Existence and uniqueness of periodic solutions for a kind of Liénard equation with two deviating arguments

Qiyuan Zhou\textsuperscript{a,}\textsuperscript{*}, Fei Long\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, Hunan University of Arts and Science, Changde, Hunan 415000, China
\textsuperscript{b}College of Mathematics and Econometrics, Hunan University, Changsha 410082, P.R. China

Received 24 July 2006; received in revised form 26 September 2006

Abstract

In this paper, a kind of Liénard equation with two deviating arguments of the form

\[ x''(t) + f(x(t))x'(t) + g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_2(t))) = p(t) \]

is considered. We derive some new sufficient conditions for checking uniqueness of \( T \)-periodic solutions of this equation. By using coincidence degree theory, we also establish some new results on the existence and uniqueness of \( T \)-periodic solutions for this equation, which are new and complement previously known results.

\( © 2006 \) Elsevier B.V. All rights reserved.

\textit{MSC:} 34C25; 34D40

\textit{Keywords:} Liénard equation; Deviating argument; Periodic solution; Existence; Uniqueness; Coincidence degree

1. Introduction

Consider the Liénard equation with two deviating arguments of the form

\[ x''(t) + f(x(t))x'(t) + g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_2(t))) = p(t), \quad (1.1) \]

where \( f, \tau_1, \tau_2, p : R \rightarrow R \) and \( g_1, g_2 : R \times R \rightarrow R \) are continuous functions, \( \tau_1, \tau_2 \) and \( p \) are \( T \)-periodic, \( g_1 \) and \( g_2 \) are \( T \)-periodic in the first argument, and \( T > 0 \). In recent years, some results on the existence of periodic solutions of Eq. (1.1) have appeared by applying continuation theorem, see paper [2,4–8]. However, the work of these papers were only subject to study the \textit{existence} of periodic solutions of Eq. (1.1). Since it is difficult to establish sufficient conditions for checking uniqueness of \( T \)-periodic solutions of Eq. (1.1). Therefore, there exist few results for the \textit{existence and uniqueness} of periodic solutions of Eq. (1.1). Thus, it is worth while to study the problem of the periodic solutions of Eq. (1.1) in this case.

The main purpose of this paper is to establish sufficient conditions for the existence and uniqueness of \( T \)-periodic solutions of Eq. (1.1). The results of this paper are new and they complement previously known results.

\textsuperscript{*} This work was supported by the Hunan Provincial Natural Science Foundation of China (05JJ40009).

* Corresponding author. Tel./fax: +86 736 7186113.

E-mail address: zhouqiyuan65@yahoo.com.cn (Q. Zhou).
For ease of exposition, throughout this paper we will adopt the following notations:

$$|x|_k = \left( \int_0^T |x(t)|^k \, dt \right)^{1/k}, \quad |x|_{\infty} = \max_{t \in [0,T]} |x(t)|.$$ 

Let

$$X = \{ x | x \in C^1(R, R), \ x(t+T) = x(t), \ \text{for all} \ t \in R \}$$

and

$$Y = \{ x | x \in C(R, R), \ x(t+T) = x(t), \ \text{for all} \ t \in R \}$$

be two Banach spaces with the norms

$$\|x\|_X = \max\{|x|_{\infty}, |x'|_{\infty}\}, \quad \text{and} \quad \|x\|_Y = |x|_{\infty}.$$ 

Define a linear operator $L : D(L) \subset X \rightarrow Y$ by setting

$$D(L) = \{ x | x \in X, x'' \in C(R, R) \}$$

and for $x \in D(L)$,

$$Lx = x''.$$ (1.2)

We also define a nonlinear operator $N : X \rightarrow Y$ by setting

$$ Nx = -f(x(t))x'(t) - g_1(t, x(t - \tau_1(t))) - g_2(t, x(t - \tau_2(t))) + p(t).$$ (1.3)

It is easy to see that

$$\ker L = R, \quad \text{and} \quad \im L = \{ x | x \in Y, \int_0^T x(s) \, ds = 0 \}.$$ 

Thus, the operator $L$ is a Fredholm operator with index zero.

Define the continuous projectors $P : X \rightarrow \ker L$ and $Q : Y \rightarrow Y$ by setting

$$Px(t) = x(0) = x(T)$$

and

$$Qx(t) = \frac{1}{T} \int_0^T x(s) \, ds.$$ 

Hence, $\im P = \ker L$ and $\ker Q = \im L$. Denoting by $L_P^{-1} : \im L \rightarrow D(L) \cap \ker P$ the inverse of $L|_{D(L)\cap\ker P}$, we have

$$L_P^{-1}y(t) = -\frac{t}{T} \int_0^T (t-s)y(s) \, ds + \int_0^t (t-s)y(s) \, ds.$$ (1.4)

It is convenient to introduce the following assumption.

$(A_0)$ Assume that there exists nonnegative constants $C_1$ and $C_2$ such that

$$|f(x_1) - f(x_2)| \leq C_1|x_1 - x_2|, \quad |f(x)| \leq C_2 \text{ for all } x_1, x_2, x \in R.$$ 

The remaining part of this paper is organized as follows. In Section 2, we shall derive new sufficient conditions for checking uniqueness of $T$-periodic solutions of Eq. (1.1). In Section 3, we present some new sufficient conditions for the existence and uniqueness of $T$-periodic solutions of Eq. (1.1). In Section 4, we shall give some examples and remarks to illustrate our results obtained in the previous sections.
2. Preliminary results

In view of (1.2) and (1.3), the operator equation \( Lx = \lambda Nx \) is equivalent to the following equation:

\[
x'' + \lambda \left[ f(x(t))x'(t) + g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_2(t))) \right] = \lambda p(t),
\]

where \( \lambda \in (0, 1) \).

For convenience of use, we introduce the Continuation Theorem [2] as follows.

**Lemma 2.1.** Let \( X \) and \( Y \) be two Banach spaces. Suppose that \( L : D(L) \subset X \longrightarrow Y \) is a Fredholm operator with index zero and \( N : X \longrightarrow Y \) is \( L \)-compact on \( \Omega \), where \( \Omega \) is an open bounded subset of \( X \). Moreover, assume that all the following conditions are satisfied:

1. \( Lx \neq \lambda Nx \), for all \( x \in \partial \Omega \cap D(L) \), \( \lambda \in (0, 1) \);
2. \( Nx \notin \text{Im} L \), for all \( x \in \partial \Omega \cap \text{Ker} L \);
3. The Brouwer degree

\[
\deg\{QN, \Omega \cap \text{Ker} L, 0\} \neq 0.
\]

Then equation \( Lx = Nx \) has at least one solution on \( \Omega \).

The following lemmas will be useful to prove our main results in Section 3.

**Lemma 2.2.** If \( x \in C^2(R, R) \) with \( x(t + T) = x(t) \), then

\[
|x'(t)|^2 \leq \left( \frac{T}{2\pi} \right)^2 |x''(t)|^2.
\]

**Proof.** Lemma 2.2 is a direct consequence of the Wirtinger inequality, and see [3,8] for its proof. \( \square \)

**Lemma 2.3.** Assume that the following conditions are satisfied:

(A1) one of the following conditions holds:

1. \( g_i(t, u_1) - g_i(t, u_2))u_1 - u_2) > 0 \), for \( i = 1, 2, u_1, u_2 \in R, \forall t \in R \) and \( u_1 \neq u_2 \);
2. \( g_i(t, u_1) - g_i(t, u_2))u_1 - u_2) < 0 \), for \( i = 1, 2, u_1, u_2 \in R, \forall t \in R \) and \( u_1 \neq u_2 \);

(A2) there exists a constant \( d > 0 \) such that one of the following conditions holds:

1. \( x(g_1(t, x) + g_2(t, x) - p(t)) > 0 \), for all \( t \in R, |x| \geq d \);
2. \( x(g_1(t, x) + g_2(t, x) - p(t)) < 0 \), for all \( t \in R, |x| \geq d \).

If \( x(t) \) is a \( T \)-periodic solution of (2.1), then

\[
|x|_{\infty} \leq d + \sqrt{T}|x'|_2.
\]

**Proof.** Let \( x(t) \) be a \( T \)-periodic solution of (2.1). Set

\[
x(t_{\max}) = \max_{t \in R} x(t), \quad x(t_{\min}) = \min_{t \in R} x(t), \quad \text{where} \ t_{\max}, t_{\min} \in R.
\]

Then we have

\[
x'(t_{\max}) = 0, \quad x''(t_{\max}) \leq 0, \quad \text{and} \quad x'(t_{\min}) = 0, \quad x''(t_{\min}) \geq 0.
\]

In view of (2.1), (2.4) implies that

\[
\frac{x''(t_{\max})}{\lambda} \geq 0,
\]
and
\[ g_1(t_{\min}, x(t_{\min} - \tau_1(t_{\min}))) + g_2(t_{\min}, x(t_{\min} - \tau_2(t_{\min}))) - p(t_{\min}) = -\frac{x''(t_{\min})}{\kappa} \leq 0. \]  
(2.6)

Since \( g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_2(t))) - p(t) \) is a continuous function on \( R \), it follows from (2.5) and (2.6) that there exists a constant \( t_1 \in R \) such that:
\[ g_1(t_1, x(t_1 - \tau_1(t_1))) + g_2(t_1, x(t_1 - \tau_2(t_1))) - p(t_1) = 0. \]  
(2.7)

Now we show that the following claim is true.

**Claim.** If \( x(t) \) is a \( T \)-periodic solution of (2.1), then there exists a constant \( t_2 \in R \) such that
\[ |x(t_2)| \leq d. \]  
(2.8)

Assume, by way of contradiction, that (2.8) does not hold. Then
\[ |x(t)| > d \quad \text{for all} \quad t \in R, \]  
(2.9)

which, together with (A2) and (2.7), implies that one of the following relations holds:

\[ x(t_1 - \tau_1(t_1)) > x(t_1 - \tau_2(t_1)) > d, \]  
(2.10)

\[ x(t_1 - \tau_2(t_1)) > x(t_1 - \tau_1(t_1)) > d, \]  
(2.11)

\[ x(t_1 - \tau_1(t_1)) < x(t_1 - \tau_2(t_1)) < -d, \]  
(2.12)

\[ x(t_1 - \tau_2(t_1)) < x(t_1 - \tau_1(t_1)) < -d. \]  
(2.13)

Suppose that (2.10) holds, in view of (A1)(1), (A1)(2), (A2)(1) and (A2)(2), we will consider four cases as follows:

**Case (i):** If (A2)(1) and (A1)(1) hold, according to (2.10), we obtain
\[ 0 < g_1(t_1, x(t_1 - \tau_1(t_1))) + g_2(t_1, x(t_1 - \tau_2(t_1))) - p(t_1) \]
\[ < g_1(t_1, x(t_1 - \tau_1(t_1))) + g_2(t_1, x(t_1 - \tau_2(t_1))) - p(t_1), \]
which contradicts (2.7). This contradiction implies that (2.8) is true.

**Case (ii):** If (A2)(1) and (A1)(2) hold, according to (2.10), we obtain
\[ 0 < g_1(t_1, x(t_1 - \tau_1(t_1))) + g_2(t_1, x(t_1 - \tau_1(t_1))) - p(t_1) \]
\[ < g_1(t_1, x(t_1 - \tau_1(t_1))) + g_2(t_1, x(t_1 - \tau_2(t_1))) - p(t_1), \]
which contradicts (2.7). This contradiction implies that (2.8) is true.

**Case (iii):** If (A2)(2) and (A1)(1) hold, according to (2.10), we obtain
\[ 0 > g_1(t_1, x(t_1 - \tau_1(t_1))) + g_2(t_1, x(t_1 - \tau_1(t_1))) - p(t_1) \]
\[ > g_1(t_1, x(t_1 - \tau_1(t_1))) + g_2(t_1, x(t_1 - \tau_2(t_1))) - p(t_1), \]
which contradicts (2.7). This contradiction implies that (2.8) is true.

**Case (iv):** If (A2)(2) and (A1)(2) hold, according to (2.10), we obtain
\[ 0 > g_1(t_1, x(t_1 - \tau_2(t_1))) + g_2(t_1, x(t_1 - \tau_2(t_1))) - p(t_1) \]
\[ > g_1(t_1, x(t_1 - \tau_1(t_1))) + g_2(t_1, x(t_1 - \tau_2(t_1))) - p(t_1), \]
which contradicts (2.7). This contradiction implies that (2.8) is true.

Suppose that (2.11) (or (2.12), or (2.13)) holds; using methods similar to those used in Cases (i)–(iv), we can show that (2.8) is true. This completes the proof of the above claim.
Let \( t_2 = mT + t_0 \), where \( t_0 \in [0, T] \) and \( m \) is an integer. Then, using the Schwarz inequality and the relation

\[
|x(t)| = |x(t_0) + \int_{t_0}^{t} x'(s) \, ds| \leq d + \int_{0}^{T} |x'(s)| \, ds, \quad t \in [0, T].
\]

we obtain

\[
|x|_{\infty} = \max_{t \in [0, T]} |x(t)| \leq d + \sqrt{T} |x'_{2}|
\]

This completes the proof of Lemma 2.3. □

**Lemma 2.4.** Let \((A_0), (A_1)\) and \((A_2)\) hold. Assume that the following condition is satisfied:

(A3) There exist constants \( b_1 \) and \( b_2 \) such that

\[
C_2 \frac{T}{2\pi} + (b_1 + b_2) \frac{T^2}{2\pi} < 1, \quad \text{and} \quad |g_i(t, x_1) - g_i(t, x_2)| \leq b_1 |x_1 - x_2|
\]

for all \( t, x_i \in R, \ i = 1, 2 \).

If \( x(t) \) is a \( T \)-periodic solution of Eq. (1.1). Then

\[
|x'|_{2} \leq \frac{[(b_1 + b_2)d + \max\{|g_1(t, 0)| + |g_2(t, 0)| : 0 \leq t \leq T\} + |p|_{\infty}]T}{1 - [C_2T/2\pi + (b_1 + b_2)T^2/2\pi]} := D. \tag{2.14}
\]

**Proof.** Let \( x(t) \) be a \( T \)-periodic solution of Eq. (1.1). From \((A_1)\) and \((A_2)\), we can easily show that \((2.3)\) also holds. Multiplying \( x''(t) \) and Eq. (1.1) and then integrating it from 0 to \( T \), in view of (2.2), (2.3), (A3) and the inequality of Schwarz, we have

\[
|x''|_{2}^2 = -\int_{0}^{T} f(x(t))x'(t)x''(t) \, dt - \int_{0}^{T} g_1(t, x(t - \tau_1(t)))x''(t) \, dt
\]

\[
- \int_{0}^{T} g_2(t, x(t - \tau_2(t)))x''(t) \, dt + \int_{0}^{T} p(t)x''(t) \, dt
\]

\[
\leq C_2 \frac{T}{2\pi} |x''|_{2}^2 + \int_{0}^{T} |g_1(t, x(t - \tau_1(t))) - g_1(t, 0) + |g_1(t, 0)|| \cdot |x''(t)| \, dt
\]

\[
+ \int_{0}^{T} |g_2(t, x(t - \tau_2(t))) - g_2(t, 0) + |g_2(t, 0)|| \cdot |x''(t)| \, dt + \int_{0}^{T} |p(t)| \cdot |x''(t)| \, dt
\]

\[
\leq C_2 \frac{T}{2\pi} |x''|_{2}^2 + b_1 \int_{0}^{T} |x(t - \tau_1(t))| \cdot |x''(t)| \, dt + b_2 \int_{0}^{T} |x(t - \tau_2(t))| \cdot |x''(t)| \, dt
\]

\[
+ \int_{0}^{T} |g_1(t, 0)| \cdot |x''(t)| \, dt + \int_{0}^{T} |g_2(t, 0)| \cdot |x''(t)| \, dt + \int_{0}^{T} |p(t)| \cdot |x''(t)| \, dt
\]

\[
\leq C_2 \frac{T}{2\pi} |x''|_{2}^2 + (b_1 + b_2)|x|_{\infty}\sqrt{T}|x''|_{2} + \max\{|g_1(t, 0)| + |g_2(t, 0)| : 0 \leq t \leq T\}
\]

\[
+ |p|_{\infty}\sqrt{T}|x''|_{2},
\]

\[
\leq [C_2 \frac{T}{2\pi} + (b_1 + b_2) \frac{T^2}{2\pi}] |x''|_{2}^2 + [(b_1 + b_2)d + \max\{|g_1(t, 0)| + |g_2(t, 0)| : 0 \leq t \leq T\}
\]

\[
+ |p|_{\infty}\sqrt{T}|x''|_{2},
\]

\[
\tag{2.15}
\]
Thus, in view of Mean Value Theorem of Integrals, it follows that there exists a constant $C_1$ such that

$$
|x''|_2 \geq \frac{[(b_1 + b_2)d + \max\{|g_1(t, 0)| + |g_2(t, 0)| : 0 \leq t \leq T\} + |p|_{\infty}}{1 - [C_2T/2\pi + (b_1 + b_2)T^2/2\pi]}.$$

(2.16)

Since $x(0) = x(T)$, there exists a constant $\zeta \in [0, T]$ such that

$$
x'(\zeta) = 0,
$$

and

$$
|x'(t)| = |x'(\zeta) + \int_{\zeta}^{t} x''(s) \, ds| \leq \sqrt{T}|x''|_2 \quad \text{for all } t \in [0, T].
$$

(2.17)

Thus, in view of (2.16) and (2.17), we have

$$
|x'|_{\infty} \leq \frac{[(b_1 + b_2)d + \max\{|g_1(t, 0)| + |g_2(t, 0)| : 0 \leq t \leq T\} + |p|_{\infty}T]}{1 - [C_2T/2\pi + (b_1 + b_2)T^2/2\pi]} := D.
$$

This completes the proof of Lemma 2.4. □

**Lemma 2.5.** Let (A1) and (A2) hold. Assume that the following condition is satisfied:

(A4) Suppose that (A0) hold, and there exist nonnegative constants $b_1$ and $b_2$ such that

$$
C_1DT \frac{T}{2\pi} + C_2 \frac{T}{2\pi} + (b_1 + b_2) \frac{T^2}{2\pi} < 1, \quad \text{and} \quad |g_i(t, x_1) - g_i(t, x_2)| \leq b_i|x_1 - x_2|,
$$

for all $t, x_1, x_2 \in R, i = 1, 2$. Then Eq. (1.1) has at most one $T$-periodic solution.

**Proof.** Suppose that $x_1(t)$ and $x_2(t)$ are two $T$-periodic solutions of Eq. (1.1). Set $Z(t) = x_1(t) - x_2(t)$. Then, we obtain

$$
Z''(t) + (f(x_1(t)))x'_1(t) - f(x_2(t))x'_2(t) + (g_1(t, x_1(t - \tau_1(t))) - g_1(t, x_2(t - \tau_1(t))))
+ (g_2(t, x_1(t - \tau_2(t))) - g_2(t, x_2(t - \tau_2(t)))) = 0.
$$

(2.18)

Since $x_1(t)$ and $x_2(t)$ are $T$-periodic, integrating (2.18) from 0 to $T$, we obtain

$$
\int_{0}^{T} [(g_1(t, x_1(t - \tau_1(t))) - g_1(t, x_2(t - \tau_1(t)))) + (g_2(t, x_1(t - \tau_2(t))) - g_2(t, x_2(t - \tau_2(t))))] \, dt = 0.
$$

Thus, in view of Mean Value Theorem of Integrals, it follows that there exists a constant $\gamma \in [0, T]$ such that:

$$
g_1(\gamma, x_1(\gamma - \tau_1(\gamma))) - g_1(\gamma, x_2(\gamma - \tau_1(\gamma))) + g_2(\gamma, x_1(\gamma - \tau_2(\gamma))) - g_2(\gamma, x_2(\gamma - \tau_2(\gamma))) = 0.
$$

(2.19)

From (A1), (2.19) implies that

$$
Z(\gamma - \tau_1(\gamma))Z(\gamma - \tau_2(\gamma)) = (x_1(\gamma - \tau_1(\gamma)) - x_2(\gamma - \tau_1(\gamma)))(x_1(\gamma - \tau_2(\gamma)) - x_2(\gamma - \tau_2(\gamma))) \leq 0.
$$

Since $Z(t) = x_1(t) - x_2(t)$ is a continuous function on $R$, it follows that there exists a constant $\zeta \in R$ such that:

$$
Z(\zeta) = 0.
$$

(2.20)

Let $\zeta = nT + \tilde{\gamma}$, where $\tilde{\gamma} \in [0, T]$ and $n$ is an integer. Then, (2.20) implies that there exists a constant $\tilde{\gamma} \in [0, T]$ such that

$$
Z(\tilde{\gamma}) = Z(\zeta) = 0.
$$

(2.21)
Hence,

\[ |Z(t)| = |Z(\gamma)| + \int_{\gamma}^{T} |Z'(s)| \, ds \leq \int_{0}^{T} |Z'(s)| \, ds, \quad t \in [0, T], \]

and

\[ |Z|_{\infty} \leq \sqrt{T} |Z'|_{2}. \] (2.22)

Multiplying \( Z''(t) \) and (2.18) and then integrating it from 0 to \( T \), from (2.2), (2.14), (2.22) and Schwarz inequality, we get

\[
\begin{aligned}
|Z''|_{2} & = - \int_{0}^{T} (f(x_1(t))x_1'(t) - f(x_2(t))x_2'(t))Z''(t) \, dt - \int_{0}^{T} (g_1(t, x_1(t - \tau_1(t))) - g_1(t, x_1(t - \tau_2(t))))Z''(t) \, dt \\
& \quad - g_1(t, x_2(t - \tau_1(t))))Z''(t) \, dt - \int_{0}^{T} (g_2(t, x_1(t - \tau_2(t))) - g_2(t, x_2(t - \tau_2(t))))Z''(t) \, dt \\
& \leq \int_{0}^{T} |f(x_1(t))||x_1'(t) - x_2'(t)||Z''(t)| \, dt + \int_{0}^{T} |f(x_1(t)) - f(x_2(t))||x_2'(t)||Z''(t)| \, dt \\
& \quad + b_1 \int_{0}^{T} |x_1(t - \tau_1(t)) - x_2(t - \tau_1(t))||Z''(t)| \, dt + b_2 \int_{0}^{T} |x_1(t - \tau_2(t)) - x_2(t - \tau_2(t))||Z''(t)| \, dt \\
& \leq \int_{0}^{T} C_1 |x_1'(t) - x_2'(t)||Z''(t)| \, dt + \int_{0}^{T} C_2 |x_1(t) - x_2(t)||D||Z''(t)| \, dt \\
& \quad + b_1 \int_{0}^{T} |Z(t - \tau_1(t))||Z''(t)| \, dt + b_2 \int_{0}^{T} |Z(t - \tau_2(t))||Z''(t)| \, dt \\
& \leq C_2 |Z'|_{2}|Z''|_{2} + C_1 DT |Z'|_{2}|Z''|_{2} + (b_1 + b_2)|Z|_{\infty}|\sqrt{T}|Z''|_{2} \\
& \leq \left[ C_1 DT \frac{T}{2\pi} + C_2 \frac{T}{2\pi} + (b_1 + b_2) \frac{T^2}{2\pi} \right] |Z'|_{2}. \quad \text{(2.23)}
\end{aligned}
\]

Since \( Z(t), Z'(t) \) and \( Z''(t) \) are \( T \)-periodic and continuous functions, in view of (A4), (2.22) and (2.23), we have

\[ Z(t) \equiv Z'(t) \equiv Z''(t) \equiv 0 \quad \text{for all } t \in R. \]

Thus, \( x_1(t) \equiv x_2(t) \), for all \( t \in R \). Therefore, Eq. (1.1) has at most one \( T \)-periodic solution. The proof of Lemma 2.5 is now complete. \( \square \)

3. Main results

Theorem 1. Let (A1), (A2) and (A4) hold. Then Eq. (1.1) has a unique \( T \)-periodic solution.

Proof. By Lemma 2.5, it is easy to see that Eq. (1.1) has at most one \( T \)-periodic solution. Thus, to prove Theorem 1, it suffices to show that Eq. (1.1) has at least one \( T \)-periodic solution. To do this, we shall apply Lemma 2.1. Firstly, we will claim that the set of all possible \( T \)-periodic solutions of Eq. (2.1) are bounded.
Let \( x(t) \) be a \( T \)-periodic solution of Eq. (2.1). Multiplying \( x''(t) \) and Eq. (2.1) and then integrating it from 0 to \( T \), in view of (2.2), (2.3), (A4) and the inequality of Schwarz, we have

\[
|x''|^2 = -\lambda \int_0^T f(x(t))x'(t)x''(t) \, dt - \lambda \int_0^T g_1(t, x(t - \tau_1(t)))x''(t) \, dt
\]

\[
- \lambda \int_0^T g_2(t, x(t - \tau_2(t)))x''(t) \, dt + \lambda \int_0^T p(t)x''(t) \, dt
\]

\[
\leq C_2 \frac{T}{2\pi} |x''|^2 + \int_0^T [\left|g_1(t, x(t - \tau_1(t))) - g_1(t, 0)\right| + |g_1(t, 0)|] \cdot |x''(t)| \, dt
\]

\[
+ \int_0^T [\left|g_2(t, x(t - \tau_2(t))) - g_2(t, 0)\right| + |g_2(t, 0)|] \cdot |x''(t)| \, dt + \int_0^T |p(t)| \cdot |x''(t)| \, dt
\]

\[
\leq [C_2 \frac{T}{2\pi} + (b_1 + b_2) \frac{T^2}{2\pi}] |x''|^2 + [(b_1 + b_2)d + \max\{|g_1(t, 0)| + |g_1(t, 0)| : 0 \leq t \leq T\}]
\]

\[
+ |p|_\infty \sqrt{T} |x''|_2,
\]

which, together with (A4), implies that there exist positive constants \( D_1 \) and \( D_2 \) such that

\[
|x''|_2 < D_1,
\]

and

\[
|x'|_2 < D_2, \quad |x|_\infty < D_2.
\]

Since \( x(0) = x(T) \), there exists a constant \( \tilde{\xi} \in [0, T] \) such that

\[
x'(\tilde{\xi}) = 0,
\]

and

\[
|x'(t)| = |x'(\tilde{\xi}) + \int_\tilde{\xi}^t x''(s) \, ds| \leq \sqrt{T} |x''|^2 < \sqrt{T} D_1, \quad \text{for all } t \in [0, T].
\]

Therefore, in view of (3.3) and (3.4), there exists a positive constant \( M_1 > \sqrt{T} D_1 + D_2 \) such that

\[
\|x\|_X \leq |x|_\infty + |x'|_\infty < M_1.
\]

If \( x \in \Omega_1 = \{x|x \in \text{Ker } L \cap X, \text{ and } Nx \in \text{Im } L\} \), then there exists a constant \( M_2 \) such that

\[
x(t) \equiv M_2, \quad \text{and} \quad \int_0^T [g_1(t, M_2) + g_2(t, M_2) - p(t)] \, dt = 0.
\]

Thus,

\[
|x(t)| \equiv |M_2| < d \quad \text{for all } x(t) \in \Omega_1.
\]

Let \( M = M_1 + d + 1. \) Set

\[
\Omega = \{x|x \in X, |x|_\infty < M, |x'|_\infty < M\}.
\]

It is easy to see from (1.3) and (1.4) that \( N \) is \( L \)-compact on \( \Omega \). We have from (3.5) and (3.6) and the fact \( M > \max\{M_1, d\} \) that the conditions (1) and (2) in Lemma 2.1 hold.
Furthermore, define continuous functions $H_1(x, \mu)$ and $H_2(x, \mu)$ by setting

$$H_1(x, \mu) = -(1 - \mu)x - \mu \cdot \frac{1}{T} \int_0^T [g_1(t, x) + g_2(t, x) - p(t)] \, dt, \quad \mu \in [0, 1],$$

$$H_2(x, \mu) = (1 - \mu)x - \mu \cdot \frac{1}{T} \int_0^T [g_1(t, x) + g_2(t, x) - p(t)] \, dt, \quad \mu \in [0, 1].$$

If $(A_2)(1)$ holds, then

$$xH_1(x, \mu) \neq 0, \quad \text{for all } x \in \partial \Omega \cap \text{Ker } L.$$ 

Hence, using the homotopy invariance theorem, we have

$$\deg\{QN, \Omega \cap \text{Ker } L, 0\} = \deg\left\{- \frac{1}{T} \int_0^T [g_1(t, x) + g_2(t, x) - p(t)] \, dt, \Omega \cap \text{Ker } L, 0 \right\}$$

$$= \deg\{-x, \Omega \cap \text{Ker } L, 0\} \neq 0.$$

If $(A_2)(2)$ holds, then

$$xH_2(x, \mu) \neq 0 \quad \text{for all } x \in \partial \Omega \cap \text{Ker } L.$$ 

Hence, using the homotopy invariance theorem, we obtain

$$\deg\{QN, \Omega \cap \text{Ker } L, 0\} = \deg\left\{- \frac{1}{T} \int_0^T [g_1(t, x) + g_2(t, x) - p(t)] \, dt, \Omega \cap \text{Ker } L, 0 \right\}$$

$$= \deg\{x, \Omega \cap \text{Ker } L, 0\} \neq 0.$$ 

In view of all the discussions above, we conclude from Lemma 2.1 that Theorem 3.1 is proved. \qed

### 4. Example and remark

**Example 4.1.** Let $g_1(t, x) = g_2(t, x) = (1/96\pi)x$, for all $t, x \in R$. Then the Liénard equation

$$x''(t) + \frac{1}{8}(\sin x(t))x'(t) + g_1(t, x(t - \sin^2 t)) + g_2(t, x(t - \cos^2 t)) = \frac{1}{6\pi}e^{\cos t - 1}$$

(4.1)

has a unique $2\pi$-periodic solution.

**Proof.** By (4.1), we have $d = 1$, $b_1 + b_2 = (1/48\pi)$, $C_1 = C_2 = \frac{1}{8}$, $\tau_1(t) = \sin^2 t$, $\tau_2(t) = \cos^2 t$, $T = 2\pi$ and $p(t) = (1/6\pi)e^{\cos t - 1}$, then

$$\frac{[(b_1 + b_2)\delta + \max\{|g_1(t, 0)| + |g_2(t, 0)| : 0 \leq t \leq T\} + |p|_{\infty}T}{1 - [C_2T/2\pi + (b_1 + b_2)T^2/2\pi]} := D = \frac{1/48\pi \times 2\pi}{1 - \frac{1}{8} - \frac{1}{48}} = \frac{2}{41},$$

$$C_1 DT = \frac{T}{2\pi} + C_2 \frac{T}{2\pi} + (b_1 + b_2) \frac{T^2}{2\pi} = \frac{\pi}{82} + \frac{7}{48} < 1.$$ 

It is obvious that the assumptions $(A_1), (A_2)$ and $(A_4)$ hold. Hence, by Theorem 3.1, Eq. (4.1) has a unique $2\pi$-periodic solution. \qed

**Remark 4.1.** Eq. (4.1) is a very simple version of Liénard equation. Since $f(x) = \frac{1}{8}\sin x$, $\tau_1(t) = \sin^2 t$ and $\tau_2(t) = \cos^2 t$, all the results in [1,2,4–8] and the references therein are not applicable to Eq. (4.1) to obtain the existence and uniqueness of $2\pi$-periodic solutions. This implies that the results of this paper are essentially new.
References