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Exact Markov inequalities for the Hermite and Laguerre weights ☆

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Abstract

Denote by π_n the set of all real algebraic polynomials of degree at most n and let $U_n := \{e^{-x^2} p(x) : p \in \pi_n\}$, $V_n := \{e^{-x} p(x) : p \in \pi_n\}$. We prove the following exact Markov inequalities:

$$\|u^{(k)}\|_{\mathbb{R}} \leqslant \|u_{*,n}^{(k)}\|_{\mathbb{R}} \|u\|_{\mathbb{R}}, \quad \forall u \in U_n, \quad \forall k \in \mathbb{N},$$

and

$$\|v^{(k)}\|_{\mathbb{R}_+} \leqslant \|v_{*,n}^{(k)}\|_{\mathbb{R}_+} \|v\|_{\mathbb{R}_+}, \quad \forall u \in V_n, \quad \forall k \in \mathbb{N},$$

where $\|\cdot\|_{\mathbb{R}}$ ($\|\cdot\|_{\mathbb{R}_+}$) is the supremum norm on $\mathbb{R}$ ($\mathbb{R}_+ := [0, \infty)$) and $u_{*,n}$ ($v_{*,n}$) is the Chebyshev polynomial from U_n (V_n).

MSC: 41A17; 26D10

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1. Introduction

Denote by π_n the set of all real algebraic polynomials of degree not exceeding n, and by $\|\cdot\|_I$ the supremum norm for a given interval $I \subseteq \mathbb{R}$, $\|f\|_I := \sup_{x \in I} |f(x)|$.

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In 1892 Markov [13] proved that if $f \in \pi_n$ satisfies $\|f\|_{[-1,1]} \leqslant 1$ then for every $k = 1, \ldots, n$

$$\|f^{(k)}\|_{[-1,1]} \leqslant T_n^{(k)}(1),$$

where the equality is attained only for the Chebyshev polynomial $T_n(x) := \cos(n \arccos x)$ (up to a factor -1).

It is well known that the Markov inequality and the Chebyshev polynomial play an important role in the theory of approximations with algebraic polynomials. There are a lot of results on Markov-type inequalities (see, e.g. [2,4,18,21,22], and the references therein). In connection with the research in the field of the weighted approximation by polynomials, Markov-type inequalities have been proved for various weights, norms and sets over which the norm is taken (cf. [16,23,25,24,9,20,7,17,14,12,10]). In the case of supremum norm on an infinite interval there are only two exact Markov-type inequalities (see [11,5]). They are of the form

$$\|(wp)'\| \leqslant C_n(w)\|wp\|, \quad \forall p \in \pi_n, \tag{1}$$

where $w(x) = e^{-x^2}$ on $\mathbb{R}$ or $w(x) = e^{-x}$ on $\mathbb{R}_+ := [0, \infty)$. The equality in (1) is attained only for the corresponding weighted Chebyshev polynomial (up to a constant factor).

The aim of this paper is to extend the above inequalities to derivatives of arbitrary order. Note that such a extension was obtained in [15] for polynomials which have only real zeros.

Next we formulate our main results. Denote by U_n the space of all weighted polynomials of the form $u(x) = e^{-x^2} p(x)$, where $p \in \pi_n$. We shall use the notation $u_{*,n}$ for the Chebyshev polynomial from U_n. Precisely, $u_{*,n}$ is the unique polynomial from U_n which has norm equal to 1 and there exist $n+1$ points $t_0 < \cdots < t_n$ such that $u_{*,n}(t_k) = (-1)^{n-k}$ for $k = 0, \ldots, n$.

Theorem 1. *Let $u \in U_n$. Then for every natural number k, the inequality*

$$\|u^{(k)}\|_{\mathbb{R}} \leqslant \|u_{*,n}^{(k)}\|_{\mathbb{R}} \|u\|_{\mathbb{R}}$$

holds. The equality is attained if and only if $u(x) = cu_{*,n}(x)$.

Let V_n be the space of all weighted polynomials of the form $v(x) = e^{-x} p(x)$, where $p \in \pi_n$, and $v_{*,n}$ be the Chebyshev polynomial from V_n.

Theorem 2. *Let $v \in V_n$. Then for every natural number k, the inequality*

$$\|v^{(k)}\|_{\mathbb{R}_+} \leqslant \|v_{*,n}^{(k)}\|_{\mathbb{R}_+} \|v\|_{\mathbb{R}_+}$$

holds. The equality is attained if and only if $v(x) = cv_{*,n}(x)$.

In the proofs of the above theorems we use some ideas of Bojanov [2], who gave a new proof of the inequality of Markov for algebraic polynomials.

2. Markov inequality for the weight e^{-x^2} on $\mathbb{R}$

For the sake of simplicity in this section we shall write $\|\cdot\|$ instead of $\|\cdot\|_{\mathbb{R}}$. To start with we note that every non-zero polynomial from U_n has at most n real zeros, counting the multiplicities and if $u \in U_n$ then $u' \in U_{n+1}$. Next we list some of the results of [15], which will be needed in the sequel. Let $\mathcal{U}_n := \{u \in U_n : u$ has n simple real zeros$\}$. It is easily seen that if $u \in \mathcal{U}_n$ then

$u' \in \mathcal{U}_{n+1}$. Moreover, if $x_1 < \cdots < x_n$ are the zeros of u and $t_0 < \cdots < t_n$ are the zeros of u', then $t_0 < x_1 < t_1 < \cdots < t_{n-1} < x_n < t_n$.

The following theorem from [15] gives the solution of a problem about interpolation at extremal points for polynomials from $\mathcal{U}_n$ (cf. [6,19,8,1]).

Theorem A. *Given positive numbers $h_0, \ldots, h_n$, there exists a unique $u \in \mathcal{U}_n$ and a unique set of points $t_0 < \cdots < t_n$ such that*

$$\begin{aligned} u(t_k) &= (-1)^{n-k} h_k, \quad k = 0, \ldots, n, \\ u'(t_k) &= 0, \quad k = 0, \ldots, n. \end{aligned} \tag{2}$$

Since every $u \in \mathcal{U}_n$ has exactly $n+1$ extremal points $t_0 < \cdots < t_n$, Theorem A shows that the parameters $h_i(u) := |u(t_i)|$, $i = 0, \ldots, n$, determine u uniquely (up to multiplication by -1). Given $\mathbf{h} = (h_0, \ldots, h_n)$ where $h_j > 0$ for $j = 0, \ldots, n$, we shall use the notation $u(\mathbf{h}; \cdot)$ for the unique solution of (2). Clearly, $u_{*,n} = u(\mathbf{1}; \cdot)$, where $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^{n+1}$.

In [3] Bojanov and Rahman proposed a method for derivation of estimates for functionals in the set of algebraic polynomials, having only real zeros. This method was applied in [15] to prove the following:

Theorem B. *Let u_1 and u_2 be polynomials from $\mathcal{U}_n$. Suppose that*

$$0 < h_i(u_1) \leqslant h_i(u_2) \quad for \;\; i = 0, \ldots, n.$$

Then for every natural number k, the inequalities

$$0 < h_j(u_1^{(k)}) \leqslant h_j(u_2^{(k)}), \quad j = 0, \ldots, n+k, \tag{3}$$

hold. In particular,

$$\|u_1^{(k)}\| \leqslant \|u_2^{(k)}\|. \tag{4}$$

Moreover, the equality in (3) (for some j) and (4) is attained if and only if $h_i(u_1) = h_i(u_2)$ for all $i = 0, \ldots, n$.

Consequently, the absolute values of the local extrema of the kth derivative of a weighted polynomial $u \in \mathcal{U}_n$ are strictly increasing functions of $h_0(u), \ldots, h_n(u)$.

In the next lemma we study a Birkhoff-type interpolation problem for weighted polynomials.

Lemma 1. *Let k and m be natural numbers. Given points $t_1 < \cdots < t_m$, ξ and arbitrary values $\{a_j\}_1^{m+2}$, there exists a unique polynomial $g \in U_{m+1}$ for which*

$$g(t_j) = a_j, \quad j = 1, \ldots, m, \quad g^{(k)}(\xi) = a_{m+1}, \quad g^{(k+1)}(\xi) = a_{m+2}. \tag{5}$$

Proof. Conditions (5) can be considered as a system of linear equations for the coefficients in the representation

$$g(x) = e^{-x^2} \sum_{i=0}^{m+1} b_i x^i.$$

In order to prove the existence and the uniqueness of the solution of (5), it is sufficient to prove that the corresponding homogeneous system

$$g(t_j) = 0, \quad j = 1, \dots, m, \quad g^{(k)}(\xi) = 0, \quad g^{(k+1)}(\xi) = 0 \tag{6}$$

has only the trivial solution. The proof goes by induction on k, for arbitrary m, $t_1 < \cdots < t_m$ and ξ.

Let $k = 1$. If $\xi = t_j$ for some $j \in \{1, \dots, m\}$ then g has $m+2$ zeros, counting the multiplicities, hence $g \equiv 0$. So, we may assume $\xi \notin \{t_1, \dots, t_m\}$. By Rolle's theorem, $g'(x)$ changes its sign at some points $\xi_i \in (t_i, t_{i+1})$ for $i = 1, \dots, m-1$. But $g(x) \to 0$ for $x \to \pm\infty$, hence $g'(x)$ has also zeros $\xi_0 < t_0$ and $\xi_m > t_m$.

If $\xi \notin \{\xi_0, \dots, \xi_m\}$, then according to (6), ξ is at least double zero of g'. Thus $g' \in U_{m+2}$ has $m+3$ zeros counting the multiplicities. It follows that g is a constant, i.e. $g \equiv 0$, provided $m \geqslant 1$.

Otherwise, if $\xi = \xi_j$ for some $j \in \{0, \dots, m\}$ then g' must change its sign at ξ. Taking in view (6), we conclude that g' has at least triple zero at ξ, which also implies $g \equiv 0$.

Let $k \geqslant 2$. Assume the assertion holds for $k-1$. Let g satisfy (6) for some $t_1 < \cdots < t_m$ and ξ. Consider the polynomial $g_1(x) := g'(x)$. Clearly, $g_1 \in U_{m+2}$, g_1 vanishes at some points $\xi_0 < \cdots < \xi_m$ and $g_1^{(k-1)}(\xi) = g_1^{(k)}(\xi) = 0$. Then by the inductional hypothesis $g_1 \equiv 0$, hence $g \equiv 0$. The lemma is proved. □

Lemma 2. *Let $u \in U_n$, $\|u\| = 1$. Let $t_1 < \cdots < t_m$ $(m \leqslant n)$ be the points for which $|u(t_k)| = 1$. If $g \in U_n$ vanishes at $t_1, \dots, t_m$ then*

$$\|u + \varepsilon g\| = 1 + o(\varepsilon) \quad \text{as} \ \ \varepsilon \to 0.$$

Proof. We can choose $\delta > 0$ so that

$$t_j \notin (t_i - \delta, t_i + \delta)$$

for $i \neq j$ $(i, j = 1, \dots, n)$. Since $u + \varepsilon g$ tends uniformly to u on $\mathbb{R}$ as $\varepsilon \to 0$ there exists an $\varepsilon_0 > 0$ such that

$$|u(x) + \varepsilon g(x)| < 1 \quad \text{for } x \notin \bigcup_{i=1}^{n} [t_i - \delta, t_i + \delta],$$

provided $0 < \varepsilon < \varepsilon_0$. Hence

$$\|u + \varepsilon g\| = \max_{i=1,\dots,m} \|u + \varepsilon g\|_{[t_i - \delta, t_i + \delta]}.$$

Let i be a fixed number from $\{1, \dots, m\}$. Without loss of generality we may assume that $u(t_i) = 1$. We define $x_i(\varepsilon) \in (t_i - \delta, t_i + \delta)$ as the solution of $u(x) + \varepsilon g(x) = 1$, farthest from t_i. (It is possible $x_i(\varepsilon) = t_i$.)

Let $\Delta_i(\varepsilon) := \{x : |x - t_i| \leqslant |x_i(\varepsilon) - t_i|\}$. Clearly

$$\|u + \varepsilon g\|_{[t_i - \delta, t_i + \delta]} = \|u + \varepsilon g\|_{\Delta_i(\varepsilon)}.$$

Let $u'(t_i) = \cdots = u^{(2l-1)}(t_i) = 0$, $u^{(2l)}(t_i) < 0$. (Recall that t_i is a local maximum of u.) We can assume that $u^{(2l)}(x) \leqslant c < 0$ for $x \in [t_i - \delta, t_i + \delta]$, provided δ is sufficiently small. We have

$$u(t_i + x_i(\varepsilon) - t_i) + \varepsilon g(t_i + x_i(\varepsilon) - t_i) = 1$$

and by Taylor's formula we get

$$1 + \frac{u^{(2l)}(\xi_i^1)}{(2l)!}(x_i(\varepsilon) - t_i)^{2l} + \varepsilon g'(\xi_i^2)(x_i(\varepsilon) - t_i) = 1,$$

where $\xi_i^1, \xi_i^2 \in \Delta_i(\varepsilon)$. Hence

$$(x_i(\varepsilon) - t_i)^{2l-1} = -\frac{(2l)! g'(\xi_i^2)\varepsilon}{u^{(2l)}(\xi_i^1)} = O(\varepsilon),$$

which implies $x_i(\varepsilon) - t_i = O(\varepsilon^{\frac{1}{2l-1}})$. For each $x \in \Delta_i(\varepsilon)$ we have

$$u(x) + \varepsilon g(x) = 1 + \frac{u^{(2l)}(\eta_i^1)}{(2l)!}(x - t_i)^{2l} + \varepsilon g'(\eta_i^2)(x - t_i) = 1 + O(\varepsilon^{\frac{2l}{2l-1}}),$$

which finishes the proof of Lemma 2. □

In the next lemma we prove a property of the polynomials from $\mathcal{U}_n$, which is well known for algebraic polynomials.

Lemma 3. *Each zero η of the derivative of a weighted polynomial $u(x) = ce^{-x^2}(x - x_1)\cdots(x - x_n)$ $(c \neq 0)$ is a strictly increasing function of x_k in the domain $x_1 < \cdots < x_n$.*

Proof. Denote for brevity $\omega(x) = (x - x_1)\cdots(x - x_n)$. Since

$$\frac{u'(x)}{u(x)} = -2x + \frac{\omega'(x)}{\omega(x)}$$

and $u'(\eta) = 0$, we get

$$-2\eta + \sum_{i=1}^{n} \frac{1}{\eta - x_i} = 0.$$

Differentiating the last identity with respect to x_k we obtain

$$\left(2 + \sum_{i=1}^{n} \frac{1}{(\eta - x_i)^2}\right) \frac{\partial \eta}{\partial x_k} = \frac{1}{(\eta - x_k)^2},$$

which implies $\frac{\partial \eta}{\partial x_k} > 0$. Lemma 3 is proved. □

An immediate consequence of Lemma 3 is the following:

Corollary 3. *Let u_1 and u_2 be two polynomials from $\mathcal{U}_n$ having zeros $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$, respectively. Suppose that*

$$x_i \leqslant y_i, \quad i = 1, \ldots, n,$$

with at least one strict inequality. Then the zeros $t_1 < \cdots < t_{n+1}$ of $u_1'(x)$ and the zeros $\tau_1 < \cdots < \tau_{n+1}$ of $u_2'(x)$ satisfy

$$t_i < \tau_i, \quad i = 1, \ldots, n+1.$$

Our next result is a weighted analogue of the famous Markov's lemma concerning the zeros of the algebraic polynomials.

Lemma 4. *Assume that the zeros* $x_1 < \cdots < x_n$ *of* $u_1 \in \mathcal{U}_n$ *and* $y_1 < \cdots < y_{n-1}$ *of* $u_2 \in \mathcal{U}_{n-1}$ *satisfy the interlacing conditions*

$$x_1 \leqslant y_1 \leqslant x_2 \leqslant \cdots \leqslant x_{n-1} \leqslant y_{n-1} \leqslant x_n.$$

Then the zeros $t_1 < \cdots < t_{n+1}$ *of* u_1' *and the zeros* $\tau_1 < \cdots < \tau_n$ *of* u_2' *interlace strictly, that is,*

$$t_1 < \tau_1 < t_2 < \cdots < t_n < \tau_n < t_{n+1}.$$

Proof. We will prove only the inequalities

$$t_i < \tau_i \quad \text{for } i = 1, \ldots, n. \tag{7}$$

(The remaining ones can be established in a similar way.) Set

$$y_k(\varepsilon) := \begin{cases} y_k & \text{for } k = 1, \ldots, n-1, \\ \frac{1}{\varepsilon} & \text{for } k = n. \end{cases}$$

The inequalities

$$x_1 \leqslant y_1(\varepsilon) \leqslant x_2 \leqslant \cdots \leqslant y_{n-1}(\varepsilon) \leqslant x_n < y_n(\varepsilon) \tag{8}$$

hold true, provided ε is a sufficiently small positive number.

Let us define $u_\varepsilon(x) := u_2(x)(1 - \varepsilon x)$. Clearly, $y_k(\varepsilon),\ k = 1, \ldots, n$, are the zeros of u_ε and let $\tau_1(\varepsilon) < \cdots < \tau_{n+1}(\varepsilon)$ be the zeros of u_ε'. Corollary 3 and (8) imply

$$t_i < \tau_i(\varepsilon) \quad \text{for } i = 1, \ldots, n+1. \tag{9}$$

Note that $\tau_i(\varepsilon) \to \tau_i,\ \ i = 1, \ldots, n$, because $u_\varepsilon^{(k)}$ tends uniformly to $u_2^{(k)}$ on $\mathbb{R}$ as $\varepsilon \to 0$. According to Lemma 3, each of $\tau_i(\varepsilon)$ increases strictly when ε decreases. Letting $\varepsilon \downarrow 0$ in (9) we obtain (7). Lemma 4 is proved. □

In the next lemma we compare the norms of the derivatives of the weighted Chebyshev polynomials for different n.

Lemma 5. *For every natural number k the inequality*

$$\|u_{*,n-1}^{(k)}\| < \|u_{*,n}^{(k)}\| \tag{10}$$

holds true.

Proof. Let $u_{*,n-1}(x) = e^{-x^2}(\alpha_{n-1}x^{n-1} + \cdots)$, where $\alpha_{n-1} > 0$. For every $\varepsilon > 0$ we consider the polynomial $u_\varepsilon(x) = u_{*,n-1}(x) - \varepsilon x^n e^{-x^2}$. It is easily seen that for each $j \geqslant 0$ we have

$$\|u_\varepsilon^{(j)} - u_{*,n-1}^{(j)}\| \to 0 \quad \text{as } \varepsilon \to 0. \tag{11}$$

Let us fix a point b greater than all zeros of $u_{*,n-1}$. Clearly, $u_{*,n-1}(b) > 0$. Hence, for sufficiently small ε, u_ε has $n-1$ simple zeros in $(-\infty, b)$ (close to the zeros of $u_{*,n-1}$) and $u_\varepsilon(b) > 0$. But the leading coefficient of $u_\varepsilon(x)$ is negative, hence u_ε must have another real zero $x(\varepsilon) > b$. Since b can be arbitrarily large, it follows that $x(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

Let us denote the points of the local extrema of the oscillating polynomial u_ε by $t_0(\varepsilon) < \cdots < t_n(\varepsilon)$ and those of $u_{*,n-1}$ by $t_0 < \cdots < t_{n-1}$. We have $t_n(\varepsilon) \to \infty$ while (from (11)) $t_i(\varepsilon) \to t_i$ as $\varepsilon \to 0$ for $i = 0, \dots, n-1$. Also $u_\varepsilon(t_n(\varepsilon)) \to -0$ and $u_\varepsilon(t_i(\varepsilon)) \to (-1)^{n-1-i}$ for $i = 0, \dots, n-1$.

According to Theorem A, $u_\varepsilon(x) = -u(\mathbf{h}_0(\varepsilon); x)$, where $\mathbf{h}_0(\varepsilon) := (h_0(u_\varepsilon), \dots, h_n(u_\varepsilon))$. If $\mathbf{h}_1(\varepsilon) := (h_0(u_\varepsilon), \dots, h_{n-1}(u_\varepsilon), 1/2)$ then by Theorem B $\|u_\varepsilon^{(k)}\| < \|u^{(k)}(\mathbf{h}_1(\varepsilon); \cdot)\|$, provided ε is sufficiently small. Letting $\varepsilon \to 0$ we obtain

$$\|u_{*,n-1}^{(k)}\| \leqslant \|u^{(k)}(\mathbf{h}_1; \cdot)\|, \tag{12}$$

where $\mathbf{h}_1 = (1, \dots, 1, 1/2) \in \mathbb{R}^{n+1}$. Using again the strict monotonicity we get

$$\|u^{(k)}(\mathbf{h}_1; \cdot)\| < \|u_{*,n}^{(k)}\|. \tag{13}$$

Inequality (10) is a direct consequence from (12) and (13). Lemma 5 is proved. □

Proof of Theorem 1. An equivalent setting is to prove that $u_{*,n}$ is the unique solution of the extremal problem

$$\|u^{(k)}\| \to \sup \quad \text{over all } u \in U_n, \ \|u\| \leqslant 1. \tag{14}$$

Let u be a fixed extremal polynomial to problem (14). Note that $\|u\| = 1$. We claim that $|u(x)|$ attains its maximal value at least at n points. Indeed, assume that $t_1 < \cdots < t_m$ $(m \leqslant n-1)$ are all points such that $|u(t_k)| = 1$. Let $M_k := \|u^{(k)}\| = |u^{(k)}(\xi)|$. According to Lemma 1 there exists $g \in U_{m+1} \subseteq U_n$ satisfying the conditions

$$g(t_j) = 0, \quad j = 1, \dots, m, \quad g^{(k)}(\xi) = \operatorname{sign} u^{(k)}(\xi). \tag{15}$$

(For $g^{(k+1)}(\xi)$ we can take any value.)

Consider the polynomial $u_\varepsilon(x) := (u(x) + \varepsilon g(x))/\|u + \varepsilon g\|$. Clearly, $u_\varepsilon \in U_n$ and $\|u_\varepsilon\| = 1$. It follows from Lemma 2 and (15) that

$$|u_\varepsilon^{(k)}(\xi)| = \frac{|u^{(k)}(\xi) + \varepsilon g^{(k)}(\xi)|}{1 + o(\varepsilon)} = \frac{M_k + \varepsilon}{1 + o(\varepsilon)} > M_k,$$

provided ε is a sufficiently small positive number. The last inequality contradicts with the extremality of u. The claim is proved.

Note that the equation

$$|u(t)| = 1 \tag{16}$$

cannot have more than $n+1$ solutions. Otherwise, $u'(x)$ would have $n+2$ zeros, so $u'(x) \equiv 0$, a contradiction.

Furthermore, if there exist exactly $n+1$ points at which (16) holds, then it is easily seen that $u \equiv \pm u_{*,n}$, so Theorem 1 will be proved.

It remains to exclude the case when (16) has exactly n solutions. Assume the contrary and let $t_1 < \cdots < t_n$ be all the points at which $|u(x)|$ attains its maximal value.

Our next goal is to show that they are alternation points for u, i.e. $u(t_k) = \sigma(-1)^k$ for $k = 1, \dots, n$, where $\sigma \in \{-1, 1\}$. Assume the contrary. Then there exists an index i for which

$u(t_i)u(t_{i+1}) > 0$, hence u' has a zero $\gamma \in (t_i, t_{i+1})$. Consequently, $\{t_k\}_1^n$ and γ are all the zeros of $u' \in U_{n+1}$. If $\omega(x) := e^{-x^2}(x - t_1) \cdots (x - t_n)$, then the zeros of u' and ω interlace, hence by Lemma 4, the zeros of $u^{(k+1)}$ and $\omega^{(k)}$ interlace strictly. As $u^{(k+1)}(\xi) = 0$, we conclude that $\omega^{(k)}(\xi) \neq 0$. Then, for sufficiently small $\varepsilon > 0$, one of the polynomials $(u \pm \varepsilon\omega)/\|u \pm \varepsilon\omega\|$ will have larger norm of the kth derivative than u, which is a contradiction.

So, the extremal polynomial u has n alternation points, hence at least $n - 1$ simple zeros. If $u \in U_{n-1}$ then u has to coincide with $\pm u_{*,n-1}$, but this is impossible in view of Lemma 5. It follows that u is a weighted polynomial of exact degree n, hence u must have n simple real zeros. Taking into account Theorem B, we conclude that $u = \pm u_{*,n}$, which is a contradiction. Theorem 1 is proved. □

3. Markov inequality for the weight e^{-x} on $\mathbb{R}_+$

In this section we abbreviate the notation $\|\cdot\|_{\mathbb{R}_+}$ to $\|\cdot\|$. The approach is similar to that in Section 2, but the analysis is somewhat simpler, due to the translation invariance property of V_n, that is, $v(x + c) \in V_n$ for every $v \in V_n$ and $c \in \mathbb{R}$.

Lemma 6. *Let k and m be natural numbers. Given points $t_1 < \cdots < t_m$ in $[0, \infty)$ and values $\{a_j\}_0^m$, there exists a unique polynomial $g \in V_m$ for which*

$$g(t_j) = a_j, \quad j = 1, \ldots, m, \quad g^{(k)}(0) = a_0.$$

Proof. As in Lemma 1, we will show that the homogeneous system of equations

$$v(t_j) = 0, \quad j = 1, \ldots, m, \quad v^{(k)}(0) = 0 \tag{17}$$

admits only the trivial solution $v \equiv 0$ in V_m.

Let v be a solution of (17). By Rolle's theorem, $v'(x)$ has at least one zero $\xi_i \in (t_i, t_{i+1})$ for $i = 1, \ldots, m$, where $t_{m+1} := \infty$. Repeating this argument, we conclude that $v^{(k)}$ vanishes at some points $\xi_1^{(k)} < \cdots < \xi_m^{(k)}$ in $(0, \infty)$. Because of (17), $v^{(k)} \in V_m$ has $m + 1$ zeros in $[0, \infty)$, which implies $v^{(k)} \equiv 0$.

Now, let $v(x) = e^{-x} p(x)$, where $p(x)$ is an algebraic polynomial of degree $\leqslant m$. It is easily seen that $v^{(k)}(x) = e^{-x} q(x)$, where $q(x) = \sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} p^{(s)}(x)$. But $q \equiv 0$, hence the degree of p is less than m. Taking in view (17), we conclude that $p \equiv 0$. The lemma is proved. □

Lemma 7. *Let $v \in V_n$, $\|v\| = 1$. Let $m \leqslant n$ and $t_1 < \cdots < t_m$ be the points for which $|v(t_k)| = 1$. If $g \in V_n$ vanishes at $t_1, \ldots, t_m$ then*

$$\|v + \varepsilon g\| = 1 + o(\varepsilon) \quad \text{if} \quad \varepsilon \to 0.$$

Proof. As in Lemma 2, it is sufficient to consider $v + \varepsilon g$ on small neighbourhoods of the points t_i, $i = 1, \ldots, m$. If $t_i > 0$ then the estimation of the norm of $v + \varepsilon g$ around t_i is completely analogous to that in Lemma 2. It remains to estimate $v + \varepsilon g$ around t_1 if $t_1 = 0$. Let $\delta < t_2$ be a sufficiently small, fixed positive number. Our goal is to prove that $\|v + \varepsilon g\|_{[0,\delta]} = 1 + o(\varepsilon)$ as $\varepsilon \to 0$. Without loss of generality we may assume $v(0) = 1$ and, as a consequence, $v'(0) \leqslant 0$. If $v'(0) < 0$ then it is easy to see that $\|v + \varepsilon g\|_{[0,\delta]} = 1$.

Suppose now $v'(0) = 0$. Set $x(\varepsilon) := \sup\{x \in [0, \delta) : v(x) + \varepsilon g(x) = 1\}$. It follows that $\|v + \varepsilon g\|_{[0,\delta]} = \|v + \varepsilon g\|_{[0,x(\varepsilon)]}$. Furthermore, arguing as in Lemma 2, we get $x(\varepsilon) = O(\varepsilon^{\frac{1}{s-1}})$, provided $v'(0) = \cdots = v^{(s-1)}(0) = 0$, $v^{(s)}(0) \neq 0$ for some $s \geqslant 2$. Consequently, if $x \in [0, x(\varepsilon)]$ then $v(x) + \varepsilon g(x) = 1 + O(\varepsilon^{\frac{s}{s-1}})$, which finishes the proof of Lemma 7. □

Proof of Theorem 2. As in Theorem 1, it is sufficient to prove that $v_{*,n}$ is the unique solution of the extremal problem

$$\|v^{(k)}\| \to \sup \quad \text{over all } v \in V_n, \quad \|v\| \leqslant 1. \tag{18}$$

Let v be a fixed extremal polynomial to problem (18). Clearly, $\|v\| = 1$ and the equation

$$|v(t)| = 1 \tag{19}$$

cannot have more than $n + 1$ solutions on $[0, \infty)$. We claim that $|v(x)|$ attains its maximal value at exactly $n + 1$ points. On the contrary, we assume that Eq. (19) has exactly $m \leqslant n$ solutions $t_1 < \cdots < t_m$ in $[0, \infty)$. There exists a point $\xi \in [0, \infty)$ such that $M_k := \|v^{(k)}\| = |v^{(k)}(\xi)|$. Without loss of generality we suppose that $\xi = 0$. (Otherwise, we can consider $v_1(x) := v(x + \xi) \in V_n$. We have $\|v_1\| \leqslant \|v\| = 1$ and $|v_1^{(k)}(0)| = M_k$, hence v_1 is also extremal in (18), which implies $\|v_1\| = 1$. In addition, the equation $|v_1(x)| = 1$ also has less than $n + 1$ solutions in $[0, \infty)$.) Lemma 6 ensures the existence of a $g \in V_m \subseteq V_n$ such that

$$g(t_j) = 0, \quad j = 1, \ldots, m, \qquad g^{(k)}(0) = \operatorname{sign} v^{(k)}(0). \tag{20}$$

If $v_\varepsilon(x) := (v(x) + \varepsilon g(x))/\|v + \varepsilon g\|$ then $v_\varepsilon \in V_n$ and $\|v_\varepsilon\| = 1$. Using Lemma 7 and (20) (as in the proof of Theorem 1) we conclude that $|v_\varepsilon^{(k)}(0)| > M_k$, provided $\varepsilon > 0$ is sufficiently small. This is a contradiction, which proves the claim.

Let us denote the points at which $|v(x)|$ attains its maximal value by $t_0 < \cdots < t_n$. Next we will prove that they are alternation points for v, which implies $v = \pm v_{*,n}$. Assume the contrary, i.e. there exists $i \in \{0, \ldots, n - 1\}$ such that $v(t_i)v(t_{i+1}) > 0$. Then v' has a zero in (t_i, t_{i+1}). Since $v'(t_k) = 0$ for $k = 1, \ldots, n$ and $v' \in V_n$, we conclude that $v' \equiv 0$, which is a contradiction. Theorem 2 is proved. □

Remark. In fact $\|v_{*,n}^{(k)}\| = |v_{*,n}^{(k)}(0)|$. Otherwise, a proper translation of $v_{*,n}$ will produce a different extremal polynomial in (18).

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