On the asymptotic representation of the Euler gamma function by Ramanujan

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Abstract

The problem of approximation to the Euler gamma function on the basis of some Ramanujan’s formulas is considered. The function $h(x) = (g(x))^{6} - (8x^{3} + 4x^{2} + x)$, where $g(x) = (e/x)^{x} / \sqrt{x} e^{x}$, is studied. It is proved that on the interval $(1, \infty)$ the function $h(x)$ is increasing monotonically from $h(1) = 0.0111976 \ldots$ to $h(\infty) = 1/30 = 0.0333 \ldots$. © 2001 Published by Elsevier Science B.V.

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1. Introduction. Statement of the problem

There is the following record in “The lost notebook and other unpublished papers” of Srinivasa Ramanujan [6, p. 339]:

“If $x \geqslant 0$,

$$
\Gamma(1 + x) = \sqrt{\pi} \left( \frac{x}{e} \right)^{x} \left( 8x^{3} + 4x^{2} + x + \frac{\theta_{1}}{30} \right)^{1/6},
$$

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1 The story of this problem is described in [2, p. 48 (Question 754)].
where \( \theta \) is a positive proper fraction

\[
\theta_0 = \frac{30}{\pi^2} = 0.9675
\]

\[
\theta_{1/12} = 0.8071, \quad \theta_{7/12} = 0.3058
\]

\[
\theta_{2/12} = 0.6160, \quad \theta_{8/12} = 0.3014
\]

\[
\theta_{3/12} = 0.4867, \quad \theta_{9/12} = 0.3041
\]

\[
\theta_{4/12} = 0.4029, \quad \theta_{10/12} = 0.3118
\]

\[
\theta_{5/12} = 0.3509, \quad \theta_{11/12} = 0.3227
\]

\[
\theta_{6/12} = 0.3207, \quad \theta_1 = 0.3359
\]

\[
\theta_\infty = 1.
\]

\[
\sqrt{\pi} \left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1}{100}\right)^{1/6} < \Gamma(1 + x) < \sqrt{\pi} \left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1}{30}\right)^{1/6}.
\]

Considering this record, Anderson et al. in [1, p. 476] defined the function

\[
h(x) = (g(x))^6 - (8x^3 + 4x^2 + x), \quad (1.1)
\]

where

\[
g(x) = \left(\frac{e^x \Gamma(1 + x)}{x^{\frac{1}{\sqrt{\pi}}}}\right), \quad (1.2)
\]

and formulated the conjecture: to prove that “\( h(x) \) is increasing from \((1, \infty)\) into \((\frac{1}{100}; \frac{1}{30})\)”.

The present paper gives the proof of the theorem.

**Theorem 1.** The function \( h(x) \) is increasing monotonically from \((1, \infty)\) onto \((h(1), h(\infty))\) with \( h(1) = e^{6/\pi^2} - 13 = 0.0111976 \ldots \) and \( h(\infty) = 1/30 = 0.0333 \ldots \).

For this purpose, three lemmas are proved.

**Lemma 1.** For \( x \geq x_0 = 2.4 \), the function \( h(x) \) satisfies the inequalities \( \frac{1}{100} \leq h(x) < \frac{1}{30} \), and \( h(x) \to \frac{1}{30}, \quad x \to \infty \).

**Lemma 2.** For \( x \geq x_1 = 4.21 \), the function \( h(x) \) is monotonically increasing.

**Lemma 3.** For \( 1 < x \leq \max(x_0, x_1) = 4.21 \), the function \( h(x) \) is monotonically increasing.

The basis for proving are the asymptotic formulas of Stirling for the functions \( y = \log \Gamma(x) \) and \( \psi(x) = (d/dx)\log \Gamma(x) \), and computer calculations for \( 1 < x \leq 4.21 \).

**2. The representation of the function \( h(x) \) via a special integral**

Taking logarithms of both sides of (1.2) and using the fact that \( \Gamma(x + 1) = x\Gamma(x) \), we have

\[
\log g(x) = x - x \log x + \log x - \log \sqrt{\pi} + \log \Gamma(x).
\]

(2.1)
We substitute in (2.1), the Stirling formula for \( \log x \) [3, pp. 342–343]
\[
\log x = (x - \frac{1}{2}) \log x - x + \log \sqrt{2\pi} + J(x),
\]
(2.2)
where
\[
J(x) = \int_0^\infty \frac{\sigma(u) \, du}{(x + u)^2},
\]
(2.3)
\[
\sigma(u) = \int_0^u \rho(t) \, dt,
\]
(2.4)
\[
\rho(t) = \frac{1}{2} - \{t\},
\]
(2.5)
and \( \{t\} \) is the fractional part of the number \( t \). We obtain
\[
\log g(x) = \log \sqrt{2x} + J(x),
\]
or
\[
g(x) = \sqrt{2x} e^{J(x)}.
\]
From this and from (1.1), we have the following convenient representation for \( h(x) \):
\[
h(x) = 8x^3 e^{6J(x)} - (8x^3 + 4x^2 + x).
\]
(2.6)

3. Asymptotic representation for the integral \( J(x) \)

We notice that from the definition of \( \sigma(u) \) from (2.4) and (2.5)
\[
\sigma(u + 1) = \sigma(u),
\]
\[
\sigma(u) = \sigma(\{u\}) = \int_0^{\{u\}} \left( \frac{1}{2} - t \right) \, dt = \frac{1}{2} \{u\}(1 - \{u\}).
\]
Hence for \( 0 \leq u \leq 1 \),
\[
\sigma(u) = \frac{1}{2} u(1 - u); \quad \sigma(0) = \sigma(1) = 0; \quad \sigma'(u) = \frac{1}{2} - u.
\]
We expand the function \( \sigma(u) \) in the Fourier series
\[
\sigma(u) = \sum_{n=-\infty}^{+\infty} c(n) e^{2\pi i nu},
\]
(3.1)
with coefficients
\[
c(n) = \int_0^1 \sigma(u) e^{-2\pi i nu} \, du, \quad n \neq 0,
\]
(3.2)
\[
c(0) = \int_0^1 \sigma(u) \, du = \int_0^1 \frac{1}{2} [u(1 - u)] \, du = \frac{1}{12}.
\]
(3.3)
Integrating (3.2) by parts, we obtain
\[
c(n) = -\frac{1}{4\pi^2 n^2}, \quad n \neq 0.
\]
From this and from (3.1) and (3.3),

\[
\sigma(u) = \frac{1}{12} - \sum_{n=-\infty \atop n \neq 0}^{\infty} \frac{1}{4\pi^2 n^2} e^{2\pi i n u} = \frac{1}{12} - \sum_{n=1}^{\infty} \frac{1}{2\pi^2 n^2} \cos 2\pi n u. \tag{3.4}
\]

Substituting the last expression into (2.3), we get

\[
J(x) = \frac{1}{12} \int_0^\infty \frac{du}{(u+x)^2} - \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^\infty \frac{\cos 2\pi n u}{(u+x)^2} du = \frac{1}{12x} - \frac{1}{2\pi^2} \sum_{n=1}^{\infty} J_0(x; n) \frac{n^2}{n^2},
\]

where

\[
J_0(x; n) = \int_0^\infty \frac{\cos 2\pi n u}{(u+x)^2} du.
\]

Integrating \(J_0(x; n)\) by parts (this process is detailed in [4]), we obtain for \(J(x)\) the well-known classical asymptotic formula [7, pp. 62–66], which is usually deduced by other, more complicated ways:

\[
J(x) = \sum_{k=1}^{n-1} \frac{B_{2k}}{2k(2k-1)x^{2k-1}} + R_n(x), \tag{3.5}
\]

where

\[
|R_n(x)| \leq \frac{|B_{2n}|}{2n(2n-1)x^{2n-1}}, \tag{3.6}
\]

and \(B_{2k}\) are Bernoulli coefficients [6, p. 3]

\[
B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \ldots
\]

Setting \(n = 3\), we find from (3.5), the following convenient asymptotic expression for the integral \(J(x)\):

\[
J(x) = \frac{1}{12x} - \frac{1}{360x^3} + R_3(x), \tag{3.7}
\]

where

\[
0 < R_3(x) < \frac{1}{1260x^5}. \tag{3.8}
\]

4. Asymptotic representation for the function \(h(x)\)

Taking into consideration (3.7) and (3.8), one can represent \(e^{\zeta J(x)}\) in the form

\[
e^{\zeta J(x)} = e^{\zeta/(2x)} e^{-x}, \tag{4.1}
\]

where

\[
\zeta = \zeta(x) = \frac{1}{60x^3} - R, \tag{4.2}
\]

\[
R = R(x), \quad 0 < R < \frac{1}{210x^5}. \tag{4.3}
\]
It is easily seen that for $x \geqslant 1$ from (4.2) and (4.3),

$$0 < x \leqslant \frac{1}{60x^3}.$$  

Expanding $e^{-\sqrt{x}}$ in the Taylor series, we get the following simple inequalities

$$1 - x \leqslant e^{-\sqrt{x}} \leqslant 1 - x + \frac{x^2}{2},$$

that is

$$1 - \frac{1}{60x^3} + R \leqslant e^{-\sqrt{x}} \leqslant 1 - \frac{1}{60x^3} + R + \frac{1}{2} \left( \frac{1}{60x^3} - R \right)^2,$$

or

$$1 - \frac{1}{60x^3} \leqslant e^{-\sqrt{x}} \leqslant 1 - \frac{1}{60x^3} + \frac{1}{210x^5} + \frac{9}{39200x^6}.$$

From this and from (4.1) we have

$$e^{1/2x} \left( 1 - \frac{1}{60x^3} \right) \leqslant e^{6/(x)} \leqslant e^{1/2x} \left( 1 - \frac{1}{60x^3} + \frac{1}{210x^5} + \frac{9}{39200x^6} \right).$$

Using the Taylor series for $e^{1/(2x)}$, we find the following bounds for $e^{6/(x)}$:

$$e^{6/(x)} \geqslant \left( 1 - \frac{1}{60x^3} \right) \left( 1 + \frac{1}{2x} + \frac{1}{2!2x^2} + \frac{1}{3!2x^3} + \frac{1}{4!2x^4} + \frac{1}{5!2x^5} \right),$$

$$e^{6/(x)} \leqslant \left( 1 - \frac{1}{60x^3} + \frac{1}{210x^5} + \frac{9}{39200x^6} \right) \left( 1 + \frac{1}{2x} + \frac{1}{2!2x^2} + \cdots \right).$$

From (4.4) we have the lower bound for $h(x)$:

$$h(x) = 8x^3 e^{6/(x)} - (8x^3 + 4x^2 + x)$$

$$\geqslant \left( 8x^3 + 4x^2 + x + \frac{1}{6} + \frac{1}{48x} + \frac{1}{480x^2} \right) \left( 1 - \frac{1}{60x^3} \right) - (8x^3 + 4x^2 + x)$$

$$\geqslant \frac{1}{30} - \frac{11}{240x} - \frac{7}{480x^2} - \frac{1}{360x^3} - \frac{1}{2880x^5} - \frac{1}{28800x^6}.$$  

(4.6)

From the last inequality it follows that for $x \geqslant 2.4$,

$$h(x) > 0.0114 > h(1).$$  

(4.7)

To obtain the upper bound for $h(x)$ we shall consider (4.5). We denote

$$S(x) = 1 - \frac{1}{60x^3} + \frac{1}{210x^5},$$

$$T(x) = 1 + \frac{1}{2x} + \frac{1}{8x^2} + \frac{1}{48x^3} + \frac{1}{384x^4} + \frac{1}{3840x^5},$$

$$\delta(x) = \frac{9}{39200x^6} T(x) + \left( S(x) + \frac{9}{39200x^6} \right) \left( \frac{1}{6!2x^6} + \frac{1}{7!2x^7} + \cdots \right),$$

(4.8)
so one can rewrite (4.5) in the form
\[ e^{\delta(x)} \leq S(x)T(x) + \delta(x). \]  \hfill (4.9)

Taking into account that
\[
\frac{1}{6!(2x)^6} + \frac{1}{7!(2x)^7} + \cdots \leq \frac{3/2}{6!(2x)^6} = \frac{1}{30720x^6},
\]
and also that for \( x \geq 1 \),
\[ S(x) \leq 1; \quad T(x) \leq \frac{33}{20}, \]
we have from (4.8)
\[ 0 \leq \delta(x) \leq \frac{297}{784000x^6} + \left(1 + \frac{9}{39200x^6}\right) \frac{1}{30720x^6} \leq \frac{21}{50000x^6}. \]  \hfill (4.10)

From (2.6), (4.9), (4.10) the following inequality holds for \( h(x) \):
\[ h(x) \leq 8x^3 S(x) T(x) - (8x^3 + 4x^2 + x) + \delta_1, \]  \hfill (4.11)
where
\[ 0 \leq \delta_1 = 8x^3 \delta(x) \leq \frac{21}{6250x^3}. \]

Taking into consideration that in (4.11)
\[
8x^3 S(x) T(x) - (8x^3 + 4x^2 + x)
\]
\[ = \left(8x^3 + 4x^2 + x + \frac{1}{6} + \frac{1}{48x} + \frac{1}{480x^2}\right) \left(1 - \frac{1}{60x^3} + \frac{1}{210x^5}\right) - (8x^3 + 4x^2 + x)
\]
\[ = \frac{1}{6} + \frac{1}{48x} + \frac{1}{480x^2} - \frac{8}{60} - \frac{4}{60x} - \frac{1}{60x^2} - \frac{1}{360x^3} - \frac{1}{2880x^4}
\]
\[ - \frac{1}{28800x^5} + \frac{4}{105x^2} + \frac{2}{105x^3} + \frac{1}{210x^4} + \frac{1}{1260x^5} + \frac{1}{10080x^6} + \frac{1}{100800x^7}
\]
\[ = \frac{1}{30} - \frac{11}{240x} + \frac{79}{3360x^2} + \delta_2(x),
\]
where
\[ 0 \leq \delta_2(x) \leq \frac{41}{2520x^3} + \frac{89}{20160x^4} + \frac{17}{22400x^5} + \frac{1}{10080x^6} + \frac{1}{100800x^7} \leq \frac{1}{46x^3}, \]
we find for \( h(x) \) the following upper bound
\[ h(x) \leq \frac{1}{30} - \frac{11}{240x} + \frac{79}{3360x^2} + \frac{21}{6250x^3} + \frac{1}{46x^3} \leq \frac{1}{30} - \frac{11}{240x} + \frac{79}{3360x^2} + \frac{251}{10000x^3}. \]

Rewriting the last inequality in the form
\[ h(x) \leq \frac{1}{30} - \left(\frac{11}{240x} - \frac{79}{3360x^2} - \frac{251}{10000x^3}\right), \]  \hfill (4.12)
it is easy to test that for $x \geq 1.04$ the value of the expression in the parentheses of the right part of (4.12) is a positive number. Consequently,

$$h(x) \leq \frac{1}{30}$$

for $x \geq 1.04$; and besides, from (4.6) and (4.12) we have

$$h(x) \to \frac{1}{30}$$

for $x \to +\infty$. From this and from (4.7) we obtain the assertion of Lemma 1.

5. The new asymptotic formula for the Euler gamma function

Now, we show that for $x \to \infty$ one can represent the function $h(x)$ in the form

$$h(x) = \frac{1}{30} + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n} + A_{n+1}(x),$$

(5.1)

where

$$A_{n+1}(x) = O\left(\frac{1}{x^{n+1}}\right),$$

and point out the algorithm for the calculation of the expansion coefficients $a_n$ for any natural number $n$.

It is readily seen from (4.6), (4.12), (5.1) that

$$a_1 = -\frac{11}{240} = 27(B_2)^4 + 12B_2B_4.$$  

The second coefficient can be easily found by the same procedure

$$a_2 = \frac{79}{3360} = 8 \left(\frac{(3B_2)^5}{5!} + \frac{(3B_2)^2B_4}{4} + \frac{B_6}{5}\right).$$

To obtain the formula of the $\mu$th coefficient of expansion (5.1) we insert (3.5) into (2.6). We have

$$h(x) = 8x^3 e^6 \sum_{k=1}^{n-1} \frac{B_{2k}}{2k(2k-1)x^{2k-1}} + \frac{1}{2!} \left(\frac{6B_{2k}}{2k(2k-1)x^{2k-1}}\right)^2 + \cdots - (8x^3 + 4x^2 + x).$$

(5.2)

Or

$$h(x) = e^{6R_n(x)} 8x^3 \sum_{k=1}^{n-1} \left(1 + \frac{6B_{2k}}{2k(2k-1)x^{2k-1}} + \frac{1}{2!} \left(\frac{6B_{2k}}{2k(2k-1)x^{2k-1}}\right)^2 + \cdots \right) - (8x^3 + 4x^2 + x)$$

$$= e^{6R_n(x)} 8x^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} \frac{(3B_2/(1 \ast 1))^j_1(3B_4/(2 \ast 3))^j_2(3B_6/(3 \ast 5))^j_3 \cdots (3B_{2n-2}/((n-1)(2n-3)))^j_{n-1}}{j_1!j_2!j_3! \cdots j_{n-1}!}$$

$$\times x^{-(j_1+3j_2+\cdots+(2n-3)j_{n-1})} - (8x^3 + 4x^2 + x).$$

(5.3)
Equating the coefficients by the equal powers of \( x \) in (5.1) and (5.3), we obtain the coefficient by \( x^{-\mu}; \mu = 1, 2, 3, \ldots, n \),

\[
a_\mu = 8 \sum_{j_1 + 2j_2 + \cdots + (2k-1)j_k = \mu+3, \atop 2 \leq 2k \leq \mu+4} \frac{(3B_2)^{j_1}}{j_1!} \frac{(3B_4/(2 \ast 3))^{j_2}}{j_2!} \frac{(3B_6/(3 \ast 5))^{j_3}}{j_3!} \cdots \frac{(3B_{2k}/(2k-1))^{j_k}}{j_k!},
\]

(5.4)

where \( B_2, B_4, \ldots, B_{2k} \) are the Bernoulli coefficients.

We notice that representation (5.1) holds for \( x \to \infty \). It follows from (1.2), (2.1), (4.13), (5.1), (5.4) that we have found the new asymptotic representation of the Euler gamma function:

\[
\Gamma(x+1) = \sqrt{\pi} \left( \frac{x}{e} \right)^x \left( 8x^3 + 4x^2 + x + \frac{1}{30} - \frac{11}{240x} + \frac{79}{3360x^2} \right.
\]

\[
+ \frac{3539}{201600x^3} - \frac{9511}{403200x^4} - \frac{10051}{716800x^5} + \frac{4747887}{1277337600x^6}
\]

\[
+ \frac{a_7}{x^7} + \cdots + \frac{a_n}{x^n} + A_{n+1}(x) \right)^{1/6},
\]

(5.5)

where \( A_{n+1}(x) = O(1/x^{n+1}) \).

The obtained below uniform by \( x \) and by \( n \) estimate of the value of the remainder \( A_n(x) \) permits to use (5.5) for the calculation of the Euler gamma function for concrete values \( x \) and \( n \).

### 6. The uniform estimate of the remainder

We shall estimate uniformly by \( x \) and \( n \) the value of \( A_{n+1}(x) \). From (2.6) and (5.1) we have

\[
h(x) = 8x^3e^{6h(x)} - (8x^3 + 4x^2 + x) = \frac{1}{30} + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n} + A_{n+1}(x),
\]

(6.1)

where \( A_{n+1}(x) = O(1/x^{n+1}) \). We use formula (5.2). We shall assume that \( x \geq 2n \geq 10 \). Since [7, p. 5]

\[
B_{2n} = (-1)^{n-1} \frac{2(2n)!}{(2\pi)^{2n}} \zeta(2n);
\]

(2n)! < \( n^{2n} \) and \( 1 < \zeta(2n) < 2 \), then

\[
|B_{2n}| \leq 4 \left( \frac{n}{2\pi} \right)^{2n}.
\]

(6.2)

From this and from (3.5) we obtain

\[
6|R_n(x)| \leq \frac{3|B_{2n}|}{n(2n-1)x^{2n-1}} \leq \frac{1}{2},
\]

(6.3)

\[
e^{6R_n(x)} = 1 + 6R_n(x) + \frac{1}{2!} (6R_n(x))^2 + \cdots = 1 + 12\theta R_n(x),
\]

(6.4)
where $0 < \theta < 1$. Since from (3.6), (3.7) for $x \geq 1$, $e^{6/J(x)} < e^{1/(2x)} \leq e^{1/2}$, then
\[
\prod_{k=1}^{n-1} e^{3B_{2k}/(k(2k-1)x^{2k-1})} = e^{6/J(x) - 6R_n(x)} \leq e^{1/2 + 1/2} = e. \tag{6.5}
\]
From (6.5) and (6.6) we find that
\[
e^{6/J(x)} = \prod_{k=1}^{n-1} e^{3B_{2k}/(k(2k-1)x^{2k-1})} (1 + 12\theta R_n(x)) = \prod_{k=1}^{n-1} e^{3B_{2k}/[k(k-1)x^{2k-1}]} + 12\theta n eR_n(x),
\]
where $|\theta| < 1$. From this and from (6.1) we have
\[
8x^3 \prod_{k=1}^{n-1} e^{3B_{2k}/(k(2k-1)x^{2k-1})} + 96\theta n e^3 R_n(x) - (8x^3 + 4x^2 + x)
\]
\[= \frac{1}{30} + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n} + A_{n+1}(x). \tag{6.6}
\]
It is easy to see from (6.3) that for $n \geq 5$,
\[
|x^3 R_n(x)| \leq \frac{|B_{2n}|}{2n(2n - 1)x^{2n-4}} = O \left( \frac{1}{x^{n+1}} \right). \tag{6.7}
\]
Let
\[
\prod_{k=1}^{n-1} e^{3B_{2k}/(k(2k-1)x^{2k-1})} = A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \cdots + \sum_{j=0}^{\infty} A_j x^j.
\]
Then
\[
8x^3 \prod_{k=1}^{n-1} e^{3B_{2k}/(k(2k-1)x^{2k-1})} = 8A_0 x^3 + 8A_1 x^2 + 8A_2 x + \sum_{j=0}^{n} \frac{8A_{j+3}}{x^j} + A^*_n(x), \tag{6.8}
\]
where
\[
A^*_n(x) = \sum_{j=n+1}^{\infty} \frac{8A_{j+3}}{x^j}.
\]
It follows from (6.6)–(6.8) that
\[
8A_3 = \frac{1}{30}; \quad a_j = 8A_{j+3}; \quad j = 1, 2, \ldots, n;
\]
and
\[
|A_{n+1}(x)| \leq |A^*_n(x)| + 96e^3 |R_n(x)| \leq |A^*_n(x)| + 96e \frac{|B_{2n}|}{2n(2n - 1)x^{2n-7}}. \tag{6.9}
\]
We shall estimate the value of $A^*_n(x)$,
\[
|A^*_n(x)| \leq \sum_{j=n+1}^{\infty} \frac{|8A_{j+3}|}{x^j}. \tag{6.10}
\]
Using (6.8), first we estimate $|A_j|$. We denote
\[
b_k = \frac{3B_{2k}}{k(2k - 1)} \tag{6.11}
\]
and change the variable in (6.8), setting $y = 1/x$. We get
\[ f(y) = \prod_{k=1}^{n-1} e^{b_k y^{2k-1}} = e^{b_1 y + b_2 y^3 + \cdots + b_{n-1} y^{2n-3}} = A_0 + A_1 y + A_2 y^2 + \cdots. \] (6.12)

Thus $A_j, j = 1, 2, \ldots, n$; are the coefficients of the Taylor series for the function $f(y)$. Therefore [7, pp. 133–136] for every $a > 0$ we have
\[ A_j = \frac{1}{2\pi i} \int_{|s|=a} \frac{f(s)}{s^{j+1}} \, ds. \]

From this
\[ |A_j| \leq \frac{1}{2\pi} \int_0^{2\pi} a|f(ae^{i\phi})| \, |d\phi| \leq \max_{0 \leq \phi \leq 2\pi} |f(ae^{i\phi})| a^{-j}. \] (6.13)

On the other hand, from (6.12)
\[ |f(ae^{i\phi})| \leq e^{|b_1| |a| + |b_2| |a|^3 + \cdots + |b_{n-1}| |a|^{2n-3}} = e^{\sum_{k=1}^{n-1} |b_k| a^{2k-1}}. \] (6.14)

We estimate the value $\sum_{k=1}^{n-1} |b_k| a^{2k-1}$. From (6.2) and (6.11) we have
\[ |b_k| < \frac{12 k^2}{k(2k-1)(2\pi)^{2k}}. \] (6.15)

Hence
\[ \sum_{k=1}^{n-1} |b_k| a^{2k-1} \leq \frac{6}{\pi} \sum_{k=1}^{n-1} \left( \frac{ka}{2\pi} \right)^{2k-1} \frac{1}{2k-1}. \] (6.16)

Let $a = 1/n$. We have for $k \leq n-1$,
\[ \frac{ka}{2\pi} \leq \frac{n-1}{2\pi n} < \frac{1}{2\pi}. \]

From this and from (6.16)
\[ \sum_{k=1}^{n-1} |b_k| a^{2k-1} \leq \frac{6}{\pi} \sum_{k=1}^{n-1} \left( \frac{1}{2\pi} \right)^{2k-1} \frac{1}{2k-1} < 1. \]

Thus, the following inequality holds in (6.14) with $a = 1/n$:
\[ |f(ae^{i\phi})| \leq e. \]

This inequality and (6.13) yield
\[ |A_j| \leq en^j. \] (6.17)

Since by assumption $x \geq 2n$, from (6.10) and (6.17) we obtain the following estimate for $A_{n+1}^*(x)$,
\[ |A_{n+1}^*(x)| \leq \sum_{j=n+1}^{\infty} \frac{|8A_{j+3}|}{x^j} < 8e \sum_{j=n+1}^{\infty} \left( \frac{n}{x} \right)^j n^3 \left( \frac{n}{x} \right)^{n+1} \frac{1}{1 - n/x} \leq 16e \frac{n^{n+4}}{x^{n+1}}. \]

From this and from (6.2) and (6.9) we find the estimate
\[ |A_{n+1}(x)| \leq 16e \frac{n^{n+4}}{x^{n+1}} + 96e \frac{|B_{2n}|}{2n(2n-1)x^{2n-7}} \leq 16e \frac{n^{n+4}}{x^{n+1}} + 192e \frac{n^{2n-1}}{(2\pi)^2n(2n-1)x^{2n-7}}. \]
Thus we have proved the statement: the following estimate for the remainder \( \Delta_{n+1}(x) \) in expansion (5.1) and (5.5) with coefficients (5.4) holds when \( x \geq 2n \),

\[
|\Delta_{n+1}(x)| \leq 16e\pi^{n+4}x^{-(n+1)} + \frac{192e}{n(2n-1)} \left( \frac{n}{2\pi} \right)^{2n} x^{-(2n-7)}.
\]

7. The monotonicity of the function \( h(x) \)

We differentiate (1.1) and (1.2):

\[
h'(x) = 6g'(x)g^5(x) - (24x^2 + 8x + 1),
\]

\[
g'(x) = g(x) \left( \frac{1}{x} + \psi(x) - \log x \right).
\]

Differentiating Stirling’s formula (2.2) and (2.3), we get

\[
\psi(x) = \log x - \frac{1}{2x} + J'(x),
\]

where

\[
J'(x) = -2 \int_0^\infty \frac{\sigma(u) \, du}{(x+u)^3}.
\]

Hence one can rewrite (7.1) and (7.2) in the form

\[
g'(x) = g(x) \left( \frac{1}{2x} + J'(x) \right),
\]

\[
h'(x) = 48x^3 e^{6/(x)} \left( \frac{1}{2x} + J'(x) \right) - (24x^2 + 8x + 1).
\]

To prove the statement of Lemma 2, one needs to prove that with \( x \geq x_1 \), \( h'(x) > 0 \), or that in (7.4)

\[
\frac{1}{2x} + J'(x) > \left( \frac{1}{2x} + \frac{1}{6x^2} + \frac{1}{48x^3} \right) e^{-6/(x)}.
\]

From (3.7), (3.8)

\[
-6J(x) = -\frac{1}{2} + \frac{1}{60x^3} - \frac{\eta_1}{210x^5},
\]

where \( 0 \leq \eta_1 \leq 1 \). Inserting (7.6) into (7.5), we have

\[
\frac{1}{2x} + J'(x) > \left( \frac{1}{2x} + \frac{1}{6x^2} + \frac{1}{48x^3} \right) e^{-1/(2x)} e^{1/(60x^2)} e^{-\eta_1/(210x^5)},
\]

\( 0 \leq \eta_1 \leq 1 \).

To prove inequality (7.7), we will simplify it in the following way. We contribute the right part of this inequality, by setting \( \eta_1 = 0 \), and obtain a new inequality, from the validity of which follows the validity of (7.7):

\[
e^{-1/(60x^2)} \left( \frac{1}{2x} + J'(x) \right) > \left( \frac{1}{2x} + \frac{1}{6x^2} + \frac{1}{48x^3} \right) e^{-1/(2x)}.
\]
We substitute Taylor’s series for the exponential functions in the obtained inequality. We diminish the left and add the right part of the last inequality, using the relations:

\[ 1 - \beta \leq e^{-\beta} \leq 1 - \beta + \frac{\beta^2}{2!} - \frac{\beta^3}{3!} + \frac{\beta^4}{4!} \]

As a result, we get the following inequality, the validity of which gives the validity of (7.7):

\[ \left( 1 - \frac{1}{60x^3} \right) \left( \frac{1}{2x} + J'(x) \right) > \left( \frac{1}{2x} + \frac{1}{6x^2} + \frac{1}{48x^3} \right) \left( 1 - \frac{1}{2x} + \frac{1}{8x^2} - \frac{1}{48x^3} + \frac{1}{384x^4} \right). \]  

(7.8)

Now, we represent the integral \( J'(x) \) from (7.3) as the asymptotic expansion in powers of \( 1/x \) (this process is detailed in [4])

\[ J'(x) = -\frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \frac{\eta_2}{120x^8}, \]  

(7.9)

\( 0 \leq \eta_2 \leq 1. \) It is apparent that

\[ J'(x) > -\frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6}. \]

With the help of the last inequality we diminish the left part of (7.8) and we obtain the following inequality, the validity of which gives the validity of (7.8):

\[ \left( 1 - \frac{1}{60x^3} \right) \left( \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} \right) \]

\[ > \left( \frac{1}{2x} + \frac{1}{6x^2} + \frac{1}{48x^3} \right) \left( 1 - \frac{1}{2x} + \frac{1}{8x^2} - \frac{1}{48x^3} + \frac{1}{384x^4} \right). \]  

(7.10)

Multiplying the expression in the parentheses in (7.10), we have

\[ -\frac{1}{252x^6} + \frac{1}{720x^5} - \frac{1}{7200x^3} + \frac{1}{15120x^9} > \frac{1}{2304x^5} + \frac{1}{18432x^7}. \]

or

\[ \frac{11}{11520} > \frac{1}{252x} + \frac{89}{460800x^2} - \frac{1}{15120x^4}. \]

The last inequality holds when \( x \geq 4.21. \) Therefore, with \( x \geq x_1 = 4.21 \) also inequality (7.5) holds. From this and from (7.4) we draw the conclusion that for \( x \geq x_1 = 4.21, \) the function \( h(x) \) is monotonically increasing. This completes the proof of Lemma 2.

**8. The function \( h(x) \) behavior on the interval \( 1 < x < 4.21 \)**

Since from (7.1) and (7.2)

\[ h'(1) = 6 \frac{e^6}{\pi^3} (1 - \gamma) - 33 = 0.00558319 \ldots > 0, \]  

(8.1)

then from the above it follows that to prove Lemma 3, it will suffice to prove that

\[ h'(x) > 0 \]

for any \( x, 1 < x \leq \max(x_0, x_1) = x_1 = 4.21. \)
By the Lagrange formula [8, p. 138] for \( h'(x) \) with \( d > 0 \),
\[
h'(x + d) - h'(x) = dh''(x + \vartheta d)
\] (8.2)
\( 0 \leq \vartheta \leq 1 \). Since from (8.1), \( h'(1) > 0 \) and
\[
h'(1 + d) = h'(1) + dh''(1 + \vartheta d) \geq h'(1) - d|h''(1 + \vartheta d)|,
\]
then to prove that \( h'(x) > 0 \) at the point \( x = \bar{x} = 1 + d \), it will suffice to prove that
\[
h'(1) - d|h''(1 + \vartheta d)| > 0, \quad 0 \leq \vartheta \leq 1.
\] (8.3)

We estimate the value \( h''(x) \). From (7.4) we have
\[
h''(x) = 48x^3e^{6J(x)}\left(\frac{1}{x^2} + \frac{6}{x}J'(x) + 6(J'(x))^2 + J''(x)\right) - (48x + 8),
\]
or
\[
\frac{h''(x)e^{-6J(x)}}{48x^3} = \frac{1}{x^2} + \frac{6}{x}J'(x) + 6(J'(x))^2 + J''(x) - \left(\frac{1}{x^2} + \frac{1}{6x^3}\right)e^{-6J(x)}.
\] (8.4)

It follows from (7.9) that
\[
\frac{6}{x}J'(x) = -\frac{1}{2x^3} + \frac{\vartheta_1}{20x^5},
\] (8.5)
\[
6(J'(x))^2 = \frac{1}{24x^4} - \frac{\vartheta_1}{120x^6} + \frac{\vartheta_1^2}{2400x^8},
\] (8.6)
where \( 0 \leq \vartheta_1 \leq 1.01 \).

To estimate the value of \( J''(x) \), we write out for it the asymptotic representation in powers of \( 1/x \), just as it was made earlier for \( J'(x) \). We differentiate (7.3):
\[
J''(x) = 6 \int_0^\infty \frac{\sigma(u) \, du}{(x + u)^4}.
\] (8.7)

We substitute the Fourier series for the function \( \sigma(u) \) from (3.4) into (8.7) and take the obtained integral by parts. We have
\[
J''(x) = 6 \int_0^\infty \frac{1}{(x + u)^4} \left(\frac{1}{12} - \sum_{n=1}^\infty \frac{\cos 2\pi nu}{2\pi^2 n^2}\right) \, du
\]
\[
= \frac{1}{2} \int_0^\infty \frac{du}{(u + x)^4} - \frac{3}{\pi^2} \sum_{n=1}^\infty \frac{1}{n^2} \int_0^\infty \frac{\cos 2\pi nu \, du}{(u + x)^4}
\]
\[
= \frac{1}{6x^3} - \frac{3}{\pi^2} \sum_{n=1}^\infty \frac{1}{n^2} \left(\frac{1}{\pi^2 n^2 x^2} - \frac{30}{2\pi^3 n^3} \int_0^\infty \frac{\sin 2\pi nu \, du}{(u + x)^7}\right).
\] (8.8)

Since
\[
0 < \int_0^\infty \frac{\sin 2\pi nu \, du}{(u + x)^7} = \frac{1}{2\pi n} \int_0^\infty \frac{\sin u \, dv}{(u + v/(2\pi))^7} < \frac{1}{2\pi n} \int_0^\pi \frac{\sin u \, dv}{(u + v/(2\pi))^7} \leq \frac{1}{2\pi n x^7},
\]
then we obtain from (8.8):

\[ J''(x) = \frac{1}{6x^3} - \frac{3}{\pi^4 x^5} \sum_{n=1}^{\infty} \frac{1}{n^4} + \frac{45 \vartheta_2}{\pi^6 x^7} \sum_{n=1}^{\infty} \frac{1}{n^6}, \]  

(8.9)

where \( 0 \leq \vartheta_2 \leq 1 \). Taking into account that

\[ \sum_{n=1}^{\infty} \frac{1}{n^4} = \zeta(4) = \frac{\pi^4}{90}; \quad \sum_{n=1}^{\infty} \frac{1}{n^6} = \zeta(6) = \frac{\pi^6}{45 \times 21}, \]

we find from (8.9)

\[ J''(x) = \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{\vartheta_3}{21x^7}; \quad 0 \leq \vartheta_3 \leq 1, \]

or

\[ J''(x) = \frac{\vartheta_4}{6x^3}; \quad 0 \leq \vartheta_4 \leq 1. \]  

(8.10)

To estimate the value of \( e^{-\frac{6J(x)}{x}} \), we use (7.6):

\[ e^{-\frac{6J(x)}{x}} \leq e^{-1/(2x) + 1/(60x^3)} \]

\[ = \left(1 - \frac{1}{2x} + \frac{1}{8x^2} - \frac{1}{48x^3} + \cdots\right) \left(1 + \frac{1}{60x^3} + \frac{1}{7200x^6} + \cdots\right) \]

\[ = 1 - \frac{1}{2x} + \frac{1}{8x^2} - \frac{1}{48x^3} + \frac{\vartheta_5}{384x^5} \left(1 + \frac{1}{60x^3} + \frac{1}{7200x^6} + \cdots\right) \]

\[ = 1 - \frac{1}{2x} + \frac{1}{8x^2} - \frac{1}{48x^3} + \frac{\vartheta_6}{384x^5}, \]

where \( 0 \leq \vartheta_5 \leq 1, 0 \leq \vartheta_6 \leq 1.02 \). From the last equality, it follows that

\[ e^{-\frac{6J(x)}{x}} \left(\frac{1}{x^2} + \frac{1}{6x^3}\right) = \frac{1}{x^2} - \frac{1}{2x^3} + \frac{1}{6x^4} - \frac{1}{12x^5} + \frac{1}{2x^6} - \frac{1}{6x^7} + \frac{\vartheta_6}{384x^7} + \frac{\vartheta_6}{6 \times 384x^8}. \]  

(8.11)

From (8.4)–(8.6), (8.10), (8.11) we get

\[ h''(x) e^{-\frac{6J(x)}{x}} = \frac{1}{48x^3} \left[ \frac{1}{x^2} - \frac{1}{2x^3} + \frac{\vartheta_1}{20x^5} + \frac{1}{24x^4} - \frac{\vartheta_1}{120x^6} \right] + \frac{\vartheta_2}{2400x^8} + \frac{\vartheta_4}{6x^3} - \frac{1}{x^2} - \frac{1}{6x^3} + \frac{1}{2x^4} + \frac{1}{12x^5} + \frac{1}{8x^4} + \frac{1}{48 \times 6x^6} - \frac{\vartheta_6}{384x^7} - \frac{\vartheta_6}{6 \times 384x^8} \]

\[ = -\frac{1}{6x^3} + \frac{\vartheta_1}{20x^5} + \frac{1}{288x^6} - \frac{\vartheta_1}{120x^6} + \frac{\vartheta_2}{2400x^8} + \frac{\vartheta_4}{6x^3} - \frac{\vartheta_6}{384x^7} - \frac{\vartheta_6}{6 \times 384x^8}, \]

\( 0 \leq \vartheta_1 \leq 1.01, \quad 0 \leq \vartheta_4 \leq 1, \quad 0 \leq \vartheta_6 \leq 1.02. \)
From this and from (3.7) and (3.8)
\[ |h''(x)| \leq e^{6J(x)} \left( \frac{1}{6} + \frac{1.01}{120x^3} + \frac{1.01}{384x^4} + \frac{1.01}{2304x^5} \right) \leq 9e^{6J(x)} \leq 9\sqrt{e} \leq 15. \] (8.12)

We consider the segment 1 \( \leq x \leq 4.3. \) Using (7.1) and (7.2), we calculate \( h'(4.3). \) We obtain (the program “Maple V Release 5” was used)
\[ h'(4.3) = 0.0017968 \ldots . \] (8.13)

We see from (8.1), (8.12) and (8.13) that the rough estimate for the \( h''(x) \) and the proximity to zero of the \( h'(x) \) permit the use of the Lagrange formula (8.2) to check inequality (8.3) only with very small step.

Let 1 \( \leq x \leq 4.3. \) Taking into account (8.12) and (8.13), we choose step \( d, \)
\[ d = 0.0001 \leq \frac{h'(4.3)}{15} = 0.00011978 \ldots \leq \frac{h'(4.3)}{|h''(x)|}. \]

According to (8.2)
\[ h'(1 + jd) = h'(1 + (j - 1)d) + dh''(1 + \vartheta d), \quad 0 \leq \vartheta \leq 1, \quad j = 1, 2, 3, \ldots, 3.3 \frac{3}{d}. \]

Since
\[ h'(1 + jd) \geq h'(1 + (j - 1)d) - d|h''(1 + \vartheta d)| \geq h'(1 + (j - 1)d) - 0.0001 \times 15, \]
then the proof that
\[ h'(1 + (j - 1)d) - 0.0001 \times 15 > 0, \] (8.14)
for all \( j = 1, 2, 3, \ldots, 33000, \) establishes that \( h'(x) > 0, \) when 1 \( \leq x \leq 4.3. \)

Inequality (8.14) is proved by the computation by (7.1) and (7.2) the values
\[ h'(1 + (j - 1)d), \quad j = 1, 2, 3, \ldots, 33000, \]
and check of inequality (8.14). The “Maple V Release 5” calculated \( 33000 + 1 \) values of the function \( h'(x) - 0.0015, \) by \( x \) from 1 to 4.3 by step 0.0001. We present here the first ten and the last eleven of these values. Let \( f_j = 10^3(h'(1 + (j - 1)d) - 0.0015). \) We have
\[ f_1 = 4.08319, \quad f_2 = 4.08392, \quad f_3 = 4.08440, \quad f_4 = 4.08508, \quad f_5 = 4.08581, \]
\[ f_6 = 4.08638, \quad f_7 = 4.08707, \quad f_8 = 4.08773, \quad f_9 = 4.08829, \quad f_{10} = 4.08910, \ldots \]
\[ f_{32991} = 0.2963, \quad f_{32992} = 0.3045, \quad f_{32993} = 0.2968, \quad f_{32994} = 0.2954, \quad f_{32995} = 0.2956, \]
\[ f_{32996} = 0.2938, \quad f_{32997} = 0.3015, \quad f_{32998} = 0.3024, \quad f_{32999} = 0.2999, \quad f_{33000} = 0.2960, \]
\[ f_{33001} = 0.2968. \]

All these 33001 values are positive. According to the Lagrange formula it means that \( h'(x) > 0 \) also between these points. This proves Lemma 3.
9. Conclusion

The proof of Theorem 1 follows immediately from the proofs of Lemmas 1–3.

It should be noted that applying some program of symbolic computations (“Maple”, “Mathematika”) one can calculate a lot of new coefficients of asymptotic representation (5.5) by Ramanujan for the Euler gamma function.

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References