mathbb{P}(y \prec x)$ and $\mathbb{P}(z \prec y)$ are both less than β, then we are in the third case, where $y \prec x$ and $z \prec y$ are mutually exclusive events the sum of whose probabilities is at least $(5-\sqrt{5})/5$. Hence $\mathbb{P}(x \prec y \prec z) \leqslant 1/\sqrt{5}$. We now compare x with y, and y with z, and determine which of $y \prec x$, $z \prec y$ and $x \prec y \prec z$ holds. Each of these has probability at most $1/\sqrt{5} < (1-\beta)^2$, so again we are done by induction. □

The error in the proof of Theorem 8.1 in [6] is that, in the case where $\{x, y, z\}$ forms an antichain, the events $y \prec x$ and $z \prec y$ are not mutually exclusive. Still, as mentioned after Theorem 3.3, it should not be too hard to get improved bounds in this case, and such progress would automatically lead to an improvement over the upper bound on R_0 in Theorem 4.2.

Turning now to lower bounds, we have already seen, in Example 2, Linial's [24] example L_n, with $R(L_n) \to (\log(1+\sqrt{5})-1)^{-1} \simeq 1.4404$ as $n \to \infty$. We strongly suspect that this is actually the true value of R_0, i.e., that large finite segments of the infinite ladder L are the least efficient to sort in this sense. In [24], Linial showed that this is true, in a strong sense, if we restrict to the class of width 2 partial orders. The proof is very sweet.

Theorem 4.3. *For each integer $m \geqslant 1$, every width two partial order with fewer than F_{m+1} linear extensions can be sorted using at most $m-1$ comparisons.*

Proof. We work by induction on m; the result is trivial for $m=1$ and obvious for $m=2$. Take $k \geqslant 3$, and suppose the result is true for all $m < k$. Let P be a width 2 partial order with fewer than F_{k+1} linear extensions. As usual, we may assume that P has two minimal elements.

Take a decomposition of P into two chains, and consider the two minimal elements, which are the bottom elements of the chains. Label these x and y so that $\mathbb{P}(x \prec y) \geqslant 1/2$. Suppose the chain C with bottom element x is $x = x_1 < x_2 < x_3 < \cdots$, and let r be the largest integer such that $\mathbb{P}(x_r \prec y) \geqslant 1/2$. If r is the top element of the chain C, then there is only one linear extension with $x_r \prec y$, so at most two linear extensions in all, and we have already covered this case. Therefore we may assume that there is a further element x_{r+1} in C above x_r.

Now, we have either (i) the number of linear extensions of P with $x_r \prec y$ is less than F_k, or (ii) the number of linear extensions of P with $y \prec x_r$ is less than F_{k-1}, since the sum of these two numbers of linear extensions is $e(P) < F_{k+1}$.

In case (i), we compare y with x_r. If we find that $x_r \prec y$, we have reduced the number of linear extensions below F_k, and if we find that $y \prec x_r$, then we have reduced the number of linear extensions below $F_{k+1}/2 \leqslant F_k$. In either case, the induction hypothesis implies that we can complete the sorting with at most another $k-2$ comparisons, and we are done.

In case (ii), our first step is to compare y with x_{r+1}. If we find that $x_{r+1} \prec y$, then we are done as above since we are down to less than $F_{k+1}/2$ linear extensions by choice of r.

Thus we may assume that we find that $y \prec x_{r+1}$. Now we observe that the set of linear extensions of P with $x_r \prec y \prec x_{r+1}$ is no larger than the set of linear extensions with $y \prec x_r$, since every linear extension in the first set has y immediately above x_r, and so swapping y and x_r gives an injection from the first set to the second. We compare y with x_r; whatever result we get, the number of linear extensions remaining is at most the number of linear extensions of P with $y \prec x_r$. This number is less than F_{k-1} so, by the induction hypothesis, the sorting can be completed using at most a further $k-3$ comparisons, and we are done.

This completes the proof. □

Since computing numbers of linear extensions of width 2 partial orders can be done in polynomial time — see, for instance, Atkinson and Chang [2] — the above proof does provide an algorithm for comparison sorting, starting from a width 2 partial order P, in time polynomial in the number of elements of P, and using at most $C \log e(P)$ comparisons, where $C \simeq 1.4404$ is the best possible constant. As we shall see in Section 5, this is much more than can currently be said for the general case.

4.4. Other questions about balancing constants

As we saw in Section 2, there are some reasonable classes $\mathscr{Q}$ of partial orders for which $b(\mathscr{Q})$ is even greater than $\frac{1}{3}$, i.e., for some $\alpha > \frac{1}{3}$, every partial order in the class contains an α-balanced pair. The theme of the results in Section 2.2 is that this is the case when $\mathscr{Q}$ is a class of partial orders all of which have many elements and small height.

A long-standing open question, posed by Kahn and Saks [19], is whether the same phenomenon occurs if our class $\mathscr{Q}$ contains only partial orders of large width. To be precise, let $\mathscr{W}_k$ denote the family of finite partial orders of width at least k: Kahn and Saks conjecture that $b(\mathscr{W}_k) \to \frac{1}{2}$ as $k \to \infty$. Komlós's result, Theorem 2.5, can be seen as evidence in favour of this conjecture.

Changing tack slightly, consider the set $B = \{b(P) : P \text{ a finite partial order}\}$. It seems that almost nothing is known about the structure of B. We pose a number of

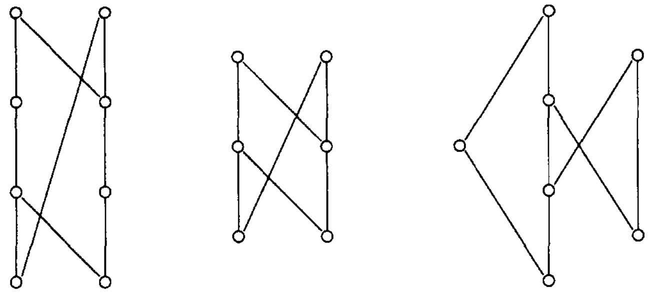

Fig. 9. Three partial orders P with $b(P)$ fairly small.

questions, none of which seem to have been considered before. Probably most of them are hard.

The set B contains the numbers 0 (P a chain) and $\frac{1}{3}$ ($P = T$). The 1/3–2/3 Conjecture says that $\frac{1}{3}$ is the lowest number after 0; what happens above that? Might there be a substantial gap before the next member of B? One possibility is that there are no other members of B below $\frac{16}{45} \simeq 0.3556$, achieved by the 8-element partial order A in Fig. 9. Other members of B (maybe the next ones?) are $\frac{5}{14} \simeq 0.3571$, acheived by the 6-element partial order H, and $\frac{14}{39} \simeq 0.3590$, achieved by the segment M_7 of the infinite ladder M. This last example first appears in this context in Saks [26], who mentions it as the width 3 partial order with the smallest known balance constant. See Fig. 9.

More generally, the sequence $b(M_{4k+3})$, $k = 0, 1, 2, \ldots$, of numbers in B increases towards the limit $(7 - \sqrt{17})/8 \simeq 0.3596$. Is this the lowest limit point of the set B?

Is the set B dense in some interval $(\frac{1}{2} - \varepsilon, \frac{1}{2})$? If so, what is the largest value of ε for which this is true? Is it even possible that B contains every rational in some interval?

All the same questions can be asked about the set $B' = \{b(P)\colon P \text{ a thin partial order}\}$; the fact that we know the second lowest member after 0, namely $\alpha = (5 - \sqrt{5})/10$, does not seem to help with most of the questions, but it may be that the methods of [6] can be adapted to show that there is some positive ε such that $B' \cap (\alpha, \alpha + \varepsilon)$ is empty.

Another possibility is to ask the above questions for the family of width 2 (finite) partial orders; this might be a more tractable area of study.

5. Algorithms

If we are *really* interested in designing a sorting algorithm to operate efficiently in the presence of partial information, are the results of the previous sections any help to us?

One might contemplate an algorithm along the following lines: given a partial order $(X, <)$, choose the incomparable pair (x, y) with the value of $\mathbb{P}(x \prec y)$ closest to 1/2, and make that comparison; either outcome gives us a new partial order on X, and we repeat until we have found the linear order $\prec$.

Certainly this is an algorithm for which we can make some performance guarantees. The catch is that it involves calculating $\mathbb{P}(x \prec y)$ for some incomparable pairs at each stage. In a practical setting, there will almost certainly be some limitations on the amount of computation we are able to do between comparisons, so the question arises of how hard it is to compute $\mathbb{P}(x \prec y)$.

Let us start with the bad news: computing the probability exactly is #P-hard. This means that it is at least as hard as computing, for instance, the number of Hamiltonian circuits in a graph, or the number of satisfying assignments of a Boolean formula. This follows from a result of Brightwell and Winkler [7], stating that counting the number of linear extensions $e(P)$ of a partial order P is #P-complete. Indeed, given an oracle for calculating $\mathbb{P}(x \prec y)$ for any pair (x, y) of elements in a partial order, and a partial order $P = (X, <)$, we can find a sequence of partial orders $P = P_0, P_1, P_2, \ldots, P_m$, (with $m \leqslant n$), such that P_m is a linear order and each P_i is obtained from P_{i-1} by adding in the single extra relation $x_i < y_i$ and taking the transitive closure. Then $e(P_i)/e(P_{i-1}) = \mathbb{P}(x_i \prec y_i)$ for each i, and $1/e(P)$ can now be obtained as the product of all the $\mathbb{P}(x_i \prec y_i)$. This clearly yields a polynomial time algorithm for calculating $e(P)$, given an oracle for $\mathbb{P}(x \prec y)$.

There is however some good news; there are polynomial time *randomised* algorithms that approximate the number of linear extensions to within any desired constant factor, with probability of success as close as desired to 1. The first such algorithm was due to Dyer, Frieze and Kannan [11], and was a consequence of their general approximation algorithm for the volume of a convex body in n dimensions: the translation is achieved by observing, as we did earlier, that the volume of the *order polytope* of the partial order P with ground-set $\{1, \ldots, n\}$ is exactly $e(P)/n!$.

An algorithm more tailored to the special case of approximating $e(P)$, for an n-element partial order P, was investigated by Karzanov and Khachiyan [20]. We briefly sketch their approach. As we saw above (essentially), to approximate $e(P)$, it is enough to be able to approximate $\mathbb{P}(x \prec y)$ for elements x, y. To do this, it is enough to be able to sample approximately uniformly from the set of all linear extensions of P. Karzanov and Khachiyan examine a Markov chain on the set of linear extensions. Call two linear extensions *neighbours* if they differ (as permutations of $[n]$) by an adjacent transposition. Now define a Markov chain which steps from any linear extension to each neighbour with probability $1/(2n-2)$, and otherwise stays still. Karzanov and Khachiyan show that this chain is *rapidly mixing*, i.e., that it approaches its stationary distribution, which is the uniform distribution, in time about $O(n^6)$. The method above then yields an algorithm approximating $e(P)$ to within a multiplicative factor $(1+\varepsilon)$, with probability at least $\frac{3}{4}$, in time about $O(n^9)$ (see also Brightwell and Winkler [7]). Dyer and Frieze [10] gave several improvements to the techniques, leading to a lower bound of about $O(n^6)$ on the time required to approximate $e(P)$.

A slightly different Markov chain has recently been proposed by Bubley and Dyer [9], which they show leads to an algorithm approximating $e(P)$ to within a multiplicative factor $(1+\varepsilon)$, with probability at least $\frac{3}{4}$, in time $O(n^5 \log^2 n\, \varepsilon^{-2} \log(n/\varepsilon))$ — in fact they show that the original Karzanov–Khachiyan chain achieves this bound also.

What all this means is that, if one is prepared to accept a randomised algorithm (and a running time of about $O(n^5)$), then it is possible to find at least a fairly balanced pair in polynomial time, and run the sorting algorithm proposed at the beginning of this section.

A completely different approach was taken by Kahn and Kim [17]. They give a deterministic algorithm to sort, starting from partial information given by an n-element partial order P, in time polynomial in n, using at most $C \log e(P)$ comparisons, where C is a moderate-sized constant. Their approach to this is based on consideration not of the number $e(P)$ of linear extensions, but of a related parameter which is more computationally tractable.

For a partial order $P=(X,<)$, with $X=[n]$, define the *chain polytope*

$$\mathscr{C}(P) = \{\mathbf{x} \in [0,1]^n : \sum_{i \in C} x_i \leqslant 1 \text{ for every chain } C \text{ of } P\}.$$

Stanley [27] proved the striking result that $\mathscr{C}(P)$ has the same volume as the order polytope $\mathscr{O}(P)$, which is just $e(P)/n!$. The *entropy* $H(P)$ of P is the minimum, over all points $\mathbf{x} \in \mathscr{C}(P)$, of $-\frac{1}{n}\sum_i \log x_i$. For instance, if P is a chain, then $H(P)=\log n$: the minimum is achieved by setting all the x_i equal to $1/n$.

Finding the entropy is equivalent to maximizing $\prod_i x_i$ over $\mathscr{C}(P)$. This maximum is obviously at most the volume of $\mathscr{C}(P)$: the first indication that this might be a worthwhile line of enquiry is the result from [17] that it is at least $(n!/n^n)$ times the volume. Combining this with Stanley's result shows that the entropy of P is close to $-(1/n)\log(e(P)/n!)$, provided $e(P)$ is not too small (in this context, 'small' means $O(C^n)$).

Kahn and Kim [17] show, among other things, that entropy has the following pleasant properties:

- $\log e(P) \geqslant \frac{1}{12} n(\log n - H(P))$.
- There is a deterministic polynomial time algorithm to find a pair (x,y) to compare such that, whatever the outcome of the comparison, the entropy of the partial order has been increased by at least $1/5n$.

The algorithm is now apparent: starting from P_0, keep making comparisons to increase the entropy until $\log n - H(P)$ is reduced to 0, i.e., P becomes a chain. The number of comparisons required is at most $5n(\log n - H(P_0)) \leqslant 60 \log e(P_0)$.

6. Conclusion

This survey contains a fair number of results bearing on the 1/3–2/3 Conjecture. However, it would be misleading to suggest that any of the approaches outlined are likely to lead to a resolution of the full conjecture. Indeed, I would say quite the opposite, that no line of attack has yet been suggested which has any realistic hope of proving the conjecture. Nevertheless, I remain convinced that the conjecture is true, and I would very much like to see it proved.

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