J. Differential Equations 248 (2010) 2199–2226

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Existence and uniqueness of traveling waves for non-monotone integral equations with applications

Jian Fang [a,b], Xiao-Qiang Zhao [b,*]

[a] Department of Mathematics, Harbin Institute of Technology, Harbin 150001, PR China
[b] Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, NL A1C 5S7, Canada

ARTICLE INFO

Article history:
Received 18 April 2008
Revised 2 August 2009
Available online 13 February 2010

MSC:
35B30
35R10
45M20
92D25

Keywords:
Integral equations
Spreading speeds
Traveling waves
Existence and uniqueness

ABSTRACT

A class of integral equations without monotonicity is investigated. It is shown that there is a spreading speed $c^* > 0$ for such an integral equation, and that its limiting integral equation admits a unique traveling wave (up to translation) with speed $c \geqslant c^*$ and no traveling wave with $c < c^*$. These results are also applied to some nonlocal reaction–diffusion population models.

1. Introduction

In this work, we consider the following integral equation

$$u(t,z) = u_0(t,z) + \int_0^t \int_{\mathbb{R}} F\big(u(t-s, z-y), s, y\big)\, dy\, ds, \quad t \geqslant 0,\ z \in \mathbb{R}, \tag{1.1}$$

and its limiting form

* Corresponding author.
E-mail address: zhao@mun.ca (X.-Q. Zhao).

0022-0396/$ – see front matter
doi:10.1016/j.jde.2010.01.009

$$u(t,z)=\int_0^\infty\int_{\mathbb{R}} F\big(u(t-s,z-y),s,y\big)\,dy\,ds,\quad t\geqslant 0,\ z\in\mathbb{R}. \tag{1.2}$$

Thieme and Zhao [21] showed that the spreading speed of (1.1) coincides with the minimal wave speed of monotone traveling waves for (1.2) in the case where $F(u,s,y)$ is monotone in u. Note that certain population models may be described by (1.1) with appropriate choices of F (see, e.g., [5,19]). Also some reaction–diffusion models in population biology can be rewritten as the form (1.1), and the existence and uniqueness of traveling waves for these reaction–diffusion equations are equivalent to those for (1.2) (see examples in [21,25]). The main purpose of this paper is to study the nonexistence, existence and uniqueness (up to translation) of traveling waves for (1.2) in the case where $F(u,s,y)$ is non-monotone in u.

Throughout this paper, a traveling wave solution of (1.2) always refers to a pair (U,c), where U is a bounded, continuous, nonnegative and nonconstant function from $\mathbb{R}$ to $\mathbb{R}$ such that $u(t,z):=U(z+ct)$ satisfies (1.2). Clearly, $U(x)$ satisfies the following wave profile equation

$$u(x)=\int_0^\infty\int_{\mathbb{R}} F\big(u(x-y-cs),s,y\big)\,dy\,ds,\quad x\in\mathbb{R}. \tag{1.3}$$

We first recall some existing methods on the existence of traveling waves for non-monotone equations. Wu and Zou [23] and Huang and Zou [13] studied the existence of traveling waves for time-delayed reaction–diffusion equations by the exponential ordering and iteration method. Ou and Wu [17] established the persistence of wavefronts for time-delayed nonlocal reaction–diffusion equations via the perturbation method. But these results are valid only for small delays. Faria, Huang and Wu [9], also using the perturbation method, obtained the existence of traveling waves with large wave speed for time-delayed nonlocal reaction–diffusion equations. In [15], Ma employed Schauder's fixed point theorem to prove the existence of traveling waves with speed $c>c^*$ for the following time-delayed nonlocal reaction–diffusion equation

$$u_t(t,z)=Du_{xx}(t,z)-g\big(u(t,z)\big)+h\big(u(t,z)\big)\int_{\mathbb{R}} f\big(u(t-r,y)\big)J(z-y)\,dy,\quad t\geqslant 0,\ z\in\mathbb{R}. \tag{1.4}$$

However, the nonexistence of traveling waves with speed $c<c^*$ was not addressed in [15]. More recently, Hsu and Zhao [12] investigated the spreading speed and traveling waves for the non-monotone integro-difference equation

$$u_{n+1}(x)=\int_{\mathbb{R}} f\big(u_n(y)\big)k(x-y)\,dy,\quad x\in\mathbb{R},\ n\geqslant 0. \tag{1.5}$$

In particular, they also used Schauder's fixed point theorem to get the existence of traveling waves, but their constructions of convex subset are quite different from those in [15].

Regarding the uniqueness of traveling wave solutions, Diekmann and Kapper [6] studied the integral equation

$$u(x)=\int_{\mathbb{R}} g\big(u(y)\big)k(x-y)\,dy,\quad x\in\mathbb{R}. \tag{1.6}$$

They used the powerful Tauberian Ikehara's theorem for Laplace transforms to get the exact asymptotic behavior of monotone wave profiles at $-\infty$, and for non-monotone wave profiles, they provided a nice idea to estimate the asymptotic behavior of wave profiles. Carr and Chamj [3] employed this Tauberian method to study the integro-differential equation

$$u_t = J * u - u + f(u), \tag{1.7}$$

where $J * u(z) = \int_{\mathbb{R}} J(z-y)u(y)\,dy$, J has compact support and f is monostable. A different approach was developed by Chen and Guo [4] to establish the uniqueness of traveling waves for the lattice equation

$$\dot{u}_j = g(u_{j+1}) + g(u_{j-1}) - 2g(u_j) + f(u_j), \quad j \in \mathbb{Z}, \tag{1.8}$$

where g is increasing and f is monostable. They estimated the asymptotic behavior of the wave profile U by analyzing the limit of $\frac{U'(x)}{U(x)}$ as $x \to \infty$. Ma and Zou [16] used the same idea to prove the uniqueness of traveling waves with speed $c > c^*$ for the following time-delayed lattice equation

$$u_t(x,t) = D\big[u(x+1,t) + u(x-1,t) - 2u(x,t)\big] - du(x,t) + b\big(u(x,t-r)\big), \quad x \in \mathbb{R}, \tag{1.9}$$

where $b \in C^1(\mathbb{R})$ and $b(0) = 0 = dK - b(K)$ for some $K > 0$ under monostable assumption. As discussed in [3], the key point in these two methods is to estimate the asymptotic behavior of wave profiles at $-\infty$.

The rest of this paper is organized as follows. In Section 2, the spreading speed c^* for (1.1), as a complement of [21], is presented; the nonexistence of traveling waves with speed $c < c^*$ for (1.2) is then obtained by the result on spreading speeds; the existence of traveling waves with speed $c > c^*$ is established via Schauder's fixed point theorem, and the existence of the traveling wave with speed c^* is proved by a limiting argument. In Section 3, the exact asymptotic behavior of wave profiles at $-\infty$ is investigated, and then the uniqueness of traveling waves is proved by similar arguments as in [3,6]. Finally, these results are applied to some population models.

2. The existence

Assume that $F(u,s,y)$ is continuous in $u \in \mathbb{R}_+$ and Borel measurable in $(s,y) \in \mathbb{R}_+ \times \mathbb{R}$. We further impose the following conditions on F:

(A) There exists a function $k : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ such that:
- (A1) $k^* := \int_0^\infty \int_{\mathbb{R}} k(s,y)\,dy\,ds \in (1,\infty)$.
- (A2) $0 \leqslant F(u,s,y) \leqslant uk(s,y)$, $\forall u, s \geqslant 0$, $y \in \mathbb{R}$.
- (A3) For every compact interval I in $(0,\infty)$, there exists some $\epsilon > 0$ such that

$$F(u,s,y) \geqslant \epsilon k(s,y), \quad \forall u \in I,\ s \geqslant 0,\ y \in \mathbb{R}.$$

- (A4) For every $\epsilon > 0$, there exists some $\delta > 0$ such that

$$F(u,s,y) \geqslant (1-\epsilon)uk(s,y), \quad \forall u \in [0,\delta],\ s \geqslant 0,\ y \in \mathbb{R}.$$

As stated in [21], assumption (A2) implies that $F(0,s,y) = 0$, $\forall s \geqslant 0$, $y \in \mathbb{R}$. Though we do not assume that F is differentiable at $u = 0$, (A2) and (A4) together imply that k is something like the derivative of F at $u = 0$. With this in mind, (A2) also states that F is dominated by its linearization at $u = 0$.

Assumptions (A1) and (A2) imply that $\int_0^\infty \int_{\mathbb{R}} F(u,s,y)\,dy\,ds < \infty$, $\forall u \geqslant 0$. Define $\breve{F} : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\breve{F}(u) = \int_0^\infty \int_{\mathbb{R}} F(u,s,y)\,dy\,ds. \tag{2.1}$$

Let

$$F_+(u,s,y) = \max_{v \in [0,u]} F(v,s,y), \quad \forall s \geqslant 0,\ y \in \mathbb{R}.$$

Define $\breve{F}_+$ as in (2.1) with $F = F_+$. Suppose there exists $u_+^* > 0$ such that

$$\breve{F}_+\big(u_+^*\big) = u_+^* \quad \text{and} \quad \breve{F}_+(u) > u, \quad \forall u \in \big(0, u_+^*\big).$$

In addition, we need the following assumptions on F:

(A5) There exists $\Lambda > 0$ such that

$$\big|F(u_1,s,y) - F(u_2,s,y)\big| \leqslant \Lambda |u_1 - u_2| k(s,y), \quad \forall u_1, u_2 \in \big[0, u_+^*\big],\ s \geqslant 0,\ y \in \mathbb{R}.$$

(A6) There exist $\delta_0 \in (0, u_+^*]$, $\sigma > 1$ and $a > 0$ such that

$$F(u,s,y) \geqslant \big(u - a u^\sigma\big) k(s,y), \quad \forall u \in [0, \delta_0],\ s \geqslant 0,\ y \in \mathbb{R}.$$

(A7) There exists some $L > 0$ such that

$$\big|F(u,s,y_1) - F(u,s,y_2)\big| \leqslant L \big|k(s,y_1) - k(s,y_2)\big|, \quad \forall u \in \big[0, u_+^*\big],\ s \geqslant 0,\ y_1, y_2 \in \mathbb{R}.$$

Let

$$F_-(u,s,y) := \min_{v \in [u, u_+^*]} F(v,s,y), \quad \forall s \geqslant 0,\ y \in \mathbb{R}.$$

Then we can define $\breve{F}_-$ similarly. It then follows that $F_\pm$ are both continuous and nondecreasing in u. Clearly, $\breve{F}$ and $\breve{F}_\pm$ are all functions from $[0, u_+^*]$ to $[0, u_+^*]$ and

$$F_-(u,s,y) \leqslant F(u,s,y) \leqslant F_+(u,s,y), \quad \forall u \in \big[0, u_+^*\big],\ s \geqslant 0,\ y \in \mathbb{R}.$$

Lemma 2.1. *Let $F_\pm$, $\breve{F}_\pm$ and u_+^* be defined as above. Assume that F satisfies* (A1)–(A6)*. Then the following statements are valid*:

(1) *$F_\pm$ both satisfy* (A1)–(A6)*.*
(2) *There exist $u^* \in (0, u_+^*]$ and $u_-^* \in (0, u^*]$ such that*

$$\breve{F}\big(u^*\big) = u^*, \qquad \breve{F}(u) > u, \quad \forall u \in \big(0, u^*\big),$$

and

$$\breve{F}_-\big(u_-^*\big) = u_-^*, \qquad \breve{F}_-(u) > u, \quad \forall u \in \big(0, u_-^*\big).$$

Proof. (1) For F_+, it suffices to prove that (A2) and (A5) hold. By the definition of F_+, it follows that

$$F_+(u,s,y) = \max_{v\in[0,u]} F(v,s,y) \leqslant \max_{v\in[0,u]} vk(s,y) = uk(s,y),$$

and for any $u_+^* \geqslant u_1 > u_2 \geqslant 0$,

$$\begin{aligned} F_+(u_1,s,y) - F_+(u_2,s,y) &= \max_{v\in[0,u_1]} F(v,s,y) - \max_{v\in[0,u_2]} F(v,s,y) \\ &\leqslant \max\Big\{0, \max_{v\in[u_2,u_1]} F(v,s,y) - \max_{v\in[0,u_2]} F(v,s,y)\Big\} \\ &\leqslant \max_{v\in[u_2,u_1]} F(v,s,y) - F(u_2,s,y) \\ &\leqslant \max_{v\in[u_2,u_1]} \Lambda(v-u_2)k(s,y) \\ &= \Lambda(u_1-u_2)k(s,y). \end{aligned}$$

For F_-, it suffices to prove that (A3)–(A6) hold. (A3) is obvious. Let

$$\Omega_1 := \big\{(s,y)\in\mathbb{R}_+\times\mathbb{R}\colon k(s,y)=0\big\}, \qquad \Omega_2 := \mathbb{R}_+\times\mathbb{R}\setminus\Omega_1.$$

Then F satisfying (A2) implies

$$F(u,s,y)=0, \quad \forall u\geqslant 0,\ (s,y)\in\Omega_1,$$

and

$$\frac{F(u,s,y)}{k(s,y)} \leqslant u, \quad \forall u\geqslant 0,\ (s,y)\in\Omega_2.$$

Since F satisfies (A3)–(A4), it follows that for any $\epsilon>0$, there exist $\delta>0$ and $\eta>0$ such that

$$\frac{F(u,s,y)}{k(s,y)} \geqslant (1-\epsilon)u, \quad \forall u\in[0,\delta],\ (s,y)\in\Omega_2,$$

and

$$\frac{F(u,s,y)}{k(s,y)} \geqslant \eta, \quad \forall u\in\big[\delta,u_+^*\big],\ (s,y)\in\Omega_2.$$

Note that

$$\lim_{u\to 0^+}\frac{F(u,s,y)}{k(s,y)} = 0 \quad \text{uniformly for } (s,y)\in\Omega_2.$$

Then there exists $\delta_1\in(0,\delta)$ such that

$$\sup_{(s,y)\in\Omega_2}\frac{F(u,s,y)}{k(s,y)} < \eta, \quad \forall u\in[0,\delta_1].$$

Thus, for any $u\in[0,\delta_1]$,

$$F_-(u,s,y) = \min_{v\in[u,\delta]} F(v,s,y) \geqslant \min_{v\in[u,\delta]} (1-\epsilon)vk(s,y) = (1-\epsilon)uk(s,y),$$

and for any $u_+^* \geqslant u_1 > u_2 \geqslant 0$,

$$\begin{aligned} F_-(u_1,s,y) - F_-(u_2,s,y) &= \min_{v\in[u_1,u_+^*]} F(v,s,y) - \min_{v\in[u_2,u_+^*]} F(v,s,y) \\ &\leqslant \max\Big\{0, \min_{v\in[u_1,u_+^*]} F(v,s,y) - \min_{v\in[u_2,u_1]} F(v,s,y)\Big\} \\ &\leqslant F(u_1,s,y) - \min_{v\in[u_2,u_1]} F(v,s,y) \\ &\leqslant \max_{v\in[u_2,u_1]} \Lambda(u_1-v)k(s,y) \\ &= \Lambda(u_1-u_2)k(s,y). \end{aligned}$$

Note that there exist $\delta_0 \in (0, u_+^*]$, $\sigma > 1$ and $a > 0$ such that

$$F(u,s,y) \geqslant \big(u - au^\sigma\big)k(s,y), \quad \forall u \in [0,\delta_0],\ s \geqslant 0,\ y \in \mathbb{R},$$

and the function $u - au^\sigma$ is increasing when u is sufficiently small. It then follows that there exist $\delta_2 \in (0,\delta_0)$ and $\delta_3 \in (0,\delta_2)$ such that for any $u \in [0,\delta_3]$,

$$F_-(u,s,y) = \min_{v\in[u,\delta_2]} F(v,s,y) \geqslant \min_{v\in[u,\delta_2]} \big(v - av^\sigma\big)k(s,y) = \big(u - au^\sigma\big)k(s,y).$$

(2) Since F and F_- satisfy (A1) and (A4), there exists $\delta_4 > 0$ such that $\breve{F}(u) > u$ and $\breve{F}_-(u) > u$ when $u \in (0,\delta_4)$. Since

$$\breve{F}_-(u) \leqslant \breve{F}(u) \leqslant \breve{F}_+(u) \leqslant \breve{F}_+\big(u_+^*\big) = u_+^*, \quad \forall u \in \big[0, u_+^*\big],$$

it follows that such u^* and u_-^* exist. □

Since $F_\pm$ are both nondecreasing, the following two equations

$$u(t,z) = u_0(t,z) + \int_0^t \int_{\mathbb{R}} F_-\big(u(t-s,z-y),s,y\big)\,dy\,ds \tag{2.2}$$

and

$$u(t,z) = u_0(t,z) + \int_0^t \int_{\mathbb{R}} F_+\big(u(t-s,z-y),s,y\big)\,dy\,ds \tag{2.3}$$

admit the comparison principle (see [20, Lemma 3.2]).

We use the same definitions of $\mathcal{K}(c,\lambda)$ and c^* as in [21]. Let

$$\mathcal{K}(c,\lambda) := \int_0^\infty \int_{\mathbb{R}} e^{-\lambda(y+cs)}k(s,y)\,dy\,ds \tag{2.4}$$

and

$$c^* := \inf\{c \geqslant 0\colon \mathcal{K}(c,\lambda) < 1 \text{ for some } \lambda > 0\}. \tag{2.5}$$

We call $\mathcal{K}(c,\lambda) = 1$ as the characteristic equation. Note that the definitions of $\mathcal{K}(c,\lambda)$ and c^* involve only the function k. Thus, for Eqs. (1.1), (2.2) and (2.3), they have the same c^* and $\mathcal{K}(c,\lambda)$. In order to analyze $\mathcal{K}(c,\lambda)$, we make a couple of assumptions concerning k.

(B) $k : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a Borel measurable function such that:
 (B1) For any $c \geqslant 0$, there exists some $\lambda^\sharp = \lambda^\sharp(c) \in (0,\infty]$ such that $\mathcal{K}(c,\lambda) < \infty$ for $\lambda \in [0,\lambda^\sharp)$ and $\lim_{\lambda \uparrow \lambda^\sharp(c)} \mathcal{K}(c,\lambda) = \infty$.
 (B2) There exist numbers $\sigma_2 > \sigma_1 > 0$, $\rho > 0$ such that

$$k(s,y) > 0, \quad \forall s \in (\sigma_1,\sigma_2),\ |y| \in [0,\rho).$$

 (B3) k is isotropic, i.e., $k(s,y) = k(s,-y)$, $\forall s \geqslant 0$, $y \in \mathbb{R}$.

The following properties of c^* and $\mathcal{K}(c,\lambda)$ will be used.

Proposition 2.1. *(See [21, Lemmas 2.1 and 2.2 and Proposition 2.3].) Let* (B) *hold. Then $c^* \in (0,\infty)$, and the following statements are valid*:

(1) $\lim_{c\to\infty} \mathcal{K}(c,\lambda) = 0$ *for $\lambda \in (0,\lambda^\sharp(0)]$ with $\lambda^\sharp(c)$ from assumption* (B1).
(2) $\liminf_{c\to 0} \mathcal{K}(c,\lambda) \geqslant k^*$ *uniformly in $\lambda \geqslant 0$.*
(3) *For every $c > 0$, there is some $\lambda > 0$ such that $\mathcal{K}(c,\lambda) < \mathcal{K}(c,0) = k^*$.*
(4) *For $\lambda > 0$, $\mathcal{K}(c,\lambda)$ is a decreasing convex function of c.*
(5) *For $c \geqslant 0$, $\mathcal{K}(c,\lambda)$ is a convex function of λ.*
(6) *For any $c > c^*$, there exist $\lambda_1 < \lambda_2 < \lambda^\sharp(c)$ such that $\mathcal{K}(c,\lambda_1) = 1 = \mathcal{K}(c,\lambda_2)$ and $\mathcal{K}(c,\lambda) < 1$ for $\lambda \in (\lambda_1,\lambda_2)$.*
(7) *There exists a unique $\lambda^* \in (0,\lambda^\sharp(c))$ such that $\mathcal{K}(c^*,\lambda^*) = 1$ and $\mathcal{K}(c^*,\lambda) > 1$ for $\lambda \neq \lambda^*$. Moreover, c^* and λ^* are uniquely determined as the solutions of the system*

$$\mathcal{K}(c,\lambda) = 1, \qquad \frac{\partial \mathcal{K}}{\partial \lambda}(c,\lambda) = 0.$$

We say u_0 in (1.1) is admissible if for every $c, \lambda > 0$ with $\mathcal{K}(c,\lambda) < 1$, there exists some $\gamma > 0$ such that

$$u_0(t,z) \leqslant \gamma e^{\lambda(ct-|z|)}, \quad \forall t \geqslant 0,\ z \in \mathbb{R}. \tag{2.6}$$

We say $\breve{F}$ has the property (P) provided that:

(P) If $v, w \in (0,u_+^*]$ with $v \leqslant u^* \leqslant w$, $v \geqslant \breve{F}(w)$ and $w \leqslant \breve{F}(v)$, then $v = w$.

Then, by [12, Lemma 2.1] with $f = \breve{F}$, we have the following observation.

Lemma 2.2. *Either of the following two conditions is sufficient for the property* (P) *to hold*:

(P1) *$u\breve{F}(u)$ is strictly increasing for $u \in (0,u_+^*]$.*
(P2) *$\breve{F}(u)$ is nonincreasing for $u \in [u^*,u_+^*]$ and $\frac{\breve{F}^2(u)}{u}$ is strictly decreasing for $u \in (0,u^*]$, where $\breve{F}^2(u) = \breve{F}(\breve{F}(u))$.*

The following result complements [21, Proposition 2.4 and Theorem 2.5], which shows that c^* is the spreading speed of solutions to (1.1).

Theorem 2.1. *Assume that* (A1)–(A5) *and* (B) *hold. Let* $u(t,z)$ *be the unique solution of* (1.1)*. Then the following statements are valid*:

(1) *For every admissible* u_0, $u(t,z)$ *satisfies* $\lim_{t\to\infty,\,|z|\geqslant ct} u(t,z)=0$, $\forall c>c^*$.
(2) *Let* $u_0:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}_+$ *be a bounded and Borel measurable function with the property that* $u_0(t,z)\geqslant \eta>0$, $\forall t\in(t_1,t_2)$, $|z|\leqslant\eta$, *for appropriate* $t_2>t_1\geqslant 0$, $\eta>0$. *If* $u(t,z)$ *is bounded, then for each* $c\in(0,c^*)$, *there holds*

$$u_-^*\leqslant \liminf_{t\to\infty,\,|z|\leqslant ct} u(t,z)\leqslant \limsup_{t\to\infty,\,|z|\leqslant ct} u(t,z)\leqslant u_+^*.$$

(3) *Let assumptions in* (2) *hold. Assume that* $\frac{\check F(u)}{u}$ *is strictly decreasing for* $u\in(0,u_+^*]$ *and* $\check F(u)$ *has the property* (P)*. Then there holds* $\lim_{t\to\infty,\,|z|\leqslant ct} u(t,z)=u^*$, $\forall c\in(0,c^*)$.

Proof. Statement (1) is from [21, Theorem 2.1]. For statement (2), let u_+ and u_- be the solutions of (2.2) and (2.3), respectively. Then the comparison principles of (2.2) and (2.3) imply

$$u_-(t,z)\leqslant u(t,z)\leqslant u_+(t,z),\quad \forall t>0,\ z\in\mathbb{R}.$$

Since $F_\pm$ satisfy (A1)–(A5) and (B), we see from [21, Theorem 2.4] that c^* is the spreading speed for both equations (2.2) and (2.3), and hence $\lim_{t\to\infty,\,|z|\leqslant ct} u_\pm(t,z)=u_\pm^*$, $\forall c<c^*$. Thus, we reach the conclusion in (2).

For statement (3), we use similar arguments as in the proof of [12, Theorem 2.2(3)]. Using the same notations and arguments as in the proof of [21, Theorem 2.5], we see that for $0<c<\gamma<c^*$,

$$\begin{aligned} V_*(c,\gamma)&\geqslant \int_0^\infty\int_{\mathbb{R}} g\big(V_*(c,\gamma),V^*(c,\gamma),s,y\big)k(s,y)\,dy\,ds,\\ V^*(c,\gamma)&\leqslant \int_0^\infty\int_{\mathbb{R}} g\big(V^*(c,\gamma),V_*(c,\gamma),s,y\big)k(s,y)\,dy\,ds, \end{aligned}\tag{2.7}$$

where V_*, V^* and g are defined as follows

$$V_*(c,\gamma)=\inf_{c<\beta<\gamma}\ \liminf_{t\to\infty,\,|z|\leqslant\beta t} u(t,z),\qquad V^*(c,\gamma)=\sup_{c<\beta<\gamma}\ \limsup_{t\to\infty,\,|z|\leqslant\beta t} u(t,z),$$

and

$$g(v,w,s,y)=\begin{cases}\inf\{\tilde F(u,s,y):\ v\leqslant u\leqslant w\}, & \text{if } v\leqslant w,\\ \sup\{\tilde F(u,s,y):\ w\leqslant u\leqslant v\}, & \text{if } w\leqslant v,\end{cases}\tag{2.8}$$

with

$$\tilde F(u,s,x)=\begin{cases}\frac{F(u,s,x)}{k(s,x)}, & \text{if } k(s,x)>0,\\ 0, & \text{if } k(s,x)=0.\end{cases}$$

Moreover, $0 < V_*(c,\gamma) \leqslant V^*(c,\gamma)$. By the definition of g, we can find $v, w \in [V_*(c,\gamma), V^*(c,\gamma)] \subset (0, u_+^*]$ such that

$$\breve{F}(w) = \int_0^\infty \int_{\mathbb{R}} g\big(V_*(c,\gamma), V^*(c,\gamma), s, y\big) k(s,y)\,dy\,ds$$

and

$$\breve{F}(v) = \int_0^\infty \int_{\mathbb{R}} g\big(V^*(c,\gamma), V_*(c,\gamma), s, y\big) k(s,y)\,dy\,ds.$$

It then follows from (2.7) that

$$\frac{\breve{F}(w)}{w} \leqslant 1 = \frac{\breve{F}(u^*)}{u^*} \leqslant \frac{\breve{F}(v)}{v}. \tag{2.9}$$

This, together with the strict monotonicity of $\frac{\breve{F}(u)}{u}$ on $(0, u_+^*]$, shows that $v \leqslant u^* \leqslant w$. Then the property (P) implies that $v = w$. Thus, $0 < V_*(c,\lambda) = V^*(c,\lambda) \leqslant u_+^*$. From (2.7), (2.9) and the property of $\breve{F}$, we have $V_*(c,\lambda) = V^*(c,\lambda) = u^*$. By the definitions of $V_*(c,\lambda)$ and $V^*(c,\lambda)$, we arrive at

$$\lim_{t\to\infty,\, |z|\leqslant ct} u(t,z) = u^*, \quad \forall c \in \big(0, c^*\big).$$

This completes the proof. □

Now we are in a position to prove the main result of this section.

Theorem 2.2. *Let* (A) *and* (B) *hold. Then the following statements are valid*:

(1) *For any* $c \in (0, c^*)$, (1.2) *has no traveling wave* (U, c) *with* $\liminf_{x\to-\infty} U(x) < u_-^*$.
(2) *For any* $c > c^*$, (1.2) *has a traveling wave* (U, c) *with* $U(-\infty) = 0$; *for* $c = c^*$ *and any small number* $\beta > 0$, (1.2) *has a traveling wave* (U, c^*) *with* $U(0) = \beta$, $U(x) \leqslant \beta$, $\forall x < 0$; *and all these wave profiles have the following asymptotic behavior at* $+\infty$:

$$u_-^* \leqslant \liminf_{x\to+\infty} U(x) \leqslant \limsup_{x\to+\infty} U(x) \leqslant u_+^*.$$

If, in addition, $\frac{\breve{F}(u)}{u}$ *is strictly decreasing for* $u \in (0, u_+^*]$ *and* $\breve{F}(u)$ *has the property* (P), *then* $U(+\infty) = u^*$.

Proof. (1) Assume, by contradiction, that for some $c_0 \in (0, c^*)$, Eq. (1.2) has a traveling wave $u(t,z) := U(z + c_0 t)$. It then follows from Theorem 2.1(2) that

$$\liminf_{t\to\infty,\, |z|\leqslant ct} u(t,z) \geqslant u_-^*, \quad \forall c \in \big(0, c^*\big).$$

Choose $\tilde{c} \in (c_0, c^*)$ and let $z = -\tilde{c}t$. Then

$$\liminf_{t\to\infty} u(t, -\tilde{c}t) = \liminf_{t\to\infty} U\big((c_0 - \tilde{c})t\big) \geqslant u_-^*,$$

but $\liminf_{x\to-\infty} U(x) < u_-^*$, a contradiction.

(2) Let $\mathcal{C}_{u_+^*} := \{\phi \in C(\mathbb{R}, \mathbb{R}):\ 0 \leqslant \phi(x) \leqslant u_+^*,\ \forall x \in \mathbb{R}\}$. For given $c > c^*$, define a map $T : \mathcal{C}_{u_+^*} \to \mathcal{C}_{u_+^*}$ by

$$T(\phi)(x) = \int_0^\infty \int_{\mathbb{R}} F\big(\phi(x-y-cs), s, y\big)\,dy\,ds, \quad \forall x \in \mathbb{R},\ \phi \in \mathcal{C}_{u_+^*}. \tag{2.10}$$

Let $T^\pm$ be defined as in (2.10) with F replaced by $F_\pm$. It then follows that $T^\pm$ is nondecreasing with respect to the pointwise ordering on $\mathcal{C}_{u_+^*}$, and that

$$T^-(\phi) \leqslant T(\phi) \leqslant T^+(\phi), \quad \forall \phi \in \mathcal{C}_{u_+^*}. \tag{2.11}$$

Define

$$\phi^+(x) = \min\big\{u_+^* e^{\lambda_1 x}, u_+^*\big\}, \quad \forall x \in \mathbb{R},$$

and

$$\phi^-(x) = \max\big\{0, \delta\big(1 - Me^{\epsilon x}\big)e^{\lambda_1 x}\big\}, \quad \forall x \in \mathbb{R},$$

where $\lambda_1 = \lambda_1(c)$ is defined as in Proposition 2.1. Since $F_\pm$ both satisfy (A1)–(A6), by the proof of [21, Theorem 3.3], we can choose appropriate positive numbers δ, M and ϵ such that $T^+(\phi^+) \leqslant \phi^+$ and $T^-(\phi^-) \geqslant \phi^-$.

In order to apply Schauder's fixed point theorem, we need to construct a nonempty, closed and convex subset in a Banach space and define a compact operator on this subset. For a given $\lambda > 0$, let

$$X_\lambda := \Big\{\phi \in C(\mathbb{R}, \mathbb{R}):\ \sup_{x \in \mathbb{R}} \big|\phi(x)\big| e^{-\lambda x} < \infty\Big\}$$

and $\|\phi\|_\lambda = \sup_{x \in \mathbb{R}} |\phi(x)| e^{-\lambda x}$, then $(X_\lambda, \|\cdot\|_\lambda)$ is a Banach space. Note that for any given $\lambda \in (0, \lambda_1)$, $\phi^\pm$ are elements of X_λ. Thus, we can define the set

$$Y := \big\{\phi \in X_\lambda:\ \phi^- \leqslant \phi \leqslant \phi^+\big\},$$

which is nonempty, closed and convex subset of X_λ. For any $\phi \in Y$,

$$\phi^- \leqslant T^-\big(\phi^-\big) \leqslant T^-(\phi) \leqslant T(\phi) \leqslant T^+(\phi) \leqslant T^+\big(\phi^+\big) \leqslant \phi^+,$$

and hence, $T(Y) \subset Y$. Further, for any $\phi, \psi \in Y$, we have

$$\begin{aligned}
\big\|T(\phi) - T(\psi)\big\|_\lambda &\leqslant \int_0^\infty \int_{\mathbb{R}} \big|F\big(\phi(x-y-cs), s, y\big) - F\big(\psi(x-y-cs), s, y\big)\big| e^{-\lambda x}\,dy\,ds \\
&\leqslant \Lambda \int_0^\infty \int_{\mathbb{R}} \|\phi - \psi\|_\lambda k(s, y) e^{-\lambda(y+cs)}\,dy\,ds \\
&= \Lambda \mathcal{K}(c, \lambda)\|\phi - \psi\|_\lambda,
\end{aligned} \tag{2.12}$$

where $\mathcal{K}(c,\lambda) < +\infty$ since $\lambda < \lambda_1 < \lambda^\sharp(c)$. This implies that T is continuous on Y. It then follows from assumption (A7) that for any $\phi \in Y$, $x_1, x_2 \in \mathbb{R}$, there holds

$$\begin{aligned}
\bigl|T(\phi)(x_1) - T(\phi)(x_2)\bigr| &\leqslant \int_0^\infty\int_{\mathbb{R}} \bigl|F\bigl(\phi(y), s, x_1 - y - cs\bigr) - F\bigl(\phi(y), s, x_2 - y - cs\bigr)\bigr|\,dy\,ds \\
&\leqslant \int_0^\infty\int_{\mathbb{R}} L\bigl|k(s, x_1 - y - cs) - k(s, x_2 - y - cs)\bigr|\,dy\,ds \\
&\leqslant \int_0^\infty\int_{\mathbb{R}} L\bigl|k(s, x_1 - x_2 + y) - k(s, y)\bigr|\,dy\,ds \\
&= g(x_1 - x_2),
\end{aligned} \tag{2.13}$$

where $g(x) = \int_0^\infty\int_{\mathbb{R}} L|k(s, x+y) - k(s,y)|\,dy\,ds$. Clearly, $\lim_{x\to 0} g(x) = 0$. Therefore, $T(Y)$ is a family of uniformly bounded and equi-continuous functions. Thus, for any given sequence $\{\psi_n\}_{n\geqslant 1}$ in $T(Y)$, there exists a subsequence, still denoted by $\{\psi_n\}_{n\geqslant 1}$, and $\psi \in C(\mathbb{R},\mathbb{R})$ such that $\psi_n(x) \to \psi(x)$ uniformly for x in any compact subset of $\mathbb{R}$. Since $\phi^- \leqslant \psi_n \leqslant \phi^+$, we have $\phi^- \leqslant \psi \leqslant \phi^+$, and hence, $\psi \in Y$. Now it remains to show $\psi_n \to \psi$ in X_λ. Note that $\psi_n(x)e^{-\lambda x} \to \psi e^{-\lambda x}$ uniformly for x in any compact subset of $\mathbb{R}$ and $\lim_{|x|\to\infty} |\phi^+(x) - \phi^-(x)|e^{-\lambda x} = 0$. It then follows that for any $\epsilon > 0$, there exist $B > 0$ and $N > 1$ such that

$$0 \leqslant \bigl|\phi^+(x) - \phi^-(x)\bigr|e^{-\lambda x} < \epsilon, \quad \forall |x| \geqslant B,$$

and

$$\bigl|\psi_n(x) - \psi(x)\bigr|e^{-\lambda x} < \epsilon, \quad \forall |x| \leqslant B,\ n \geqslant N.$$

Thus,

$$\|\psi_n - \psi\|_\lambda < \epsilon, \quad \forall n \geqslant N.$$

Now Schauder's fixed point theorem implies that the operator T admits a fixed point U in Y. Clearly, $U(-\infty) = 0$ and U is continuous, nonnegative, nonconstant and bounded. Thus, (U, c) is a traveling wave solution connecting 0.

For $c = c^*$, we use a limiting argument (cf. [1] and [21]). Choose a sequence $\{c_j\} \subset (c^*, \infty)$ such that $\lim_{j\to\infty} c_j = c^*$. According to the above arguments, there exists a traveling wave (U_j, c_j) of (1.2) and for each j,

$$u_-^* \leqslant \liminf_{x\to+\infty} U_j(x) \leqslant \limsup_{x\to+\infty} U_j(x) \leqslant u_+^*.$$

Since each $U_j(x+h)$, $h \in \mathbb{R}$, is also such a solution, $U_j(-\infty) = 0$ and $\liminf_{x\to+\infty} U_j(x) \geqslant u_-^*$, we can assume that $U_j(0) = \beta < u_-^*$ and $U_j(x) \leqslant \beta$, $\forall x < 0$, $\forall j \geqslant 1$. For any $x_1, x_2 \in \mathbb{R}$, we have

$$\bigl|U_j(x_1) - U_j(x_2)\bigr| = \left|\int_0^\infty\int_{\mathbb{R}} F\bigl(U_j(y), s, x_1 - c_j s - y\bigr) - F\bigl(U_j(y), s, x_2 - c_j s - y\bigr)\,dy\,ds\right|$$

$$\leqslant \int_0^\infty \int_{\mathbb{R}} L \left| k(s, x_1 - c_j s - y) - k(s, x_2 - c_j s - y) \right| dy\, ds$$
$$= \int_0^\infty \int_{\mathbb{R}} L \left| k(s, y + x_1 - x_2) - k(s, y) \right| dy\, ds.$$

Thus, $\{U_j(x)\}$ is an equi-continuous and uniformly bounded sequence of functions on $\mathbb{R}$. By Ascoli's theorem and a nested subsequence argument, it follows that there exists a subsequence of $\{c_j\}$, still denoted by $\{c_j\}$, such that $U_j(x)$ converges uniformly on every bounded interval, and hence pointwise on $\mathbb{R}$ to $U^*(x)$. Note that

$$U_j(x) = \int_0^\infty \int_{\mathbb{R}} F\big(U_j(x - c_j s - y), s, y\big)\, dy\, ds, \quad \forall x \in \mathbb{R}. \tag{2.14}$$

Letting $j \to \infty$ in (2.14) and using the dominated convergence theorem, we then get

$$U^*(x) = \int_0^\infty \int_{\mathbb{R}} F\big(U^*\big(x - c^* s - y\big), s, y\big)\, dy\, ds, \quad \forall x \in \mathbb{R}, \tag{2.15}$$

and $U^*(0) = \beta$, $U^*(x) \leqslant \beta$, $\forall x < 0$.

By similar arguments as in the proof of [12, Theorem 3.1], it follows that for all these profiles with speed $c \geqslant c^*$, we have

$$u_-^* \leqslant \liminf_{x \to +\infty} U(x) \leqslant \limsup_{x \to +\infty} U(x) \leqslant u_+^*.$$

Indeed, for a fixed $c \geqslant c^*$, let $U(x) = U(z + ct)$ be a profile. Then for given $\bar{c} \in (0, c^*)$, it follows from Theorem 2.1(2) that

$$u_-^* \leqslant \liminf_{t \to \infty,\, |z| \leqslant \bar{c}t} U(z + ct) \leqslant \limsup_{t \to \infty,\, |z| \leqslant \bar{c}t} U(z + ct) \leqslant u_+^*,$$

and hence,

$$u_-^* \leqslant \liminf_{t \to \infty} U\big((c - \gamma)t\big) \leqslant \limsup_{t \to \infty} U\big((c - \gamma)t\big) \leqslant u_+^*$$

uniformly for $\gamma \in [0, \bar{c}]$. This implies that

$$u_-^* \leqslant \liminf_{t \to \infty} U(st) \leqslant \limsup_{t \to \infty} U(st) \leqslant u_+^*$$

uniformly for $s \in [c - \bar{c}, c]$. Let

$$a_n = n(c - \bar{c}), \qquad b_n = nc, \quad \forall n \geqslant 1.$$

Thus, there exists $N_0 > 1$ such that $a_{n+1} - b_n < 0, \forall n \geqslant N_0$, and hence,

$$\bigcup_{n\geqslant m}[a_n,b_n]=[a_m,+\infty),\quad \forall m\geqslant N_0.$$

It then follows that $u_-^*\leqslant \liminf_{x\to+\infty}U(x)\leqslant \limsup_{x\to+\infty}U(x)\leqslant u_+^*$. The rest part of statement (2) follows from Theorem 2.1(3) and the above arguments. □

For the case $c=c^*$ in Theorem 2.2(2), we may expect that there exists a traveling wave (U,c^*) with $U(-\infty)=0$. However, we can only show that for any small number $\beta>0$, there exists a wave profile U^* such that $\limsup_{x\to-\infty}U^*(x)<\beta$.

3. The uniqueness

In this section, we study the uniqueness of traveling waves established in Section 2. For any $c>c^*$, let $\lambda_1(c)$ be defined as in Proposition 2.1(6), and $\lambda_1(c^*):=\lambda^*$. For convenience, we use λ_1 to denote $\lambda_1(c)$. Given two real numbers c and λ, we define $l_{c,\lambda}(y):=e^{-\lambda y}\int_0^\infty k(s,y-cs)\,ds$, $\forall y\in\mathbb{R}$. Clearly, $\int_{\mathbb{R}}l_{c,\lambda}(y)\,dy=\mathcal{K}(c,\lambda)$. We impose the following condition on F and k.

(H) Assumptions (A1)–(A6) and (B) hold with $\Lambda=1$ in (A5), and $l_{c,\lambda}\in L^\infty(\mathbb{R})$ for any $c\geqslant c^*$ and $\lambda\in(0,\lambda_1)$.

Note that the assumption $\Lambda=1$ will be used only in the proof of Theorem 3.1. If $F(u,s,y)$ is differentiable in $u\geqslant 0$, then this technical assumption can be explained as $\partial_u F(u,s,y)\leqslant \partial_u F(0,s,y)$, $\forall u>0$, $s\geqslant 0$, $y\in\mathbb{R}$.

To prove the uniqueness of traveling waves for (1.2), we need a series of lemmas.

Lemma 3.1. *Assume* (H) *holds. Let c^* be defined as in* (2.5) *and u be the profile corresponding to a traveling wave connecting* 0 *of* (1.2) *with speed $c\geqslant c^*$. Then there exists a positive real number $\gamma=\gamma(c)$ such that $u(x)=o(e^{\gamma x})$ as $x\to-\infty$.*

Proof. From (A1), there exist $\epsilon>0$ and $M>1$ such that

$$k_M:=\int_0^\infty\int_{-M}^\infty(1-\epsilon)k(s,y)\,dy\,ds>1.$$

For the above ϵ, it follows from (A4) that there exists $\delta>0$ such that

$$F(u,s,x)\geqslant(1-\epsilon)uk(s,x),\quad \forall u\in[0,\delta],\ s\geqslant 0,\ x\in\mathbb{R}.$$

Since $u(-\infty)=0$, there exists $N>1$ such that for $x\leqslant -M-N$ and $x_1<x$, there holds

$$\begin{aligned}\int_{x_1}^x u(\xi)\,d\xi&=\int_{x_1}^x\int_0^\infty\int_{\mathbb{R}}F\big(u(\xi-y-cs),s,y\big)\,dy\,ds\,d\xi\\&\geqslant\int_{x_1}^x\int_0^\infty\int_{-M}^\infty F\big(u(\xi-y-cs),s,y\big)\,dy\,ds\,d\xi\\&\geqslant\int_{x_1}^x\int_0^\infty\int_{-M}^\infty(1-\epsilon)u(\xi-y-cs)k(s,y)\,dy\,ds\,d\xi.\end{aligned}\tag{3.1}$$

Consequently,

$$(1-k_M)\int_{x_1}^{x} u(\xi)\,d\xi \geqslant \int_{x_1}^{x}\int_{0}^{\infty}\int_{-M}^{\infty} (1-\epsilon)k(s,y)\big[u(\xi-y-cs)-u(\xi)\big]\,dy\,ds\,d\xi.$$

Note that for any $y\in\mathbb{R}$, $s\geqslant 0$, we have

$$\int_{x_1}^{x}\big[u(\xi-y-cs)-u(\xi)\big]\,d\xi=(y+cs)\int_{0}^{1}\big[u\big(x_1-t(y+cs)\big)-u\big(x-t(y+cs)\big)\big]\,dt.$$

It then follows from Fubini's theorem that

$$\begin{aligned}
&(1-k_M)\int_{x_1}^{x} u(\xi)\,d\xi\\
&\quad\geqslant \int_{0}^{\infty}\int_{-M}^{\infty}(1-\epsilon)k(s,y)\int_{x_1}^{x}\big[u(\xi-y-cs)-u(\xi)\big]\,d\xi\,dy\,ds\\
&\quad= \int_{0}^{\infty}\int_{-M}^{\infty}(1-\epsilon)k(s,y)(y+cs)\left\{\int_{0}^{1}\big[u\big(x_1-t(y+cs)\big)-u\big(x-t(y+cs)\big)\big]\,dt\right\}dy\,ds. \qquad (3.2)
\end{aligned}$$

By assumption (B1), we know that $\mathcal{K}(c,\lambda)$ is infinitely often differentiable for λ in some interval $[0,\delta]$ with $\delta>0$. Thus, it is easy to verify that

$$\frac{d}{d\lambda}\mathcal{K}(1,\lambda)\bigg|_{\lambda=0}=-\int_{0}^{\infty}\int_{\mathbb{R}}k(s,y)(s+y)\,dy\,ds$$

and

$$\frac{d^2}{d\lambda^2}\mathcal{K}(0,\lambda)\bigg|_{\lambda=0}=\int_{0}^{\infty}\int_{\mathbb{R}}y^2k(s,y)\,dy\,ds.$$

Since $|y|\leqslant\frac{1}{2}(1+y^2)$, $\int_0^\infty\int_{\mathbb{R}}|y|k(s,y)\,dy\,ds$ exists. It follows that for each $c>0$, there exists $a_c>0$ such that

$$\int_{0}^{\infty}\int_{\mathbb{R}}k(s,y)|y+cs|\,dy\,ds<a_c.$$

Observing each traveling wave is bounded, that is, there exists a bound $d>0$ such that $u(x)<d$, $x\in\mathbb{R}$, we see that as $x_1\to-\infty$ in (3.2),

$$\begin{aligned}
\int_{-\infty}^{x} u(\xi)\,d\xi &\leqslant \frac{1}{k_M-1}\int_0^\infty\int_{-M}^\infty (1-\epsilon)k(s,y)(y+cs)\int_0^1 u\bigl(x-t(y+cs)\bigr)\,dt\,dy\,ds\\
&\leqslant \frac{d}{k_M-1}\int_0^\infty\int_{-M}^\infty (1-\epsilon)k(s,y)|y+cs|\,dy\,ds\\
&\leqslant \frac{d}{k_M-1}\int_0^\infty\int_{\mathbb{R}} (1-\epsilon)k(s,y)|y+cs|\,dy\,ds\\
&< \frac{da_c(1-\epsilon)}{k_M-1}.
\end{aligned}\tag{3.3}$$

Since $v(x) := \int_{-\infty}^{x} u(\xi)\,d\xi$ is increasing, it follows from (3.3) that

$$\begin{aligned}
\int_{-\infty}^{x} v(\xi)\,d\xi &\leqslant \frac{1}{k_M-1}\int_0^\infty\int_{-M}^\infty (1-\epsilon)k(s,y)(y+cs)\int_0^1 v\bigl(x-t(y+cs)\bigr)\,dt\,dy\,ds\\
&\leqslant \frac{1-\epsilon}{k_M-1}\int_0^\infty\int_{-M}^\infty k(s,y)|y+cs|\int_0^1 v(x+tM)\,dt\,dy\,ds\\
&\leqslant \frac{1-\epsilon}{k_M-1}\int_0^\infty\int_{-M}^\infty k(s,y)|y+cs|v(x+M)\,dy\,ds\\
&< \frac{a_c(1-\epsilon)}{k_M-1}v(x+M).
\end{aligned}\tag{3.4}$$

Choose $r_0 > 0$ such that $\mu := \frac{a_c(1-\epsilon)}{r_0(k_M-1)} < 1$. Then for any $x \leqslant -M-N$, we have

$$v(x-r_0) \leqslant \frac{1}{r_0}\int_{x-r_0}^{x} v(\xi)\,d\xi \leqslant \frac{1}{r_0}\int_{-\infty}^{x} v(\xi)\,d\xi \leqslant \mu v(x+M).$$

Define $h(x) = v(x)e^{-\gamma_1 x}$, where $\gamma_1 = \frac{1}{M+r_0}\ln\frac{1}{\mu}$. Then we have

$$\begin{aligned}
h(x-r_0) &= v(x-r_0)e^{-\gamma_1(x-r_0)}\\
&\leqslant \mu v(x+M)e^{-\gamma_1(x+M)+\gamma_1(M+r_0)}\\
&= \mu e^{\gamma_1(M+r_0)}h(x+M)\\
&= h(x+M),
\end{aligned}\tag{3.5}$$

which shows h is bounded. Consequently, $v(x) = O(e^{\gamma_1 x})$ as $x \to -\infty$ and there exists $p > 0$ such that $v(x) \leqslant pe^{\gamma_1 x}$, $\forall x \leqslant 0$. Now we claim that for any $\gamma \in (0,\gamma_1)$, $u(x) = o(e^{\gamma x})$ as $x \to -\infty$. Indeed, define the iterative scheme $u^{(1)}(x) := v(x)$, $u^{(k)}(x) = \int_{-\infty}^{x} u^{(k-1)}(\xi)\,d\xi$, $k > 1$. It then follows that $u^{(1)}(x) = v(x) \leqslant pe^{\gamma_1 x}$ and $u^{(k)}(x) \leqslant \frac{p}{\gamma_1^{k-1}} e^{\gamma_1 x}$, $\forall k > 1$, $x \leqslant 0$. Using [6, Lemma 4.4] with $f = u$, we have $\frac{1}{k!}\int_{-\infty}^{x} (x-\xi)^k u(\xi)\, d\xi = u^{(k+1)}(x) \leqslant \frac{p}{\gamma_1^k} e^{\gamma_1 x}$, $\forall k \geqslant 0$, $x \leqslant 0$. Thus, we have

$$\int_{-\infty}^{x} e^{\lambda(x-\xi)} u(\xi)\, d\xi = \sum_{k=0}^{\infty} \frac{1}{k!}\lambda^k \int_{-\infty}^{x} (x-\xi)^k u(\xi)\, d\xi \leqslant pe^{\gamma_1 x} \sum_{k=0}^{\infty} \left(\frac{\lambda}{\gamma_1}\right)^k, \quad \forall \lambda \in (0, \gamma_1),\ x \leqslant 0.$$

This shows that $\int_{-\infty}^{x} e^{-\lambda\xi} u(\xi)\, d\xi$ is convergent for $\lambda < \gamma_1$, so is $\int_{\mathbb{R}} e^{-\lambda\xi} u(\xi)\, d\xi$. On the other hand, for any $\lambda \in (0, \min(\lambda_1, \gamma_1))$, we have

$$\begin{aligned} e^{-\lambda x} u(x) &= e^{-\lambda x} \int_0^{\infty}\int_{\mathbb{R}} F\big(u(x-y-cs), s, y\big)\, dy\, ds \\ &\leqslant e^{-\lambda x} \int_0^{\infty}\int_{\mathbb{R}} u(x-y-cs) k(s, y)\, dy\, ds \\ &= \int_{\mathbb{R}} u(x-y) e^{-\lambda(x-y)} l_{c,\lambda}(y)\, dy \\ &\leqslant \|l_{c,\lambda}\|_{\infty} \int_{\mathbb{R}} u(y) e^{-\lambda y}\, dy < +\infty. \end{aligned}$$

It then follows that for any $\gamma \in (0, \min(\lambda_1, \gamma_1))$, $u(x) = o(e^{\gamma x})$ as $x \to -\infty$. $\square$

In order to distinguish the traveling waves with different speeds, in the rest of this section, we use $u(c, x)$ to denote the profile corresponding to the traveling wave with speed c. For each $c > 0$ and λ satisfying $0 < \operatorname{Re}\lambda < \gamma$, by Lemma 3.1, we can define the Laplace transform

$$U(c, \lambda) := \int_{\mathbb{R}} e^{-\lambda x} u(c, x)\, dx.$$

Then we have the following observation.

Lemma 3.2. *Assume that* (H) *holds. Then for each* $c \geqslant c^*$, $U(c, \lambda)$ *is analytic for* $\operatorname{Re}\lambda \in (0, \lambda_1)$, *and has a singularity at* $\lambda = \lambda_1$.

Proof. Rewrite the wave profile equation (1.3) as

$$u(c, x) - \int_0^{\infty}\int_{\mathbb{R}} u(c, x-y-cs) k(s, y)\, dy\, ds = R(u)(x), \tag{3.6}$$

where

$$R(u)(x) = \int_0^{\infty}\int_{\mathbb{R}} \big[F\big(u(c, x-y-cs), s, y\big) - u(c, x-y-cs) k(s, y)\big]\, dy\, ds.$$

It then follows that

$$\left[1-\mathcal{K}(c,\lambda)\right]U(c,\lambda)=\int_{\mathbb{R}} e^{-\lambda x}R(u)(x)\,dx. \tag{3.7}$$

We first claim that, if the left-hand side of (3.7) is analytic for $\operatorname{Re}\lambda \in (0,r)$, $r<\lambda^{\sharp}(c)$, where $\lambda^{\sharp}(c)$ is defined as in assumption (B1), then there exists $\eta>0$ such that the right-hand side of (3.7) is analytic for $\operatorname{Re}\lambda \in (0,r+\eta)$. Indeed, let γ be defined as in Lemma 3.1. Since

$$0 \geqslant R(u)(x) \geqslant -a\int_0^{\infty}\int_{\mathbb{R}} u^{\sigma}(c,x-y-cs)k(s,y)\,dy\,ds,$$

by choosing η such that $\frac{\eta}{\sigma-1}<\gamma$ and $r+\eta<\lambda^{\sharp}(c)$, we see that $u(c,y)e^{-\frac{\eta}{\sigma-1}y}$ is bounded by a positive number M, and for any $\lambda \in (0,r+\eta)$,

$$\begin{aligned}
\left|\int_{\mathbb{R}} e^{-\lambda x}R(u)(x)\,dx\right| &= -\int_{\mathbb{R}} e^{-\lambda x}R(u)(x)\,dx \\
&\leqslant a\int_{\mathbb{R}} e^{-\lambda x}\int_0^{\infty}\int_{\mathbb{R}} u^{\sigma}(c,x-y-cs)k(s,y)\,dy\,ds\,dx \\
&= a\int_{\mathbb{R}}\int_0^{\infty}\int_{\mathbb{R}} e^{-\lambda(x+y+cs)}u^{\sigma}(c,y)k(s,x)\,dx\,ds\,dy \\
&= a\mathcal{K}(c,\lambda)\int_{\mathbb{R}} e^{-\lambda y}u^{\sigma}(c,y)\,dy \\
&= a\mathcal{K}(c,\lambda)\int_{\mathbb{R}} e^{-(\lambda-\eta)y}u(c,y)\big(u(c,y)e^{-\frac{\eta}{\sigma-1}y}\big)^{\sigma-1}\,dy \\
&\leqslant aM^{\sigma-1}\mathcal{K}(c,\lambda)U(c,\lambda-\eta)<+\infty.
\end{aligned} \tag{3.8}$$

Note that $U(c,\lambda)$ has a singularity at $\lambda=\lambda_1$. This is because the right-hand side of (3.7) is identically 0 if $U(c,\lambda_1)<+\infty$, and hence $u(c,x)$ is identically 0 due to the assumption (A2) on F. Now we use a property of Laplace transform [22, p. 58]. Since u is positive, there exists a real number $B=B(c)$ such that $U(c,\lambda)$ is analytic for $\operatorname{Re}\lambda \in (0,B)$ and has a singularity at $\lambda=B$. Next we show $B=\lambda_1$. Firstly, $B\leqslant\lambda_1$. Otherwise, taking $\lambda=\lambda_1$ in (3.7), we know $u(c,x)$ is identically 0, a contradiction. Since the abscissa of convergence of $U(c,\lambda)$ is different from that of the right-hand side of (3.7), we have that B must be the smallest positive root of the characteristic equation $\mathcal{K}(c,\lambda)=1$, and hence $B=\lambda_1$. □

With the help of the above results, we can estimate the exact asymptotic behavior of wave profiles.

Lemma 3.3. *Let $u(c,x)$ be a wave profile with $c>c^*$. Then there exists a positive number θ such that $\lim_{x\to-\infty} u(c,x)e^{-\lambda_1 x}=\theta$.*

Proof. Let λ_2 be defined as in Proposition 2.1. Choose $\epsilon > 0$ such that $\lambda_1(1+\epsilon) < \lambda_2$ and $1+\epsilon < \sigma$. Let $\beta \in (\lambda_1, \lambda_1(1+\epsilon))$. Then $l_{c,\frac{\beta}{1+\epsilon}} \in L^\infty(\mathbb{R})$. Therefore, there exists $M_1 > 0$ such that $u^\sigma(c,x) \leqslant M_1 u^{1+\epsilon}(c,x)$ and

$$\begin{aligned} e^{-\frac{\beta}{1+\epsilon}x} u(c,x) &\leqslant e^{-\frac{\beta}{1+\epsilon}x} \int_0^\infty \int_{\mathbb{R}} u(c,x-y-cs)k(s,y)\,dy\,ds \\ &= \int_{\mathbb{R}} u(c,x-y) e^{-\frac{\beta}{1+\epsilon}(x-y)} l_{c,\frac{\beta}{1+\epsilon}}(y)\,dy < M_1. \end{aligned}$$

For the remainder $R(u)(x)$ in (3.6), we have the following estimate:

$$\begin{aligned} 0 \geqslant R(u)(x) &\geqslant -a \int_0^\infty \int_{\mathbb{R}} u^\sigma(c,x-y-cs)k(s,y)\,dy\,ds \\ &= -aM_1 \int_0^\infty \int_{\mathbb{R}} u^{1+\epsilon}(c,x-y-cs)k(s,y)\,dy\,ds \\ &= -aM_1 \int_0^\infty \int_{\mathbb{R}} \big(u(c,x-y-cs)e^{-\frac{\beta}{1+\epsilon}(x-y-cs)}\big)^{1+\epsilon} e^{\beta(x-y-cs)} k(s,y)\,dy\,ds \\ &\geqslant -aM_1^{2+\epsilon}\mathcal{K}(c,\beta)e^{\beta x} \\ &= -Me^{\beta x}, \end{aligned} \tag{3.9}$$

where $M := aM_1^{2+\epsilon}\mathcal{K}(c,\beta)$. In the rest of this proof, we use the similar arguments as in the proof of [6, Theorem 6.3]. Consider the iterative scheme $v^{(0)}(x) := e^{-\beta x}R(u)(x)$ and

$$v^{(n)}(x) := \int_0^\infty \int_{\mathbb{R}} v^{(n-1)}(x-y-cs)e^{-\beta(y+cs)}k(s,y)\,dy\,ds + e^{-\beta x}R(u)(x), \quad n \geqslant 1.$$

Since $R(u)(x) \leqslant 0$, the sequence $v^{(n)}(x)$, $n \geqslant 0$, is nonincreasing. Note that

$$\begin{aligned} v^{(n)}(x) &\geqslant \inf_{x\in\mathbb{R}} v^{(n-1)}(x)\mathcal{K}(c,\beta) + \inf_{x\in\mathbb{R}} v^{(0)}(x) \\ &\geqslant \inf_{x\in\mathbb{R}} v^{(0)}(x)\big(1+\mathcal{K}(c,\beta)+\cdots+\big(\mathcal{K}(c,\beta)\big)^n\big) \\ &\geqslant -M\big(1+\mathcal{K}(c,\beta)+\cdots+\big(\mathcal{K}(c,\beta)\big)^n\big), \quad \forall n \geqslant 1, \end{aligned} \tag{3.10}$$

where $\mathcal{K}(c,\beta) < 1$. It then follows that the sequence $v^{(n)}(x)$, $n \geqslant 0$, is bounded, and hence there is a limit function $v(x) \leqslant 0$ such that $v(x)$ is bounded and satisfies the following non-homogeneous linear equation

$$\phi(x) = \int_0^\infty \int_{\mathbb{R}} \phi(x-y-cs)e^{-\beta(y+cs)}k(s,y)\,dy\,ds + e^{-\beta x}R(u)(x), \tag{3.11}$$

which is also satisfied by $u(c,x)e^{-\beta x}$. Thus, $w(x) := u(c,x) - v(x)e^{\beta x}$ is nonnegative but not identically 0. Let $k(y) := \int_0^\infty k(s, y-cs)\,ds$. Then w satisfies the homogeneous linear equation

$$w(x) = \int_{\mathbb{R}} w(x-y)k(y)\,dy. \tag{3.12}$$

Since $u(c,x)$ and $v(x)$ are both bounded, $w(x)$ satisfies the estimates $0 \leqslant w(x) \leqslant C(1+e^{\beta x})$ for some constant C. As argued in the proof of [6, Theorem 6.3], $w(x) = \theta e^{\lambda_1 x}$ for some constant $\theta > 0$, and hence

$$\lim_{x\to-\infty} u(c,x)e^{-\lambda_1 x} = \lim_{x\to-\infty}\bigl[w(x)e^{-\lambda_1 x} + v(x)e^{(\beta-\lambda_1)x}\bigr] = \theta + 0 = \theta.$$

This completes the proof. $\square$

Now we are ready to prove the main result of this section.

Theorem 3.1. *Assume* (H) *holds. Then for each $c > c^*$, there is at most one* (*up to translation*) *traveling wave of* (1.2) *connecting* 0.

Proof. Let $u_1(c,x)$ and $u_2(c,x)$ be two wave profiles with $c > c^*$. Set

$$w(c,x) = \bigl|u_1(c,x) - u_2(c,x)\bigr|e^{-\lambda_1 x}.$$

Then, after propriate translation of u_1, we have the following properties of $w(c,x)$:

(1) $w(c,\pm\infty) = 0$.
(2) There exists x_0 such that $w(c,x_0) = \max_{x\in\mathbb{R}} w(c,x)$.

Property (1) follows from Lemma 3.3, and the existence of maximum value is implied by property (1). Note that

$$\int_0^\infty\int_{\mathbb{R}} \cdots \int_0^\infty\int_{\mathbb{R}} \prod_{i=1}^n k(s_i, y_i)e^{\lambda_1 \sum_{i=1}^n (y_i + cs_i)}\,dy_1\,ds_1 \cdots dy_n\,ds_n = 1, \quad \forall n \in \mathbb{N}. \tag{3.13}$$

Using the Lipschitz constant $\Lambda = 1$, we then have

$$\begin{aligned} w(c,x_0) &\leqslant \int_0^\infty\int_{\mathbb{R}} w(c, x_0 - y - cs)k(s,y)e^{-\lambda_1(y+cs)}\,dy\,ds \\ &\leqslant \int_0^\infty\int_{\mathbb{R}} \cdots \int_0^\infty\int_{\mathbb{R}} w\Biggl(c, x_0 - \sum_{i=1}^n (y_i + cs_i)\Biggr) \\ &\quad \times \prod_{i=1}^n k(s_i, y_i)e^{-\lambda_1 \sum_{i=1}^n (y_i + cs_i)}\,dy_1\,ds_1 \cdots dy_n\,ds_n \\ &\leqslant w(c,x_0), \quad \forall n \in \mathbb{N}. \end{aligned} \tag{3.14}$$

One can choose $(\bar{y}, \bar{s}) \in \mathbb{R} \times \mathbb{R}_+$ such that $\bar{y} + c\bar{s} \neq 0$ and $k(\bar{s}, \bar{y}) \neq 0$, which, together with inequality (3.14), indicates that $w(c, x_0 - n(\bar{y} + c\bar{s})) = w(c, x_0)$, $\forall n \in \mathbb{N}$. This implies that $w(c, x_0) = 0$ since $w(c, \pm\infty) = 0$. More precisely, we have $u_1(c, x) \equiv u_2(c, x)$. □

4. Applications

In this section, we apply Theorems 2.1, 2.2 and 3.1 to some nonlocal reaction–diffusion population models, which can be transformed into the form (1.1). Firstly, we investigate a general population model, then give a detailed conclusion on spreading speeds and traveling waves for a specific case. Finally, we present a complement result on uniqueness of traveling waves for an epidemic model.

In literature, the equation for mature members of some age-structured population is described by the reaction–diffusion equation of the following form

$$\frac{\partial u}{\partial t} = D\Delta u - g(u) + \int_0^\infty \int_{\mathbb{R}} f\big(u(t-s, z-y)\big) J(s, y)\, dy\, ds, \tag{4.1}$$

where $D > 0$, $f, g \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $J \in L^1(\mathbb{R}_+ \times \mathbb{R})$ with $J(s, y) = J(s, -y)$, $\forall s \geqslant 0$, $y \in \mathbb{R}$. When $J(s, y) = \delta(s - \tau)$, $\forall y \in \mathbb{R}$ with $\tau > 0$, (4.1) reduces to the model studied in [21], which is a generalization of the model derived in [18] (see also [10]); when $g(u) = du$, (4.1) reduces to the model investigated in [11,14]; when $g(u) = \beta u^2$, (4.1) reduces to the model studied in [7]. However, the spreading speeds and traveling waves for (4.1) are still unsolved in the general case.

For any $\alpha > 0$, let $g_\alpha(u) = \alpha u - g(u)$ and $\Gamma_\alpha(t, z)$ be the Green function of $\partial_t u(t, z) = Du_{zz}(t, z) - \alpha u(t, z)$. Then, by the standard variation of constant formula, (4.1) can be transformed into the following form

$$\begin{aligned} u(t, z) = &\int_{\mathbb{R}} \Gamma_\alpha(t, z-y) u(0, y)\, dy + \int_0^t \int_{\mathbb{R}} \Gamma_\alpha(t-s, z-y) g_\alpha\big(u(s, y)\big)\, dy\, ds \\ &+ \int_0^t \int_{\mathbb{R}} \Gamma_\alpha(t-s, z-y) \left(\int_0^\infty \int_{\mathbb{R}} f\big(u(s-r, y-x)\big) J(r, x)\, dx\, dr \right) dy\, ds. \end{aligned} \tag{4.2}$$

By changing the order of the variables of integration, we can simplify (4.2) into the form

$$\begin{aligned} u(t, z) = &u_0(t, z) + \int_0^t \int_{\mathbb{R}} \Gamma_\alpha(t-s, z-y) g_\alpha\big(u(s, y)\big)\, dy\, ds \\ &+ \int_0^t \int_{\mathbb{R}} f\big(u(t-s, z-y)\big) \left(\int_0^s \int_{\mathbb{R}} \Gamma_\alpha(s-r, y-x) J(r, x)\, dx\, dr \right) dy\, ds, \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} u_0(t, z) = &\int_{\mathbb{R}} \Gamma_\alpha(t, z-y) u(0, y)\, dy \\ &+ \int_0^t \int_{\mathbb{R}} \Gamma_\alpha(t-s, z-y) \left(\int_s^\infty \int_{\mathbb{R}} f\big(u(s-r, y-x)\big) J(r, x)\, dx\, dr \right) dy\, ds. \end{aligned}$$

Let

$$k_1(s,y) = \Gamma_\alpha(s,y)$$

and

$$k_2(s,y) = \int_0^s \int_{\mathbb{R}} \Gamma_\alpha(s-r, y-x) J(r,x)\, dx\, dr.$$

Then (4.3) takes the form

$$u(t,z) = u_0(t,z) + \int_0^t \int_{\mathbb{R}} g_\alpha\big(u(t-s, z-y)\big) k_1(s,y)\, dy\, ds + \int_0^t \int_{\mathbb{R}} f\big(u(t-s, z-y)\big) k_2(s,y)\, dy\, ds. \tag{4.4}$$

Thus, $u(t,z)$ satisfies

$$u(t,z) = u_0(t,z) + \int_0^t \int_{\mathbb{R}} F^\alpha\big(u(t-s, z-y), s, y\big)\, dy\, ds, \quad z \in \mathbb{R},\ t \geqslant 0, \tag{4.5}$$

with

$$F^\alpha(u,s,y) = g_\alpha(u) k_1(s,y) + f(u) k_2(s,y).$$

For any $\phi \in L^1(\mathbb{R}_+ \times \mathbb{R})$ with the property that there exists some $\lambda^\diamond > 0$ such that

$$\int_0^\infty \int_{\mathbb{R}} e^{-\lambda^\diamond y} \phi(s,y)\, dy\, ds < +\infty,$$

we define

$$\mathcal{K}_\phi(c,\lambda) := \int_0^\infty \int_{\mathbb{R}} e^{-\lambda(y+cs)} \phi(s,y)\, dy\, ds, \quad \forall c > 0,\ \lambda \in [0, \lambda^\diamond),$$

and

$$\phi^* := \mathcal{K}_\phi(c,0), \quad \forall c > 0.$$

From [21, Propositions 4.2 and 4.1(2)], we see that

$$\mathcal{K}_{k_1}(c,\lambda) = \mathcal{K}_{\Gamma_\alpha}(c,\lambda) = \int_0^\infty e^{-(c\lambda + \alpha - D\lambda^2)s}\, ds, \quad k_1^* = \frac{1}{\alpha},$$

and

$$\mathcal{K}_{k_2}(c,\lambda) = \mathcal{K}_J(c,\lambda)\int_0^\infty e^{-(c\lambda+\alpha-D\lambda^2)s}\,ds, \quad k_2^* = \frac{J^*}{\alpha}.$$

We impose the following condition on J:

(L1) For any $c \geqslant 0$, there exists $\lambda^\sharp = \lambda^\sharp(c) \in (0,+\infty]$ such that $\mathcal{K}_J(c,\lambda) < +\infty$ whenever $\lambda \in (0,\lambda^\sharp)$ and $\lim_{\lambda\uparrow\lambda^\sharp(c)} \mathcal{K}_J(c,\lambda) = +\infty$.

Assumption (L1) implies that for any $c > 0$, $\mathcal{K}_k(c,\lambda) = 1$ admits two real roots $\lambda_1 < \lambda_2 < \lambda^\sharp$, where $k(s,y) = g'_\alpha(0)k_1(s,y) + f'(0)k_2(s,y)$. In addition, we assume that:

(L2) For any $c > c^*$ and $\lambda \in (0,\lambda_1)$, the function $e^{-\lambda x}\int_0^\infty k_2(s,x-cs)\,ds$ is in $L^\infty(\mathbb{R})$.

We will impose the following conditions on f and g.

(L3) $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and bounded, $g : \mathbb{R}_+ \to \mathbb{R}_+$ is continuously differentiable and strictly increasing, $f(0) = g(0) = 0$, $f(u) > 0$, $g(u) > 0$, $\forall u > 0$, and the equation $f(u) = g(u)$ has a smallest positive solution $\bar{u}$.
(L4) $f'(0)$, $f''(0)$ and $g''(0)$ exist, $g'(u) \geqslant g'(0)$, $\forall u > 0$, $f'(0)J^* > g'(0)$ and

$$\bigl|f(u) - f(v)\bigr| \leqslant f'(0)|u - v|, \quad \forall u, v > 0.$$

Define

$$F_+^\alpha(u,s,y) := \max_{v\in[0,u]} F^\alpha(v,s,y), \qquad \breve{F}_+^\alpha(u) := \int_0^\infty\int_{\mathbb{R}} F_+^\alpha(u,s,y),$$

and

$$\breve{F}^\alpha(u) := \int_0^\infty\int_{\mathbb{R}} F^\alpha(u,s,y).$$

Note that f is bounded and g is strictly increasing. It then follows that there exists $M > 0$ such that $\max_{v\in[0,u]} f(v)J^* < g(u)$, $\forall u \geqslant M$. For the above M, there exists $\alpha_0 > 0$ such that for any $\alpha \geqslant \alpha_0$, $\max_{v\in[0,u]} g_\alpha(v) = g_\alpha(u)$, $\forall u \in [0,M]$. Thus, we have

$$\begin{aligned}
\breve{F}_+^\alpha(M) &= \int_0^\infty\int_{\mathbb{R}} F_+^\alpha(M,s,y)\,dy\,ds \\
&= \int_0^\infty\int_{\mathbb{R}} \max_{v\in[0,M]}\bigl[g_\alpha(M)k_1(s,y) + f(M)k_2(s,y)\bigr]\,dy\,ds \\
&\leqslant \int_0^\infty\int_{\mathbb{R}} \Bigl[\max_{v\in[0,M]} g_\alpha(M)k_1(s,y) + \max_{v\in[0,M]} f(M)k_2(s,y)\Bigr]\,dy\,ds
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\alpha}\Big[g_\alpha(M) + \max_{v\in[0,M]} f(v)J^*\Big] \\
&= M + \frac{1}{\alpha}\Big[\max_{v\in[0,M]} f(v)J^* - g(M)\Big] \\
&\leqslant M,
\end{aligned}
$$

which implies that there exists $u_+^{\alpha,*} \in (0, M)$ such that

$$\breve{F}_+^\alpha\big(u_+^{\alpha,*}\big) = u_+^{\alpha,*} \quad \text{and} \quad \breve{F}_+^\alpha(u) > u, \quad \forall u \in \big(0, u_+^{\alpha,*}\big).$$

Then, as in Lemma 2.1, we can define $u^{\alpha,*}$ and $u_-^{\alpha,*}$. Actually, $u^{\alpha,*} = \bar{u}$ since $\bar{u}$ is the smallest spatially homogeneous equilibrium.

Lemma 4.1. *Assume* (L1)–(L4) *hold. Then there exists* $\alpha_1 > 0$ *such that for any* $\alpha > \alpha_1$, *assumption* (H) *is satisfied with* $F(u, s, y) = F^\alpha(u, s, y)$ *and* $k(s, y) = g'_\alpha(0)k_1(s, y) + f'(0)k_2(s, y)$.

Proof. Note that g is continuous differentiable and increasing. It follows that there exists $\alpha_1 \geqslant \alpha_0$ such that

$$\max_{u\in[0,u_+^{\alpha,*}]} g'(u) \leqslant \max_{u\in[0,M]} g'(u) \leqslant 2\alpha - g'(0), \quad \forall \alpha > \alpha_1.$$

Then for any $u_+^{\alpha,*} \geqslant u \neq v \geqslant 0$, there holds

$$g'(0) \leqslant \frac{g(u) - g(v)}{u - v} \leqslant 2\alpha - g'(0).$$

This implies that

$$\big|g_\alpha(u) - g_\alpha(v)\big| \leqslant g'_\alpha(0)|u - v|,$$

and hence, (A5) with $\Lambda = 1$ holds. Since the other assumptions can be easily verified, we omit the details here. $\square$

Lemma 4.2. *Assume* (L1)–(L4) *holds. For each* $\alpha > 0$, *let* c_α^* *be defined as in* (2.5) *with* $k(s, y) = g'_\alpha(0)k_1(s, y) + f'(0)k_2(s, y)$. *Then* c_α^* *is independent of* α.

Proof. Note that assumptions (L1)–(L4) imply that the assumption (B) with $k(s, y) = g'_\alpha(0)k_1(s, y) + f'(0)k_2(s, y)$ holds and that the condition in Proposition 2.1(7) is satisfied. Thus, c_α^* is determined by the positive root of the following system

$$\mathcal{K}_k(c,\lambda) = 1, \qquad \frac{d}{d\lambda}\mathcal{K}_k(c,\lambda) = 0. \tag{4.6}$$

Since

$$
\begin{aligned}
\mathcal{K}_k(c,\lambda) &= g'_\alpha(0)\mathcal{K}_{k_1}(c,\lambda) + f'(0)\mathcal{K}_{k_2}(c,\lambda) \\
&= \begin{cases} \frac{1}{\alpha + c\lambda - D\lambda^2}[g'_\alpha(0) + f'(0)\mathcal{K}_J(c,\lambda)], & \alpha + c\lambda - D\lambda^2 > 0, \\ +\infty, & \alpha + c\lambda - D\lambda^2 \leqslant 0, \end{cases}
\end{aligned}
$$

and $g'_\alpha(0) = \alpha - g'(0)$, it then follows that system (4.6) is equivalent to

$$\begin{cases} -g'(0) + f'(0)\mathcal{K}_J(c,\lambda) = c\lambda - D\lambda^2, \\ f'(0)\dfrac{d}{d\lambda}\mathcal{K}_J(c,\lambda) = c - 2D\lambda. \end{cases} \tag{4.7}$$

Thus, c^*_α is independent of α. $\square$

Since c^*_α is independent of α, we denote c^*_α by c^*. Define

$$u^*_- := \sup_{\alpha>\alpha_1} u^{\alpha,*}_- \quad \text{and} \quad u^*_+ := \inf_{\alpha>\alpha_1} u^{\alpha,*}_+.$$

We further assume that:

(L5) There exists some $\bar{\alpha} > \alpha_1$ such that $\frac{f(u)J^* - g(u)}{u}$ is strictly decreasing for $u \in (0, u^{\bar{\alpha},*}_+]$ and $\check{F}^{\bar{\alpha}}$ has the property (P).

As argued in [21], the wave profile equation of (4.1) is equivalent to that of the limiting equation of (4.5). By Theorems 2.1, 2.2 and 3.1, as applied to integral equation (4.5), we have the following result.

Theorem 4.1. *Let $\bar{u}$, $u^*_\pm$ be defined as above and $u(t,z;\phi)$ be the unique solution of* (4.1) *through ϕ. Assume* (L1)–(L4) *hold. Then the following statements are valid*:

(1) *For any $\phi \in C(\mathbb{R}_- \times \mathbb{R}, \mathbb{R}_+)$ with compact support,*

$$\lim_{t\to\infty,\ |z|\geqslant ct} u(t,z) = 0, \quad \forall c > c^*;$$

and for any bounded $\phi \in C(\mathbb{R}_- \times \mathbb{R}, \mathbb{R}_+) \setminus \{0\}$,

$$u^*_- \leqslant \liminf_{t\to\infty,\ |z|\leqslant ct} u(t,z;\phi) \leqslant \limsup_{t\to\infty,\ |z|\leqslant ct} u(t,z;\phi) \leqslant u^*_+, \quad \forall c \in \big(0, c^*\big).$$

If, in addition, (L5) *holds, then $\lim_{t\to\infty,\ |z|\leqslant ct} u(t,z) = \bar{u}$.*

(2) *For any $c \in (0, c^*)$,* (4.1) *has no traveling wave (U, c) with $\liminf_{x\to-\infty} U(x) < u^*_-$.*

(3) *For any $c > c^*$,* (4.1) *has a unique traveling wave (U, c) with $U(-\infty) = 0$; and for $c = c^*$ and any small number $\beta > 0$,* (4.1) *has a traveling wave solution (U, c^*) such that $U(x) \leqslant \beta$, $\forall x \leqslant 0$; and all these traveling waves satisfy*

$$u^*_- \leqslant \liminf_{x\to\infty} U(x) \leqslant \limsup_{x\to\infty} U(x) \leqslant u^*_+.$$

Further, if $f(u)$ is nondecreasing for $u \in [0, \bar{u}]$, then $U(x)$ is nondecreasing in x for all $c \geqslant c^*$ and $U(-\infty) = 0$ for $c = c^*$; and if (L5) *holds, then $\lim_{x\to\infty} U(x) = \bar{u}$ for all $c \geqslant c^*$.*

Proof. For the first part of statement (1), we use the comparison principle (cf. [20]) of the linearized equation of (4.1) at $u \equiv 0$:

$$u_t(t,z) = Du_{zz}(t,z) - g'(0)u(t,z) + \int_0^\infty \int_{\mathbb{R}} f'(0)u(t-s, z-y)J(s,y)\,dy\,ds. \tag{4.8}$$

For any $c > c^*$, we choose $\bar{c} \in (c^*, c)$, then there exists $\bar{\lambda} > 0$ such that $\mathcal{K}(\bar{c}, \bar{\lambda}) < 1$. This implies that for any $\gamma > 0$, $\gamma e^{\bar{\lambda}(\bar{c}t - |z|)}$ is an upper solution of (4.8). Since ϕ has compact support, there exists $\overline{\gamma} > 0$ such that

$$\bar{u}(t,z) := \overline{\gamma} e^{\bar{\lambda}(\bar{c}t - |z|)} > \phi(t,z), \quad \forall t \leqslant 0,\ z \in \mathbb{R}.$$

Note that solutions of (4.1) are lower solutions of (4.8) since $g(u) \geqslant g'(0)u$ and $f(u) \leqslant f'(0)u$. It then follows that $u(t,z) \leqslant \bar{u}(t,z)$, where $u(t,z)$ is the solution through ϕ of (4.1). Thus, for any $c > c^*$, there holds

$$\lim_{t\to\infty,\, |z|\geqslant ct} u(t,z) \leqslant \lim_{t\to\infty,\, |z|\geqslant ct} \bar{u}(t,z) = \lim_{t\to\infty,\, |z|\geqslant ct} \overline{\gamma} e^{\bar{\lambda}(\bar{c}t - |z|)} \leqslant \lim_{t\to\infty} \overline{\gamma} e^{\bar{\lambda}(\bar{c}t - ct)} = 0.$$

For any bounded $\phi \in C(\mathbb{R}_- \times \mathbb{R}, \mathbb{R}_+) \setminus \{0\}$, $u_0(t,z)$ in (4.5) satisfies the conditions in Theorem 2.1. Thus, for any $\alpha > \alpha_1$, we have

$$u_-^{\alpha,*} \leqslant \liminf_{t\to\infty,\, |z|\leqslant ct} u(t,z) \leqslant \limsup_{t\to\infty,\, |z|\leqslant ct} u(t,z) \leqslant u_+^{\alpha,*}, \quad \forall c \in \big(0, c^*\big).$$

By taking supremum and infimum for $\alpha > \alpha_1$ in the above inequality, we arrive at

$$u_-^{*} \leqslant \liminf_{t\to\infty,\, |z|\leqslant ct} u(t,z) \leqslant \limsup_{t\to\infty,\, |z|\leqslant ct} u(t,z) \leqslant u_+^{*}, \quad \forall c \in \big(0, c^*\big).$$

Clearly, assumptions (L1)–(L4) imply that (A), (B) and (H) hold with $F = F^\alpha$ and $k(s,y) = g'_\alpha(0)k_1(s,y) + f'(0)k_2(s,y), \forall \alpha > \alpha_1$. Thus, by Theorems 2.2 and 3.1, we obtain the existence and uniqueness of traveling waves. As argued above, for all wave profiles we have

$$u_-^{*} \leqslant \liminf_{x\to+\infty} U(x) \leqslant \limsup_{x\to+\infty} U(x) \leqslant u_+^{*}.$$

The rest parts follow directly from Theorems 2.2 and 3.1. □

Next we consider the following specific case of (4.1):

$$u_t(t,z) = Du_{zz}(t,z) - du(t,z) + \epsilon \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi\gamma}} e^{-\frac{(z-y)^2}{4\gamma}} b\big(u(t-\tau, y)\big)\, dy, \quad z \in \mathbb{R}, \tag{4.9}$$

where $b(u) = pue^{-qu}$. For biological interpretations of these parameters, we refer to [18]. We should also mention that the local form of (4.9) (i.e., $\gamma = 0$) was studied in [8].

Choosing $\alpha = d$, we transform (4.9) into the form of (4.5) with

$$u_0(t,z) = \int_{\mathbb{R}} \frac{e^{-dt}}{\sqrt{4\pi Dt}} e^{-\frac{(z-y)^2}{4Dt}} u(0,y)\, dy$$

and

$$F^\alpha(u,s,y) = F^d(u,s,y) = \epsilon pue^{-qu} k_3(s,y),$$

where

$$k_3(s,y)=\begin{cases}\frac{e^{-d(s-\tau)}}{\sqrt{4\pi(D(s-\tau)+\gamma)}}e^{-\frac{y^2}{4(D(s-\tau)+\gamma)}}, & s>\tau,\\ 0, & s\in[0,\tau].\end{cases}$$

Note that $\mathcal{K}_{k_3}(c,\lambda)=e^{\lambda^2\gamma-\lambda c\tau}\int_0^\infty e^{-(d+c\lambda-D\lambda_2)s}\,ds$ and $k_3^*=\frac{1}{d}$. Letting $\beta:=\frac{\epsilon p}{d}$, we have

$$\breve{F}^d(u)=\beta u e^{-qu}\quad\text{and}\quad \breve{F}^d_+(u)=\begin{cases}\breve{F}^d(u), & u\in[0,\frac{1}{q}],\\ \breve{F}^d(\frac{1}{q}), & u\in(\frac{1}{q},\infty).\end{cases}$$

It then follows that

$$u^{d,*}_+=\begin{cases}\frac{1}{q}\ln\beta, & \beta\in(1,e],\\ \frac{\beta}{qe}, & \beta\in[e,\infty),\end{cases}\qquad \breve{F}^d_-(u)=\begin{cases}\breve{F}^d(u), & u\in[0,u_1],\\ \breve{F}^d(u^{d,*}_+), & u\in[u_1,u^{d,*}_+],\end{cases}$$

and

$$u^{d,*}_-=\breve{F}^d\big(u^*_+\big),\qquad u^{d,*}=\frac{1}{q}\ln\beta,$$

where u_1 is the root of $\breve{F}^d(u)=\breve{F}^d(u^{d,*}_+)$ other than $u^{d,*}_+$.

Now we are ready to describe the spatial dynamics of (4.9) in terms of the parameter $\beta:=\frac{\epsilon p}{d}$.

Theorem 4.2. *Let $u(t,z;\phi)$ be the solution of* (4.9) *through ϕ, and define $u^{d,*}_\pm$ and $u^{d,*}$ as above. Then the following statements are valid*:

- (i) *For each $\beta>1$,* (4.9) *admits a spreading speed $c^*>0$ which coincides with the minimal wave speed of traveling waves connecting zero, and all traveling waves connecting zero and with wave speed $c>c^*$ are unique up to translation.*
- (ii) *For each $\beta\in(1,e^2]$, the upward convergence holds, and hence, all wave profiles at $+\infty$ have the same limit, which is the positive equilibrium $u^{d,*}$; for each $\beta>e^2$, there exists $\tau_0=\tau_0(\beta)>0$ such that the upward convergence cannot hold for $\tau>\tau_0$, and in this case for any bounded $\phi\in C([-\tau,0]\times\mathbb{R},\mathbb{R}_+)\setminus\{0\}$, there holds*

$$u^{d,*}_-\leqslant\liminf_{t\to\infty,\,|z|\leqslant ct}u(t,z;\phi)\leqslant\limsup_{t\to\infty,\,|z|\leqslant ct}u(t,z;\phi)\leqslant u^{d,*}_+,\quad\forall c\in\big(0,c^*\big).$$

Proof. Statement (i) follows directly from Theorem 4.1 because (L1)–(L4) are satisfied with $g(u)=du$, $f(u)=\epsilon p u e^{-qu}$ and $J(s,y)=\frac{\delta(s-\tau)}{\sqrt{4\pi\gamma}}e^{-\frac{(z-y)^2}{4\gamma}}$ when $\beta>1$. It remains to prove statement (ii). Note that when $\beta\in(1,e]$, $\breve{F}^d$ satisfies the condition (P1) in Lemma 2.2, and when $\beta\in(e,e^2]$, $\breve{F}^d$ satisfies the condition (P2) in Lemma 2.2. It then follows form Lemma 2.2 that $\breve{F}^d$ has the property (P) when $\beta\in(1,e^2]$, and hence, (L5) is also satisfied since $\frac{f(u)}{u}=\epsilon p e^{-qu}$ is strictly decreasing for $u>0$. Thus, when $\beta\in(1,e^2]$, for any bounded $\phi\in C([-\tau,0]\times\mathbb{R},\mathbb{R}_+)\setminus\{0\}$, we have the upward convergence

$$\lim_{t\to\infty,\,|z|\leqslant ct}u(t,z;\phi)=u^{d,*},\quad\forall c\in\big(0,c^*\big).$$

However, for any given $\beta>e^2$, such convergence does not hold any more for large delay τ. Indeed, the upward convergence implies that $u^{d,*}$ is globally attractive for the following spatially homogeneous equation

$$u'(t) = -du(t) + \epsilon p u(t-\tau) e^{-qu(t-\tau)}, \quad t > 0. \tag{4.10}$$

On the other hand, [24, Theorem 2.3(ii)] implies that $u^{d,*}$ is an unstable equilibrium of (4.10) when $\beta > e^2$ and $\tau > \tau_0$, where $\tau_0 = \tau_0(\beta) > 0$ is the first Hopf bifurcation point given by

$$\tau_0 = \frac{\arccos \frac{1}{1-\ln\beta}}{d\sqrt{(\ln\beta)^2 - 2\ln\beta}}.$$

The rest part follows from Theorem 4.1(1). $\square$

When $\tau > \tau_0$, the upward convergence does not hold. In this case, we see from Theorem 4.1(3) that for each $c > c^*$, the unique wave profile $U(x)$ satisfies

$$u_-^{d,*} \leqslant \liminf_{x\to+\infty} U(x) \leqslant \limsup_{x\to+\infty} U(x) \leqslant u_+^{d,*}.$$

This gives rise to an interesting problem: Does $U(x)$ oscillate at $+\infty$? If it oscillates, does it connect a periodic solution of (4.10)?

Remark 4.1. Capasso [2] presented the following epidemic model

$$\begin{cases} \dfrac{\partial}{\partial t}u_1(t,x) = d\Delta u_1(t,x) - a_{11}u_1(t,x) + \displaystyle\int_\Omega K(x-y)u_2(t,y)\,dy, \\ \dfrac{\partial}{\partial t}u_2(t,x) = -a_{22}u_2(t,x) + g\big(u_1(t,x)\big), \end{cases} \tag{4.11}$$

and Xu and Zhao [25] studied the spreading speed and monotone traveling waves by transforming it into the integral form (1.2). In particular, the uniqueness of monotone traveling waves was established under the assumption that g is increasing. By the uniqueness theorem developed in Section 3, even without monotonicity of g, we can obtain the uniqueness for all possible wave profiles. Thus, there is no non-monotone traveling wave when g is monotone.

Acknowledgments

We are grateful to the anonymous referee for his/her valuable suggestions which led to an improvement of our original manuscript. Jian Fang's research is supported in part by the NSF of China (grant 10771045) and the Chinese Government Scholarship. Xiao-Qiang Zhao's research is supported in part by the NSERC of Canada and the MITACS of Canada.

References

[1] K.J. Brown, J. Carr, Deterministic epidemic waves of critical velocity, Math. Proc. Cambridge Philos. Soc. 81 (1977) 431–433.
[2] V. Capasso, Asymptotic stability for an integro-differential reaction–diffusion system, J. Math. Anal. Appl. 103 (1978) 33–76.
[3] J. Carr, A. Chamj, Uniqueness of traveling waves for nonlocal monostable equations, Proc. Amer. Math. Soc. 132 (2004) 2433–2439.
[4] X. Chen, J.S. Guo, Uniqueness and existence of traveling waves for discrete quasilinear monostable dynamics, Math. Ann. 322 (2003) 123–146.
[5] O. Diekmann, Run for your life. A note on the asymptotic speed of propagation of an epidemic, J. Differential Equations 33 (1979) 58–73.
[6] O. Diekmann, H.G. Kapper, On the bounded solutions of a nonlinear convolution equation, Nonlinear Anal. 2 (1978) 721–737.
[7] J. Fang, J. Wei, X.-Q. Zhao, Spatial dynamics of a nonlocal and time-delayed reaction–diffusion system, J. Differential Equations 245 (2008) 2749–2770.

[8] T. Faria, S. Trofimchuk, Nonmonotone travelling waves in a single species reaction–diffusion equation with delay, J. Differential Equations 228 (2006) 357–376.
[9] T. Faria, W. Huang, J. Wu, Traveling waves for delayed reaction–diffusion equations with global response, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 462 (2006) 229–261.
[10] S.A. Gourley, Y. Kuang, Wavefronts and global stability in a time-delayed population model with age structure, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 459 (2003) 1563–1579.
[11] S.A. Gourley, J. So, Extinction and wavefronts propagation in a reaction–diffusion model of a structured population with distributed delay, Proc. Roy. Soc. Edinburgh Sect. A 133 (2003) 527–548.
[12] S.-B. Hsu, X.-Q. Zhao, Spreading speeds and traveling waves for non-monotone integrodifference equations, SIAM J. Math. Anal. 40 (2008) 776–789.
[13] J. Huang, X. Zou, Existence of traveling wavefronts of delayed reaction–diffusion systems without monotonicity, Discrete Contin. Dyn. Syst. Ser. A 9 (2003) 925–936.
[14] Y. Kyrychko, S.A. Gourley, M.V. Bartuccelli, Comparison and convergence to equilibrium in a nonlocal delayed reaction–diffusion model on an infinite domain, Discrete Contin. Dyn. Syst. Ser. B 5 (2005) 1015–1026.
[15] S. Ma, Traveling waves for non-local delayed diffusion equations via auxiliary equations, J. Differential Equations 237 (2007) 259–277.
[16] S. Ma, X. Zou, Existence, uniqueness and stability of traveling waves in a discrete reaction–diffusion monostable equation with delay, J. Differential Equations 217 (2005) 54–87.
[17] C. Ou, J. Wu, Persistence of wavefronts in delayed nonlocal reaction–diffusion equations, J. Differential Equations 235 (2007) 219–261.
[18] J. So, J. Wu, X. Zou, A reaction–diffusion model for a single species with age structure. I. Traveling wavefronts on unbounded domains, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 457 (2001) 1841–1853.
[19] H.R. Thieme, Density-dependent regulation of spatially distributed populations and their asymptotic speed of spread, J. Math. Biol. 8 (1979) 173–187.
[20] H.R. Thieme, Asymptotic estimates of the solutions of nonlinear integral equations and asymptotic speeds for the spread of populations, J. Reine Angew. Math. 306 (1979) 94–121.
[21] H.R. Thieme, X.-Q. Zhao, Asymptotic speeds of spread and traveling waves for integral equations and delayed reaction–diffusion models, J. Differential Equations 195 (2003) 430–470.
[22] D.V. Widder, The Laplace Transform, Princeton University Press, Princeton, NJ, 1941.
[23] J. Wu, X. Zou, Traveling wave fronts of reaction–diffusion systems with delay, J. Dynam. Differential Equations 13 (2001) 651–687, J. Dynam. Differential Equations 20 (2008) 531–533, Errata.
[24] J. Wei, Bifurcation analysis in a scalar delay differential equation, Nonlinearity 20 (2007) 2483–2498.
[25] D. Xu, X.-Q. Zhao, Asymptotic speed of spread and traveling waves for a nonlocal epidemical model, Discrete Contin. Dyn. Syst. Ser. B 5 (2005) 1043–1056.