

## HARD GRAPHS FOR THE MAXIMUM CLIQUE PROBLEM

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The maximum clique problem is one of the NP-complete problems. There are graphs for which a reduction technique exists that transforms the problem for these graphs into one for graphs with specific properties in polynomial time. The resulting graphs do not grow exponentially in order and number. Graphs that allow such a reduction technique are called soft. Hard graphs are those graphs for which none of the reduction techniques can be applied. A list of properties of hard graphs is determined.

### 1. Introduction

The maximum clique problem (MCP) is the problem to determine a largest clique in a graph  $G$ . It is one of the NP-complete problems, for which we refer to Garey and Johnson [1]. Graph theoretic terminology will be according to Harary [2]. The problem is equivalent to the independent set problem (ISP) in the complementary graph  $\bar{G}$ , which is the problem to determine a largest independent set.

We recall that a function  $f(N)$ , is defined on the set of natural numbers, is called  $O(g(N))$  if  $|f(N)| \leq m |g(N)|$  for  $N \geq N_0$ , where  $m$  and  $N_0$  are non-negative constants. If a graph  $G$  has  $N$  points an algorithm has polynomial complexity, or is shortly called polynomial, if the number of steps in the calculation is  $O(N^c)$ , where  $c$  is a positive constant.

**Definition 1.** A graph  $G$  is called *soft* if either

- (i) there exists a polynomial algorithm to solve the MCP for  $G$ , or
- (ii) there exists a transformation of the graph, taking only polynomial time, into a polynomial number of graphs of polynomial order such that a specific property is removed.

Part (ii) of this definition is the new feature brought forward in this note.

All polynomials are understood to be in  $N$ , the number of points of the original graph. In most cases that we shall deal with these transformations involve reduction techniques in which the number of points, the order, or the number of lines, the size, of the graph that is to be investigated further is reduced.

The philosophy behind this concept of soft graph can be made clear by the following example. Suppose a graph  $G$  is non-connected. Clearly we shall consider the components separately in our search for the maximum clique. By doing this we investigate graphs that have the property that they are connected. It takes only polynomial time to determine the components. There are only  $O(N)$  components and their orders are  $O(N)$  as well. These components may turn out to be hard in the sense that no polynomial algorithm is known for them. However, the original graph is soft in the sense that it has a property, non-connectedness, that certainly is not the deeper reason for the overall difficulty of the problem. We may determine the properties that are not the real cause for the overall difficulty. Their negations then determine more and more aspects of the structure of the graphs for which the problem is really difficult, the *hard* graphs for the maximum clique problem.

It is important to stress the fact that we are considering certain properties. As a result of some decomposition technique there may arise more graphs than one, each of which is to be investigated on maximum cliques. Such a decomposition not necessarily refers to a specific property. A typical example is the technique of Tarjan and Trojanowski [6].

Each point  $v$  determines a set of neighbours  $N(v)$  and we may decompose the graph into  $N$  graphs  $\langle v \cup N(v) \rangle$ . All cliques are contained in these induced subgraphs. It is clear that on repetition of this procedure one finally finds a maximum clique. Obviously, as  $v$  is adjacent to all other points of  $\langle v \cup N(v) \rangle$ ,  $v$  may be left out and one needs only determine a maximum clique in  $\langle N(v) \rangle$ . There is reduction of the order. However, each repetition introduces a factor  $O(N)$  for the number of graphs that are to be investigated. A combinatorial explosion occurs and Tarjan and Trojanowski find an  $O(2^{N/3})$  complexity. We shall call this procedure *direct decomposition*.

Reduction techniques like direct decomposition should be distinguished from a decomposition like splitting a graph into its components. That decomposition technique does not induce a combinatorial explosion for the simple reason that it cannot be repeated.

## 2. Soft graphs

The proofs of the following lemmas are quite simple and are available in a more elaborate version of this note. The proofs of Lemmas 2 and 10 are given as examples.

**Lemma 1.** *Non-connected graphs and graphs with non-connected complements are soft.*

**Definition 2.** Point  $v$  is said to *dominate* point  $w$  if  $N(v) \supseteq N(w)$ .

**Lemma 2.** *Graphs that contain a point that dominates some other point are soft.*

**Proof.** Finding a point that dominates some other point can be done by comparing  $N(v)$  and  $N(w)$  for all pairs of points  $v$  and  $w$ , i.e. in polynomial time. Let point  $v$  dominate point  $w$ , then any maximal clique that contains  $w$  has a counterpart clique in which  $w$  has been replaced by  $v$ . This clique may be maximal as well or may be part of an even large clique. For the search of a maximum clique point  $w$  may be deleted and the order of the graph is reduced.  $O(N)$  repetitions of the procedure remove all dominated points.  $\square$

**Lemma 3.** *Graphs with cutpoints are soft.*

This lemma gives an example of a reduction technique in which the order of the graphs is reduced, but the total number of points in the resulting graphs, the blocks, has increased. As the number of resulting graphs is  $O(N)$  the reduction technique does not induce a combinatorial explosion.

**Definition 3.** Cutgraphs of order  $k$  of a graph  $G$  are induced subgraphs on  $k$  points with point sets that are cut sets for  $G$ .

The obvious generalization of Lemma 3 is:

**Lemma 4.** *Graphs with cutgraphs of order smaller than or equal to a constant  $k$  are soft.*

A related result is that of Whitesides [7], that in our terminology reads:

**Lemma 5.** *Graphs with cutgraphs that are cliques are soft.*

If  $G$  is the line graph of a graph  $H$ , then cliques in  $G$  correspond to stars in  $H$ . Independent sets in  $G$  correspond to matchings in  $H$ . This was reason for the author to study decomposition methods based on turning graphs into line graphs [3]. As a result the following was found.

**Lemma 6.** *Graphs that, by addition of lines, can be turned into a line graph that is not a complete graph are soft.*

One of the forbidden induced subgraphs for a line graph is the graph  $K_{1,3}$ . This graph turns out to be non-relevant for the MCP. In [3] the author used detachments to turn graphs without induced subgraphs  $K_4 - x$  into line graphs. However, if that line of thought is not followed there are even simpler proofs of the following lemma. One technique was given by Dahlhaus, another by Schrijver.

**Lemma 7.** *Graphs that do not contain  $K_4 - x$  as induced subgraph are soft.*

The importance of graphs like  $K_4 - x$  is that they have a signalling function with respect to the fact that a line belongs to only one clique. Whenever a line is uni-cliquial this is found out by the fact that the line does not occur as the diagonal of a  $K_4 - x$ . Based on a remark of H.J. Veldman we have:

**Lemma 8.** *Graphs with uni-cliquial lines are soft.*

The reader should note the special case in which a line does not belong to any triangle and therefore forms itself a maximal clique.

Triangles have a signalling function for uni-cliquial points. If all sets of three points that contain point  $v$  induce a  $K_3$ , point  $v$  is uniclquial.

**Lemma 9.** *Graphs with uni-cliquial points are soft.*

A, perhaps superfluous, note for the Lemmas 8 and 9 is that the number of cliques that is memorized is polynomial. No combinatorial explosion is induced in the space-complexity.

Another important induced subgraph on four points is  $K_3 \cup K_1$ , that we shall call a point-triangle (configuration). The complement of this graph is  $K_{1,3}$ .

**Lemma 10.** *Graphs without point-triangles as induced subgraphs are soft.*

**Proof.** If a graph  $G$  does not contain point-triangles then  $\bar{G}$  does not contain a  $K_{1,3}$  as induced subgraph. A deep result of Minty [4] and Sbihi [5] shows that the ISP can then be solved in polynomial time in  $\bar{G}$ , which solves the MCP for  $G$ .  $G$  is soft.  $\square$

This result can be used for a further reduction technique.

**Lemma 11.** *Graphs with points that have neighbourhood graphs that do not contain point-triangles are soft.*

However, a fixed number of point-triangles can be admitted, as we have:

**Lemma 12.** *Graphs with the property that each point  $v$  has in its neighbourhood graph  $\langle N(v) \rangle$  a number of graphs  $K_3 \cup K_1$ , that is limited by a constant  $l$ , are soft.*

## Discussion

The main point of further research is the extension of the list of properties that make a graph soft. One may hope for extension of the list of properties in such a

way that one can prove that no graphs can be found that have none of them. This would establish the fact that at least one of the reduction techniques is bound to succeed at each phase of the reduction process. In a situation like that the complexity of the composite algorithm consisting of successive reductions may be investigated. Meanwhile a set of hard graphs remains. The order of the smallest hard graph known, having none of the properties mentioned in the lemmas, is 12.

## References

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