Classifications of finite highly transitive dimensional linear spaces*

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Abstract

We survey classification theorems on finite highly transitive dimensional linear spaces. Most recent results in this field rest on the classification of all finite simple groups. We depict their environment and comment on their proofs. This paper is a summary of a 'Thèse d'Agrégation de l'Enseignement Supérieur' presented at the University of Brussels in March 1991.

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1. Introduction

The notions of dimensional linear space (for short, DLS), simple matroid, geometric lattice, combinatorial geometry,

\[
\begin{array}{cccc}
\text{L} & \cdots & \text{L} \\
\hline
\end{array}
\]

diagram geometry are essentially equivalent when restricted to finite dimensions. They can be seen as a straightforward generalization of the most basic point–line–plane structure of elementary geometry and come up quite naturally when dealing with linear algebra and projective geometry.

Actually this notion was discovered and rediscovered many times by mathematicians working in very different fields, as is partly testified by the diversity of terminology. Although it already appeared (implicitly) in 1871 in Jordan’s work on permutation groups and (quite explicitly) in 1910 in Steinitz’s investigation of field extensions, it is usually credited to Whitney [111], Birkhoff [7] and Mac Lane [86] because they proved the first deep results on DLSs. Their respective motivations came from graph theory, geometrically blended lattice theory, and the study of transcendental field extensions.

Since then, close connections between DLSs and coding theory, design theory, combinatorial optimization, convexity, model theory (among other fields) emerged and are still popping up from time to time (e.g. Brickell and Davenport [11] showed a few months ago how closely DLSs are related to ideal secret sharing schemes).
Here we are especially interested in the connection between DLSs and groups, illustrated by the fact that given any abstract group $G$ and any integer $n \geq 2$, there is a DLS of dimension $n$ whose full automorphism group is isomorphic to $G$ [4].

The association of geometries with groups has proved extremely fruitful and has enriched both theories. Indeed the investigation of its automorphism group enlightens the study of any geometric structure and the most transitive geometries always play a prominent role. Although the converse idea of associating geometries with groups dates back to Klein's Erlangen programme in 1872, no significant development arose until 1955, when Tits unified and simplified the study of semi-simple groups of Lie type by presenting them as automorphism groups of certain geometries.

Later on, buildings [104, 106] and diagram geometries [14] were introduced in order to extend the benefits of such an association to all simple groups, so that most finite simple groups are now presented as the (socle of the) automorphism group of some geometry. Many other papers (among which the classics [61, 62, 68, 20]) investigate techniques for constructing various types of geometries (especially designs) from groups, provide applications to permutation group theory, and raise classification problems.

On the other hand, group theory greatly benefited geometry. In particular, after the classification of all finite simple groups was announced in 1980, many mathematicians tackled the classification of the most transitive geometries and many such problems were solved during the past five years (e.g. [72, 73, 96, 87, 2, 27, 83, 45, 97, 98, 113, 114]).

The present paper fits into this framework: we study how the simple group classification theorem can be used in order to classify finite DLSs whose automorphism group enjoys a specified transitivity property. Most of the material of this survey is spread out into 13 papers which are presented here in a uniform setting.

Section 2 introduces the terminology and notation that will be used throughout, as well as the less standard examples of transitive DLSs, and also a few group-theoretic results that are crucial to our purposes. A tight numbering will allow easy reference.

In Section 3 we gather classifications of finite DLSs whose automorphism group acts transitively on independent sets of a given size: this includes 2-homogeneity for linear spaces and basis transitivity for DLSs of any dimension. Our results there rely on a consequence of the simple group classification, namely the classification of all finite 2-transitive permutation groups.

Section 4 deals with flag transitive DLSs, and in particular chamber transitive DLSs. As often in incidence geometry, the 2-dimensional case is the hardest one: here the classification and a thorough knowledge of all finite simple groups were required to produce an almost complete classification of the finite flag transitive linear spaces. This extensive investigation was carried out by a team of six persons from the Universities of Brussels, London and Cambridge. We include only part of this work here (essentially the fundamental reduction theorem) as the revision process of this long and exacting proof is still being carried out. We will also comment on the case left open in this classification, which remains desperately open even under the additional hypothesis that the linear space be a finite projective or affine plane! Here group
theory is of very little help. Finally, a complete knowledge of all finite chamber transitive DLSs of dimension \( \geq 3 \) will follow from our classification of all (line, hyperplane)-flag transitive DLSs presented at the end of Section 4.

Section 5 investigates line primitivity in finite linear spaces.

Let us close this introduction by motivating our finiteness assumption. Indeed infinite permutation groups behave in quite a different way, so that the treatment of infinite DLSs would require totally different methods. For example, fundamental properties connecting orbit-numbers do not hold any more, in particular hyperplane transitivity does not force point transitivity, as illustrated by the \( n \)-DLS induced on a closed ball by the affine space \( AG(n, \mathbb{R}) \). Moreover, the following standard construction of an \( n \)-DLS from an \( n \)-transitive permutation group \( G \) also fails to work for infinite sets: take as set of hyperplanes the \( G \)-orbit of the fixed point set of a \( G \)-weakly closed\(^1\) subgroup in the stabilizer of some \( n \)-tuple (Mathieu designs, inversive planes and their generalizations are obtained that way, cf. Witt [112]). On an infinite set, however, it may occur that an \( n \)-point stabilizer properly contains another, as is the case for \( n = 4 \) and \( G = PGL(2, F) \), where \( F \) is a skew field with centre \( GF(2) \) in which all noncentral elements are conjugate [20].

As pointed out in Cameron [21], permutation groups were explicitly finite in the American Mathematical Society subject classification scheme in the 1970s. The recent papers on infinite permutation groups clearly show that most of the arguments used in the present work do not extend to infinite DLSs. Moreover, there is, of course, no counterpart of the classification of all finite simple groups! Hence, our methods become invalid. The results also do. For example, the famous Ostrom–Wagner theorem stating that all finite 2-transitive projective planes are Desarguesian can be seen as the common ancestor of our results. By contrast, the class of infinite 2-transitive projective planes is quite chaotic, since any finite (possibly non-Desarguesian) projective plane can be embedded into an infinite 2-transitive projective plane [76].

2. Definitions and prerequisites

We briefly recall the basic notions and results that will be used throughout the survey.

2.1. Linear space and DLS

A linear space \( S \) is a nonempty incidence structure of points and lines such that any two points are incident with exactly one line, any point being incident with at least two lines and any line with at least two points. The class of all linear spaces is denoted by the diagram

\[
\begin{align*}
| & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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Given an integer $n \geq 2$, an $n$-dimensional linear space $S$ (for short, $n$-DLS) is a firm, pure, residually connected, simple incidence structure belonging to the diagram [14]

$$
\begin{array}{cccc}
  L & \cdots & L \\
  0 & 1 & \cdots & n-1 \\
\end{array}
$$

Equivalently, $S$ can be seen as a geometric lattice $\text{Lat}(S)$ of dimension $n$ [8] or as a (possibly infinite) simple matroid $\text{Mat}(S)$ of rank $n+1$ [110]. If $n = 2$ (resp. 3), then $S$ is merely a linear space (resp. planar space). The $i$-varieties of $S$ are the varieties of type $i$; they correspond to the elements of height or dimension $i$ in $\text{Lat}(S)$ and to the flats of rank $i+1$ in $\text{Mat}(S)$. For $i = 0, 1, 2, n-1$, we rather call them points, lines, planes and hyperplanes, respectively.

### 2.2. Closure space and exchange property

Any variety can be safely identified with the set of points incident with it. If we also consider the empty set and the set $P$ of all points of $S$ as (improper) varieties (of dimension $-1$ and $n$, respectively), then the varieties of $S$ are the closed sets of a closure space satisfying the (Steinitz–MacLane) exchange property:

$$
\text{if } X \subseteq P \text{ and } y, z \in P \setminus \langle X \rangle, \text{ then } y \in \langle X \cup z \rangle \text{ forces } z \in \langle X \cup y \rangle,
$$

where $\langle \rangle$ denotes the closure operator.

Conversely, the closed sets of any closure space with the exchange property are the varieties of an $n$-DLS provided that $n+1$ is the maximum length of a chain of closed sets.

### 2.3. Independent sets and bases

Given any set $A$ of points of $S$, the variety $\langle A \rangle$ generated by $A$ is the closure of $A$ in $S$ (seen as a closure space). $A$ is an independent set if none of its points is in the closure of the other ones. A basis of an $n$-DLS $S$ is a maximal independent set, i.e. an independent $(n+1)$-set, but not an $(n+1)$-tuple. The notion of $n$ DLS can easily be defined in terms of independent sets or in terms of bases [110].

### 2.4. 0- and 1-DLSs

It is sometimes convenient to extend the definition of DLS to the dimensions 0 and 1, a 0-DLS consisting of a single point and a 1-DLS being merely a line, i.e. a set of points without any further proper variety. This allows e.g. to characterize generalized projective spaces as direct sums of projective spaces.

### 2.5. Thin varieties, trivial, circular and hypercircular DLSs

Throughout this survey we only consider finite DLSs, i.e. those on a finite number of points. The size of an $i$-variety is its number of points: it can never be less than $i+1$.
An $i$-variety with size $i+1$ is called thin. If all varieties of $S$ are thin, then $S$ is called trivial. The class of trivial linear spaces is denoted by the $c$-stroke diagram

$$|c|$$

If all lines of $S$ are thin, or equivalently if $S$ belongs to a

$$|c| |L| \cdots |L|$$

diagram, then $S$ is called circular. It is hypercircular if it belongs to

$$|c| \cdots |c| |L| .$$

An $n$-DLS is called regular if for each $i$ all $i$-varieties have the same size $s_i$. In regular DLSs the number of $j$-varieties containing a given $i$-variety $V_i$ and contained in a given $l$-variety $V_l$ ($i < j < l$) depends only on $i$, $j$, and $l$, but not on $V_i$ and $V_l$. This number $t(i,j,l)$ is given by

$$t(i,j,l) = (s_l - s_i)(s_l - s_{i+1}) \cdots (s_l - s_{j-1})/(s_j - s_i)(s_j - s_{i+1}) \cdots (s_j - s_{j-1}).$$

In particular, regular 2-DLSs are precisely the $2-(v,k,1)$ designs with $2 \leq k < v$. In this case, $v$ and $k$ are linked to the total number $b$ of lines and to the number $r$ of lines per point by the following relations:

(i) $r = (v-1)/(k-1) \geq k$.
(ii) $b = v(v-1)/k(k-1) \geq v$.
(iii) $v \geq k^2 - k + 1$.
(iv) $r > \sqrt{v}$.

2.6. $t-(v,k,1)$ designs as $t$-DLSs

Note that any $t-(v,k,1)$ design can be seen as a regular hypercircular $t$-DLS whose hyperplanes are the blocks of the design. For short we will write $S = t-(v,k,1)$ in this case, although there may be nonisomorphic $t$-designs with these parameters.

2.7. (point, hyperplane)-truncation

It is quite clear that a hypercircular $n$-DLS $S$ is uniquely determined by its (point, hyperplane)-truncation, i.e. the incidence structure of its points and hyperplanes. More generally, this holds for all $n$-DLSs and there is a well-known axiomatization of $n$-DLSs in terms of points and hyperplanes [110].
2.8. j-Truncation, erection

The j-truncation of an n-DLS S, denoted by j-S, is the j-DLS whose i-varieties (i < j ≤ n) are precisely the i-varieties of S. Conversely, S is called an erection of j-S. However, n and j-S do not uniquely determine S: trivial examples are the hypercircular DLSs, less trivial examples are obtained as follows. Take j-S = j-PG(d,q) with d ≥ j + 1. Then we get a (j + 1)-DLS S by calling hyperplane any member of a family F of subspaces of PG(d,q) of dimension ≥ j + 1 which pairwise intersect in subspaces of dimension < j, together with all j-subspaces of PG(d,q) which are not contained in any element of F.

2.9. Affine groups, 1-dimensional affine groups. \( \text{AT}^k L(1,q), \text{PSL}(2,q) \)

A permutation group G on a set Ω is called affine (or of affine type) if Ω can be identified with \( \text{GF}(q) = \text{GF}(p^d) \) (or equivalently with the point-set of \( \text{AG}(d,p) \)) and \( G \leq \text{AGL}(d,p) \). G is called n-dimensional affine if n is the smallest positive integer such that \( G \leq \text{AT}^n L(n,p^{dn}) \), the group of all semilinear transformations of \( \text{AG}(n,p^{dn}) \). In particular, we denote by \( \text{AT}^k L(1,p^d) \) the 1-dimensional affine group consisting of all permutations \( x \rightarrow ax^2 + b \) where \( a, b \in \text{GF}(p^d), a \neq 0, \sigma \in \text{Aut} \text{GF}(p^d) \) and \( k|p^d - 1 \). Similarly, \( \text{AG}^k L(1,p^d) = \text{AT}^k L(1,p^d) \cap \text{AGL}(1,p^d) \).

Let us also introduce the notation \( \text{PSL}(2,q) \) (where q is odd) for the subgroup of \( \text{PGL}(2,q) \) consisting of all automorphisms of PG(1,q) of the form \( x \rightarrow (ax^2 + b)/(cx^d + d) \) where \( ad - bc \) is a nonzero square in \( \text{GF}(q) \) and \( \sigma \in \text{Aut} \text{GF}(q) \), so that the stabilizer \( \text{PSL}(2,q) \) of the point \( \infty \) is \( \text{AT}^2 L(1,q) \).

2.10. Flags and chambers

A flag of an n-DLS S is a set of pairwise incident varieties of S; it is maximal (and called a chamber) if its cardinality is n.

Sections 2.11–2.16 provide a quick survey of flag transitive linear spaces which are not Desarguesian (affine or projective) spaces.

2.11. Generalized Netto systems

For each prime power \( q \equiv 7 \pmod{12} \), there is (up to isomorphism) exactly one Netto system \( N(3,q) \) on \( q \) points, which can be defined as follows. Let \( \epsilon \) be a primitive sixth root of unity in \( \text{GF}(q) \). The points of \( N(3,q) \) are the elements of \( \text{GF}(q) \) and the lines are the images of \{0, 1, \epsilon\} under \( \text{AT}^2 L(1,q) \). Clearly, \( N(3,7) \) is isomorphic to \( PG(2,2) \), whose full automorphism group is 2-transitive on points. From now on, we assume \( q > 7 \), so that \( \text{Aut} N(3,q) \cong \text{AT}^2 L(1,q) \) as was proved by Robinson [100]. It follows that \( \text{Aut} N(3,q) \) is 2-homogeneous but not 2-transitive on points. Moreover, \( \text{Aut} N(3,q) \) is not locally primitive (cf. Section 4.9). Indeed, the stabilizer of the point 0 in \( \text{Aut} N(3,q) \) acts on the lines through 0 in the same way as \( \text{I}^2 L(1,q) \) acts on the
nonzero squares of $GF(q)$. This action is imprimitive because $(q - 1)/2 \equiv 3 \pmod{6}$ is divisible by 3, and so the group of nonzero squares in $GF(q)$ has a proper subgroup, which means that the stabilizer of 1 in $\Gamma^2 L(1, q)$ is not a maximal subgroup.

Therefore, each Netto system $N(3, q)$ with $q > 7$ provides an example of a linear space whose full automorphism group is 2-homogeneous but neither 2-transitive nor locally primitive. Even more remarkable: the Netto systems $N(3, q)$ with $q > 7$ are the only linear spaces whose full automorphism group is 2-homogeneous but not 2-transitive (see Section 3.3).

Note that $N(3, q)$ can also be defined as the linear space with point-set $GF(q)$ and base-line $K$ under $AG^2 L(1, q)$, where $K$ is the set of third roots of unity in $GF(q)$. This generalizes to the following construction. Let $q = p^n$ with $p$ an odd prime, $n$ an integer, and let $k \geq 3$ be an odd integer such that $k(k - 1)$ divides $q - 1$. Let $K$ be the set of all $k$th roots of unity in $GF(q)$ and define a point–line incidence structure $N(k, q)$ as follows. The set of points is $GF(q)$, the lines are the images of $K$ under the group $AG^k - L(1, q)$. That group and its overgroups in $A1^{k - 1} L(1, q)$ act flag transitively on $N(k, q)$. However, $N(k, q)$ need not be a linear space; the following is a necessary and sufficient condition for $N(k, q)$ to be a linear space:

\[ (*) \text{ For any primitive } k\text{th root of unity } \varepsilon \text{ in } GF(q) \text{ the elements } \varepsilon - 1, \varepsilon^2 - 1, \ldots, \varepsilon^{k - 1} - 1 \text{ are in distinct cosets of the multiplicative group consisting of all } (k - 1)\text{th powers of nonzero elements of } GF(q). \]

In particular, if $k = 3$ then (*) holds provided $q \equiv 3 \pmod{4}$. Therefore, $N(3, q)$ is a linear space for every prime power $q \equiv 7 \pmod{12}$, namely a Netto system. When $k > 3$ we call the resulting linear spaces generalized Netto systems. If $k = 5$ then (*) is equivalent to the condition that $5^{q - 1}/4 \neq 1$ in $GF(q)$, with $q \equiv 21 \pmod{40}$. For $q < 800$ this gives four linear spaces with $q = 61, 421, 661, 701$. Just two Desarguesian projective planes arise as generalized Netto systems: $N(3, 7) = PG(2, 2)$ and $N(9, 73) = PG(2, 8)$.

2.12. Lüneburg–Tits affine translation planes

Lüneburg [84, 85] has investigated an infinite family of non-Desarguesian flag transitive affine planes $Lü(q^2)$ of order $q^2$, where $q = 2^{2e + 1} \geq 8$. These planes arise from 1-spreads in $PG(3, q)$ constructed by Tits [105] and related to the Suzuki ovoids. The point-stabilizer $G_0$ of any flag transitive group $G \leq Aut(Lü(q^2))$ normalizes the Suzuki group $^2B_2(q)$. These planes are the only non-Desarguesian affine planes whose automorphism group is transitive on the unordered pairs of intersecting lines (see Section 4.9) and, together with the two exceptional planes described in Section 2.13, are the only non-Desarguesian affine planes with a flag transitive group which is not 1-dimensional affine.

2.13. The exceptional nearfield plane and Hering’s plane

Up to isomorphism, there are only two non-Desarguesian affine planes whose automorphism group is 2-homogeneous, namely the nearfield plane $A_o$ of order 9 and
Hering's plane $A_{27}$ of order 27 [56]. The stabilizer of a point in the nearfield plane of order 9 is isomorphic to $S_5 \cdot 2^4 \cdot 2$, preserves a pairing of the 10 points at infinity and acts on this set of 5 pairs as the symmetric group $S_5$ (see e.g. [53] for the list of all flag transitive subgroups of $Aut(A_9)$). The unique flag transitive automorphism group of $A_{27}$ is isomorphic to $3^6 \cdot SL(2, 13)$ and preserves a pairing of the 28 points at infinity.

2.14. Hering's 2-(9^3, 9, 1) designs

Hering [57] has constructed two nonisomorphic flag transitive linear spaces on $3^6$ points with line size $3^2$, whose full automorphism group is 2-transitive and locally primitive but not transitive on unordered pairs of intersecting lines. This group is isomorphic in both cases to $3^6 \cdot SL(2, 13)$.

2.15. Unital

A unital of order $n$ is a linear space on $v = n^3 + 1$ points with line-size $k = n + 1$. As noticed in [47] it follows from Theorem 4.5 that the only flag transitive (and even the only line transitive) unitals are:

(i) for any prime power $q$, the Hermitian unital $U_H(q)$ of order $q$ whose points and lines are, respectively, the absolute points and nonabsolute lines of a unitary polarity in $PG(2, q^2)$, the incidence being the natural one;

(ii) for any $q = 3^{2e+1} (e \geq 0)$, the Ree unital $U_R(q)$ of order $q$ whose points and lines are, respectively, the Sylow 3-subgroups and the involutions of the Ree group $^2G_2(q)$, a point and a line being incident if and only if the involution normalizes the Sylow 3-subgroup [84].

The flag transitive automorphism groups of $U_H(q) (q \neq 2)$ are those lying between $PSL(3, q)$ and $PSL(3, q)$. They are 2-transitive and transitve on the ordered pairs of intersecting lines. Moreover, the group $PGU(3, q)$ is sharply transitive on the triples $(x, L, x')$ consisting of a flag $(x, L)$ and a point $x'$ not incident with $L$. Note also that the unital $U_H(4)$ has a remarkable property: its automorphism group is transitive on the bases, i.e. on the triples $(x, x', x'')$ of non-collinear points. As noticed in [29], any other linear space having this property (and a line-size $k \geq 3$) is a Desarguesian projective or affine space of dimension $d \geq 2$.

The flag transitive automorphism groups of $U_R(q)$ are those lying between $^2G_2(q)$ and $Aut(^2G_2(q))$. They are 2-transitive but not locally primitive.

2.16. Witt–Bose–Shrikhande spaces

Starting from the group $PSL(2, 2^n)$ with $n \geq 3$, Kantor [69] defined a flag transitive linear space $S$ as follows: the points are the subgroups of $PSL(2, 2^n)$ isomorphic to the dihedral group of order $2(2^n + 1)$, the lines are the involutions of $PSL(2, 2^n)$, a point being incident with a line if and only if the subgroup contains the involution. $S$ has $v = 2^{n-1}(2^n - 1)$ points, line-size $k = 2^{n-1}$, point-degree $r = 2^n + 1$ and contains no
Let $G$ be a group acting 2-homogeneously on a finite set $\Omega$ of $n \geq 3$ points. Kantor [65] proved that if $G$ is not 2-transitive on $\Omega$, then $G \triangleleft Aut(1, q)$ with $n - q \equiv 3 \pmod{4}$.

On the other hand, it follows from the classification of the finite simple groups that any 2-transitive group $G$ on $\Omega$ lies in the following list (see e.g. [72]):

1. $G = Sym(n)$ or $Alt(n)$, $n \geq 4$;
2. $PSL(d, q) \leq G \leq PGL(d, q)$, $n = (q^d - 1)/(q - 1)$, $d \geq 2$;
3. $PSU(3, q) \leq G \leq PGL(3, q)$, $n = q^3 + 1$;
4. $2G_2(q) \leq G \leq Aut(2G_2(q))$, $n = q^3 + 1$, $q = 3^{2e+1}$, $e \geq 0$;
5. $2B_2(q) \leq G \leq Aut(2B_2(q))$, $n = q^2 + 1$, $q = 2^{2e+1}$, $e \geq 1$;
6. $G = Sp(2e, 2)$, $n = 2^{2e-1}(2^e + 1)$, $e \geq 3$;
7. $G = PSL(2, 11)$, $n = 11$;
8. $G = Alt(7)$, $n = 15$;
9. $G = M_{11}, M_{12}, M_{22}, Aut M_{22}, M_{23}, M_{24}, n = 11$ or 12, 12, 22, 22, 23, 24 respectively;
10. $G = HS$, $n = 176$;
11. $G = Co_3$, $n = 276$;
12. $G$ is of affine type and the point-stabilizer $G_0$ is as follows:
   (a) $SL(d, q) \leq G_0 \leq GL(d, q)$, $n = q^d$, $d \geq 1$,
   (b) $Sp(2d, q) \leq G_0$, $n = q^{2d}$, $d \geq 2$,
   (c) $G_3(q) \leq G_0$, $n = q^6$, $q = 2^e$, $e \geq 1$,
   (d) $G_0 \geq E$, an extraspecial group of order $2^{e+1}$ where $n = p^e = 3^4$ or $q^2$ with $q = 3, 5, 7, 11, 23$,
   (e) the last term of the commutator series of $G_0$ is $SL(2, 5)$ and $n = q^2$ where $q = 9, 11, 19, 29, 59$. 

2.17. Finite 2-homogeneous permutation groups
Classifications of finite highly transitive dimensional linear spaces

(f) $G_0 = \text{Alt}(6)$ or $\text{Alt}(7)$ and $n = 2^4$.

(g) $G_0 = \text{SL}(2,13)$ and $n = 36$.

(h) $G_0 = \text{PSU}(3,3)$ and $n = 2^9$.

2.18. The O'Nan-Scott theorem

This theorem, first presented at the Santa Cruz Conference in 1979, classifies the finite primitive permutation groups into five major types. We will see in Section 4 how crucial it is for the investigation of finite flag transitive linear spaces. It is the basis of the reduction theorems 4.3 and 5.2, in a simplified geometric setting [15] for Theorem 4.3, and in a more complete one [80] for Theorem 5.2. We describe here the latter version, geometrically blended by the first one.

Recall first that a group $G$ is almost simple if it has a non-Abelian simple normal subgroup $N$ such that $C_G(N) = 1$, or equivalently $N \leqslant G \triangleleft \text{Aut} N$. We have already defined permutation groups of affine type in Section 2.9.

From now on, $G$ will denote a permutation group acting primitively on a finite set $\Omega$, and $\omega \in \Omega$. Suppose that $\Omega = A_1 \times \cdots \times A_n$ is a Cartesian product of $n \geq 2$ copies of some set $A$ of size $a \geq 2$. Given a subscript $j \in \{1, \ldots, n\}$ and, for every $i \neq j$, an element $a_i \in A_i$, the set of all points $(x_1, \ldots, x_n) \in \Omega$ such that $x_i = a_i$ for every $i \neq j$ is called a Cartesian line of the $j$th parallel class. The set $\Omega$ provided with all the Cartesian lines is called an $n$-dimensional Cartesian space. The group $G$ is of Cartesian type if $G$ leaves invariant the structure of some Cartesian space $A^n$ on $\Omega$. Note that, since $G$ is primitive on $\Omega$, the stabilizer $G_\omega$ of any point $\omega \in \Omega$ acts transitively on the $n$ Cartesian lines through $\omega$.

An $n$-dimensional diagonal space is a set $\Omega$ of size $a^n$ with $n$ and $a \geq 2$ provided with $n + 1$ partitions of $\Omega$ into $a$-subsets, called diagonal lines, such that $\Omega$ and any $n$ of these $n + 1$ partitions form a Cartesian space whose Cartesian lines are the diagonal lines belonging to these $n$ distinguished partitions.

Finally we shall call $G$ biregular if $\text{soc} G = T_1 \times T_2$ is a direct product of two (isomorphic) non-Abelian simple groups which both act regularly on $\Omega$. Two possibilities can occur: either $T_1 \times T_2$ is the unique minimal normal subgroup of $G$, or else both $T_1$ and $T_2$ are normal in $G$ and centralize each other.

The socle $\text{soc} G$ of a primitive group $G$ is a direct product $T_1 \times \cdots \times T_k \cong T^k$ of $k$ copies $T_1, \ldots, T_k$ of a simple group $T$. The O'Nan–Scott classification theorem reads as follows:

1. $G$ is affine, i.e. $T$ is cyclic of prime order $p$, $|\Omega| = |T|^k = p^k$, $G \leqslant \text{AGL}(k, p)$, $\Omega$ is identified with the point-set of $AG(k, p)$ or with $GF(p^k)$ and $G_\omega \cong G_0 = G \cap \text{GL}(k, p)$ acts irreducibly on $GF(p^k)^*$. In this case $G$ has a unique minimal normal subgroup, called the translation group, which is elementary Abelian and regular on $\Omega$. $G$ may possibly be of Cartesian type.

2. $G$ is almost simple, i.e. $k = 1$, $T$ is non-Abelian, $G \leqslant \text{Aut} T$ and $T_\omega \neq 1$. The possible sizes for $\Omega$ are the indices of the ordinary or novel maximal subgroups $M$ of $G$ (according as the subgroup $T \cap M \neq 1$ is or is not maximal in $T$).
(3) \(G\) has a simple diagonal action, i.e. \(k \geq 2\), \(T\) is non-Abelian, \(|\Omega| = |T|^{k-1}\), \(G \leq T^k \cdot (\operatorname{Out} T \times G^k)\), \(G_\Omega \leq \operatorname{Aut} T \times G^k\), where \(G^k\) denotes the action of \(G\) on the \(k\) copies \(T_1, \ldots, T_k\) of \(T\). This case further subdivides into:

(3a) \(G^k\) is primitive and \(G\) has a unique minimal normal subgroup. If \(k \geq 3\), then \(\Omega\) is a diagonal space of dimension \(k-1\). If \(k = 2\), then \(G\) is biregular.

(3b) \(G^k = 1\), \(k-2\) and \(G\) is biregular with two minimal normal subgroups.

(4) \(G\) has a product action, i.e. \(T\) is non-Abelian, \(\Omega\) is an \(l\)-dimensional Cartesian space \(\Gamma^l\) with \(|l|k\), \(G \leq H \wr \operatorname{Sym}(l)\), \(G_\Omega \leq H \wr \operatorname{Sym}(l)\) where \(H\) is primitive on \(\Gamma\) and \(\gamma \in \Gamma\). This case further subdivides into:

(4a) \(k = l\) and \(H\) is almost simple with socle \(T\), so that \(G \leq (\operatorname{Aut} T)^k \wr \operatorname{Sym}(k)\) and has a unique minimal normal subgroup.

(4b) \(k > l\) and \(H\) has a simple diagonal action, so that \(|\Omega| = |T|^{k-1}\) and \(|\Gamma| = |T|^{k/l-1}|T|^{k/l-1}\).

If \((H, \Gamma)\) belongs to case (3a), then \(G\) has a unique minimal normal subgroup and, unless \(k = 2l\), the Cartesian lines of \(\Omega\) are diagonal spaces of dimension \(k/l-1\).

If \((H, \Gamma)\) belongs to case (3b), then \(G\) has two minimal normal subgroups, both regular on the \((k/2)\)-dimensional Cartesian space \(\Omega\).

(5) \(G\) has a twisted wreath action. Here \(T\) is non-Abelian, \(\Omega\) is a \(k\)-dimensional Cartesian space with line-size \(|\Gamma|\), on which \(soc G\) acts regularly, and \(soc G\) is the unique minimal normal subgroup of \(G\) (Further information is provided in \([80]\)).

3. Transitivity on independent sets

3.1. Jordan transitivity

The search for basis transitive finite DLSs may be seen as a direct generalization of Jordan’s investigation in 1871 of finite permutation groups having a Jordan set. A Jordan set of a permutation group \(G\) on a point set \(P\) is a set \(J\) of (at least two) points such that the pointwise stabilizer of its complement \(P \setminus J\) is transitive on \(J\). If \(G\) is \(n\)-transitive on \(P\), then the complement of any \((n-1)\)-set is a Jordan set, called improper. If \(G\) has a proper Jordan set, then \(G\) is called a Jordan group. Hall \([54]\) stated that the complements of the Jordan sets of a finite primitive Jordan group are the varieties of a DLS \(S\) on \(P\). Of course, \(G\) is an automorphism group of \(S\) and enjoys the so-called Jordan transitivity, i.e. the pointwise stabilizer in \(G\) of any variety \(V\) of \(S\) acts transitively on the complement of \(V\). Jordan transitivity on \(S\) is quite a strong hypothesis forcing in particular transitivity on the ordered bases of \(S\).

Nevertheless, the problem of determining all finite Jordan transitive DLSs remained open (see c.g. \([54, 66, 18]\)) until 1985, when it was solved using the classification of all 2-transitive permutation groups ([72], see also [26]). Surprisingly, the latter paper is devoted to model theory.
3.2. 2-transitive 2-DLSs

The determination of 2-transitive finite linear spaces challenged geometers since 1955. The two most beautiful results in this vein are, on the one hand, the celebrated Ostrom–Wagner [93] theorem,

*If S is a finite projective plane having a 2-transitive automorphism group G, then S ≅ PG(2, q) and G ≅ PSL(3, q) for some q,*

and, on the other hand, the Cameron–Kantor [22] determination of all automorphism groups acting 2-transitively on PG(d, q) for d ≥ 3. Both proofs are nicely combinatorial and do not use group theory beyond elementary facts about Sylow subgroups of permutation groups.

Marshall Hall was very active in discussing connections between 2-designs and 2-transitive groups. In particular, he conjectured that any finite linear space with line-size 3 and a 2-transitive automorphism group should be 2-PG(d, 2) or 2-AG(d, 3). But he was only able to draw this conclusion from the stronger hypothesis of ordered basis transitivity (i.e. transitivity on ordered triangles) using a result of Bruck [54, 55].

The major reference antidating the classification of the finite simple groups was [68]. Here again the solution arose from the classification of all finite 2-transitive groups (Section 2.17).

**Theorem 3.1** (Kantor [72]). Let S be a finite nontrivial linear space and let G ≤ Aut S act 2-transitively on the points of S. Then one of the following holds:

1. S = 2-PG(d, q), G ≃ PSL(d + 1, q) or S = 2-PG(3, 2), G ≃ A_7;
2. S is the Hermitian unital U(q) of order q, G ≃ PSU(3, q);
3. S is the Ree unital U(q) of order q, G ≃ G_2(q);
4. S = 2-AG(d, q), G_0 ≤ L(1, q^d);
5. S = 2-AG(d, q), G_0 ≃ SL(d/a, q^a);
6. S = 2-AG(d, q), G_0 ≃ Sp(d/a, q^d), 4a ≤ d;
7. S = 2-AG(d, q), G_0 ≥ G_2(q^a), q even, d = 6a;
8. S = 2-AG(4, 3), G_0 ≃ SL(2, 5) or G_0 has a normal extraspecial subgroup of order 2^5;
9. S = 2-AG(6, 3), G_0 ≃ SL(2, 13);
10. S = 2-AG(2, q), G_0 ≃ SL(2, 3) or SL(2, 5), q = 5, 7, 9, 11, 19, 23, 29, 59;
11. S is the nearfield affine plane A_9 of order 9, G_0 as in (8);
12. S is Hering's affine plane A_{27} of order 27, G_0 as in (9);
13. S is one of the two Hering spaces 2-(9^3, 9, 1), G_0 as in (9).

Of course, if S is a trivial linear space, then G can be any 2-transitive group on P. Noting that the line through two points x and y must be a union of orbits of G_{xy} and that the line-size in a regular linear space does not exceed the square root of the total number of points (except for projective planes), Kantor's argument essentially amounts to a knowledge of the orbit-lengths of stabilizers of pairs of points in a 2-transitive group. However, this approach fails for the affine 1-dimensional
2-transitive groups, which were more difficult to deal with and required more ingenious arguments.

3.3. 2-homogeneous 2-DLSs

Jointly with Jean Doyen, we extended Kantor’s result to all 2-homogeneous finite linear spaces, using another result of Kantor [65] stating that any 2-homogeneous, but not 2-transitive, finite group $G$ of degree $> 3$ is 1-dimensional affine contained in $AG_2(1, p^n)$, where $p \equiv 3 \pmod{4}$ and $n$ is odd (cf. Section 2.17). If $S$ is a finite linear space on $p^n$ points preserved by $G$, then the point-set of $S$ can be identified with $GF(p^n)$ and we managed to prove that if $p | k$ then the line $L$ through 0 and 1 is a subfield of $GF(p^n)$, and that if $p \nmid k$ then $k = 3$ and the third point of $L$ is a primitive sixth root of unity in $GF(p^n)$.

After our paper was accepted for publication we heard of an independent proof by Siemons and Tamburini, and this resulted in the joint paper quoted below.

**Theorem 3.2** (Delandtsheer et al. [44]). Let $S$ be a finite nontrivial linear space and let $G \leq Aut S$ act 2-homogeneously but not 2-transitively on the points. Then $G \leq AG_2 L(1, p^n)$ for some prime $p \equiv 3 \pmod{4}$ and some odd integer $n$, and

1. $S$ is the Desarguesian affine space $AG(d, p^n)$, or
2. $S$ is the Netto system $N(3, p^n)$ and $p \equiv 7 \pmod{12}$ (see Section 2.11).

3.4. Locally 2-homogeneous erectable 2-DLSs

Another long-standing open problem was to determine all finite locally 2-homogeneous linear spaces, i.e. those having an automorphism group acting transitively on the unordered pairs of intersecting lines. Buekenhout [13] and Kantor [67] offered partial answers under rather restrictive additional assumptions on the action of point-stabilizers. Another approach is to introduce restrictions on $S$ itself. Of course, if $S$ is a finite projective plane, then the Ostrom–Wagner theorem provides the answer. However, even under the assumption that $S$ be an affine plane, the problem remained open despite the remarkable results of Schulz [102] and Czerwinski [28], who proved that if an affine plane $S$ has a locally 2-transitive automorphism group $G$, then $S$ is Desarguesian or is a Lüneburg plane, unless $S$ has odd order and $G$ contains a Baer involution.

Assuming that $S$ is not a plane, i.e. contains a proper linear subspace, we derived the following classification from Theorems 3.1 and 3.2.

**Theorem 3.3** (Delandtsheer [30]). Let $S$ be a finite nontrivial linear space admitting a proper linear subspace and let $G \leq Aut S$ be locally 2-homogeneous. Then one of the following occurs:

1. $S = 2-PG(d, q)$ and $G \geq PSL(d+1, q)$, or $G \geq A_7$ and $(d, q) = (3, 2)$;
2. $S = 2-AG(d, q)$ and $G \geq ASL(d, q)$, or $d \geq 3, q \geq 3$. 


The proof consists in showing that $S$ is the 2-truncation of a planar space, so that the following theorem applies.

**Theorem 3.4** (Delandtsheer [30]). Let $S$ be a finite nontrivial 3-DLS and let $G \leq \text{Aut } S$ act transitively on the unordered pairs of intersecting lines. Then one of the following holds:

(i) $S = 3\text{-PG}(d,q)$ and $\text{PSL}(d+1,q) \trianglelefteq G \leq \text{PGL}(d+1,q)$ ($d \geq 3$, $q \geq 2$) or $G \cong A_7$ and $(d,q) = (3,2)$;

(ii) $S = 3\text{-AG}(d,q)$ and $\text{ASL}(d,q) \leq G \leq \text{AGL}(d,q)$ ($d \geq 3$, $q \geq 3$);

(iii) $S$ is the $3\text{-}(22,6,1)$ design $\mathcal{M}_{22}$ and $G = M_{22}$ or $\text{Aut } M_{22}$;

(iv) $S$ is the $3\text{-}(q^d + 1, q + 1, 1)$ design on $PG(1,q^d)$ with base plane $PG(1,q)$ under $G \cong PSL(2,q^d)$;

(v) $S$ is the $3\text{-}(q+1,4,1)$ design on $PG(1,q)$ with base plane $K \cup \{\infty\}$, where $K$ is the set of third roots of unity in $GF(q)$ and $PSL(2,q) \trianglelefteq G \leq PZL(2,q)$.

Since the stabilizer $G_x$ of any point $x$ acts 2-homogeneously on the points of the residue $S_x$, Theorems 3.1 and 3.2 apply to the pair $(S_x, G_x)$. The proof then consists essentially of combinatorial arguments and arithmetics, supported by some knowledge about the possible pairs $(S_x, G_x)$.

As a consequence of Theorem 3.4, we also proved the following result.

**Corollary 3.5** (Delandtsheer [30]). Let $S$ be a finite nontrivial 3-DLS. Then $\text{Aut } S$ acts transitively on the pairs consisting of a plane and a line intersecting this plane if and only if one of the following occurs:

(i) $S = 3\text{-AG}(d,q)$ or $3\text{-PG}(d,q)$ with $d \geq 3$;

(ii) $S$ is the $3\text{-}(22,6,1)$ design $\mathcal{M}_{22}$;

(iii) $S$ is the Miquelian inversive plane $3\text{-}(q^2 + 1, q + 1, 1)$;

(iv) $S$ is the $3\text{-}(513,9,1)$ design constructed over $PG(1,8^3)$, with $PG(1,8)$ as a base plane under $PGL(2,8^3)$.

To close this section let us mention that we recently used the brand new near classification of finite flag transitive linear spaces (Theorem 4.5) and got a complete classification of all finite locally 2-homogeneous linear spaces (Corollary 4.8).

### 3.5. Ordered basis transitive n-DLSs

The determination of all ordered basis transitive finite linear spaces immediately follows from that of the 2-transitive ones. This quite naturally generalizes to the problem of classifying all ordered basis transitive finite DLSs dealt with by Hall [55] and Cameron [19] in very special cases, and eventually solved by Kantor [72] as another consequence of the classification of all 2-transitive groups.
Theorem 3.6 (Kantor [72]). Let $S$ be a finite nontrivial $n$-DLS. If $\text{Aut} S$ acts transitively on the ordered bases, then one of the following occurs:

(i) $S = n \cdot PG(d, q)$ or $n \cdot AG(d, q)$ with $2 \leq n < d$;
(ii) $S = M_{22}, M_{23}$ or $M_{24}$;
(iii) $S = U(4)$.

At the same time Kantor called attention to the problem of classifying (unordered) basis transitive finite DLSs, noting that the situation appears to be much more chaotic. Indeed the group $G$ does not even need to be point transitive, and strange examples arise. Li then proved that every point-orbit of $G$ is a variety of $S$ and that $S$ is a direct sum of DLSs which are both point and basis transitive. He also managed to solve the problem in dimensions 2 and 3 [78]. Our classification (Theorem 3.7) reduces the problem to the knowledge of the finite circular DLSs which are both point and basis transitive. Note also the earlier work of Astie-Vidal [3] showing that if $S$ is an $n$-DLS admitting an automorphism group $G$ which, for every integer $i$, acts transitively on the ordered pairs of bases intersecting in exactly $i$ points, then either $S$ is trivial or $S$ is a direct sum of $(d-2)$-DLSs on $d$ points.

Before stating our results, we recall a standard generalization of the notion of DLS and a (nonstandard) generalization of the direct sum of DLSs (introduced in [40]).

3.6. Pre-DLS, dual pre-DLS, direct sum and supersum

A finite pre-DLS (or matroid) $T$ is a finite closure space satisfying the exchange property (Section 2.2). $T$ is simple (and hence is a DLS) if the empty set and all singletons are closed. The simplification of $T$ is the DLS canonically associated with $T$, which is obtained by neglecting the points that are in the closure of the empty set (such points are called loops) and by identifying all points which are in the closure of each other (such points are said to be parallel). Independent sets, bases and truncations of pre-DLSs are defined in the same way as in DLSs. It is well-known that any pre-DLS $T$ is uniquely determined by its bases and that taking as bases the complements of the bases of $T$ defines a pre-DLS $T^*$ called the dual of $T$. In particular, the simplification of the dual of an $n$-DLS $S$ on $v$ points is a $(v-n-2)$-DLS $S^*$ (which is basis transitive as soon as $S$ is).

Now remember that if $\{S_i, i \in I\}$ is a family of DLSs on pairwise disjoint point-sets $P_i$, the direct sum $\bigoplus_{i \in I} S_i$ is the DLS on $\bigcup_{i \in I} P_i$ whose varieties are all unions $\bigcup_{i \in I} V_i$ where $V_i$ is a variety of $S_i$. For example, the finite-dimensional generalized projective spaces are direct sums of nondegenerate (but possibly 0- or 1-dimensional) projective spaces. If $|I| = t$ and if $S_i$ has dimension $d_i$, then $\dim(\bigoplus_i S_i) = t - 1 + \sum_i d_i$.

We now introduce supersums, which generalize direct sums, by defining a superstructure of pre-DLS on the $t$-set of all $S_i$’s, in order to select and keep only some of the varieties of $\bigoplus_i S_i$.

Let $T$ be a $d_0$-pre-DLS on $\{1, \ldots, t\}$, let $P_1, \ldots, P_t$ be pairwise disjoint sets, let $S_i$ be a $d_i$-DLS $(0 \leq d_i < +\infty)$ on $P_i$ and let $S'_i$ be a $(d_i-1)$-pre-DLS on $P_i$ such that every
basis of \( S_i \) is contained in some basis of \( S_i \), every basis of \( S_i \) contains a basis of \( S_i \) and the hyperplanes of \( S_i \) are certain colines or hyperplanes of \( S_i \). For shortness such a pair \((S_i, S_i')\) will be called a \((d, d-1)\)-DLSs association. The simplest possibility is that \( S_i \) is the \((d_i-1)\)-truncation of \( S_i \), but there are other possibilities, e.g. \( S_i \) can be any \((d_i-1)\)-DLS erected on \((d_i-2)\)-\( S_i \), the hyperplanes of \( S_i \) being certain hyperplanes and colines \((d_i-2)\)-varieties) of \( S_i \).

In order to get a DLS, assume further that if \( d_i = d_j = 0 \) for distinct \( i, j \), then the pair \( \{i, j\} \) is independent in \( T \). Then the supersum of the \((S_i, S_i')\)’s over \( T \) is the \((\sum_{i=0}^{d_1} S_i, S_i')\)-DLS \( \bigoplus_T (S_i, S_i') \) with point-set \( \bigcup_{i=1}^{d_1} P_i \) and whose bases are the sets \( B \) for which there is a basis \( J \) of \( T \) such that \( B \cap P_i \) is a basis of \( S_i \) if \( j \in J \) and \( B \cap P_j \) is a basis of \( S_j' \) if \( j \notin J \). Note that in the notation \( \bigoplus_T (S_i, S_i') \), \( T \) is a pre-DLS provided with a numbering of its points. Figure 1 suggests such a supersum for \( T = PG(2, 2) \) with points numbered as indicated:

- \( S_1 \cong S_2 \cong PG(2, 2)^* \), \( S_1' \cong S_2' \cong PG(2, 2) \);
- \( S_3 \cong S_4 \cong S_5 \cong 2-PG(3, 1) \);
- \( S_6 \cong S_7 \cong 2 \cong 2-PG(3, 1) \);
- \( S_8 \cong S_9 \cong S_10 \cong AG(3, 2) \), \( S_8' \cong S_9' \cong S_10' \cong AG(3, 2) \).

Remember that the partition DLS \( \Pi_4 \), corresponding to the lattice of partitions of a 4-set, is the punctured projective plane of order 2. All points and all lines of size \( \geq 3 \) are drawn, and the other varieties are usually omitted. The black points form a basis corresponding to the choice \( J = \{1, 3, 6\} \).

If \( T \) is the completely trivial DLS (whose only basis is the full point-set), then \( S_i \) is useless and we get back the direct sum. Our notation \( \bigoplus_T (S_i, S_i') \) thins down into \( \bigoplus_T S \) in the special case where all \( S_i \)'s are isomorphic to some \( d \)-DLS \( S \) and where
$S'_i$ is the $(d-1)$-truncation of $S_i$. This DLS coincides with the direct product $T \otimes S$ of the two matroids $T$ and $S$ defined by Lim [82], who proved that if $S$ and $T$ are two nonsingleton matroids, then $\text{Aut}(\oplus_T S) = \text{Aut} S \wr \text{Aut} T$ (the wreath product) if and only if the following two implications hold:

(i) if $S$ is a 0-DLS, then $T$ is a DLS, and

(ii) if $S$ is a direct sum, then for any pair $(x, y)$ of points in $T$, there is a dependent set containing $x$ but not $y$.

3.7. Reduction theorems for basis transitive $n$-DLSs

The notions introduced in Section 3.6 allow the following build-up of basis transitive DLSs.

(i) The simplification of the dual of a basis transitive $n$-DLS on $v$ points is a basis transitive $(v-n-2)$-DLS on $\leq v$ points. If $G$ is point primitive and basis transitive on $S$, then $G$ is also point primitive and basis transitive on the DLS $S^*$.

(ii) If $G_i$ acts basis transitively on $S_i$ for each $i$, then the direct product $\bigoplus_i G_i$ acts basis transitively on the direct sum $\bigoplus_i S_i$.

(iii) If $T$ is a point transitive and basis transitive $d_0$-pre-DLS on $t$ points, if $(S, S')$ is a $(d, d-1)$-DLS’s association sharing a common point and basis transitive automorphism group $G$, then the supersum $\bigoplus_T (S, S') := \bigoplus_T (S_i, S_i)$, where $S_i \simeq S'_i \simeq S$ is a $(d_0 + td)$-DLS admitting $G \wr \text{Aut} T$ as a basis transitive automorphism group.

Note that starting from well-known DLSs, this process can be iterated again and again ... providing quite unusual creatures.

Conversely, the following reduction theorem holds.

**Theorem 3.7** (Li [78] for (a), Delandtsheer [31, 41, 118] for (b) and (c)). Let $S$ be a finite DLS and let $G \leq \text{Aut} S$ be basis transitive. Then

(a) $S$ is a direct sum of point and basis transitive DLSs.

(b) If $S$ is point transitive, then $S$ is a supersum $\bigoplus_T (R, R')$, where $T$ is a point and basis transitive pre-$d_0$-DLS with $d_0 \geq 0$ and $(R, R')$ is a $(d, d-1)$-DLS’s association sharing a common point primitive and basis transitive automorphism group.

(c) Let $\lambda$ be the smallest dimension of a thick variety (i.e. with size $\geq \lambda + 2$). If at least one $\lambda$-variety has size $\geq 2\lambda + 1$ and if $S$ is point transitive, then $S$ is a supersum $\bigoplus_T (R, R')$, where $T$ is as in (b) and $(R, R')$ is a $(d, d-1)$-DLS’s association with $d-1 \geq \lambda$, where $R$ and $R'$ share a common $2$-homogeneous and basis transitive automorphism group. Moreover, if $\lambda = 1$, then $R' = (d-1) - R$.

The case $\lambda = 1$ in (c), i.e. the case where $S$ is noncircular, has been handled in [31], while the generalization (b) and (c) is recent. Note that whenever $R' = (d-1) - R$, the supersum $S = \bigoplus_T (R, R')$ is merely a direct product in the sense of Lim [82].

For small values of $d$ and $d_0$, we recall the following. The $(d-1)$-truncation of $R$ is the pre-DLS on the same point-set as $R$ and whose independent sets are the independent sets of size $\leq d$ in $R$. If $d_0 = 0$ the bases of $T$ are the singletons, and if $d_0 = 1$ then
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$T$ can be described as a complete regular multipartite graph $\Gamma$ (possibly with maximal coclique-size 1), the points of $T$ being the vertices of $\Gamma$ and the bases of $T$ being the edges of $\Gamma$. Let us also mention the following errata to [31]:

- p. 386, line - 14: read $r \geq 1$ instead of $r \geq 0$;
- p. 386, line - 13: read pre-$(s - 1)$-DLS instead of $(s - 1)$-dimensional geometric lattice;
- p. 393, lines -20, -19 and -18: erase the sentence ‘Then any pair ... geometric lattice.’

Note also that the classification of point and basis transitive pre-DLSs amounts to that of point and basis transitive DLSs in the following way: Let $S$ be a DLS on the point-set $P$ with $G \leq Aut S$ point and basis transitive on $S$ and let $H$ be a permutation group acting transitively on a set $\Omega$. Then $S$ defines on the point-set $\Omega \times P$ a pre-DLS $T$ whose classes of parallel points are the sets $\Omega \times \{x\}$ with $x \in P$ and whose simplification is $S$, and $K = H wr G \leq Aut T$ acts point and basis transitively on $T$. Conversely, any pair $(T, K)$, where $K$ acts point and basis transitively on $T$, is obtained in this way from the simplification $S$ of $T$ and from the actions of $K$ on $S$ and on the classes of parallel points of $T$.

3.8. Point primitive and basis transitive $n$-DLSs

As we have seen in Section 3.7, the classification of basis transitive DLSs amounts to that of point primitive and basis transitive DLSs. This work has been completed for all noncircular DLSs as well as for those of dimension $\leq 3$.

**Theorem 3.8** *(Theorem 3.7 and Delandtsheer [31]).* Let $S$ be a finite noncircular $n$-DLS and let $G \leq Aut S$ act transitively on the bases and primitively on the points of $S$. Then one of the following occurs:

(i) $S = n-PG(d, q)$ and $G \geq PSL(d + 1, q)$ $(d, q \geq 2)$ or $G = A_7$ and $(d, q) = (3, 2);

(ii) $S = n-AG(d, q)$ and $G \geq ASL(d, q)$ $(d, q \geq 2);

(iii) $S = U_4(4)$ and $G \geq PSU(3, 4)$.

The main ingredients of the proof are of course Theorems 3.1 and 3.2, which provide the structure of 2-$S$. Then the problem amounts to examining the list of pairs $(2-S, G)$ and to find all possible $G$-invariant basis transitive erections of 2-$S$, which is not so easy.

Li classified all basis transitive 2- or 3-DLSs. The results mentioned so far allow us to state the following reduced version of his theorem, noting that if the group is imprimitive then $S = \bigoplus_{t \in \mathbb{R}}(R, R')$, where $T$ is a pre-$d_0$-DLS on $t$ points and $(d_0, t) = (0, 2)$, $(0, 3)$ or $(1, 2)$.

**Theorem 3.9** *(Li [78]).* If $S$ is a finite circular 3-DLS admitting a point primitive and basis transitive automorphism group, then one of the following occurs:
(i) $S$ is the 3-(10, 4, 1) or 3-(17, 5, 1) inversive plane of order 3 or 4;
(ii) $S$ is the Mathieu–Witt design 3-(22, 6, 1);
(iii) $S = PG(2, 2)^*$. 

Other examples of point primitive and basis transitive circular $n$-DLSs are the other four Mathieu–Witt designs related to $M_{11}$, $M_{12}$, $M_{23}$ and $M_{24}$, the double circular extension of the Pentagram linear space (described in Proposition 3.10) and the duals of all aforementioned examples.

**Proposition 3.10.** There is (up to isomorphism) only one 4-DLS $S$ on 12 points whose automorphism group is the Mathieu group $M_{11}$. $S$ is the hypercircular 4-DLS whose thick hyperplanes are precisely the blocks of the Hadamard 3-(12, 6, 2) design; its top 2-dimensional residues are isomorphic to the Pentagram linear space.

The Pentagram linear space is the linear space on 10 points with line-sizes 2 and 4 admitting $\text{Sym}(5)$ as its full automorphism group.

**Proof.** Since $G = M_{11}$ acts 3-transitively on the 12 points of $S$, 3-$S$ is a 3-$(12, k, 1)$ design. Hence, $(k - 2)(k - 1)k(10 \cdot 11 \cdot 12$ and $(k - 2)(k - 1)10 \cdot 11)$, forcing $k = 3$. Thus, 3-$S = 3$-(12, 3, 1). Now $G$ has two orbits $O_1$ and $O_2$ on the 4-sets. If $A \in O_1$, then $G_A \cong 2 \cdot \text{Sym}(4)$ has only two point-orbits (of length 4 and 8, respectively), so that $\langle A \rangle = A$ because $|\langle A \rangle| < 12$. If $B \in O_2$, then $G_B \cong \text{Sym}(4)$ stabilizes $\langle B \rangle$ and has three point-orbits (of length 2, 4 and 6, respectively). If $\langle B \rangle = B$, then $S$ is the trivial 4-$(12, 4, 1)$ design, which contradicts $\text{Aut} S = M_{11}$. If $\langle B \rangle$ is the union of the orbits of length 4 and 6 of $G_B$, then $\langle B \rangle$ also contains a 4-set $A$ belonging to $O_1$, which has been seen to span only $A$, hence not $\langle B \rangle$, a contradiction. Therefore, $\langle B \rangle$ is the union of the orbits of length 2 and 4, and so is a block of the Hadamard 3-(12, 6, 2) design associated with the action of $M_{11}$ on the point-set (see [61] or [63]). \[\square\]

### 3.9. Independent $m$-set transitive $n$-DLSs

A common generalization of the problems of finding all 2-homogeneous linear spaces on the one hand and all basis transitive DLSs on the other hand is that of determining all $n$-DLSs transitive on independent $m$-sets for some positive integer $m \leq n + 1$. The problem is hopeless if $m = 1$ since any DLS induces a point-transitive DLS on each point-orbit of its automorphism groups. The case $m = 2$ would also require additional assumptions in order to be solved. Indeed, given any 2-homogeneous permutation group $G$ on $P$ and any union $O$ of orbits of subsets of size $\geq i$ of $P$ such that any two elements of $O$ intersect in at most $i$ points, we get a 2-homogeneous $(i + 1)$-DLS by calling hyperplane any element of $O$ together with any $(i + 1)$-subset which is in no element of $O$. A similar construction can also provide thick DLSs. For example, take $G = AGL(1, q^t)$, where $d$ has a proper divisor $t > 2$, in its usual action on $AG(d, q)$. Now define $S$ as follows. The point-set of $S$ is $GF(q^t)$ and the
line-set of $S$ is the orbit of the subfield $GF(q)$ under $G$. Now let $O_i$ be the orbit of $GF(q')$ under $G$ and define the planes of $S$ to be the elements of $O_i$ together with the planes of $AG(d, q)$ which are in no element of $O_i$. Then $3-S \neq 3-AG(d, q)$ although $2-S=2-AG(d, q)$ and $S$ admits a 2-transitive automorphism group.

Finally, the case $m=n+1$ was dealt with in Section 3.8. In all the remaining cases (i.e. $3 \leq m \leq n$), we have solved the problem for noncircular DLSs.

**Theorem 3.11** (Delandtsheer [32]). Let $S$ be a noncircular $n$-DLS and let $3 \leq m \leq n$. If $G \leq \text{Aut } S$ is transitive on the independent $m$-sets of $S$, then one of the following two possibilities occurs:

(i) $S=n-AG(d, q)$ or $n-PG(d, q)$, with $d \geq n$, or

(ii) $m=n=(d+1)t-1$, with $d, t \geq 1$, and $S$ is a direct sum $\bigoplus_{i=1}^{t} R_i$, where $R_i \cong R$ is a 2-homogeneous and basis transitive $d$-DLS.

Note that $R$ is known by Theorems 3.8 and 3.9: if $d \geq 2$, then $R=U_{md}(4)$ or $R=d-AG(a, q)$ or $d-PG(a, q)$ with $a \geq d \geq 2$.

Theorems 3.7 and 3.8 apply to the basis transitive $(m-1)$-truncation of $S$. Hence, the proof consists in showing that if $S$ is a supersum, then (ii) holds and that, in the other cases, $(m-1)-S$ uniquely determines $S$, which is not so easy, as suggested by the aforementioned construction for $a = m = 3$.

Finally, note that all finite $n$-DLSs which are transitive on their independent $n$-sets of which a 2-subset has been distinguished will be known from our (line, hyperplane)-flag transitivity theorem in Section 4.10.

### 4. Flag transitivity

#### 4.1. The environment

Tits contributed extensively to group theory by presenting the semi-simple groups of Lie type (1955) as well as all Lie–Chevalley groups of classical or exceptional type (1974) as chamber transitive automorphism groups of certain geometries. Later on, Buekenhout initiated a programme characterizing the sporadic simple groups by their chamber transitive action on certain diagram geometries (1979). The process of associating a geometry to a group $G$ provided with a family of subgroups was notably developed in [107], which focused on flags and residues, underlining the importance of flag transitive residually connected geometries, further investigated in [1]. This led, on the one hand, to group-theoretic results (such as geometric proofs of the uniqueness of some groups) and, on the other hand, to the construction of some interesting geometries.

Conversely, starting from geometry, much effort was aimed at the classification of chamber transitive automorphism groups of classical geometries such as Desarguesian projective spaces [58] and Desarguesian affine spaces [52]. More recently,
a systematic search for locally finite buildings of affine type with a discrete chamber transitive automorphism group [75] was motivated by possible applications to revisionism in finite group theory, illustrated by Timmesfeld's work.

On the other hand, the recent classification of all finite simple groups offers geometers a sledgehammer for the search of all geometries of a given class admitting a chamber transitive group, provided they are able to reduce their investigation to simple groups! Many recent papers, already quoted in the introduction, tackle such problems. We refer the reader to [98] for a recent survey (focusing on the important classes of $c^k \cdot A_m$, $c^k \cdot C_m$ and $c^k \cdot D_m$ diagram geometries). In this framework we will handle $L^{n-1}$ diagram geometries, i.e. $n$-DLSs. Hence, throughout Section 4, $S$ will be a finite $n$-DLS and $G$ will be an automorphism group of $S$, our aim being the classification of all chamber transitive pairs $(S, G)$. In dimension $n \geq 3$ we are able to weaken the hypothesis to (line, hyperplane)-flag transitivity and get a complete answer relying on the 2-dimensional investigation.

We first discuss the latter problem, namely the classification of all finite flag transitive linear spaces. It was considered to be untractable till the recent accomplishment of a team of three geometers and three group theorists (Buekenhout, Delandtsheer, Doyen, Kleidman, Liebeck, Saxl) prodded by the incredible optimism of Buekenhout. Finally, note that this geometric classification has been used already by Zieschang [115] to prove two theorems on finite permutation groups, all of whose two-point stabilizers have equal order $> 1$: for instance, if all these two-point stabilizers are nilpotent, then $G$ is 3/2-transitive.

### 4.2. Flag transitive projective or affine planes

The scepticism of specialists in the field rested on the intensive efforts spent on that problem in the two very special and most attractive cases where $S$ is a finite projective or affine plane (see [59, 101, 50–53, 109, 94, 95, 48, 5, 70, 71, 74, 88–91, 99, 103]). In view of the numerous and rather chaotic-looking examples constructed in some of these papers, the affine plane case turned out to be hopeless. More significant results were obtained in the projective plane case, which nevertheless had to await the completion of the classification of all finite simple groups to be... nearly settled!

**Theorem 4.1** (Kantor [73]). If $S$ is a projective plane of order $n$ having a flag transitive automorphism group $G$, then

(i) $S$ is Desarguesian and $PSL(3, n) \leq G \leq PGL(3, n)$, or

(ii) $n^2 + n + 1$ is a prime and $G$ is a sharply flag transitive Frobenius group of order $(n^2 + n + 1)(n + 1)$.

In case (ii), only two examples are known at present, namely $PG(2, 2)$ with a Frobenius group of order 7·3 and $PG(2, 8)$ with a Frobenius group of order 73·9. Any other example would necessarily be non-Desarguesian, because Higman and McLaughlin [59] proved that the only Desarguesian planes admitting a sharply flag
transitive automorphism group are \( PG(2, 2) \) and \( PG(2, 8) \). Moreover, Feit [49] proved that in case (ii), besides those two planes, any other example should have order \( n \equiv 0 \pmod{8} \), \( n \) not a power of 2 and \( d^{n+1} \equiv 1 \pmod{n^2 + n + 1} \) for every \( d \) dividing \( n \). Ho and Pott [60] got some additional information about the multiplier group of the plane.

So the existence of flag transitive non-Desarguesian projective planes remains an open problem, where the groups are so dull that group theory has to give way to arithmetics and finite field theory.

The starting point of Kantor's proof (and of any investigation about finite flag transitive linear spaces) is the following result.

**Theorem 4.2** (Higman and McLaughlin [59]). *Let \( S \) be any finite linear space. If \( G \) is flag transitive on \( S \), then \( G \) is point primitive on \( S \).*

Note that the number \( n^2 + n + 1 \) of points of a projective plane is always odd. Combining Theorem 4.2, the O'Nan–Scott theorem (Section 2.18) on primitive permutation groups, a theorem of Wagner [108] (stating that a finite projective plane is Desarguesian if its collineation group is transitive on points and contains a nontrivial perspectivity), and results on integers of the form \( u^2 + u + 1 \), Kantor was able to show that a flag transitive group \( G \) of a finite projective plane of order \( n \) is almost simple unless Theorem 4.1(ii) holds. He then used deep group-theoretic results in order to set up a list of the odd degree primitive permutation representations of all nonsporadic almost simple groups (a result obtained independently in [80], and independent of the simple group classification). This led him to a tedious case-by-case investigation using his aforementioned list together with the list of all sporadic simple groups. The simple group classification is only needed for this latter list.

### 4.3. Reduction theorem for flag transitive linear spaces

Buekenhout et al. [16] present a strategy aiming at the classification of all finite flag transitive linear spaces, a strategy which also relies on the Higman–McLaughlin theorem (Theorem 4.2) and on the O'Nan–Scott theorem (Section 2.18). We will state here the fundamental result which reduces the problem to a ... long and exacting investigation of all finite almost simple groups on the one hand, and of the irreducible non-Cartesian affine groups on the other hand! It can be seen as a generalization of Burnside's theorem reducing the search for 2-transitive groups \( G \) (which are flag transitive on \( 2-(v, 2, 1) \) designs) to the cases where \( G \) is almost simple or of affine non-Cartesian type.

**Theorem 4.3** (Buekenhout et al. [16]). *If \( S \) is a finite non-trivial linear space with point-set \( \Omega \) and if \( G \) is a flag transitive automorphism group of \( S \), then one of the following holds:*
(i) $G$ is almost simple, or
(ii) $G$ is of affine non-Cartesian type on $Ω$, or
(iii) $S = AG(2,8)$ and, up to conjugacy in $Aut S$, $G$ is one of the 5 groups below, which are of affine and Cartesian type simultaneously. They are all 1-dimensional affine, i.e. subgroups of $AGL(1,64)$. Denote by $T$ the translation group of $G$. Let $w$ be a primitive root of $GF(64)$ and let $ω$ and $α$ denote the elements of $AGL(1,64)$ defined by $ω : x \mapsto wx$ and $α : x \mapsto x^2$. The five groups are then

1. $T \cdot \langle ω^7 \rangle = AG^7L(1,64)$ of order $2^6 \cdot 9$,
2. $T \cdot \langle ω^2 \rangle$ of order $2^6 \cdot 9$,
3. $T \cdot \langle ω^7, α^3 \rangle$ of order $2^6 \cdot 18$,
4. $T \cdot \langle ω^7, α^2 \rangle$ of order $2^6 \cdot 27$,
5. $T \cdot \langle ω^7, α \rangle = AGL^7L(1,64)$ of order $2^6 \cdot 54$.

In cases (1) and (2), the groups are sharply flag transitive; they are permutation isomorphic but not conjugate in $Aut S$.

The proof first employs the Higman–McLaughlin theorem and then the O'Nan–Scott theorem to conclude that if $G$ is neither almost simple nor affine non-Cartesian, then either $G$ preserves a Cartesian structure on $P$ or else $G$ has a simple diagonal action. In the latter case there is a non-Abelian simple group $T$ and an integer $n ≥ 2$ such that $soc G ≥ T^n$ and the point-stabilizer $N_x := (soc G)_x$ is isomorphic to $T$, so that $v = |T|^n - 1$. Since the length of the $N_x$-orbits on the lines through $x$ divides both $r$ and $|T|$, and hence both $v - 1$ and $v$, the $x$-stabilizer $N_x$ also stabilizes all lines through $x$. The Feit–Thompson theorem then ensures that $N_x$ contains an involution fixing at least two points, ... and so forced to stabilize all lines through any of these two points, which leads to a contradiction.

The argument ruling out the Cartesian action is much longer, although a combinatorial argument reduces the investigation to the 2-dimensional Cartesian spaces on the one hand and to the 3-dimensional Cartesian spaces having at most 63 points on the other.

4.4. The cube root bound as a starter

The following simple observation [16] is crucial to our purposes. Let $(S,G)$ be a finite flag transitive linear space. Since $G_x$ acts transitively on the $r$ lines through $x$, its order is divisible by $r$. Therefore,

$$v^{1/2} < r = (v - 1)/(k - 1)|(|G_x|, v - 1)$$ (1)

and so

$$v < (|G_x|, v - 1)^2.$$ (2)

In particular, since $v = |G|/|G_x|$, we get

$$3\sqrt[3]{|G|} < |G_x|$$ (the cube root bound). (3)
Hence, the problem ‘Can a given finite abstract group $G$ act flag transitively on some linear space?’ can be approached as follows. Theorem 4.3 provides a negative answer as soon as $G$ does not have a unique minimal normal subgroup which is either simple or elementary Abelian. Recalling the Higman–McLaughlin theorem and the combinatorics above, the next step is to look for all maximal subgroups $(G_x)$ of $G$ satisfying the cube root bound and such that there are integers $r$ and $k$ for which (1)–(3) hold. Surprisingly, only very few maximal subgroups of $G$ pass this test. Of course, other arguments (sometimes quite involved) are then necessary in order to settle completely the question.

We illustrate this process with the two papers where we handled the cases $G \cong 2B_2(q)$, $PSL(2,q)$, $PSU(3,q)$, $Alt(n)$ and $Sym(n)$. These groups, together with most of the sporadic simple groups, were the first groups $G$ for which we completely settled the problem.

Theorem 4.4 (Delandtsheer [33, 34]). Let $G \cong 2B_2(q)$, $PSL(2,q)$, $PSL(3,q)$, $Alt(n)$ or $Sym(n)$. If $G$ acts flag transitively on a finite nontrivial linear space $S$, then one of the following occurs:

(i) $G \cong PSL(2,2^n)$ $(n \geq 3)$ and $S$ is the Witt–Bose–Shrikhande space $W(2^n)$;
(ii) $G \cong PSL(3,q)$ and $S = PG(2,q)$;
(iii) $G \cong PSU(3,q)$ and $S = U_3(q)$;
(iv) $G = Alt(7)$ and $S = PG(3,2)$;
(v) $G = Alt(8)$ and $S = PG(3,2)$.

4.5 The classification of flag transitive linear spaces

Let $G$ act flag transitively on a finite linear space $S$. In view of the reduction theorem (Theorem 4.3) and the classification of all finite simple groups, we can distinguish the following cases:

(1) The socle of $G$ is a Lie–Chevalley group of classical or exceptional type. In these cases much is known about the structure of the maximal subgroups of $G$, allowing for an exacting and difficult case-by-case analysis carried out essentially by J. Saxl and P. Kleidman.

(2) The socle of $G$ is alternating or sporadic. In the first case very little additional group theory is needed; however, much of the knowledge on the maximal subgroups of the sporadic simple groups is used in the long analysis of the second case. This work was carried out essentially by F. Buekenhout and A. Delandtsheer.

(3) $G$ is non-Cartesian affine but not 1-dimensional. The methods used here are rather similar to those used above but some representation theory is needed as well. The analysis is mainly due to M. Liebeck.

(4) $G$ is non-Cartesian 1-dimensional affine. Here group theory is of very little help and the profusion of rather chaotic—looking examples leads us to surmise that a classification is hopeless. We will go back to this case in Section 4.6.

The classification reads as follows.
Theorem 4.5 (Buekenhout et al. [17]). If $S$ is a finite nontrivial linear space admitting a (point, line)-flag transitive automorphism group and if $G$ is not 1-dimensional affine, then the pair $(S, G)$ is known. In particular, one of the following holds:

1. $S = 2\cdot A^G(d, q)$ and $G$ is 2-transitive nonsolvable, except possibly for $(d, q) = (4, 3)$ or $(2, q)$ where $q = 5, 7, 9, 11, 19, 23, 29$ or 59; more precisely, one of the following holds:
   1a. $G$ is 2-transitive and
   1a.1. $G \cong SL(2, q^d)$, or
   1a.2. $G \cong Sp_{2d}(q^{d/2})$, or
   1a.3. $G \cong G_2(q^{d/6})$, $q$ is even, or
   1a.4. $G \cong E$, where $E$ is an extraspecial group of order $2^{n+1}$ where $n = \log_2 q$ and $(d, q) = (4, 3), (2, 3), (2, 5), (2, 7), (2, 11)$ or $(2, 23)$, or
   1a.5. $G \cong SL(2, 5), d = 2$ and $q = 9, 11, 19, 29, 59$, or
   1a.6. $G \cong SL(2, 13)$ and $(d, q) = (6, 3)$;
2. $(d, q) = (4, 3), (2, 3), (2, 5), (2, 7), (2, 11)$ or $(2, 23)$, or
   1b. $d = 2, q = 11$ or $23$ and $G$ is one of the three solvable flag transitive groups described in [52, pp. 455–457].
3. $S$ is a non-Desarguesian translation affine plane:
   2a. $S$ is a Lüneburg–Tits plane $L_\mu(q^2)$ of order $q^2$, where $q = 2^{2e+1}$ ($e \geq 1$), and $2B_3(q) \cong G \cong Aut(2B_3(q))$;
   2b. $S$ is Hering’s plane $A_{27}$ of order 27 and $G \cong SL(2, 13)$;
   2c. $S$ is the nearfield plane $A_9$ of order 9 and $G$ is one of the seven flag transitive subgroups of $Aut A_9 \cong 3^4 \cdot Sym(5) \cdot 2^4 \cdot 2$, described in [53, pp. 197–201].
4. $S$ is one of the two Hering spaces on $9^3$ points with line-size 9 and $G = 3^6 \cdot SL(2, 13)$.
5. $S = 2\cdot PG(d, q)$ and $PSL(d + 1, q) \cong G \subseteq PGF(d + 1, q)$, or $G = A_7$ and $(d, q) = (3, 2)$.
6. $S$ is a Hermitian unital $U_4(q)$ and $PSU(3, q) \cong G \subseteq PGU(3, q)$.
7. $S$ is a Ree unital $U_6(q)$ and $2G_2(q) \cong G \subseteq Aut^2 G_2(q)$, with $q = 3^{2e+1}$, $e > 0$.
8. $S$ is a Witt–Bose–Shrikhande space $W(q)$ and $PSL(2, q) \cong G \subseteq PGF(2, q)$, with $q = 2^e \geq 8$.

4.6. The 1-dimensional affine case

Here $v = p^d$, $p$ a prime number, and the point-set $\Omega$ of the linear space $S$ may be identified with the field $GF(p^d)$ as well as with the vector space $V = GF(p)^d$ and with the corresponding affine space $AG(d, p)$. The group $G \subseteq AGL(1, p^d)$ contains the translation group $T \cong p^d$ and two cases are to be distinguished, according as some nontrivial translation preserves a line $L$ or not.

1. The spread case: $T_L \neq 1$ and $k = p^l$. If $T_L \neq 1$, then $L$ is a union $O_1 \cup \cdots \cup O_n$ of $T_L$-orbits. Since $O_2$ contains at least $p \geq 2$ points, any translation mapping $O_1$ onto $O_2$ preserves $L$. This provides a contradiction unless $n = 1$. Hence, $L$ is a point-orbit of $T_L$, and so the lines of $S$ are subspaces (of dimension $l = \log_p |T_L|$) of the affine space $AG(d, p)$. Lines through 0 form a spread (i.e. they induce a spread of $(l - 1)$-subspaces
in the space $\text{PG}(d-1, p)$ at infinity of $V$). This explains the terminology. All affine spaces belong to this case.

(2) The nonspread case: $T_L = 1$. In this case, we can show that

(2.1) $p^r|b = p^s(p^r - 1)/k(k-1)$, hence $(p,k)=1$,

(2.2) $p \geq 3$,

(2.3) $G_0 \cap GL(1,p^s)$ has odd order,

(2.4) $G_L$ fixes a unique point $p(L)$, called the pole of $L$,

(2.5) $\Omega := G_L \cap AGL(1,p^s)$ is cyclic and semiregular on $\Omega \setminus p(L)$.

**Proof.** (2.1) If $T_L = 1$, then all $T$-orbits on the line-set have length $p^d$; hence, $p^r|b = p^s(p^r - 1)/k(k-1)$ and our assertion follows.

(2.2) If $p = 2$, then the translation mapping $x$ to $y$ is involutory, hence preserves the line $\langle x, y \rangle$, contradicting $T_L = 1$. Hence, $p \geq 3$.

(2.3) If $G_0 \cap GL(1,p^s)$ has even order, then it contains the involutory homothety $i$, so that $G$ contains the group $\langle T, i \rangle$ consisting of $p^s$ translations and $p^s$ conjugates of involutions with $(p^s - 1)/2$ cycles. Since the total number of pairs of points is $p^s(p^s - 1)/2$ and since the product of two distinct involutions is a nontrivial translation, any two points $a$ and $b$ are interchanged by an involution $i_{ab} \in \langle T, i \rangle$. Let $c$ be a third point on the line $\langle a, b \rangle$. Then $i_{ab} \cdot i_{bc}$ is the translation $t_{ab}$ mapping $a$ onto $b$. Since both $i_{bc}$ and $i_{ab}$ preserve the line $\langle a, b \rangle$, the translation $t_{ab}$ also does, a contradiction.

(2.4) If $T_L = 1$, then $G_L$ belongs to the translation complement of $G$, and so $G_L$ fixes a point, say $0$, which cannot belong to $L$ because of the flag transitivity of $G$. If $G_L$ fixes another point $x \neq 0$, then $G_L \leq G_{0x} \leq G_{\langle 0, x \rangle}$. Since any two line-stabilizers are conjugate in $G$, it follows that the stabilizer of the line $\langle 0, x \rangle$ fixes the two points $0$ and $x$, contradicting the flag transitivity of $G$. Hence, $G_L$ fixes a unique point $0 = p(L)$, called the pole of $L$.

(2.5) Let $p(L) = 0$. Then there is some integer $m|p^d - 1$ such that $Z_L = G^m L(1,p^d)$. Hence, $Z_L$ acts semi-regularly on $GF(p^d)^\times$. 

4.7. The nonspread case: generalized Netto systems

In contrast to the spread case, only one family of examples of the nonspread type is known, namely the generalized Netto systems (Section 2.11). We will now prove that the assumption $G \leq AGL(1,p^s)$ indeed forces $S$ to be a generalized Netto system (in the nonspread case).

**Theorem 4.6** If $(S, G)$ is a nontrivial flag transitive linear space of nonspread type with $G \leq AGL(1,p^s)$, then $S \cong N(k, p^s)$.

**Proof.** Let $L$ be a line with pole $0$, so that $G_L \leq G_0 \leq GL(1,p^s) =: H$. We now use properties (2.1)–(2.5) stated in Section 4.6. By (2.5), $G_L$ is regular on $L$, so that $|G_L| = k$ and $G$ is sharply flag transitive. Since $r = (p^d - 1)/(k - 1)$, we deduce that $G_0 = H^{k-1}$ and $G = AG^{k-1} L(1,p^d)$. By (2.3), $r$ is odd. Using (2.2) and (2.1), this forces
\[ b/p^d = (p^d - 1)/k(k - 1) \] to be an odd integer, say \( 2n + 1 \) with \( n \geq 0 \) (note that \( n = 0 \) would force \( S \) to be a projective plane).

Since \( G \) is sharply flag transitive, \( G_L = H^{(p^d - 1)/k} = H^{(2n + 1)(k - 1)} =: K \) is the group of \( k \)th roots of unity in \( GF(p^d) \). Hence, \( L = aK \) for some \( a \in GF(p^d) \) and the line-set \( L \) of \( S \) is the \( G \)-orbit of \( aK \). Now the element \( x \rightarrow x^{-1} \) of \( AGL(1, p^d) \) takes the incidence structure \((\Omega, L, \varepsilon)\) onto an isomorphic incidence structure \((\Omega, L', \varepsilon)\) with base-line \( K \) instead of \( aK \). Hence, we may assume without loss of generality that \( L = K \), so that \( S \) is a generalized Netto system. \( \square \)

This, however, does not settle the problem of the existence of \( S \), i.e. whether given \( n, k, p^d \) as above, the incidence structure \((\Omega, L, \varepsilon)\) is indeed a linear space. Let \( L = K = \{1, \varepsilon, \ldots, \varepsilon^{k-1}\} \), where \( \varepsilon \) is a primitive \( k \)th root of unity in \( GF(q) \). Then the set of differences of elements of \( L \) is

\[ \Delta L = \pm K \{\varepsilon - 1, \varepsilon^2 - 1, \ldots, \varepsilon^{(k-1)/2} - 1\} = K \{\varepsilon - 1, \varepsilon^2 - 1, \ldots, \varepsilon^{k-1}\}, \]

so that the family of differences of pairs of elements belonging to a common line \( L \) is

\[ A = H^{k-1} \{\varepsilon - 1, \varepsilon^2 - 1, \ldots, \varepsilon^{k-1}\}. \]

The following statements are equivalent:

(a) \((\Omega, L, \varepsilon)\) is a linear space;
(b) \( A \) is a difference set for the additive group of \( GF(q) \);
(c) \( \varepsilon - 1, \varepsilon^2 - 1, \ldots, \varepsilon^{k-1} - 1 \) belong to distinct cosets of \( H^{k-1} \);
(d) \( \varepsilon - 1, \varepsilon^2 - 1, \ldots, \varepsilon^{(k-1)/2} - 1 \) belong to distinct cosets of \( H^{(k-1)/2} \).

(The equivalence between (c) and (d) follows from the fact that \(- (\varepsilon^i - 1) \varepsilon^{k-i} = \varepsilon^{k-i} - 1 \) for any \( i \leq (k-1)/2 \), \( K \leq H^{k-1} \) and \(- 1 \notin H^{k-1} \).)

Write \( q = p^d \). If \( k = 3 \), then (d) holds provided \( q \equiv 3 \) (mod 4). Therefore, \( N(3,q) \) is a linear space for every prime power \( q \equiv 7 \) (mod 12). These spaces are indeed the Netto systems.

If \( k = 5 \), then (d) is equivalent to the condition that \( 5(q-1)/4 \neq 1 \) in \( GF(q) \), with \( q \equiv 21 \) (mod 40) (see e.g. [6, p. 326]). For \( q < 800 \) this gives four linear spaces with \( q = 61, 421, 661, 701 \). Note that the total number \( q \) of points is prime in these instances. This is also true for any flag transitive but not 2-transitive finite projective plane, as seen in Theorem 4.1. Remember, however, that the only known examples are the Desarguesian planes \( N(3,7) = PG(2,2) \) and \( N(9,73) = PG(2,8) \).

Finally, note that Kantor's inflation trick described in Section 4.8 will enable us to derive the existence of an \( N(k, q^n) \) linear space from that of \( N(k, q) \) for any \( n \) such that \((k - 1, (q^n - 1)/(q - 1)) = 1 \).

4.8. The spread case: Kantor's constructions

Many examples of non-Desarguesian flag transitive affine planes belonging to the 1-dimensional affine spread case were constructed by several authors, already quoted at the very beginning of Section 4.2. Kantor was particularly active in producing large families of examples. Moreover, he recently extended his constructions to flag
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transitive 2-\((q^n, q, 1)\) designs with \(n \geq 3\) and \(G \leq AGL(1, q^n)\), but distinct from \(AG(n, q)\) [74]. In the same paper he went on inflating both the number of examples and the size of \(v/k\) by the following process, which we call Kantor’s inflation trick.

Suppose that \(G\) is a flag transitive automorphism group of a nontrivial linear space \(S\) with point-set \(\Omega\) and let \(L\) be a line of \(S\). If the stabilizer \(G_L\) acts transitively on some linear space \(S’\) with point-set \(L\), then all the images of any line \(L’\) of \(S’\) under \(G\) are the lines of a flag transitive linear space \(S”\) on the point-set \(\Omega\).

Start, for example, with \(\Omega = GF(q^n), \ L = GF(q)\) and \(G = AG^{mL}(1, q^n)\): they define a flag transitive linear space \(S\), namely \(S = AG(n, q)\), if and only if 
\[(m, (q^n-1)/(q-1))-1,\] and then \(G_L = AG^{mL}(1, q)\). If there is a generalized Netto system \(S’ = N(m+1, q)\), then by Theorem 4.6 we just get more generalized Netto systems \(S” = N(m+1, q^n)\) for those values of \(n\) satisfying the above gcd condition. In the spread case, however, this construction provides lots of new examples of flag transitive linear spaces which admit a spread of proper linear subspaces which are not Desarguesian affine spaces (with the superstructure of an \(AG(n, q)\) induced on the point-set). For any development of this we refer the reader to [74].

4.9. Locally 2-homogeneous linear spaces

The flag transitive linear spaces \(S\) obtained by application of Kantor’s inflation trick are reducible in the following sense: \(S\) has a proper linear subspace whose \(G\)-orbit is the line-set of another flag transitive linear space on the same point-set and with the same group \(G\). This suggests that we define some kind of irreducibility for flag transitive 1-dimensional affine linear spaces, but even so the profusion of examples leaves us little hope for a complete classification. A rather strong kind of irreducibility is local primitivity, which requires that \(G_0\) be primitive on the lines through \(0\). We proved that this condition is equivalent to the primality of the point degree \(r\).

Theorem 4.7 (Delandtsheer [39]). Let \(S\) be a finite linear space. If \(G \leq Aut S\) is flag transitive and \(G \leq AGL(1, v)\), then

(i) \(G\) is locally primitive if and only if the point degree \(r\) is a prime number,

(ii) \(G\) is not transitive on the unordered pairs of intersecting lines, except in the following cases:

(a) \(S\) is the trivial 2-(\(v, 2, 1\)) design with \(v = 3, 4\) or 8 and \(G = AGL(1, 3), AGL(1, 4), AGL(1, 4)\) or \(AG(L, 1, 4)\); 
(b) \(S = PG(2, 2)\) and \(G = AG^2(L, 1, 7)\); 
(c) \(S = AG(2, 4)\) and \(G = AG(L, 1, 16)\).

Then using Theorems 4.5, 3.3 and 3.4 we derived a complete classification of finite locally 2-homogeneous linear spaces, i.e. those flag transitive pairs \((S, G)\) where the point-stabilizer \(G_x\) is 2-homogeneous on the lines through \(x\).

Corollary 4.8 (Delandtsheer [39]). Let \(S\) be a finite linear space. If \(G \leq Aut S\) is transitive on the unordered pairs of intersecting lines of \(S\), then one of the following occurs:

...
(i) \( S \) is trivial; in this case \( G \) is any 3-homogeneous group of degree \( v \), except possibly if \( v \leq 5 \), or \( v = 8 \) and \( G = AGL(1, 8) \), or \( v = 32 \) and \( G = AGL(1, 32) \); 
(ii) \( S = 2-PG(d, q) \) and \( G \supset PSL(d + 1, q) \), or \( S = 2-PG(3, 2) \) and \( G = A_7 \) or \( S = PG(2, 2) \) and \( G = AG^2L(1, 7) \); 
(iii) \( S = 2-AG(d, q) \) and \( G \supset ASL(d, q) \); 
(iv) \( S = \mathbb{L}(q^2) \), the Lüneburg–Tits affine plane of order \( q^2 \), and \( G \supset q^4 \cdot 2B_2(q) \), \( q = 2^{2e + 1} (e \geq 1) \); 
(v) \( S = U_{16}(q) \), the Hermitian unital of order \( q \), and \( G \supset PSU(3, q) \), \( q \geq 3 \).

Let us now turn to flag transitive \( n \)-DLSs with \( n \geq 3 \).

4.10. (Line, hyperplane)-flag transitive \( n \)-DLSs

Given distinct nonnegative integers \( i, j, l \leq m - 1 \), call \((i, j)\)-flag (resp. \((i, j, l)\)-flag) any flag consisting of varieties of dimensions \( i \) and \( j \) (resp. \( i, j \) and \( l \)). We got a complete classification of \((1, n - 1)\)-flag transitive \( n \)-DLSs \((n \geq 3)\) and we derived classifications of \((i, j, l)\)-flag transitive \( n \)-DLSs and of \((i, j)\)-flag transitive thick \( n \)-DLSs. This completely settles problems on hypercircular \( n \)-DLSs tackled in [61, 12].

**Theorem 4.9** (Delandtsheer [37, 38]). Let \( S \) be a finite nontrivial \( n \)-DLS with \( n \geq 3 \) and let \( G \) be an automorphism group of \( S \).

(a) If \( G \) is (line, hyperplane)-flag transitive, then one of the following holds:

1. \( S = n-PG(d, q) \) with \( d \geq n \geq 3 \) and \( PSL(d + 1, q) \supset G \leq PGL(d + 1, q) \), or \( S = PG(3, 2) \) and \( G = A_7 \); 
2. \( S = n-AG(d, q) \) and \( ASL(d, q) \supset G \leq AGL(d, q) \), or \( S = AG(3, 2) \) and \( G = AGL(1, 2^3) \), or \( S = AG(3, 8) \) and \( A^2GL(1, 8^3) \supset G \leq AGL(1, 8^3) \); 
3. \( S \) is one of the Mathieu–Witt designs \( 4-(11, 5, 1) \), \( 5-(12, 6, 1) \), \( 3-(22, 6, 1) \), \( 4-(23, 7, 1) \) or \( 5-(24, 8, 1) \) and \( G \) is the associated Mathieu group \( M_{11}, M_{12}, M_{22} \) or \( Aut M_{22}, M_{23} \) or \( M_{24} \).
4. \( S \) is the \( 3-(q^d + 1, q + 1, 1) \) design on the projective line \( PG(1, q^d) \) with a projective subline \( PG(1, q) \) as base block, \( PSL(2, q^d) \supset G \leq PGL(2, q^d) \) and \( q \geq 3 \); 
5. \( S \) is the \( 3-(q + 1, 4, 1) \) design on the projective line \( PG(1, q) \) with \( \{ \infty \} \cup K \) as base block, where \( K \) is the set of all third roots of unity in \( GF(q) \), \( PSL(2, q) \supset G \leq PSL(2, q) \) and \( q \equiv 7 \pmod{12} \).

(b) If \( G \) is \((i, j, n - 1)\)-flag transitive for some \( i, j \) satisfying \( 0 \leq i < j < n - 1 \), then \((S, G)\) is as in (a).

(c) If \( j-S \) is nontrivial and \( G \) is \((1, j)\)-flag transitive for some \( j \) satisfying \( 1 < j \leq n - 1 \), then \((S, G)\) is as in (1) or (2) of statement (a).

(d) Given \( i \) such that \( 1 \leq i \leq n - 3 \), if \( S \) has a constant parameter \( t(1, i, i + 1) \) (see 2.5) and if \( G \) is \((i, n - 1)\)-flag transitive, then \((S, G)\) is as in (a).

(e) Given \((i, j)\) such that \( 1 \leq i \leq j - 2 \leq n - 3 \), if \( j-S \) is nontrivial and has a constant parameter \( t(1, i, i + 1) \) and if \( G \) is \((i, j)\)-flag transitive, then \((S, G)\) is as in (c).
Hence, all chamber transitive $n$-DLSs with $n \geq 3$ are known. The start of the proof and the key to relate different types of flag transitivity is to consider residues and truncations of $S$ and use the following consequence of Block [9] and Dowling and Wilson [46]: any hyperplane transitive automorphism group of a finite $n$-DLS ($n \geq 2$) is point transitive. Similarly, the last two statements (d) and (e) are easily deduced from the following fact: if $1 \leq i \leq n - 2$, then in any finite $n$-DLS in which $t(1, i, i+1)$ is a constant, transitivity on $i$-varieties forces transitivity on lines [77].

Note that a classification of the (point, hyperplane)-flag transitive finite $n$-DLSs would include the long-standing and still open problem of classifying all (point, block)-flag transitive $n-(v, k, 1)$ designs. The situation for $(0, 1)$-flag transitive $n$-DLSs is still worse since it encapsulates not only the tantalizing case $n=2$ with a 1-dimensional affine group, but also the hopeless case $n \geq 3$ with a 2-transitive group, illustrated in Section 3.9.

If $(S, G)$ is a (line, hyperplane)-flag transitive pair, then by Kung's theorem $(2-S, G)$ is a (point, line)-flag transitive linear space and $(3-S, G)$ is a (line, plane)-flag transitive planar space, so that Theorem 4.5 is one of the main ingredients of our proof, while the 3-dimensional classification is a crucial step, which we shortened by the use of Theorems 3.3 and 3.4. As the reader may surmise, the difficulties arise from the 1-dimensional affine case and from the DLSs with trivial 2- or 3-truncations.

5. Line primitivity

5.1. Introduction

The first two steps towards the classification of finite flag transitive linear spaces are Higman–McLaughlin's theorem asserting that flag transitivity forces point primitivity, and its consequence, the reduction theorem (Theorem 4.3), stating that a flag transitive group is almost simple or is affine on the point-set. If the flag transitivity hypothesis is weakened to line transitivity, known to force point transitivity by [9], then both statements become invalid as illustrated by groups generated by a Singer cycle in projective planes on a nonprime number of points (and lines). Delandtsheer and Doyen [42] proved that the only line transitive but point imprimitive finite linear spaces with line-size less than 8 are $PG(2, 4)$ and two 2-(91, 6, 1) designs due to Mills and McCalla (see [25]). Actually, Theorem 5.1 shows that all such spaces have a relatively small number $b$ of lines with respect to their number $v$ of points, namely $v \leq b \leq v^{3/2}$.

**Theorem 5.1** (Delandtsheer and Doyen [43]). Let $S$ be a finite linear space with line-size $k$ admitting a line transitive automorphism group $G$ which preserves a nontrivial partition $C$ of the point-set into $c$ classes of size $s$. Given any line $L$, denote by $n$ the number of unordered pairs of points of $L$ contained in the same class of $C$. Then there is a positive integer $m$ such that

$$s = \binom{k}{2} - n \equiv m \quad \text{and} \quad c = \binom{k}{2} - m \equiv n.$$
Since $n$ and $m \geq 1$, the number $v = sc$ of points is bounded above by $(k-2)^2(k+1)^2/4$. Moreover, this upper bound is reached only for $k = 8$, but well reached, since there are precisely 446 line transitive but point imprimitive 2-(729, 8, 1) designs! (see [92]).

On the other hand, Camina and Gagen [24] proved that in the very special case where the line-size $k$ divides the total number $v$ of points, line transitivity indeed coincides with flag transitivity (and so forces point primitivity).

We investigate here line primitivity, supporting the conjecture that line primitivity forces point primitivity and reducing the investigation of line primitive groups to the almost simple case. To that end, from now on we let $S$ be a finite linear space with point-set $P$ and line-set $L$, and $G \leq Aut S$ act primitively on $L$. The following fact is crucial to our purposes: if $G$ acts primitively on a set $C$ of $c > 1$ classes of some partition of $P$, then $G$ acts faithfully on $C$, and so $G$ is a subgroup of $Sym(c)$. This enables us to compare the three faithful actions of $G$ on $P$, $L$ and $C$, those on $L$ and $C$ being distinct and primitive in any case.

5.2 Reduction theorem

**Theorem 5.2** (Delandtsheer [36]). *Let $S$ be a finite nontrivial linear space other than a projective plane. If $G$ acts primitively on the lines of $S$, then $G$ is almost simple.*

Line primitivity is fairly well under control in finite projective planes $S$ since Kantor [73] proved the following generalization of Theorem 4.1: if $G \leq Aut S$ is point primitive, then either $S = PG(2, n)$ and $PSL(3, n) \leq G$, or else $n^2 + n + 1$ is prime and $G$ is a regular group or a Frobenius group whose order divides $(n^2 + n + 1)(n+1)$ or $(n^2 + n + 1)n$.

In particular, line primitivity is equivalent to point primitivity in any finite projective plane, a property which does not extend to any linear space $S$. Indeed, if $S$ is trivial, then line primitivity obviously forces point primitivity while point primitivity does not even force line transitivity, i.e. 2-homogeneity on points.

Note also that assuming line primitivity is strictly stronger than just assuming line transitivity, as illustrated by the examples mentioned in Section 5.1 and by the trivial linear space $S$ on a point-set $P$ on which some 2-homogeneous group $G$ preserves the structure of a nontrivial linear space $S'$. By contrast, line transitivity suffices to guarantee line primitivity if $S$ is a projective space, a Hermitian unital of order $\geq 3$ or a Ree unital of order $\geq 27$. Moreover, it follows from Theorems 5.2 and 4.5 that these are the only finite nontrivial linear spaces admitting a flag transitive and line primitive automorphism group.

The proof of Theorem 5.2 rests on crossed applications of the O'Nan–Scott theorem (Section 2.18) to the primitive actions of $G$ on both $L$ and $C$, and consists of repeated comparisons of the transitive actions on $P$, $L$ and $C$ (of cardinalities $v$, $b$ and $c$, respectively). By Kantor's aforementioned theorem on projective planes, we may assume $b \neq v$, so that there is always a prime number $p$ dividing $b$ but neither $v$ nor $c$. From the transitivity of the socle $T_1 \times \cdots \times T_n \cong T^n$ of $G$ on $L$, we get that $b$ divides $|T|^n$,
and so $p$ divides $|T|$. Since $p \nmid c$, the O'Nan–Scott theorem shows that either $G$ is almost simple or $G$ preserves the structure of a Cartesian space on $C$ with almost simple stabilizers of Cartesian lines (case (4a) of Section 2.18). In the latter case, the following five preliminary steps, together with combinatorial arguments, lead to a contradiction:

1. $G$ is imprimitive on $P$;
2. The stabilizers of distinct lines in $soc G$ are distinct and nontrivial;
3. the set of all lines fixed by the $T_1$-stabilizer of a line $L$ is precisely the orbit of $L$ under $1 \times T_2 \times \ldots \times T_n$;
4. for every $i=1,\ldots,n$ and every point $x$, there is an involution in $T_i$ fixing $x$ but moving some line through $x$;
5. in $soc G$, each point-stabilizer contains some line-stabilizer.

5.3. **Primitive actions of rank $\leq 7$ on the line-set**

The following theorem generalizes a previous (unpublished) result of Di Martino where the line-rank was assumed to be 3.

**Theorem 5.3** (Delandtsheer [35]). If $G \leq Aut S$ is primitive of rank $\leq 7$ on $L$, then $G$ is primitive on $P$.

The proof rests on counting arguments involving the number of flag orbits and uses [23] asserting that if a subgroup $H$ of $Aut S$ has equally many point-orbits and line-orbits, then the number of lines of any line-orbit $L_j$ intersecting a given line $L \in L_i$ is

$$|L_j|k/r + r(r-k-1)\delta_{ij},$$

where $\delta_{ij}$ is the Kronecker symbol.

5.4. **Line primitivity and line size $< 30$**

All observations made so far support the conjecture that line primitivity forces point primitivity. At least they enabled us to prove this for all finite linear spaces with line-size $k < 30$. Note that this conjecture has also been proved under the assumption $(v-1,k) \leq 4$ [79].

**Theorem 5.4** (Delandtsheer [36]). Let $S$ be a finite linear space with line-size $< 30$. If $G \leq Aut S$ is line primitive, then $G$ is also point primitive.

Once again our proof is a mixed bag of combinatorics and group theory. It appeals to several results on primitive actions, due e.g. to Manning (see e.g. [123], Praeger [122], Cooperstein [117], Pogorelov [121] and Guralnick [120]. Nevertheless, the groundbreaking arguments are combinatorial and use Theorem 5.1 together with the following combinatorial version of Higman and McLaughlin [59]: the point-set of a finite linear space with constant line-size cannot be partitioned into classes of equal size $> 1$ in such a way that, for some integer $e$, each line intersects each class in $0$ or $e$ points.
5.5. Conclusion

Actually, our work on finite line primitive linear spaces leads us to surmise that such spaces are rather rare, but we do hope this will be proved by ingenious elementary arguments rather than by using the sledgehammer method suggested by Theorem 5.2 and the classification of all finite simple groups. After all, this is the best one could wish for the results presented in this survey, despite the already long story of unsuccessful attempts by such bare means.

Note added in proof. Since March 1991, Fang and Li [118], improving [43], proved that the number of line-transitive but point-imprimitive finite linear spaces on \( v \) points and with line-size \( k \) is finite as soon as \( k/(k,v) \) is bounded. They also proved that this number is zero when \( k/(k,v) \leq 4 \). This has been extended to \( k/(k,v) \leq 6 \) and further work has been done on line-transitive linear space with small \( k/(k,v) \) in [116].

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