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Versal deformations in orbit spaces

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Abstract

Given an orbit space M/Γ and an equivalence relation defined in it by means of the action of a group G , we obtain a miniversal deformation of an orbit through a miniversal deformation in M with regard to a suitable group action of $G \times \Gamma$. We show some applications to the perturbations of m -tuples of subspaces and (C, A) -invariant subspaces.

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1. Introduction

One approach to the study of local perturbations of matrices, pairs of matrices, pencils, etc. is Arnold's technique described in [1], where versal or miniversal deformations (see Definition 2.2) are obtained through transverse manifolds to the orbits of appropriate Lie group actions (see [2–4,6,8]). In most of these cases the object to be perturbed belongs to a linear manifold; otherwise, the explicit obtention of the versal deformation seems not to be, in general, possible. However, many objects appearing in several problems of geometry and linear control, such as invariant subspaces of a square matrix, flag manifolds, (A, B) or (C, A) invariant subspaces etc.,

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do not belong to a linear manifold. In all these cases the corresponding manifolds are orbit spaces, that is, manifolds the elements of which are the orbits of the action of a Lie group on a space of matrices. A first approach in the obtention of miniversal deformations of this kind of manifolds has been made in [4] where the authors study the deformation of invariant subspaces with regard to a fixed endomorphism.

Here we generalize the above work to a general orbit space. In fact, if M is a manifold, G, Γ are Lie groups acting on M so that the orbit space M/Γ is a manifold, and G induces a group action on M/Γ , then in Theorem 2.3 we relate a versal deformation of an element of $x \in M$ with a versal deformation of its orbit $x\Gamma$. The interest of this relation is clear whenever we know how to obtain explicitly a versal deformation of the elements of M . This is the case when M and G are linear manifolds (Theorem 2.6).

Theorem 2.6 can be applied to a wide range of situations in which we consider a perturbation of an object represented not by a single matrix, pair of matrices etc., but by a continuous orbit of matrices, as for example an m -tuple of subspaces, (conditioned) invariant subspaces, etc. Section 3 will present some of these applications.

In this paper we use the following notation.

\mathbb{F} is the field of either the complex or the real numbers. $M_{p,q}$ denotes the set of $p \times q$ matrices with entries in \mathbb{F} and $M_{p,q}^*$ the set of the full rank ones. $M_{p,p}^*$ is the linear group $GL(p)$. If E is a vector space, $Gr_d(E)$ denotes the Grassmann manifold of d -dimensional subspaces of E . Throughout the paper, we will denote by I the identity element of a group. If M is a manifold and $x \in M$, (M, x) denotes an open neighbourhood of x .

2. Versal deformation in orbit spaces

First we recall the definition of a versal and a miniversal deformation. Let M be a smooth manifold and x an element of M .

Definition 2.1. A local deformation of $x \in M$ is a smooth map $\phi : \mathcal{U} \rightarrow (M, x)$, where $\mathcal{U} \subset \mathbb{F}^n$ is a neighbourhood of the origin, (M, x) is a neighbourhood of the element $x \in M$, and $\phi(0) = x$.

Let G be a Lie group acting smoothly on M on the left. We denote the action of $g \in G$ on $x \in M$ by $(g, x) \mapsto gx$.

Definition 2.2. A local deformation of $x \in M$ is called *versal* (with regard to the action of G) if for any other deformation $\psi : \mathcal{V} \rightarrow (M, x)$, $\mathcal{V} \subset \mathbb{F}^n$, $0 \in \mathcal{V}$, there exist a neighbourhood $\mathcal{V}' \subset \mathcal{V}$, $0 \in \mathcal{V}'$, a smooth map $h : \mathcal{V}' \rightarrow \mathcal{U}$, with $h(0) = 0$ and a deformation $\theta : \mathcal{V}' \rightarrow (G, I)$ of the identity element of G ($\theta(0) = I$) such that $\psi(v) = \theta(v)\phi(h(v))$ for every $v \in \mathcal{V}'$. We indicate this relation by

$$\psi = \theta(\phi \circ h).$$

To say it shortly, we call a local deformation simply a deformation. Versal deformations having a minimal number of parameters are called it miniversal.

We are going to consider the following situation.

Let Γ be a Lie group acting on M on the right. We assume that the orbit space

$$M/\Gamma := \{x\Gamma \mid x \in M\}$$

has a differentiable structure such that the natural projection

$$\pi : M \longrightarrow M/\Gamma$$

is a submersion and that the actions of G and Γ on M are *compatibles*, that is to say, that $(gx)\gamma = g(x\gamma)$ for all $g \in G, \gamma \in \Gamma, x \in M$.

We consider in M the action of $G \times \Gamma$ defined by

$$(g, \gamma)x = (gx)\gamma = g(x\gamma),$$

where the product in $G \times \Gamma$ is defined by

$$(g, \gamma)(g', \gamma') = (g'g, \gamma\gamma'),$$

and the action of G on M/Γ is defined by

$$g(x\Gamma) = (gx)\Gamma$$

(this is well defined because the actions of G and Γ are compatibles).

Our goal is to show that a versal deformation of an orbit $x\Gamma \in M/\Gamma$ can be obtained through a versal deformation of $x \in M$.

In fact, we have the following basic result.

Theorem 2.3. *Let $\psi : \mathcal{U} \longrightarrow (M, x)$ be a deformation of x in M . Then,*

- (i) $\pi \circ \psi$ is a deformation of $\pi(x)$ in M/Γ .
- (ii) Any deformation φ of $\pi(x)$ in M/Γ can be written as $\pi \circ \psi$, with ψ a deformation of x in M .
- (iii) ψ is versal if and only if $\varphi = \pi \circ \psi$ is versal.

Proof

- (i) Is obvious.
- (ii) Follows from the existence of local sections of π .
- (iii) Let us assume that φ is versal. In order to prove that ψ is versal, we have to show that for any other deformation $\psi' : \mathcal{V}' \longrightarrow (M, x)$ there exist $\theta : \mathcal{V}' \longrightarrow (G \times \Gamma, (I, I)), h : \mathcal{V}' \longrightarrow \mathcal{U}$ with $\mathcal{V}' \subset \mathcal{V}, 0 \in \mathcal{V}'$, such that $\psi' = \theta(\psi \circ h)$.

To prove this, let us consider the deformation φ' defined by $\varphi' = \pi \circ \psi'$.

Since φ is a versal deformation, there exist $\theta_p : \mathcal{V}' \longrightarrow (G, I)$ and $h : \mathcal{V}' \longrightarrow \mathcal{U}$ with $\mathcal{V}' \subset \mathcal{V}, 0 \in \mathcal{V}'$, such that

$$\begin{aligned}
\varphi'(z) &= \theta_p(z)((\varphi \circ h)(z)) \\
&= \theta_p(z)((\pi \circ \psi \circ h)(z)) \\
&= \pi(\theta_p(z)(\psi \circ h)(z)) \quad \text{for any } z \in \mathcal{V}'.
\end{aligned}$$

Therefore taking into account that by definition, $\varphi'(z) = \pi(\psi'(z))$, we conclude that there exists a unique $\gamma \in \Gamma$ such that

$$\theta_p(z)((\psi \circ h)(z))\gamma = \psi'(z).$$

Thus, it makes sense to define $\theta_q : \mathcal{V}' \rightarrow (\Gamma, I)$ by $\theta_q(z) := \gamma$. It can be checked that θ_q is smooth. So, if we define $\theta : \mathcal{V}' \rightarrow (G \times \Gamma, (I, I))$ by

$$\theta(z) = (\theta_p(z), \theta_q(z))$$

it follows

$$\psi'(z) = \theta(z)(\psi \circ h)(z) \quad \text{for any } z \in \mathcal{V}',$$

and the statement that ψ is versal is proved.

Conversely. Let us assume now that ψ is versal. In order to prove that φ is a versal deformation, let $\varphi' : \mathcal{V} \rightarrow (M/\Gamma, \pi(x))$ be any deformation of $\pi(x)$ and $\sigma : (M/\Gamma, \pi(x)) \rightarrow (M, x)$ be a local section.

Then $\psi' = \sigma \circ \varphi'$ is a deformation of x . Hence there exist $\theta : \mathcal{V}' \rightarrow (G \times \Gamma, (I, I))$ with $\theta(z) = (\theta_p(z), \theta_q(z))$ and $h : \mathcal{V}' \rightarrow \mathcal{U}$ with $\mathcal{V}' \subset \mathcal{V}$, $0 \in \mathcal{V}'$, such that

$$\psi' = \theta(\psi \circ h).$$

Then, for every $z \in \mathcal{V}'$

$$\begin{aligned}
\varphi'(z) &= (\pi \circ \psi')(z) \\
&= \pi(\theta(z)(\psi \circ h)(z)) \\
&= \pi(\theta_p(z)(\psi \circ h)(z)\theta_q(z)) \\
&= \theta_p(z)((\pi \circ \psi \circ h)(z)) \\
&= \theta_p(z)((\varphi \circ h)(z)).
\end{aligned}$$

This shows that φ is versal and the theorem is proved. \square

Corollary 2.4. *With the above notation, φ is miniversal if and only if ψ is miniversal.*

We are going to apply the above theorem to the following particular case where M is an open and dense subset of a linear subvariety of $M_{p,q}$, G a subgroup of $\text{Gl}(p)$ which is an open and dense subset of a linear subvariety of $M_{p,p}$ and Γ a subgroup of $\text{Gl}(q)$ which is an open and dense subset of a linear subvariety of $M_{q,q}$. We suppose

that G (respectively, Γ), acts on M on the left (respectively on the right) by matrix multiplication.

Since the elements of M , G and Γ are now matrices, we denote them by capital letters.

As we have just showed, a miniversal deformation of an orbit $\pi(X) \in M/\Gamma$ with regard to the action of G is the projection of a miniversal deformation of $X \in M$ with regard to the action of $G \times \Gamma$, which, as Arnold showed, is given by a parametrization of $T_X(GX\Gamma)^\perp$, where \perp denotes the orthogonal with regard to the inner product $\langle X, Y \rangle = \text{trace}(XY^*)$ and $T_X(GX\Gamma)$, the tangent space to the orbit $GX\Gamma$ in the point X .

Lemma 2.5. $T_X(GX\Gamma)^\perp$ is the set

$$\mathcal{W} = \{W \in \overline{M} \mid \text{trace}(PXW^*) = 0, \text{trace}(XQW^*) = 0 \ \forall P \in \overline{G}, Q \in \overline{\Gamma}\},$$

where the upper bar stands for the topological closure in the respective linear subspace.

Proof. Since M , G and Γ are open and dense subsets of linear manifolds, it follows

$$\begin{aligned} T_X(M) &= \overline{M}, \\ T_I(G) &= \overline{G}, \\ T_I(\Gamma) &= \overline{\Gamma}. \end{aligned}$$

Let $\Phi_X : G \times \Gamma \rightarrow M$ be the map defined by $\Phi_X(P, Q) = PXQ$. It is well known that

$$T_X(G \times \Gamma) = \text{Im } d\Phi_{X,(I,I)}.$$

So, we are lead to compute

$$\begin{aligned} \Phi_X((I, I) + \varepsilon(P, Q)) &= (I + \varepsilon P)X(I + \varepsilon Q) \\ &= X + \varepsilon(PX + XQ) + \varepsilon^2 PXQ, \end{aligned}$$

where $P \in \overline{G}$ and $Q \in \overline{\Gamma}$ and we conclude that $W \in T_X(GX\Gamma)^\perp$ if and only if

$$\langle PX + XQ, W \rangle = \text{trace}(PXW^*) + \text{trace}(XQW^*) = 0$$

for all $P \in \overline{G}$ and $Q \in \overline{\Gamma}$. This proves the lemma. \square

As a consequence of Theorem 2.3 and Lemma 2.5 we have the following result generalizing Theorem 3.9 of [4].

Theorem 2.6. *With the above notation, a miniversal deformation of an orbit $X\Gamma$ in M/Γ is given by*

$$\pi(X + W) = (X + W)\Gamma, \quad W \in \mathcal{W},$$

where \mathcal{W} is (a neighbourhood of the origin of) the set of matrices $W \in \overline{M}$ such that

$$\text{trace}(PXW^*) = \text{trace}(XQW^*) = 0$$

for all $P \in \overline{G}$ and $Q \in \overline{\Gamma}$ (in fact, for all P, Q of a respective basis).

As a typical application of the deformation theory we have the following.

Corollary 2.7. *Let $O(X\Gamma)$ be the orbit of $X\Gamma$ under the action of G . Then*

$$\dim O(X\Gamma) = \dim(M/\Gamma) - \dim \mathcal{W}.$$

3. Applications

As we have pointed out in the introduction, Theorem 2.6 can be applied to a wide set of situations. In this section we focus on two cases. In the first one we consider a product of Grassman manifolds while in the second one we consider a submanifold of a single Grassman manifold (the set of (C, A) -invariant subspaces with fixed Brunovsky restricted indices).

3.1. Deformation of m -tuples of subspaces

A natural equivalence relation in the set of m -tuples of subspaces of an n -dimensional vector space E over \mathbb{F} is defined by

$$(V_1, \dots, V_m) \sim (V'_1, \dots, V'_m)$$

if there exists $\varphi \in \text{Aut}_{\mathbb{F}}(E)$ such that $\varphi(V_i) = V'_i$ for $1 \leq i \leq m$. Of course, $\{\dim V_i, 1 \leq i \leq m\}$ is a set of invariants for the class of (V_1, \dots, V_m) . Therefore, it makes sense to consider the above equivalence relation restricted to the manifold $\text{Gr}_{r_1}(E) \times \dots \times \text{Gr}_{r_m}(E)$, $r_i \leq n$.

The problem of finding a complete set of invariants for the above equivalence relation is an open problem in general (in fact it is a “wild” problem for $m \geq 5$ and a “time infinite” problem for $m = 4$). Nevertheless, Theorem 2.6 provides a formula for the local moduli of the above equivalence relation that we can compute explicitly for $m \leq 3$, as we shall see later, or for small dimensions of E .

In order to apply Theorem 2.6 we state the above equivalence relation in terms of a group action in a quotient space. In fact,

$$\text{Gr}_{r_1}(E) \times \dots \times \text{Gr}_{r_m}(E) \cong M/\Gamma,$$

where

$$M = \{(X_1 \cdots X_m) \mid X_i \in M_{n,r_i}^*\} \quad \text{and} \quad \Gamma = \{\text{diag}(S_1, \dots, S_m) \mid S_i \in \text{Gl}(r_i)\}$$

(the columns of each X_i form a basis of $V_i \in \text{Gr}_{r_i}(E)$).

Then, the above equivalence relation can be stated in M/Γ as $X\Gamma \sim X'\Gamma$ if there exists $S \in \text{Gl}(n)$ such that $SX\Gamma = X'\Gamma$, or equivalently, if there exists $(S, T) \in \text{Gl}(n) \times \Gamma$ such that $SXT = X'$.

Following Theorem 2.6, a miniversal deformation of an m -tuple of subspaces (V_1, \dots, V_m) is given, according to the above representation, by $\pi(W)(= W\Gamma)$ with

$$\begin{aligned}
 W &= \left\{ (W_1 \cdots W_m) \in M \mid \text{tr} \sum_{i=1}^m P X_i W_i^* = 0, \text{tr} \sum_{i=1}^m X_i Q_i W^* = 0, \right. \\
 &\quad \left. \forall P \in M_{n,n}, Q_i \in M_{r_i, r_i} \right\} \\
 &= \left\{ (W_1 \cdots W_m) \in M \mid \sum_{i=1}^m X_i W_i^* = 0, \text{tr} X_i Q_i W^* = 0, \forall Q_i \in M_{r_i, r_i} \right\},
 \end{aligned}$$

where X_i is a basis of V_i .

We can compute W explicitly if we represent (V_1, \dots, V_m) by a “canonical” matrix $(X_1 \cdots X_m)$. In this case, W leads to a “local” canonical form. In order to show this, we consider the case $m = 3$ in which we know a complete set of invariants of each equivalence class. There is no loss of generality in assuming $E = \mathbb{F}^n$. We have the following proposition.

Proposition 3.1. *Given $(V_1, V_2, V_3) \in \text{Gr}_{r_1}(\mathbb{F}^n) \times \text{Gr}_{r_2}(\mathbb{F}^n) \times \text{Gr}_{r_3}(\mathbb{F}^n)$, the set of integers $k_1 = \dim V_1 \cap V_2 \cap V_3$, $k_2 = \dim V_1 \cap V_2$, $k_3 = \dim V_1 \cap V_3$, $k_4 = \dim V_2 \cap V_3$, $k_5 = \dim V_1 \cap (V_2 + V_3)$, $k_6 = \dim V_2 \cap (V_1 + V_3)$ and $k_7 = \dim V_3 \cap (V_1 + V_2)$ is a complete set of invariants of the class of (V_1, V_2, V_3) . Moreover, the matrix*

$$X = \left(\begin{array}{cccc|cccc|cccc}
 I_{k_1} & 0 & 0 & 0 & 0 & I_{k_1} & 0 & 0 & 0 & 0 & I_{k_1} & 0 & 0 & 0 & 0 \\
 0 & I_{\bar{k}_2} & 0 & 0 & 0 & 0 & I_{\bar{k}_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & I_{\bar{k}_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\bar{k}_3} & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\bar{k}_4} & 0 & 0 & 0 & 0 & 0 & I_{\bar{k}_4} & 0 \\
 0 & 0 & 0 & I_{\bar{k}_5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\bar{k}_6} & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & I_{\bar{r}_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\bar{r}_2} & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\bar{r}_3} & 0
 \end{array} \right)$$

with $\bar{k}_2 = k_2 - k_1$, $\bar{k}_3 = k_3 - k_1$, $\bar{k}_4 = k_4 - k_1$, $\bar{k}_5 = k_5 - k_2 - k_3 + k_1$, $\bar{k}_6 = k_6 - k_2 - k_4 + k_1$, $\bar{r}_1 = r_1 - k_5$, $\bar{r}_2 = r_2 - k_6$ and $\bar{r}_3 = r_3 - k_7$ is a canonical representative of the class of (V_1, V_2, V_3) , or equivalently, of the orbit $\text{Gl}(n)X\Gamma$ with $\Gamma = \{\text{diag}(S_1, S_2, S_3) \mid S_i \in \text{Gl}(r_i), i = 1, 2, 3\}$.

Proof. Let e_1, \dots, e_{10} be sets of linearly independent vectors such that

$$\begin{aligned} V_1 \cap V_2 \cap V_3 &= [e_1], \\ V_1 \cap V_2 &= [e_2] \oplus V_1 \cap V_2 \cap V_3, \\ V_1 \cap V_3 &= [e_3] \oplus V_1 \cap V_2 \cap V_3, \\ V_2 \cap V_3 &= [e_4] \oplus V_1 \cap V_2 \cap V_3, \\ V_1 \cap (V_2 + V_3) &= [e_5] \oplus (V_1 \cap V_2 + V_1 \cap V_3), \\ V_2 \cap (V_1 + V_3) &= [e_6] \oplus (V_2 \cap V_1 + V_2 \cap V_3), \\ V_1 &= [e_7] \oplus V_1 \cap (V_2 + V_3), \\ V_2 &= [e_8] \oplus V_2 \cap (V_1 + V_3), \\ V_3 &= [e_9] \oplus V_3 \cap (V_1 + V_2), \\ \mathbb{F}^n &= [e_{10}] \oplus (V_1 + V_2 + V_3). \end{aligned}$$

We remark that

$$V_3 \cap (V_1 \cap (V_2 + V_3) + V_2 \cap (V_1 + V_3)) = V_3 \cap (V_1 + V_2).$$

Therefore, the set $e_1 \cup \dots \cup e_{10}$ is a basis of \mathbb{F}^n while $e_1 \cup e_2 \cup e_3 \cup e_5 \cup e_7$, $e_1 \cup e_2 \cup e_4 \cup e_6 \cup e_8$ and $e_1 \cup e_3 \cup e_4 \cup e_9$ are bases of V_1 , V_2 and V_3 , respectively. Taking the components of the last bases in the first one, we have the canonical form of the proposition.

In order to see that k_1, \dots, k_7 is a complete set of invariants it remains to observe that $\dim(V_1 + V_2 + V_3) = r_1 + \dim(V_2 + V_3) - k_7 = r_1 + r_2 + r_3 - k_4 - k_7$. \square

Let us compute the miniversal deformation of the above canonical form. For this, let $W = (W_1 W_2 W_3)$ satisfying the equations

$$\begin{cases} X_1 W_1^* + X_2 W_2^* + X_3 W_3^* = 0, \\ \text{tr } X_1 Q_1 W_1^* = 0, \\ \text{tr } X_2 Q_2 W_2^* = 0, \\ \text{tr } X_3 Q_3 W_3^* = 0, \end{cases}$$

where Q_i run over a basis of M_{r_i, r_i} , $i = 1, 2, 3$. The last three equations give

$$W = \left(\begin{array}{cccccc|cccccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

On the other side, the first equation gives

$$W = \left(\begin{array}{cccc|cccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & U & 0 & V & 0 & 0 & -U & 0 & -V \\ L & 0 & M & 0 & 0 & 0 & 0 & 0 & 0 & -L & -M & 0 \\ 0 & 0 & 0 & 0 & Y & 0 & Z & 0 & 0 & -Y & 0 & -Z \\ N & 0 & Q & 0 & 0 & 0 & 0 & 0 & 0 & -N & -Q & 0 \\ R & S & 0 & 0 & -R & -S & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

Hence, we have proved the following theorem. \square

Theorem 3.2. *With the above notation, $\text{Im}(X + W)$ is a miniversal deformation of (V_1, V_2, V_3) .*

Example 3.3. Let $n = 4$ and $r_1 = r_2 = r_3 = 2$. Let $V_1 = [e_1, e_2]$, $V_2 = [e_2, e_3]$ and $V_3 = [e_3, e_4]$ where e_1, e_2, e_3, e_4 is the usual basis of \mathbb{F}^n . According to the above proposition, a complete list of invariants of the triple (V_1, V_2, V_3) is $k = (0, 1, 0, 1, 1, 2, 1)$. A miniversal deformation of (V_1, V_2, V_3) is

$$\text{Im} \left(\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & z & -z & 0 \\ s & 0 & -s & 0 & 0 & 1 \end{array} \right).$$

Counting the number of free parameters of the above form we find that the orbit of (V_1, V_2, V_3) has dimension $(8 - 4) - 2 = 2$.

Moreover, the above form exhibits, as usual, bifurcation diagrams for small perturbations of the triple (V_1, V_2, V_3) . Here we have the class of (V_1, V_2, V_3) for $s = z = 0$ and the three different adherent classes for the rest of parameters:

$$\begin{aligned} k &= (0, 0, 0, 1, 1, 2, 2) \quad \text{for } s = 0, \quad z \neq 0, \\ k &= (0, 1, 0, 0, 2, 2, 1) \quad \text{for } s \neq 0, \quad z = 0, \\ k &= (0, 0, 0, 0, 2, 2, 2) \quad \text{for } s \neq 0, \quad z \neq 0. \end{aligned}$$

3.2. Deformation of (C,A) -invariant subspaces

In this section we extend the work in [4], where A -invariant subspaces are considered, to the set of (C, A) -invariant subspaces of an observable pair (C, A) .

We recall that a subspace V of \mathbb{F}^n is (C, A) -invariant if $A(V \cap \text{Ker } C) \subset V$ or equivalently if there exists J such that $(A + JC)V \subset V$. We also recall that two A -invariant subspaces U, V are said to be equivalents ($U \sim V$) if there exists S

$\in \text{Gl}(n)$ such that $S(U) = V$ and $AS = SA$ (so that the A -invariance of U and the Jordan form of the restriction of A to U is preserved by S for all A -invariant subspace U).

Then, a natural extension of this definition to the set of (C, A) -invariant subspaces is the following.

We say that U and V are equivalent ($U \sim V$) if there exists $S \in \text{Gl}(n)$ such that $S(U) = V$, $\tilde{S} \in \text{Gl}(n + p)$ such that $\tilde{S}(\mathbb{R}^n \times \{0\}) = \mathbb{R}^n \times \{0\}$ (the state space is fixed by \tilde{S}) and the diagram

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\begin{pmatrix} A \\ C \end{pmatrix}} & \mathbb{R}^{n+p} \\ S \downarrow & & \downarrow \tilde{S} \\ \mathbb{R}^n & \xrightarrow{\begin{pmatrix} A \\ C \end{pmatrix}} & \mathbb{R}^{n+p} \end{array}$$

commutes. That is to say, there exists $S \in \text{Gl}(n)$, $L \in \text{Gl}(p)$ and $K \in M_{n,p}$ such that

$$\begin{pmatrix} S & K \\ O & L \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} A \\ C \end{pmatrix} S$$

or

$$S^{-1}AS = A + JC \quad \text{and} \quad CS = LC,$$

where $J = S^{-1}K$.

By duality this equivalent relation can be translated to the set of (A, B) -invariant subspaces. We remark that these equivalent relations preserves the (C, A) -invariance ((A, B) -invariance) of subspaces. Moreover, they have some further interesting properties. For example, if $U \sim V$, the controllability indices of the corresponding restricted systems (see [7]) are the same. In particular, if U is a controllability subspace, V is a controllability subspace as well.

One can check that the Brunovsky indices of the restriction of (C, A) to a (C, A) -invariant subspace S is a set of invariants of the class of S for the above equivalence relation. Therefore it makes sense to restrict ourselves to the set of (C, A) -invariant subspaces of a fixed dimension with fixed Brunovsky restricted indices, which is a smooth manifold having the structure of an orbit space (see [5]). Let us recall it.

Let (C, A) be a Brunovsky observable pair, where $A \in M_{n,n}$ and $C \in M_{q,n}$ and with observability indices $k_1 \geq k_2 \geq \dots \geq k_r$. Let $h_1 \geq h_2 \geq \dots \geq h_s$ be the observability indices of a restriction of (C, A) , that is to say, $s \leq r, h_i \leq k_i$. Let $M(k, h)$ be the set of full rank matrices X such that

$$X = \begin{pmatrix} X_{11} & \dots & X_{1r} \\ \dots & & \dots \\ X_{s1} & \dots & X_{sr} \end{pmatrix}.$$

X_{ij} being matrices of the form

$$X_{ij} = \begin{pmatrix} x_1 & 0 & \dots & 0 \\ x_2 & x_1 & \dots & 0 \\ \dots & x_2 & \dots & x_1 \\ x_{k_i-h_j+1} & \dots & \dots & x_2 \\ 0 & x_{k_i-h_j+1} & \dots & \dots \\ 0 & 0 & \dots & x_{k_i-h_j+1} \end{pmatrix} \quad \text{if } k_i \geq h_j,$$

$$X_{ij} = 0 \quad \text{if } k_i < h_j,$$

$i = 1, \dots, r, j = 1, \dots, s.$

We set $\Gamma := M(h, h).$

In [5] it is shown that the map $X \mapsto \text{Im } X$ induces a bijection between the orbit space M/Γ and the set of d -dimensional (C, A) -invariant subspaces such that the observability indices of the corresponding restrictions of (C, A) are $h = (h_1, \dots, h_s).$ We denote this set by $\text{Inv}(k, h).$

Let $G = M(k, k) = \{S \in \text{Gl}(p) \mid \text{there exist } J \text{ and } L \text{ such that } A = S^{-1}AS + JC \text{ and } CS = LC\}.$

Since $S(\text{Im } X) = \text{Im}(SX),$ the equivalence relation defined on the set of (C, A) -invariant subspaces induces an equivalence relation on M/Γ defined by $X\Gamma \sim X'\Gamma$ if $X\Gamma = PX'\Gamma$ with $P \in G.$

In the above reference it is also proved that M/Γ is a differentiable manifold such that the natural projection $\pi : M \rightarrow M/\Gamma$ is a submersion. A differentiable structure is introduced in $\text{Inv}(k, h)$ through the above identification.

This identification gives also the following formula for the dimension:

$$\dim \text{Inv}(k, h) = \sum_{\substack{1 \leq i \leq r \\ 1 \leq j \leq s}} \sup\{k_i - h_j + 1, 0\} - \sum_{1 \leq i, j \leq s} \sup\{k_i - h_j + 1, 0\}.$$

Then, according to Theorem 2.6 and the identification above, a miniversal deformation of a subspace $V \in \text{Inv}(k, h)$ is given by

$$\text{Im}(X + W), \quad W \in \mathcal{W}, \quad X \in M(k, h),$$

where $V = \text{Im } X$ and \mathcal{W} is (a neighbourhood of the origin of) the set of matrices $W \in \overline{M}(k, h)$ such that

$$\text{trace}(PXW^*) = \text{trace}(XQW^*) = 0$$

for all P, Q belonging to a basis of $\overline{M}(k, k)$ and $\overline{M}(h, h)$ respectively.

We illustrate this with the following examples.

Example 3.4. Let $V \in \text{Inv}((4, 2), (3, 1)).$ A miniversal deformation of V is given by the subspaces spanned by the columns of the matrices:

$$X + W = \left(\begin{array}{cccc|cccc} a & 0 & 0 & c & w_1 & 0 & 0 & w_3 \\ b & a & 0 & d & w_2 & w_1 & 0 & w_4 \\ 0 & b & a & e & 0 & w_2 & w_1 & w_5 \\ 0 & 0 & b & f & 0 & 0 & w_2 & w_6 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & g & 0 & 0 & 0 & w_7 \\ 0 & 0 & 0 & h & 0 & 0 & 0 & w_8 \end{array} \right) + \left(\begin{array}{cccc|cccc} w_1 & 0 & 0 & w_3 & w_2 & w_1 & 0 & w_4 \\ w_2 & w_1 & 0 & w_5 & 0 & w_2 & w_1 & w_6 \\ 0 & w_2 & w_1 & w_7 & 0 & 0 & w_2 & w_8 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & w_7 & 0 & 0 & 0 & w_8 \\ 0 & 0 & 0 & w_8 & 0 & 0 & 0 & w_8 \end{array} \right),$$

where W is (a neighbourhood of the origin of) the set of matrices satisfying the above conditions. In this case, a basis for $\overline{G} = \overline{M}((4, 2), (4, 2))$ is formed by the set of matrices

$$\left\{ P_i = \left(\begin{array}{cccc|cccc} e_{1,i} & 0 & 0 & 0 & e_{2,i} & 0 & 0 & 0 \\ 0 & e_{1,i} & 0 & 0 & e_{3,i} & e_{2,i} & 0 & 0 \\ 0 & 0 & e_{1,i} & 0 & e_{4,i} & e_{3,i} & 0 & 0 \\ 0 & 0 & 0 & e_{1,i} & 0 & e_{4,i} & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & e_{5,i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e_{5,i} & 0 & 0 \end{array} \right), \text{ with } e_{i,j} = 0 \text{ if } i \neq j \text{ and } e_{i,i} = 1 \right\}$$

and similarly for $\Gamma = M((3, 1), (3, 1))$.

From the set of equations

$$\begin{aligned} \text{trace}(P_i X W^*) &= 0, \quad i = 1, \dots, 5, \\ \text{trace}(X Q_j W^*) &= 0, \quad j = 1, \dots, 5, \end{aligned}$$

we deduce

$$\left\{ \begin{array}{l} 3aw_1 + 3bw_2 + cw_3 + dw_4 + ew_5 + fw_6 = 0, \\ gw_3 + hw_4 = 0, \\ gw_4 + hw_5 = 0, \\ gw_5 + hw_6 = 0, \\ gw_7 + hw_8 = 0, \\ aw_1 + bw_2 = 0, \\ aw_3 + bw_4 = 0, \\ aw_4 + bw_5 = 0, \\ aw_5 + bw_6 = 0, \\ cw_3 + dw_4 + ew_5 + fw_6 + gw_7 + hw_8 = 0. \end{array} \right.$$

We know (see [5]) that $\begin{pmatrix} b & f \\ 0 & h \end{pmatrix}$ is a full rank matrix. Therefore $h \neq 0$ and $b \neq 0$.

If $\text{rank} \begin{pmatrix} g & h \\ a & b \end{pmatrix} = 2$, then $w_3 = w_4 = w_5 = w_6 = 0$ and it follows that

$$W = \left(\begin{array}{ccc|c} w_1 & 0 & 0 & 0 \\ -\frac{a}{b}w_1 & w_1 & 0 & 0 \\ 0 & -\frac{a}{b}w_1 & w_1 & 0 \\ 0 & 0 & -\frac{a}{b}w_1 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & w_7 \\ 0 & 0 & 0 & -\frac{g}{h}w_7 \end{array} \right).$$

If $\text{rank} \begin{pmatrix} g & a \\ h & b \end{pmatrix} = 1$, then $gb = ah$ and

$$W = \left(\begin{array}{ccc|c} w_1 & 0 & 0 & w_3 \\ -\frac{a}{b}w_1 & w_1 & 0 & -\frac{a}{b}w_3 \\ 0 & -\frac{a}{b}w_1 & w_1 & \frac{a^2}{b^2}w_3 \\ 0 & 0 & -\frac{a}{b}w_1 & -\frac{a^3}{b^3}w_3 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & w_7 \\ 0 & 0 & 0 & -\frac{g}{h}w_7 \end{array} \right).$$

Since $\dim \text{Inv}((4, 2), (3, 1)) = 3$, we see that

$$\dim O(V) = \begin{cases} 1 & \text{if } \text{rank} \begin{pmatrix} g & a \\ h & b \end{pmatrix} = 1, \\ 0 & \text{if } \text{rank} \begin{pmatrix} g & a \\ h & b \end{pmatrix} = 2. \end{cases}$$

Example 3.5. Let $V \in \text{Inv}(k, h)$ with $k = (k_1, \dots, k_r)$, $h = h_1$. Then the set of equations in Theorem 2.6 reduces to

$$\text{trace } P_i X W^* = 0,$$

$1 \leq i \leq \dim \overline{M}(k, k)$. Since the number of unknowns is $\dim \overline{M}(k, h)$ we conclude that in the generic case, if

$$\dim \overline{M}(k, k) \geq \dim \overline{M}(k, h)$$

it follows that

$$\dim W = 0.$$

So, generically, if the above inequality holds, $\dim O(V) = \text{Inv}(k, h)$ and $O(V)$ is open and dense in $\text{Inv}(k, h)$; that is, “almost” all pairs of subspaces $V, V' \in \text{Inv}(k, h)$ are equivalent. The proof of these statements will be clear through the following particular case where $k = (4, 3)$ and $h = 3$. Then, if we take the basis of $\overline{G} = \overline{M}(k, k)$ as in the previous example, the set of equations

$$\text{trace } P_i \begin{pmatrix} a & 0 & 0 \\ b & a & 0 \\ 0 & b & a \\ 0 & 0 & b \\ \dots & \dots & \dots \\ c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} 0 & 0 & x & y & | & 0 & 0 & z \\ 0 & x & y & 0 & | & 0 & z & 0 \\ x & y & 0 & 0 & | & z & 0 & 0 \end{pmatrix} = 0,$$

$i = 1, \dots, 4$, is

$$\left. \begin{array}{l} ax + by = 0 \\ cx = 0 \\ cy = 0 \\ cz = 0 \end{array} \right\},$$

which, clearly, has only, in the generic case, the solution $x = y = z = 0$.

Remark 3.6. Analogously to what happens with the set of invariant subspaces of an endomorphism, the determination of the equivalence classes in $\text{Inv}(k, h)$ is an open problem. However, as the former examples show, from Corollary 2.7, some insight can be derived with regard to this problem. In fact, if we consider the following two partitions of $\text{Inv}(k, h)$:

- (i) $\text{Inv}(k, h) = \cup_V O(V)$ where V runs over the set of nonequivalent set of subspaces $V \in \text{Inv}(k, h)$ (that is, the partition formed by the set of equivalence classes).
- (ii) $\text{Inv}(k, h) = \cup_k \mathcal{W}_k$ where \mathcal{W}_k is the set of $V \in \text{Inv}(k, h)$ such that the dimension of a miniversal deformation of V is k , so that $0 \leq k \leq \dim \text{Inv}(k, h)$. Note that some \mathcal{W}_k can be empty.

Then if $V \in \mathcal{W}_k$ and $V' \in \mathcal{W}_{k'}$ with $k \neq k'$ we know that V and V' are not equivalent. Hence a necessary condition in order that two (C, A) -invariant subspaces could be equivalent is that they belong to the same \mathcal{W}_k .

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