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Periodic Solutions of Spatially Periodic, Even Hamiltonian Systems

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1. INTRODUCTION AND MAIN RESULTS

Let (M, ω) be a symplectic manifold and G a compact Lie group which acts symplectically on M, that is $\omega(gu, gv) = \omega(u, v)$ for $g \in G$, $u, v \in T_x M$. Let $H: M \times \mathbb{R} \to \mathbb{R}$ be a time dependent Hamiltonian function of class C^2 which is 2π-periodic in $t \in \mathbb{R}$ and invariant under the action of G: $H(gx, t) = H(x, t)$ for $g \in G$, $x \in M$, $t \in \mathbb{R}$. The time dependent Hamiltonian vector field X_t associated with H is defined by $\omega(\cdot, X_t) = dH_t$. We are interested in periodic solutions of the Hamiltonian system

$$\dot{z}(t) = X_t(z(t)). \tag{1.1}$$

This is an old problem and there has been a lot of work devoted to various aspects of it. In this paper we consider the following situation. Suppose the fixed point set $M^G = \{x \in M: gx = x \text{ for all } g \in G\}$ is discrete. Then the invariance of H under the action of G implies that the constant functions $z_0(t) \equiv x_0 \in M^G$ solve (1.1). These solutions are called trivial and we want to find nontrivial 2π-periodic solutions. The basic idea is to combine local information on H near a trivial periodic solution, given by its Conley–Zehnder or Maslov index, with global information on the topology of M and the behavior of H at infinity (if M is not compact).

In this generality the problem is complicated. We shall restrict our attention to the special case where $M = T^* T^N$ is the cotangent space of the N-torus, $G = \mathbb{Z}/2$ and the action of G on M is induced by the antipodal

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map on $\mathbb{R}^{2N}$. Before stating our result we discuss the result of our first paper [BaW] in this direction. There we considered the case where $M = T^{2N}$ is the $2N$-dimensional torus with its standard symplectic structure. The action of the group $G = \mathbb{Z}/2 = \{1, \tau\}$ on $T^{2N} = \mathbb{R}^{2N}/\mathbb{Z}^{2N}$ is also induced by the antipodal map $x \mapsto -x$ on $\mathbb{R}^{2N}$. This action has precisely 2^{2N} fixed points on M, namely all elements of the form $x = (x_1, ..., x_{2N})$ mod $\mathbb{Z}^{2N}$ with $x_i \in \{0, 1/2\}$, $i = 1, ..., 2N$. Thus there are already as many trivial stationary solutions of (1.1) forced by the symmetry as the Arnold conjecture predicts; cf. [A], [CoZ1]. To a 2π-periodic solution z of (1.1) on T^{2N} we may associate its Conley–Zehnder index $i(z) \in \mathbb{Z}$ and its nullity $v(z) \in \mathbb{N}_0$; see [CZ] or [L1, 2] for definitions. The main result of [BaW] is the following.

THEOREM 1.1. *Suppose all trivial solutions of* (1.1) *in* $(T^{2N})^G$ *are nondegenerate. Then* (1.1) *has at least* $\max |i(z_0)| - N$ *pairs of nontrivial 2π-periodic solutions where the maxiumum is taken over all trivial stationary solutions z_0 of* (1.1).

The nondegeneracy assumption in Theorem 1.1 can be weakened but this is not essential. The proof in [BaW] is based on a variational principle. For a 2π-periodic smooth map $z = (p, q) : S^1 = \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^{2N}$ we set

$$f(z) := \int_{S^1} \dot{q} \cdot p \, dt - \int_{S^1} H(p, q, t) \, dt. \tag{1.2}$$

This induces a function on $E = W^{1/2, 2}(S^1, \mathbb{R}^{2N})$ and on $(E_0 \oplus E_0^\perp)/\mathbb{Z}^{2N} \cong T^{2N} \times E_0^\perp$ where $E_0 \cong \mathbb{R}^{2N}$ is the space of constant maps. Critical points of f correspond to 2π-periodic solutions of (1.1). The invariance of H with respect to the reflection τ on T^{2N} implies that f is even, more precisely $f(\tau x, -u) = f(x, u)$ for $x \in T^{2N}$, $u \in E_0^\perp$. The main new problem in finding nontrivial critical points of f is the presence of the trivial critical points which are also fixed under the action of τ.

In this paper we consider the case where $M = T^* T^N \cong \mathbb{R}^N \times T^N$ is the cotangent space of the torus T^N with the natural symplectic structure. The action of the group $G = \mathbb{Z}/2 = \{1, \tau\}$ on M is again induced by the antipodal map on $\mathbb{R}^{2N}$. By abuse of notation we shall not distinguish between elements of M and $\mathbb{R}^{2N}$. We also write $x \mapsto -x$ for the action on M induced by the antipodal map on $\mathbb{R}^{2N}$. The fixed point set M^G contains precisely the 2^N elements of the form $x = (0, q)$ with $q = (q_1, ..., q_N)$ mod $\mathbb{Z}^N$ and $q_i \in \{0, 1/2\}$ for $i = 1, ..., N$. In addition to these fixed points the noncompactness of M causes problems. Therefore we need assumptions on the behavior of $H(p, q, t)$ as $|p| \to \infty$. We require the following hypotheses:

(H_1) H: $T^* T^N \times \mathbb{R} \to \mathbb{R}$ is of class C^2 and 2π-periodic in $t \in \mathbb{R}$.

(H_2) H is even: $H(-x, t) = H(x, t)$.

(H_3) There are constants $\mu > 1$ and $R > 0$ such that

$$0 < \mu H(p, q, t) \leqslant p \cdot H_p(p, q, t) \qquad \text{if} \quad |p| \geqslant R.$$

(H_4) There are constants $a > 0$, $s \leqslant \mu$ such that

$$|H_q(p, q, t)| \leqslant a(1 + |p|^s) \qquad \text{for all} \quad t \in \mathbb{R}, \ (p, q) \in M.$$

(H_5) All trivial solutions are nondegenerate.

For a 2π-periodic solution z_0 of (1.1) we let $i(z_0) \in \mathbb{Z}$ be its Conley–Zehnder index and $\nu(z_0) \in \mathbb{N}_0$ its nullity.

THEOREM 1.2. *Suppose* (H_1)–(H_5) *are satisfied. Let* z_0 *be a trivial solution of* (1.1).

(a) *If* $i(z_0) > N$ *then* (1.1) *has at least* $m = i(z_0) - N$ *pairs of nontrivial* 2π-*periodic solutions* $\pm z_1, ..., \pm z_m$. *If they are nondegenerate then their Conley–Zehnder indices are given by* $i(z_j) = i(z_0) - m - 1 + j = N - 1 + j$ *for* $j = 1, ..., m$.

(b) *If* $i(z_0) < 0$ *then* (1.1) *has at least* $m = |i(z_0)|$ *pairs of nontrivial* 2π-*periodic solutions* $\pm z_1, ..., \pm z_m$. *If they are nondegenerate then their Conley–Zehnder indices are given by* $i(z_j) = i(z_0) + j$ *for* $j = 1, ..., m$.

The result on the Conley–Zehnder indices can be used to distinguish the nontrivial solutions which we obtain. Suppose, for instance, w_0 and z_0 are trivial solutions with $i(w_0) < 0$ and $i(z_0) > N$. We can apply (a) of Theorem 1.2 to z_0 and (b) to w_0. Then we obtain $i(z_0) - N$ pairs of nontrivial solutions with Conley–Zehnder indices $N, N+1, ..., i(z_0) - 1$ and $|i(w_0)|$ such pairs with indices $i(w_0) + 1, i(w_0) + 2, ..., 0$. Thus we have $i(z_0) - N + |i(w_0)|$ different pairs of nontrivial solutions. This argument requires of course that the nontrivial solutions are nondegenerate.

The nondegeneracy assumptions in Theorem 1.2 can be weakened. We expect the result to be true without assumption (H_5). On the other hand, (H_5) is only an assumption on a finite number of known solutions which can be checked for a given Hamiltonian system. The statements on the Conley–Zehnder indices of the nontrivial periodic solutions can be generalized to the degenerate case provided the nontrivial solutions are isolated. Let z_j be one of the nontrivial solutions with Conley–Zehnder index $i(z_j)$ and nullity $\nu(z_j)$. In the case of 1.2(a) we have the inequalities

$$i(z_j) \leqslant i(z_0) - m - 1 + j \leqslant i(z_j) + \nu(z_j) \qquad \text{for} \quad j = 1, ..., m.$$

Similarly, in 1.2(b) we have the inequalities

$$i(z_j) \leqslant i(z_0) + j \leqslant i(z_j) + v(z_j) \qquad \text{for} \quad j = 1, ..., m.$$

Theorems 1.1 and 1.2 are quite different in spirit from other papers on Hamiltonian systems on T^{2N} or $T^* T^N$, or equivalently on spatially periodic Hamiltonian systems on $\mathbb{R}^{2N}$. In [CoZ1] Conley and Zehnder proved the existence of $2^{2N} 2\pi$-periodic solutions if $M = T^{2N}$, provided all periodic solutions are non-degenerate, and $2N+1$ without this assumption. Similarly, if $M = T^* T^N$ and if H satisfies certain growth conditions then one can prove the existence of 2^N periodic solutions (respectively $N+1$, if degenerate solutions are allowed); cf. [R3], [Fe], [FoM], for instance. These results have been generalized to the case $M = T^{2N_1} \times T^* T^{N_2} \times \mathbb{R}^{2N_3}$ by K. C. Chang in [Ch3], Theorem IV.2.5. If H is even as in Theorems 1.1 and 1.2 the papers quoted above do not yield any nontrivial solution. The presence of the many trivial solutions makes it difficult to find nontrivial solutions. Although we shall not treat the general case as in [Ch3] we want to mention that Theorems 1.1 and 1.2 can be combined to a more general result on even Hamiltonian systems on $T^{2N_1} \times T^* T^{N_2} \times \mathbb{R}^{2N_3}$. This does not need any new ideas, it merely complicates the notation.

The proof of Theorem 1.2 is also based on a variational approach using the functional f from (1.2). As in [BaW] we need some abstract critical point theory for even functionals defined on the product of a torus and a Hilbert space. Unfortunately, the technicalities are more complicated here. First, we cannot make an Amann–Zehnder type reduction to a finite-dimensional functional $\tilde{f}$. Second, we have to modify the Hamiltonian H for $|p|$ large because otherwise f will not be defined on all of E and will not satisfy the Palais–Smale condition even if restricted to $W^{1,2}(S^1, \mathbb{R}^{2N}) \subset E$. In section 2 we shall develop some abstract critical point theory for f generalizing our earlier result in [BaW] for $\tilde{f}$. We want to emphasize that due to the presence of the fixed points of the action of $\mathbb{Z}/2$ it does not seem possible to use standard tools like the genus or the cohomological index (cf. [FaR]). The main result of section 2, Theorem 2.2, will be proved in section 3 using Borel cohomology. Finally, in section 4 we prove Theorem 1.2. Here we shall follow the technical setting up of the problem as in Felmer's paper [Fe] although our final result is quite different in nature from the one in [Fe].

As a consequence of the abstract results of section 2 we obtain another existence theorem for periodic solutions of (1.1).

THEOREM 1.3. *Suppose* (H_1)–(H_4) *are satisfied with* $\mu > 2$ *in* (H_3). *Then* (1.1) *has infinitely many pairs of nontrivial* 2π*-periodic solutions* $\pm z_1$, $\pm z_2, \ldots$. *If* z_0 *is a trivial solution with Maslov index* $i(z_0)$ *and if all solutions* *are nondegenerate then the Maslov indices are given by* $i(z_j) = i(z_0) + j$ *for* $j \geqslant 1$.

If one is just interested in the existence of the infinitely many nontrivial periodic solutions then Theorem 1.3 follows from a special case of Theorem 2.2 which can be proved with the help of the genus. We sketch this in section 3.

2. SOME ABSTRACT CRITICAL POINT THEORY

Let E be a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and $B: E \times E \to \mathbb{R}$ a continuous symmetric bilinear form. This induces a quadratic form $Q: E \to \mathbb{R}$, $Q(z) = \frac{1}{2} B(z, z)$, and a bounded linear self adjoint map $L: E \to E$ which satisfies $B(z_1, z_2) = \langle L z_1, z_2 \rangle$. We assume that 0 is isolated in the spectrum of L and $M(0) := \ker L$ is finite dimensional. We call $\nu(Q) = \nu(L) = \dim M(0)$ the nullity of Q and L. We decompose $E = E^- \oplus M(0) \oplus E^+$ according to the spectrum of L. There exists $\alpha > 0$ such that $\pm Q(z) \geqslant \alpha \|z\|^2$ for $z \in E^\pm$. We also assume that E^+ and E^- are orthogonal sums of finite dimensional subspaces $M(\pm l)$, $l \in \mathbb{N}$, which are invariant under L, $E^+ = \bigoplus_{l \geqslant 1} M(l)$ and $E^- = \bigoplus_{l \leqslant -1} M(l)$. We set $E_k := \bigoplus_{|l| \leqslant k} M(l)$ and $E_k^\pm \cap E^\pm$, $k \in \mathbb{N}$. These data are fixed throughout this section.

Now we want to associate a relative Morse index to a nondegenerate compact perturbation of the quadratic form Q. We write $\mathcal{K}(E)$ for the space of self adjoint, compact linear maps $E \to E$. For $C \in \mathcal{K}(E)$ let $Q_C: E \to \mathbb{R}$, $Q_C(z) := \frac{1}{2} \langle Cz, z \rangle$, be the associated quadratic form. Then it is not difficult to see the following proposition.

PROPOSITION 2.1. *For* $C \in \mathcal{K}(E)$ *with* $Q + Q_C$ *nondegenerate there exists* $k_C \in \mathbb{N}$ *such that for* $k \geqslant k_C$ *the difference of Morse indices*

$$\mu((Q + Q_C)\,|_{E_k}) - \mu(Q\,|_{E_k}) = \mu((Q + Q_C)\,|_{E_k}) - \dim E_k^-$$

is independent of k.

For such $C \in \mathcal{K}(E)$ we may therefore define the relative Morse index

$$\mu_r(Q + Q_C) := \mu((Q + Q_C)\,|_{E_k}) - \mu(Q\,|_{E_k}), \qquad k \geqslant k_C.$$

Clearly we have $\nu(Q) = \dim M(0)$ and $\mu_r(Q) = 0$. It is very well possible that the relative Morse index $\mu_r(Q + Q_C)$ is negative.

It is also possible to deal with degenerate critical points (and degenerate periodic solutions of Hamiltonian systems).

We are interested in critical points of functionals $f \in C^2(E, \mathbb{R})$ of the form $f = Q + g$ where $g \in C^2(E, \mathbb{R})$ is such that the gradient $g' : E \to E$ is a compact nonlinear operator. For a critical point $z \in E$ of f the Hessian $C = g''(z) \in \mathscr{K}(E)$, so the nullity and the relative Morse index of $Q + Q_C$ are defined. We set $\nu(f, z) := \nu(Q + Q_C)$ and $\mu_r(f, z) := \mu_r(Q + Q_C)$. The basic assumptions on f and g are as follows.

(A1) $g \in C^2(E, \mathbb{R})$ and the gradient g' is compact.

(A_2) g is even: $g(-z) = g(z)$ for every $z \in E$.

(A_3) There are N linearly independent elements $e_1, ..., e_N \in E_0$ such that $g(z + \sum_{i=1}^N l_i e_i) = g(z)$ for all $z \in E$, $l_1, ..., l_N \in \mathbb{Z}$.

We set $T^N := \operatorname{span}\{e_1, ..., e_N\}/\mathbb{Z}^N$, $Z := (\operatorname{span}\{e_1, ..., e_N\})^\perp$ and $X := Z \cap (E^- \oplus E_0)$. Assumption ($A_4$) implies that f induces a C^2-function $T^N \times Z \to \mathbb{R}$ which we continue to denote by f. For simplicity of notation we write z both for elements of E and for elements of $T^N \times Z = E/\mathbb{Z}^N$.

(A_4) $f = Q + g$ satisfies the $(PS)^*$-condition with respect to the sequence (E_k). This means that any sequence $(z_{k_i})_{i \in \mathbb{N}}$ with $k_i \to \infty$, $z_{k_i} \in E_{k_i}$, $f(z_{k_i})$ bounded and $(f|_{E_{k_i}})'(z_{k_i}) \to 0$ has a subsequence which converges in $T^N \times Z$ to a critical point of f.

The antipodal map $z \mapsto -z$ on E induces an involution on $T^N \times Z$ which has precisely 2^N fixed points, namely all elements of the form $z = (x, 0) \in T^N \times Z$ with $x = \sum_{i=1}^N x_i e_i \bmod \mathbb{Z}^N$, $x_i \in \{0, 1/2\}$ for $i = 1, ..., N$. Since f is invariant under this symmetry by (A_3) these fixed points must be critical points of f. They are called trivial critical points.

For the main result of this section we need one more assumption on f.

(A_5) There exists a complement Y of X in Z and an integer $n \geqslant -1$ such that f is bounded below on $T^N \times (X_n \oplus Y) \subset T^N \times Z$. Here $X_n := X \cap E_n$, $X_{-1} := \{0\}$.

THEOREM 2.2. *Suppose* (A_1)–(A_5) *hold. Let z_0 be a nondegenerate trivial critical point and assume that all other trivial critical points z with $f(z) < f(z_0)$ are nondegenerate. Then f has at least $m := \mu_r(f, z_0) + \dim X_n - \dim E_0$ pairs of nontrivial critical points $\pm z_1, ..., \pm z_m$ with critical values between inf $f(T^N \times (X_n \oplus Y))$ and $f(z_0)$. If the nontrivial solutions are nondegenerate then their relative Morse indices are given by $\mu_r(f, z_j) = \mu_r(f, z_0) - m - 1 + j = \dim E^0 - \dim X_n - 1 + j$ for $j = 1, ..., m$.*

We postpone the proof of Theorem 2.2 to the next section. A few remarks are in order. We believe that the result is also true without the nondegeneracy assumption on the trivial critical points below the level $f(z_0)$. As with the corresponding hypothesis (H_5) in section 1 this nondegeneracy assumption is a generic assumption on f at a finite number of known points and can be checked for a given f.

If the nontrivial critical points are possibly degenerate then we only obtain two inequalities relating the relative Morse indices $\mu_r(f, z_j)$ and the nullities $\nu(f, z_j)$, namely

$$\mu_r(f, z_j) \leqslant \mu_r(f, z_0) - m - 1 + j \leqslant \mu_r(f, z_j) + \nu(f, z_j) \qquad \text{for} \quad j = 1, ..., m.$$

This follows from a slight modification of the proof of Theorem 2.2 in section 3 using critical groups; see section 3.

It is also possible to use a dimension argument as in the proof of Proposition 1.2. In fact, the critical points z_j are limits of almost critical points $z_{k,j} \in E_k$ as $k \to \infty$ and $f''(z_{k,j}) \to f''(z_j)$. Thus if $f''(z_j)$ is positive (or negative) definite on a subspace of E, so is $f''(z_{k,j})$ for k large.

Condition (A_5) can be somewhat weakened. It is possible to replace the subspace $X_n \oplus Y$ of Z by a more general subspace W of Z. Then the dimensions and codimensions of the intersections of W with E^+ and X come into play which would only complicate the statement of the result.

The proof of Theorem 2.2 uses machinery from algebraic topology, namely Borel cohomology for the group $\mathbb{Z}/2$. This is due to the fact that we have to find nontrivial critical points on level sets of f which may contain some of the trivial critical points. Theorem 2.2 can be proved in a more standard way using the genus of Krasnoselski and Yang if there are no other trivial critical points on levels below $f(z_0)$. Moreover, in that case we obtain a better bound for the number of nontrivial critical points.

THEOREM 2.3. *Suppose* (A_1)–(A_5) *hold. Let* z_0 *be a trivial critical point and assume that there are no other trivial critical points* z *with* $f(z) < f(z_0)$. *Then* f *has at least* $m + N = \mu_r(f, z_0) + \dim X_n - \dim E_0 + N$ *pairs of nontrivial critical points* $\pm z_1, ..., \pm z_{m+N}$ *with critical values between* $\inf f(T^N \times (X_n \oplus Y))$ *and* $f(z_0))$. *If the nontrivial solutions are nondegenerate then their relative Morse indices are given by*

$$\mu_r(f, z_j) = \mu_r(f, z_0) - m - 1 - N + j \qquad \textit{for} \quad j = 1, ..., m + N.$$

A sketch of the proof of Theorem 2.3 follows at the end of section 3. It is possible to improve the lower bound for the number of nontrivial critical points in Theorem 2.2 if one knows the number of trivial critical points z with $f(z) < f(z_0)$. Theorem 2.3 is the simplest example in this direction. It is also possible to replace the torus T^N by other compact manifolds M with an action of the group $G = \mathbb{Z}/2 = \{1, \tau\}$ and to investigate the existence of nontrivial critical points of even functionals $f: M \times Z \to \mathbb{R}$. This can be applied to investigate even Lagrangian or Hamiltonian systems on cotangent bundles T^*M instead of T^*T^N. We plan to pursue this in a sequel to this paper.

We conclude this section with two corollaries of Theorem 2.2 which are adapted to our application to Hamiltonian systems. There we have $\dim M(l) = 2N$ for all $l \in \mathbb{Z}$ which we assume from now on. Then the relative Morse index of f and the Morse index of $f_k := f|_{E_k}$ at the trivial critical point z_0 are related by the equation

$$\mu(f_k, z_0) = \mu_r(f, z_0) + 2kN \qquad \text{for} \quad k \text{ large.} \tag{2.1}$$

In our application z_0 is a stationary, hence, 2π-periodic solution of a Hamiltonian system and f is the associated action functional. To such a periodic solution one can associate its Conley–Zehnder index $i(z_0) \in \mathbb{Z}$ and its nullity $v(z_0) \geqslant 0$. It is easy to see that $v(z_0) = v(f, z_0)$ is the same as the nullity of z_0 as a critical point of f. Moreover, Long [Lo1] proved that the Conley–Zehnder index satisfies the equation (cf. Lemma 4.2)

$$\mu(f_k, z_0) = i(z_0) + (2k+1)N \qquad \text{for} \quad k \text{ large.} \tag{2.2}$$

From (2.1) and (2.2) we deduce

$$\mu_r(f, z_0) = i(z_0) + N. \tag{2.3}$$

Corollary 2.4. *Suppose the assumptions of Theorem 2.2 hold with $n = -1$ in (A_5), so f is bounded below on $T^N \times Y \subset T^N \times Z$. Then f has at least $\mu_r(f, z_0) - 2N = i(z_0) - N$ pairs of nontrivial critical points $\pm z_j$ with values between $\inf f(T^N \times Y)$ and $f(z_0)$. If they are nondegenerate then their Conley–Zehnder indices are given by $i(z_j) = i(z_0) - m - 1 + j = N - 1 + j$ for $j = 1, ..., m := i(z_0) - N$.* ∎

The corollary is only useful if the Maslov index satisfies $i(z_0) > N$, that is, if the relative Morse index satisfies $\mu_r(f, z_0) > 2N$. We now state a result which applies to the case where $i(z_0)$ is negative. For this we replace (A_5) by a dual assumption.

(A_6) There exists an integer $n \geqslant 0$ such that f is bounded above on $T^N \times (X \oplus E_n^+)$. Here $E_0^+ = \{0\}$.

Corollary 2.5. *Suppose (A_1)–(A_4) and (A_6) hold. Let z_0 be a nondegenerate trivial critical point such that all other trivial critical points z with $f(z) > f(z_0)$ are nondegenerate. Then f has at least*

$$m := -\mu_r(f, z_0) + (2n+1)N = -i(z_0) + 2nN$$

pairs of nontrivial critical points $\pm z_1, ..., \pm z_m$ *with critical values between* $f(z_0)$ *and* $\sup f(T^N \times (X \oplus E_n^+))$. *If they are nondegenerate then their Conley–Zehnder indices are given by* $i(z_j) = i(z_0) + j$ *for* $j = 1, ..., m$.

In the special case when f is bounded above on $T^N \times X$, z_0 is nondegenerate and the Conley–Zehnder index $i(z_0)$ is negative Corollary 2.5 yields $|i(z_0)|$ pairs of nontrivial critical points.

Proof of Corollary 2.5. We apply Theorem 2.2 to $\tilde{f} := -f$. Only some bookkeeping is needed for this. We set $\tilde{E}^+ := E^-$, $\tilde{E}^- := E^+$, $\tilde{M}(l) := M(-l)$ etc. It follows that

$$\begin{aligned} \mu(\tilde{f}_k, z_0) &= \dim E_k - \mu(f_k, z_0) \\ &= (2k+1)\, 2N - \mu(f_k, z_0) \end{aligned}$$

hence

$$\begin{aligned} \mu_r(\tilde{f}, z_0) &= \mu(\tilde{f}_k, z_0) - \dim \tilde{E}_k^- \\ &= 2N - \mu_r(f, z_0). \end{aligned}$$

With $\tilde{Z} = Z$, $\tilde{X} = Z \cap (E^+ \oplus E_0)$ and $\tilde{Y} = E^-$ assumption (A_6) implies that $\tilde{f}$ is bounded below on $T^N \times (\tilde{X}_n \oplus \tilde{Y})$. Theorem 2.2 yields at least

$$\mu_r(\tilde{f}, z_0) + \dim \tilde{X}_n - \dim \tilde{E}_0 = -\mu_r(f, z_0) + (2n+1)N$$

pairs of critical points of $\tilde{f}$ with critical values between $\inf \tilde{f}(T^N \times (\tilde{X}_n \oplus \tilde{Y}))$ and $\tilde{f}(z_0)$. The statement on the Conley–Zehnder indices follows from the corresponding statement on the relative Morse indices in 2.2 and the relation (2.3). ∎

3. PROOF OF THEOREM 2.2

In order to make this paper readable we first recall some notions from topology. With $H^*(A, B; \mathbb{F}_2) = H^*(A, B)$ we denote the Alexander–Spanier cohomology of a pair (A, B) of topological spaces with coefficients in the field $\mathbb{F}_2$ of two elements. Consequently $H^*(A, B)$ is a vector space over $\mathbb{F}_2$. Given a C^2-function $f: X \to \mathbb{R}$ on a manifold X and a critical point $x \in X$ of f the critical groups (vector spaces) of f at x are defined by

$$C^*(f, x) := H^*(f^c, f^c \backslash \{x\}) \qquad \text{where} \quad c = f(x).$$

Another way to define $C^*(f, x)$ for an isolated critical point x of f is by setting

$$C^*(f, x) = H^*(f^{c-\varepsilon} \cup U, f^{c-\varepsilon}),$$

where U is a neighborhood of x which contains no other critical point and such that the negative gradient flow φ^t associated to f satisfies $\varphi^t(U) \subset f^{c-\varepsilon} \cup U$ for all $t \geqslant 0$. In other words, the pair $(f^{c-\varepsilon} \cup U, f^{c-\varepsilon})$ is an index pair in the sense of Conley index theory. For an isolated critical point it is always possible to find $\varepsilon > 0$ and U as above. In fact they may be chosen arbitrarily small. If x is nondegenerate then $\dim C^k(f, x) = \delta_{k\mu}$ where $\mu = \mu(f, x)$ is the Morse index of f at x. If x is an isolated but possibly degenerate critical point then $C^k(f, x) = 0$ for $k \notin [\mu, \mu + \nu]$ where ν is the nullity of x. In the range $k \in [\mu, \mu + \nu]$ the critical vector space $C^k(f, x)$ can have arbitrary but finite dimension.

Now we consider spaces A with an action of the group $G = \mathbb{Z}/2 = \{1, \tau\}$, that is, with an involution τ on A. We write $A^G = \{x \in A : \tau x = x\}$ for the fixed point set and $A/G = A/(x \sim \tau x)$ for the orbit space. If the action is free, that is, if $\tau x \neq x$ for all $x \in A$, then the projection $A \to A/G$ is a fibre map. Moreover, A/G is a manifold if A is one. This is not the case in general. The Borel construction helps in the non-free case. We fix a contractible free G-space EG, for instance, the unit sphere in an infinite-dimensional normed $\mathbb{R}$-vector space where G acts via the antipodal map τ. The orbit space $BG := EG/G$ is then (homotopy equivalent to) the infinite projective space $\mathbb{R}P^\infty$. For a G-space A the product $EG \times A$ is a free G-space even if $A^G \neq \emptyset$. The orbit space $EG \times_G A := (EG \times A)/G$ or the bundle $EG \times_G A \to EG/G = BG$ is called the Borel construction for A. Given a pair $B \subset A$ of G-spaces the Borel cohomology of (A, B) is defined by

$$h^*(A, B) = H^*_G(A, B) := H^*(EG \times_G A, EG \times_G B).$$

The cup product in cohomology induces a product

$$\smile : h^k(A, B) \otimes h^l(A, C) \to h^{k+l}(A, B \cup C)$$

which turns $h^*(A)$ into a ring with unit $1_A \in h^0(A)$ and $h^*(A, B)$ into a graded commutative module over $h^*(A)$. The constant map $p_A : A \to pt$ turns $h^*(A, B)$ into a module over $R := h^*(pt)$ by setting $\alpha\xi := p_A^*(\alpha) \smile \xi$ for $\alpha \in R$, $\xi \in h^*(A, B)$. The coefficient ring R is isomorphic to $H^*(\mathbb{R}P^\infty) \cong \mathbb{F}_2[\omega]$ with $\omega \in h^1(pt) = H^1(\mathbb{R}P^\infty) \cong \mathbb{F}_2$. If A is a free G-space then $EG \times_G A$ is homotopy equivalent to A/G, hence $h^*(A, B) \cong H^*(A/G, B/G)$ for free G-spaces $B \subset A$. In particular, if G acts on the sphere S^{n-1} via the antipodal map then $h^*(S^{n-1}) \cong H^*(\mathbb{R}P^{n-1}) \cong \mathbb{F}_2[\omega]/\omega^n$ and the homomorphism

$$\mathbb{F}_2[\omega] \cong R = h^*(pt) \cong h^*(B^n) \to h^*(S^{n-1}) \cong \mathbb{F}_2[\omega]/\omega^n \tag{3.1}$$

which is induced by the inclusion $S^{n-1} \hookrightarrow B^n$ or by the constant map $S^{n-1} \to pt$ is the canonical quotient map.

If G acts on a manifold X and $f \in C^2(X, \mathbb{R})$ is invariant, that is $f(\tau x) = f(x)$ for $x \in X$, then one can define the equivariant critical groups of f at a critical orbit $Gx = \{x, \tau x\}$ by setting

$$C_G^*(f, Gx) := h^*(f^c \backslash Gx) \qquad \text{where} \quad c = f(x).$$

If $x \neq \tau x$ then using excision it follows that

$$\begin{aligned} C_G^*(f, Gx) &\cong h^*(f^c \cap U, f^c \cap U \backslash Gx) \\ &\cong H^*((f^c \cap U)/G, (f^c \cap U \backslash Gx)/G) \\ &\cong H^*(f^c/G, (f^c/G) \backslash \{Gx\}) \\ &\cong C^*(f, x), \end{aligned}$$

where U is a neighborhood of Gx such that G acts freely on U. In other words, the equivariant vector critical spaces of f at Gx are just the ordinary critical vector spaces of f at x, provided x is not a fixed point of the action.

After these preliminaries we begin with the proof of Theorem 2.2. For $k \in \mathbb{N}$ we let $f_k \colon T^N \times Z_k \to \mathbb{R}$ be the restriction of f to $T^N \times Z_k$, where $Z_k := E_k \cap Z$. For $j = 1, ..., m := \mu_r(f, u_0) + \dim X_n - \dim E_0$ we first define a sequence of almost critical values $c_{k,j}$, $k > n$. The definition of $c_{k,j}$ works as in [BaW]. We may assume that $z_0 = 0 \in T^* T^N$ and that $f(z_0) = 0$. We remind the reader that we do not distinguish between elements of T^N (or $T^* T^N$) and those of $\mathbb{R}^N$ (or $\mathbb{R}^N \times \mathbb{R}^N$). Setting $B := \{x \in T^N : \|x\| \leqslant 1/4\}$ and $S := \partial B = \{x \in T^N : \|x\| = 1/4\}$ we consider the following homomorphisms in Borel cohomology which are all induced by inclusions:

$$\begin{tikzcd} h^N(B, B \backslash \{0\}) \arrow[r, "i_1^*"] & h^N(B) \arrow[r] & h^N(B \backslash \{0\}) \cong 0 \\ h^N(T^N, T^N \backslash \{0\}) \arrow[u, "i_2^*"] \arrow[d, "i_3^*"] & & \\ h^N(T^N). & & \end{tikzcd}$$

The map i_2^* is an excision isomorphism and i_1^* is onto by exactness of the top row. Consequently there exists an element $\xi \in h^N(T^N, T^N \backslash \{0\})$ such that $i_1^* \circ i_2^*(\xi) = \omega^N \cdot 1_B \in h^N(B)$. We set $\eta := i_3^*(\xi) \in h^N(T^N)$ and write $\eta = \xi |_{T^N}$ since we think of η as the restriction of ξ to T^N. Similar notation will be used below.

For $k>n$ we set

$$d_k := \sum_{l=n+1}^{k} \dim M(-l) - 1$$

and define our almost critical values by

$$\begin{aligned} c_{k,j} &:= \inf\{c \in \mathbb{R} : \omega^{d_k+j}\pi^*(\eta)\,|_{f_k^c} \neq 0\} \\ &= \sup\{c \in \mathbb{R} : \omega^{d_k+j}\pi^*(\eta)\,|_{f_k^c} = 0\}. \end{aligned}$$

Here $\pi: T^N \times Z \to T^N$ is the projection, so $\pi^*(\eta) \in h^N(T^N \times Z)$, $\pi^*(\eta)\,|_{f_k^c} \in h^N(f_k^c)$ and $\omega^{d_k+j}\pi^*(\eta)\,|_{f_k^c} \in h^{d_k+j+N}(f_k^c)$. We need a uniform bound for the $c_{k,j}$.

Lemma 3.1. *There exist $b<0$ such that for $k>n$ large*

$$a := \inf f\,|_{T^N \times (X_n \oplus Y)} \leqslant c_{k,1} \leqslant \cdots \leqslant c_{k,m} \leqslant b.$$

Proof. For any $\varepsilon>0$ we have

$$f_k^{a-\varepsilon} \subset T^N \times (Z_k \backslash X_n \oplus Y_k) \simeq T^N \times S^{d_k-1} \tag{3.2}$$

because $d_k = \dim Z_k - \dim X_n \oplus Y_k$. Since G acts freely on $Z_k \backslash X_n \oplus Y_k \simeq S^{d_k-1}$ we have

$$\begin{aligned} h^r(T^N \times (Z_k \backslash X_n \oplus Y_k)) &\cong h^r(T^N \times S^{d_k-1}) \\ &\cong H^r((T^N \times S^{d_k-1})/G) \\ &\cong 0 \end{aligned}$$

for $r \geqslant N+d_k$. It follows that $\omega^{d_k+j}\pi^*(\eta)\,|_{T^N \times (Z_k \backslash X_n \oplus Y_k)} = 0$, hence $\omega^{d_k+j}\pi^*(\eta)\,|_{f_k^{a-\varepsilon}} = 0$ for every $j \geqslant 1$. This implies $c_{k,j} \geqslant a-\varepsilon$ for $k>n$, $j \geqslant 1$, any $\varepsilon>0$.

In order to see that $c_{k,1}, ..., c_{k,m}$ are bounded away from 0 let F^- be the negative eigenspace of $f''(0) = L + g''(0)$. Then we have for $\varepsilon>0$ small

$$b := \tfrac{1}{2} \sup\{f(z) : z \in F^-,\ \|z\| = \varepsilon\} < 0.$$

We fix such an $\varepsilon>0$. Let F_k^- be the negative eigenspace of $f_k''(0) = L\,|_{E_k} + (g \mid E_k)''\,(0)$. For k large we obtain

$$\sup\{f_k(z) = f(z) : z \in F_k^-,\ \|z\| = \varepsilon\} \leqslant b. \tag{3.3}$$

Therefore f_k^b contains the sphere S_k of radius ε in F_k^-. Now we look at the commutative diagram

$$\begin{array}{ccc} h^{\mu(f_k,0)-1}(T^N) & \xrightarrow{\pi^*} & h^{\mu(f_k,0)-1}(T^N\times Z) \\ \downarrow & & \downarrow \\ h^{\mu(f_k,0)-1}(B) & \xrightarrow{(\pi\mid S_k)^*} & h^{\mu(f_k,0)-1}(S_k). \end{array}$$

Observe that the projection $\pi\colon T^N\times Z\to T^N$ maps S_k into $B=\{x\in T^N : \|x\|\leqslant 1/4\}$, provided $\varepsilon\leqslant 1/4$. The vertical homomorphisms are induced by inclusions. Since by construction $\eta\mid_B=\omega^N\cdot 1_B$ we obtain

$$\begin{aligned}\omega^{\mu(f_k,0)-N-1}\pi^*(\eta)\mid_{S_k} &= (\pi\mid_{S_k})^*\,(\omega^{\mu(f_k,0)-N-1}\eta\mid_B)\\ &=\omega^{\mu(f_k,0)-1}1_{S_k}\\ &\neq 0.\end{aligned}$$

Here we used that $\dim S_k=\mu(f_k,0)-1$, so $h^*(S_k)\cong\mathbb{F}_2[\omega]/\omega^{\mu(f_k,0)}$. Consequently $\omega^{\mu(f_k,0)-N-1}\pi^*(\eta)\mid_{f_k^b}\neq 0$ and therefore $c_{k,j}\leqslant b$ for $j\leqslant m$ because $m=\mu(f_k,0)-N-1-d_k$, k large. ∎

The proof of Theorem 2.2 can now be concluded as follows. For k fixed the sublevel sets f_k^c change topology as c passes $c_{k,j}$. Therefore there exist points $z_{k,j}\in E_k$ with

$$\|f_k'(z_{k,j})\|+|f_k(z_{k,j})-c_{k,j}|<\frac{1}{k}.$$

Since f satisfies the $(PS)^*$-condition (A_4) the almost critical points $z_{k,j}$ converge along a subsequence towards a critical point of f. However, it remains to prove that we obtain m nontrivial pairs of critical points of f. After passing to subsequences we may assume that $c_{k,j}\to c_j$ as $k\to\infty$ and $a\leqslant c_1\leqslant\cdots\leqslant c_m\leqslant b$.

LEMMA 3.2. (a) *Suppose* $c=c_j=\cdots=c_{j+l}$ *for some* $j\in\{1,...,m\}$, $0\leqslant l\leqslant m-j$. *Let*

$$A_c:=K_c-K_c^G=\{z\in T^N\times Z : f(z)=c,\ f'(z)=0,\ z\neq -z \bmod \mathbb{Z}^N\}$$

be the set of nontrivial critical points of f *at the level* c. *Then* $\omega^l\cdot 1_{A_c}\neq 0\in h^l(A_c)$.

(b) *Suppose all critical points at the level* c_j *are nondegenerate. Then there exists a critical point* $z \in A_{c_j}$ *with relative Morse index* $\mu_r(f, z) = \mu_r(f, z_0) - m - 1 + j$.

As a consequence of Lemma 3.2(a) we have $A_{c_j} \neq \emptyset$ for every $j = 1, ..., m$. Moreover, if $c = c_j = \cdots = c_{j+l}$ for some $l \geqslant 1$ then $\dim A_c \geqslant l \geqslant 1$, hence A_c contains infinitely many elements. This proves the first statement in Theorem 2.2. The second statement on the relative Morse indices follows from 3.2(b). It remains to prove Lemma 3.2.

Proof of Lemma 3.2(a). Arguing indirectly we assume that $\omega^l \cdot 1_{A_c} = 0$. The continuity of Alexander–Spanier cohomology implies that $\omega^l \cdot 1_V = 0 \in h^l(V)$ for some invariant neighborhood V of A_c. Since the trivial critical points at the level c are nondegenerate by assumption the set K_c is the topological sum $K_c = A_c + K_c^G$. In addition, for $x \in K_c^G$ there exist $\varepsilon_x > 0$, $k_x \in \mathbb{N}$ and an invariant neighborhood W_x of x in $T^N \times Z$ with the following property. For $\varepsilon < \varepsilon_x$ and $k \geqslant k_x$ the pair $W_x \cap f_k^{c+\varepsilon}$, $\partial W_x \cap f_k^{c+\varepsilon}$ is homotopy equivalent (via odd maps) to the pair $(B^{\mu(f_k, x)}, S^{\mu(f_k, x)-1})$. This can be achieved as follows. Let F_x^+ respectively F_x^- be the positive respectively negative eigenspace of $f''(x) = L + g''(x)$. Then we set $W_x := x + B_{\delta_1}(F^+) \times B_{\delta_2}(F^-)$ where δ_1, $\delta_2 > 0$ are chosen such that

$$f(x + B_{\delta_1}(F^+) \times \partial B_{\delta_2}(F^-)) < f(x) - 2\varepsilon_x \qquad \text{for some} \quad \varepsilon_x > 0.$$

We may also assume that $\langle \nabla f(x+z), P_x^+ v \rangle > 0$ is bounded away from 0 uniformly for $z \in \partial B_{\delta_1}(F^+) \times B_{\delta_2}(F^-)$ with $x + z \notin f^{c-\varepsilon_x}$. Here $P_x^+ : E \to F_x^+$ is the orthogonal projection. Then one may use the negative gradient flow of f_k to produce a homotopy equivalence

$$(W_x \cap f_k^{c+\varepsilon_x}, \partial W_x \cap f_k^{c+\varepsilon_x}) \simeq (B^{\mu(f_k}, x), S^{\mu(f_k, x)-1})$$

provided k is large.

As a consequence of the construction of W_x we obtain from (3.1) that the homomorphism $h^*(W_x \cap f_k^{c+\varepsilon_x}) \to h^*(\partial W_x \cap f_k^{c+\varepsilon_x})$ which is induced by the inclusion, is onto. Setting $W := \bigcup_{x \in K_c^G} W_x$ we may assume that $V \cap W = \emptyset$. Now $U := V \cup W$ is an invariant neighborhood of $K_c = A_c \cup K_c^G$.

Let φ_k be the negative gradient flow associated to f_k on $T^N \times Z_k$. By standard arguments it follows from the Palais–Smale condition (A_4) that there exist $\varepsilon > 0$ and $k_0 \in \mathbb{N}$ with the following property. For any $k \geqslant k_0$, any $z \in f_k^{c_{k,j+l}+\varepsilon}$ there exists $T_k(z) \geqslant 0$ such that $\varphi_k^t(z) \in f_k^{c_{k,j}-\varepsilon} \cup U$ if $t \geqslant T_k(z)$. Thus the flow φ_k yields a deformation $h_k : f_k^{c_{k,j+l}+\varepsilon} \times [0, 1] \to f_k^{c_{k,j+l}+\varepsilon}$ satisfying $h_k(z, 0) = z$ and $h_k(z, 1) \in f_k^{c_{k,j}-\varepsilon} \cup U$. Setting $M_k := f_k^{c_{k,j}-\varepsilon} \cup U$ this implies that

$$\omega^{d_k+j+l} \cdot \pi^*(\eta) \mid_{M_k} \neq 0 \qquad \text{for} \quad k \text{ large.} \tag{3.4}$$

Next we look at the Mayer–Vietoris sequence of the triad $(M_k \backslash V; M_k \backslash U, W)$:

$$\begin{aligned} h^{d_k+j+N-1}(M_k \backslash U) \oplus h^{d_k+j+N-1}(W) &\longrightarrow h^{d_k+j+N-1}(W \cap M_k \backslash U) \\ &\xrightarrow{\delta} h^{d_k+j+N}(M_k \backslash V) \\ &\longrightarrow h^{d_k+j+N}(M_k \backslash U) \oplus h^{d_k+j+N}(W). \end{aligned}$$

We claim that $\zeta := \omega^{d_k+j}\pi^*(\eta)\,|_{M_k \backslash V} = 0 \in h^{d_k+j+N}(M_k \backslash V)$. First we observe that $W \cap M_k \backslash U = W \cap f_k^{c_{k,j}-\varepsilon}$. By construction of W it follows that $h^*(W) \to h^*(W \cap M_k \backslash U)$ is onto, hence, $\delta = 0$. Next we see that $\zeta\,|_{M_k \backslash U} \in h^*(M_k \backslash U)$ is zero by definition of $c_{k,j}$ because $M_k \backslash U \subset f_k^{c_{k,j}-\varepsilon}$. Moreover, $\zeta\,|_W \in h^*(W)$ is also zero because $\eta \in h^*(T^N)$ is the restriction of $\xi \in h^*(T^N, T^N \backslash \{0\})$, hence $\omega^{d_k+j}\pi^*(\eta) \in h^*(T^N \times Z)$ comes from $h^*(T^N \times Z,\ (T^N \backslash \{0\}) \times Z)$. Thus $\zeta\,|_W = \omega^{d_k+j}\pi^*(\eta)\,|_W = 0$ because $W \subset (T^N \backslash \{0\}) \times Z$. Now $\zeta = 0$ follows from the exactness of the Mayer–Vietoris sequence.

Since $\zeta = 0$ there exists $\alpha \in h^{d_k+j+N}(M_k, M_k \backslash V)$ such that $\alpha\,|_{M_k} = \omega^{d_k+j}\pi^*(\eta)\,|_{M_k} \in h^{d_k+j+N}(M_k)$. Moreover, there exists $\beta \in h^l(M_k, V)$ such that $\beta\,|_{M_k} = \omega^l \cdot 1_{M_k}$ because $\omega^l \cdot 1_V = 0 \in h^l(V)$. Here we used the exactness of the appropriate long exact sequences. The naturality of the cup product implies that $\alpha\beta\,|_{M_k} = \omega^{d_k+j+l}\pi^*(\eta)\,|_{M_k}$ which is not zero by (3.4). This contradicts the fact that $\alpha\beta$ lies in $h^{d_k+j+l+N}(M_k, (M_k \backslash V) \cup V) = 0$.

Proof of Lemma 3.2(b). For $z \in A_{c_j}$ let F_z^+ and F_z^- be the positive respectively negative eigenspace of $f''(z)$. We have $E = F_z^+ \oplus F_z^-$ because z is nondegenerate by assumption. Now we define $V_z := z + (B_{\delta_1}(F_z^+) \times B_{\delta_2}(F_z^-))$ where $\delta_1, \delta_2 > 0$ are chosen analogous to the choice in the proof of 3.2(a). Thus

$$f(z + (B_{\delta_1}(F_z^+) \times \partial B_{\delta_2}(F_z^-))) < f(z) - 2\varepsilon_0 = c_j - 2\varepsilon_0$$

for some $\varepsilon_0 > 0$, and $\langle \nabla f(z+y), P_z^+ y \rangle > 0$ is bounded away from 0 uniformly for $y \in \partial B_{\delta_1}(F_z^+) \times B_{\delta_2}(F_z^-)$ with $z + y \notin f^{c_j - \varepsilon_0}$. Since A_c is finite the constants $\delta_1, \delta_2, \varepsilon_0$ may be chosen independently of $z \in A_c$. Now we set $V := \bigcup_{z \in A_c} V_z$. In the proof of 3.2(a) we showed that for k large there exists $\alpha \in h^{d_k+j+N}(M_k, M_k \backslash V)$ such that $\alpha\,|_{M_k} = \omega^{d_k+j}\pi^*(\eta)\,|_{M_k} \neq 0 \in h^*(M_k)$; here $M_k = f_k^{c_{k,j}-\varepsilon} \cup U$ and $U = V \cup W$ with W a neighborhood of $K_{c_j}^G$. Since all trivial points $z \in A_{c_j}$ are nondegenerate the critical points of f_k at the level $c_{k,j}$ are also nondegenerate and contained in U for k large. Moreover, we have for k large that

$$h^*(M_k, M_k \backslash V) \cong \bigoplus_{Gz} C_G^*(f_k, Gz)$$

where Gz runs through all nontrivial critical orbits of f_k at the level $c_{k,j}$. Hence there exists $z_{k,j} \in T^N \times Z_k$ with $f'_k(z_{k,j}) = 0$, $f_k(z_{k,j}) = c_{k,j}$ and

$$C^{d_k+j+N}(f_k, z_{k,j}) = C_G^{d_k+j+N}(f_k, Gz_{k,j}) \neq 0.$$

This implies that the Morse index of f_k at $z_{k,j}$ is $d_k + j + N$. Now (A_4) implies $z_{k,j} \to z_j$ along a subsequence $k \to \infty$. Since f is of class C^2 we finally obtain (for k large)

$$\begin{aligned}
\mu_r(f, z_j) &= \mu((Q + Q_{g''(z_j)}) |_{E_k}) - \dim E_k^- \\
&= \mu((Q + Q_{g''(z_{k,j})}) |_{E_k}) - \dim E_k^- \\
&= d_k + j + N - \dim E_k^- \\
&= \mu_r(f, z_0) - m - 1 + j.
\end{aligned}$$

This concludes the proof of Lemma 3.2 and of Theorem 2.2. ∎

We conclude this section with a sketch of an elementary proof of Theorem 2.3. For a space A with an action of the group $G = \mathbb{Z}/2 = \{1, \tau\}$ we let

$$\begin{aligned}
\gamma(A) := \min\{k \in \mathbb{N} \colon &\text{ there exists } \varphi \colon A \to S^{k-1} \\
&\text{continuous with } \varphi(\tau x) = -\varphi(x)\}
\end{aligned}$$

denote the genus of A. By Borsuk's theorem $\gamma(S^{n-1}) = n$ if G acts via the antipodal map on S^{n-1}. Moreover $\gamma(A) = \infty$ if $A^G \neq \emptyset$.

As in the proof of Theorem 2.2 we first define almost critical values $c_{k,j}$ of $f_k = f|_{T^N \times Z_k}$. For $k > n$ we set

$$\begin{aligned}
c_{k,j} &:= \inf\{c \in \mathbb{R} : \gamma(f_k^c) \geqslant d_k + j\} \\
&= \sup\{c \in \mathbb{R} : \gamma(f_k^c) < d_k + j\}
\end{aligned}$$

where $d_k = \sum_{l=n+1}^{k} \dim M(-l) - 1$ is as before.

Using the monotonicity of the genus it follows from (3.2) that

$$\gamma(f_k^{a-\varepsilon}) \leqslant \gamma(T^N \times S^{d_k - 1}) = d_k$$

and from (3.3) that

$$\gamma(f_k^b) \geqslant \gamma(\{z \in F_k^- : \|z\| = \varepsilon\}) = \dim F_k^- = \mu(f_k, z_0).$$

Here $a = \inf f(T^N \times (X_k \oplus Y)) \leqslant b < f(z_0)$ are as in Lemma 3.1. It follows that

$$a \leqslant c_{k,j} \leqslant b \qquad \text{for} \quad j = 1, ..., \mu(f_k, z_0) - d_k = m + N.$$

After passing to subsequences we may assume that $c_{k,j} \to c_j$ as $k \to \infty$ and $a \leqslant c_1 \leqslant \cdots \leqslant c_{m+N} \leqslant b$. If $c = c_j = \cdots = c_{j+l}$ for some $j \in \{1, ..., m+N\}$, $0 \leqslant l \leqslant m + N - j$, then $\gamma(K_c) \geqslant l$ follows by standard arguments. Since there are no trivial critical points below the level $f(z_0)$ we have $K_c^G = \emptyset$. Thus $K_c \backslash K_c^G \neq \emptyset$ for any $c = c_j$ and $K_c \backslash K_c^G$ has infinitely many elements if $c = c_j = \cdots = c_{j+l}$ for some $l \geqslant 1$.

For the statement on the relative Morse indices one repeats the above argument replacing the genus by the cohomological index for $G = \mathbb{Z}/2$ due to Yang [Y] and Fadell and Rabinowitz [FaR]. As in the proof of Lemma 3.2 one obtains nontrivial critical points $z_{k,j}$ of f_k at the level $c_{k,j}$ with Morse index $d_k + j$. We leave the details to the interested reader.

4. PROOF OF THE MAIN RESULTS

4.1. First, we have some preliminaries on the variational set up of the problem. Consider the Hilbert space $E := W^{(1/2),2}(S^1, \mathbb{R}^{2N})$. Using Fourier series, $z(t) = \sum_{k \in \mathbb{Z}} a_k e^{ikt}$, $a_k = \bar{a}_k \in \mathbb{C}^N$, belongs to E if and only if

$$\|z\|_E^2 := \sum_{k \in \mathbb{Z}} (1 + |k|)\, |a_k|^2 < \infty.$$

Then for $z_1 = (p_1, q_1)$ and $z_2 = (p_2, q_2)$ in $W^{1,2}(S^1, \mathbb{R}^{2N})$

$$B(z_1, z_2) = \int_{S^1} (p_1 \cdot \dot{q}_2 + p_2 \cdot \dot{q}_1)\, dt$$

extends to a continuous bilinear form on E, and there exists a linear bounded selfadjoint operator $L: E \to E$ defined by

$$B(z_1, z_2) = \langle Lz_1, z_2 \rangle \qquad \text{for all} \quad z_1, z_2 \in E,$$

where $\langle\ ,\ \rangle$ denotes the inner product in E. Define

$$f(z) = \tfrac{1}{2}\langle Lz, z\rangle + g(z) = \tfrac{1}{2}\langle Lz, z\rangle - \int_{S^1} H(z(t), t))\, dt.$$

If $H \in C^1$ and in addition H satisfies the following growth condition for some constants a, $b > 0$ and $s > 1$,

$$H(z, t) \leqslant a\,|z|^s + b \qquad \text{for all} \quad z \in \mathbb{R}^{2N},\ t \in \mathbb{R}, \tag{4.1}$$

then it is known (see e.g. [R3]) that $f(z)$ is of class C^1 on E and that $z \in E$ is a critical point of f if and only if z is a 2π-periodic continuously differentiable solution of (1.1).

We shall need some decompositions of E. First, let $\{e_1, ..., e_{2N}\}$ be the canonical basis in $\mathbb{R}^{2N}$. Define for $l \in \mathbb{Z}$

$$M(l) = \operatorname{span}\{\sin(lt)\,e_k - \cos(lt)\,e_{k+N},\ \cos(lt)\,e_k + \sin(lt)\,e_{k+N} \mid 1 \leqslant k \leqslant N\}$$

so that as in section 2

$$E = \bigoplus_{l \in \mathbb{Z}} M(l) = E^- \oplus M(0) \oplus E^+ \tag{4.2}$$

with

$$E^- = \bigoplus_{l<0} M(l), \qquad E^+ = \bigoplus_{l>0} M(l)$$

An equivalent norm on E is defined by

$$\|z\|^2 = \tfrac{1}{2}\langle Lz^+, z^+\rangle - \tfrac{1}{2}\langle Lz^-, z^-\rangle + |z^0|^2,$$

where $z = z^- + z^0 + z^+$ with $z^\pm \in E^\pm$, $z^0 \in M(0)$, and $|\cdot|$ denotes the usual norm in $\mathbb{R}^{2N}$.

We shall also need another decomposition of E. Define for $l \in \mathbb{N}$

$$E_p(l) = \operatorname{span}\{\sin(lt)\,e_k,\ \cos(lt)\,e_k \mid 1 \leqslant k \leqslant N\}$$

and

$$E_q(l) = \operatorname{span}\{\sin(lt)\,e_{k+N},\ \cos(lt)\,e_{k+N} \mid 1 \leqslant k \leqslant N\}.$$

Note that

$$M(0) = E_p(0) \oplus E_q(0) = \operatorname{span}\{e_1, ..., e_N, e_{N+1}, ..., e_{2N}\}.$$

Now we set

$$E_p := \bigoplus_{l=0}^{\infty} E_p(l), \qquad E_q := \bigoplus_{l=1}^{\infty} E_q(l),$$

so that

$$E = E_p \oplus E_q \oplus E_q(0). \tag{4.3}$$

Using Fourier series and (4.2), (4.3), we see the following decomposition of E also holds:

$$E = (E^- \oplus E_p(0)) \oplus E_q \oplus E_q(0).$$

For $\sigma > 1$, we shall also consider $L^\sigma \equiv L^\sigma(S^1, \mathbb{R}^{2N})$. Then the following inequality holds (e.g. [Fr]).

$$\|z\|_\sigma \leqslant C_\sigma \|z\| \qquad \text{for all} \quad z \in E \tag{4.4}$$

for some constant $C_\sigma > 0$ where $\|\cdot\|_\sigma$ is the norm in L^σ.

For $z = z^- + z_p^0$ with $z^- \in E^-$ and $z_p^0 \in E_p(0)$ there exist $\beta_0 > 0$, $\beta_- > 0$ such that

$$\|z_p^0\|_\sigma \leqslant \beta_0 \|z\|_\sigma \tag{4.5}$$

$$\|z^-\|_\sigma \leqslant \beta_- \|z\|_\sigma. \tag{4.6}$$

We refer to [Fe] for these inequalities.

4.2. In this section, we shall prove a few lemmas which will be used to prove our main results later. Setting

$$X = E_- \oplus E_p(0), \qquad Y = E_q, \qquad Z = X \oplus Y,$$

we have $E = E_q(0) \oplus Z$. Because of (H_1), $f\colon E \to \mathbb{R}$ can be reduced to a functional defined on $T^N \times Z$ which we still denote by f as in Section 2. Thus we write

$$f(x, u) = \tfrac{1}{2}\langle Lu, u\rangle + g(x, u) \qquad \text{for} \quad (x, u) \in T^N \times Z.$$

As in section 2 we set

$$X_k = \bigoplus_{l=1}^{k} M(-l) \oplus E_p(0), \qquad Y_k = \bigoplus_{l=1}^{k} E_q(l) \qquad \text{and} \qquad Z_k = X_k \oplus Y_k.$$

Finally we set as before

$$f_k := f|_{T^N \times Z_k}.$$

Our first lemma is concerned with the $(PS)^*$ condition for f with respect to $T^N \times Z_k$ as formulated in (A_4). For this purpose we also assume the following:

(H_4') There exist constants $a > 0$ and $s < \mu$ such that

$$|H_z(p, q, t)| \leqslant a(1 + |p|^s) \qquad \text{for all} \quad z = (p, q) \in \mathbb{R}^{2N}, \quad t \in \mathbb{R}.$$

Note that (H_4') is stronger than (H_4). But we shall show later that we can modify the problem to satisfy (H_4').

LEMMA 4.1. *With Z_k and f_k being given above, f satisfies the $(PS)^*$ condition on $T^N \times Z$.*

Proof. The proof is analogous to the one of Lemma 3.2 in [Fe]. Consider a sequence $(x_{k_i}, u_{k_i}) \in T^N \times Z_{k_i}$ such that $k_i \to \infty$ and

$$f_{k_i}(x_{k_i}, u_{k_i}) \leqslant C, \qquad f'_{k_i}(x_{k_i}, u_{k_i}) \to 0 \qquad \text{as} \quad i \to \infty.$$

Then $(x_{k_i}) \subset T^N$ has a convergent subsequence. To show that (u_{k_i}) has a convergent subsequence, we first show that (u_{k_i}) is bounded in $Z = X \oplus Y$. Writing $u_{k_i} = p_{k_i} + q_{k_i}$ with $p_{k_i} \in E_p$ and $q_{k_i} \in E_q$, we have for i large

$$\begin{aligned} c + \|p_{k_i}\| &\geqslant f_{k_i}(x_{k_i}, u_{k_i}) - f'_{k_i}(x_{k_i}, u_{k_i})\, p_{k_i} \\ &= \tfrac{1}{2}\langle Lu_{k_i}, u_{k_i}\rangle - \langle Lu_{k_i}, p_{k_i}\rangle \\ &\quad - \int_0^{2\pi} H(x_{k_i} + u_{k_i}, t)\, dt + \int_0^{2\pi} H_p(x_{k_i} + u_{k_i}, t)\, p_{k_i}\, dt. \end{aligned}$$

Note that $\langle Lu_{k_i}, p_{k_i}\rangle = \int_0^{2\pi} p_{k_i} \cdot \dot{q}_{k_i} = \frac{1}{2}\langle Lu_{k_i}, u_{k_i}\rangle$. Using (H_3) we get

$$c + \|p_{k_i}\| \geqslant \left(1 - \frac{1}{\mu}\right) \mu \int_0^{2\pi} H(x_{k_i} + u_{k_i}, t)\, dt.$$

By (H_3) again we obtain

$$\|p_{k_i}\|^{\mu}_{L^\mu} \leqslant C(\|u_{k_i}\| + 1), \tag{4.7}$$

where C is independent of i. Next, for i large we clearly have

$$|f'_{k_i}(x_{k_i}, u_{k_i}) u^+_{k_i}| \leqslant \|u^+_{k_i}\|.$$

This means that

$$\left| \langle Lu_{k_i}, u^+_{k_i}\rangle - \int_0^{2\pi} H_z(x_{k_i} + u_{k_i}, t) u^+_{k_i}\, dt \right| \leqslant \|u^+_{k_i}\|$$

and therefore

$$\|u_{k_i}^+\|^2 \leqslant \langle Lu_{k_i}, u_{k_i}^+\rangle \leqslant \|u_{k_i}^+\| + \left|\int_0^{2\pi} H_z(x_{k_i}+u_{k_i}, t)u_{k_i}^+\,dt\right|. \tag{4.8}$$

From (H_4) and (4.4) it follows that

$$\left|\int_0^{2\pi} H_z(x_{k_i}+u_{k_i}, t)u_{k_i}^+\,dt\right| \leqslant C(1+\|p_{k_i}\|_\mu^s)\,\|u_{k_i}^+\|.$$

Using this together with (4.7) and (4.8) we now have

$$\|u_{k_i}^+\| \leqslant C(1+\|p_{k_i}\|_\mu^s).$$

Similarly, we can prove

$$\|u_{k_i}^-\| \leqslant C(1+\|p_{k_i}\|_\mu^s)$$

and obtain with (4.4)

$$\|u_{k_i}^+\| + \|u_{k_i}^-\| \leqslant C(1+\|u_{k_i}\|^{s/\mu}).$$

Writing $p_{k_i}^0$ for the projection of p_{k_i} in $E_p(0)$ we get from (4.5) and (4.7) that

$$|p_{k_i}^0| \leqslant C\,\|p_{k_i}\|_\mu \leqslant C(1+\|u_{k_i}\|^{1/\mu}).$$

This yields

$$\|u_{k_i}\| \leqslant C(1+\|u_{k_i}\|^{s/\mu}+\|u_{k_i}\|^{1/\mu})$$

so that (u_{k_i}) is bounded in $X\times Y$.

Since $(p_{k_i}^0)$ has a convergent subsequence, it remains to prove that $(u_{k_i}^+ + u_{k_i}^-)$ has a convergent subsequence. Let $u_{k_i}^+ + u_{k_i}^- \rightharpoonup u_0$ in E. For any fixed $k\geqslant 1$, we consider $v\in M(0)\oplus X_k\oplus Y_k$. Then $f'(x_{k_i}, u_{k_i})v\to 0$ and therefore

$$\begin{aligned} 0 &= \lim_{i\to\infty}\left[\langle Lu_{k_i}, v\rangle - \int_0^{2\pi} H_z(x_{k_i}+u_{k_i}, t)v\,dt\right] \\ &= \langle Lu_0, v\rangle - \int_0^{2\pi} H_z(y_0+u_0, t)v\,dt, \end{aligned}$$

where $y_0 = \lim_{i \to \infty} (p^0_{k_i} + q^0_{k_i})$. Here we also used the hypothesis (H_4'). This shows that (y_0, u_0) is a critical point of f. Using the fact that $L^{-1}: E^- \oplus E^+ \to E^- \oplus E^+$ is a compact bounded linear operator, we finally get $u^+_{k_i} + u^-_{k_i} \to u_0$ in E. ∎

LEMMA 4.2. *Let $z_0 \in T^N \times Z$ be a trivial critical point of f. Then for k large, the Morse index of z_0 with respect to f_k in $T^N \times Z_k$ is equal to $(2k+1)N + i(z_0)$, where $i(z_0)$ is the Maslov index of z_0 considered as a 2π-periodic solution of* (1.1).

Proof. See [Lo1] or [Ch3]. ∎

Next, since we only assume (H_4) instead a (H_4'), our functional f may not be defined on all of E. However, we can follow a procedure used in [Fe] (which was in fact first employed by Rabinowitz) to define a modified Hamiltonian that satisfies (H_4'); note that (H_4') implies (4.1). Then we show that the solutions obtained for this modified problem are indeed solutions of the original problem.

Let $K > 0$ and $\eta \in C^\infty(\mathbb{R}^+, \mathbb{R}^+)$ such that $\eta(y) = 1$ if $0 \leqslant y \leqslant K$, $\eta(y) = 0$ if $y \geqslant K+1$ and $\eta'(y) \leqslant 0$ for $K \leqslant y \leqslant K+1$. Define

$$H_K(p, q, t) = \eta(|p|)\, H(p, q, t) + (1 - \eta(|p|))\, M\, |p|^\mu,$$

where $M = M(K)$ is a number satisfying

$$M \geqslant \max_{K \leqslant |p| \leqslant K+1} \frac{H(p, q, t)}{|p|^\mu}.$$

Then H_K satisfies (H_1)–(H_3) and (H_4'). Note that (H_3) is satisfied uniformly in K so that there exists a constant $C_0 > 0$ such that

$$H_K(p, q, t) \geqslant C_0(|p|^\mu - 1), \tag{4.9}$$

where C_0 is independent of K. This implies that

$$f_K(x, u) = \tfrac{1}{2}\langle Lu, u\rangle - \int_{S^1} H_K(x+u, t)\, dt$$

is well defined on $T^N \times Z$ and of class C^2. The following estimate was basically proved by Felmer in [Fe] though it was not clearly stated in this form. For its proof we refer to [Fe].

LEMMA 4.3. *For any $c>0$ there exists $A=A(c)>0$ with the following property. Whenever $(x,u)\in T^N\times Z$ is a critical point of f_K for some $K>0$ and such that $f_K(x,u)\leqslant c$ then*

$$|u(t)|\leqslant A \qquad \text{for all} \quad t\in[0,2\pi].$$

We shall apply Corollaries 2.4 and 2.5 to our problem. In order to do so we need some estimates for the functionals f and f_K on certain subspaces.

LEMMA 4.4. *Assume* (H_1)–(H_4). *Then there exists $C>0$ such that for all $K>0$, f_K is bounded from below by $-C$ on $T^N\times(\{0\}\times Y)\subset T^N\times Z$.*

Proof. When we decompose $u=u_p+u_q$ with $u_p\in E_p$ and $u_q\in E_q$ then we have for $u\in\{0\}\times Y$ that $u_q=u$ and $u_p=0$. Consequently we obtain for $(x,u)\in T^N\times(\{0\}\times Y)$

$$\begin{aligned} f_K(x,u)&=\tfrac{1}{2}\langle Lu,u\rangle-\int_{S^1} H_K(x+u,t)\,dt\\ &=-\int_{S^1} H(x+u_q,\ t)\,dt\\ &\geqslant -C>-\infty \end{aligned}$$

for some $C>0$. Here we have used the fact that H is periodic in q. ∎

LEMMA 4.5 *Assume* (H_1)–(H_4). *Then there exists $C>0$ (independent of $K>0$) such that f_K is bounded above by C on $T^N\times(X\times\{0\})\subset T^N\times Z$.*

Proof. First we observe that $\langle Lu,u\rangle\leqslant 0$ for $(x,u)\in T^N\times X$. From (H_3) and (4.9) we thus obtain

$$\begin{aligned} f_K(x,u)&=\tfrac{1}{2}\langle Lu,u\rangle-\int_{S^1} H_K(x+u,t)\,dt\\ &=-C_0\int_{S^1}|u_p|^{\mu}\,dt+C_0 2\pi\\ &\leqslant 2\pi C_0<\infty. \quad ∎ \end{aligned}$$

LEMMA 4.6. *Assume* (H_1)–(H_4) *and in addition $\mu>2$ in* (H_3). *Then for each $n\geqslant 1$ there exists $C_n>0$ such that for all $K>0$*

$$f_K\,|_{T^N\times(X\oplus\bigoplus_{l=1}^{n} M(l))}\leqslant C_n.$$

Proof. First note that by Fourier expansion $\|u_p\| = \|u_q\|$ for $u \in E^+$, where $u = u_p + u_q$ with $u_p \in E_p$ and $u_q \in E_q$. Let us fix $n \geqslant 1$. For $(x, u) \in T^N \times (X \oplus \bigoplus_{l=1}^{n} M(l))$ we write $u = u^- + u^0 + u^+ \in E^- \oplus E_p(0) \oplus E^+$. By (4.9) we have

$$\begin{aligned} f_K(x, u) &= \tfrac{1}{2}\langle Lu, u\rangle - \int_{S^1} H_K(x+u, t)\, dt \\ &\leqslant \tfrac{1}{2}\langle Lu^+, u^+\rangle - C_0 \int_{S^1} |u_p|^\mu\, dt + 2\pi C_0 \\ &= n\, \|u^+\|_2^2 - C_0 \int_{S^1} |u_p|^\mu\, dt + 2\pi C_0 \\ &= 2n\, \|u_p\|_2^2 - C_0\, \|u_p\|_\mu^\mu + 2\pi C_0 \\ &\leqslant 2n(2\pi)^{(\mu-2)/\mu}\, \|u_p\|_\mu^2 - C_0\, \|u_p\|_\mu^\mu + 2\pi C_0 \\ &\leqslant C_n \end{aligned}$$

for some C_n depending only on n (independent of K). ∎

4.3. Finally we can prove our main results in this last section.

Proof of Theorem 1.2. First we observe that under (H_1)–(H_4) f_K satisfies (A_1)–(A_4) from section 2 for any $K > 0$. Now we distinguish two cases.

Case 1. $i(z_0) > N$.

By Lemma 4.3, we may consider f_K for large K instead of f provided we can get K-independent upper bounds for the critical values of f_K. By Lemma 4.4 and Corollary 2.4, f_K has at least $i(z_0) - N$ pairs of nontrivial critical points having critical values less than

$$f_K(z_0) = -\int_{S^1} H_K(z_0, t)\, dt = -\int_{S^1} H(z_0, t)\, dt$$

which is independent of K. Thus for K large all these critical points are critical points of the original functional f. The statements on the Conley–Zehnder indices of the critical points in Corollary 2.4 and Theorem 1.2 correspond to each other.

Case 2. $i(z_0) < 0$.

Note first that Lemma 4.5 implies that f_K satisfies (A_6) for all $K > 0$ with $n = 0$ and an upper bound $2\pi C_0$ independent of K. By Corollary 2.5, f_K has at least $-i(z_0)$ pairs of nontrivial critical points with critical values less than or equal to $2\pi C_0$. Thus for K large all these critical points are critical points of f. Since we assume z_0 is nondegenerate $v(z_0)=0$ and we get $|i(z_0)|$ pairs of nontrivial 2π-periodic solutions of (1.1) with the correct Conley–Zehnder indices. ∎

Proof of Theorem 1.3. Again under (H_1)–(H_4) f_K satisfies (A_1)–(A_4) for any $K>0$. Let $n \geqslant 1$ be fixed. Lemma 4.6 implies (A_6) holds for all f_K with an upper bound C_n independent of K. By Corollary 2.5 again f_K has at least $2nN-i(z_0)$ pairs of nontrivial critical points with critical values not larger than C_n. Thus for K large these are also critical points of f. Since n is arbitrary f has infinitely many pairs of nontrivial critical points. As before the statements on the Conley–Zehnder indices in 2.5 and 1.3 correspond to each other. ∎

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