Skew Group Algebras in the Representation Theory of Artin Algebras

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INTRODUCTION

Let $A$ be a ring, $G$ a finite group acting on $A$ and $\gamma: G \times G \to U(A)$, the units of $A$, a map satisfying

1. $\gamma(g, g') \gamma(gg', g'') = g(\gamma(g', g'')) \gamma(g, g'g'')$ for $g, g', g''$ in $G$,
2. $\gamma(e, g) = 1 = \gamma(g, e)$ for $g \in G$, $e$ the identity element of $G$,
3. $\gamma(g, g')(gg')(\lambda) = g(\gamma(g', \lambda)) \gamma(g, g')$ for $g, g' \in G$.

Then the corresponding crossed product algebra $A \rtimes_\gamma G$, or $A \rtimes G$ for short, has elements $\sum_{g_i \in G} \lambda_i g_i; \lambda_i \in A$. Addition is componentwise, and multiplication is given by $\bar{g}\lambda = g(\lambda)\bar{g}$ and $\bar{g}_1\bar{g}_2 = \gamma(g_1, g_2)\bar{g}_1\bar{g}_2$. In this paper we assume that the values of $\gamma$ lie in the center $Z(A)$ of $A$. Hence the action of $G$ on $A$ is given by a group homomorphism $G \to \text{Aut}(A)$, and (3) can be left out. In the special case that $\gamma$ is the trivial map we write $AG$ instead of $A \rtimes G$, and the elements as $\sum_{g_i \in G} \lambda_i g_i$. $AG$ is then called a skew group ring.

There is a lot of literature on skew group algebras and crossed product algebras, and on the relationship with the ring $A^G$ whose elements are those elements of $A$ left fixed by $G$. Much work has been done on which properties of $A$ are inherited by $A \rtimes G$ or $A^G$. Some of the work on the relationship between these rings has its roots in trying to develop a Galois theory for noncommutative rings. We refer to [3, 7, 13–15, 19, 21, 23–25, 27, 28] and their references.

In this paper we study these constructions when $A$ is an artin algebra and $G$ a finite group of order $n$ such that $n$ is invertible in $A$. Under these assumptions the construction preserves central properties of interest in the

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representation theory of artin algebras, like finite representation type, being an Auslander algebra, etc.

General results of this type are interesting for the following reasons. They give a method for proving that an algebra $\Gamma$ has a given property if we already know that $\Lambda$ does and that $\Gamma$ can be constructed from $\Lambda$ in the above way. Also some insight is provided why some algebras which seem unrelated have a lot of common properties. For example, some hereditary algebras of type $A_{2n-3}$ and some of type $D_n$ turn out to be related this way, and also other hereditary algebras and some selfinjective algebras of type $A_{2n-3}$ and some of type $D_n$. In this connection it is also of interest to observe that various properties are not preserved by this construction, so that we deal with essentially different algebras.

Since the algebras $\Lambda$ and $\Lambda G$ have a lot of properties in common, it is of interest to find ways of describing $\Lambda G$ in terms of $\Lambda$. If $\Lambda$ is a basic algebra over an algebraically closed field $k$ and $G$ is a cyclic group, we describe how to construct the quiver of the basic algebra Morita equivalent to $\Lambda G$ in terms of the quiver for $\Lambda$. We illustrate this through several examples. It is also often possible to keep track of the relations on the quiver determining the algebra.

Many of the properties from representation theory which we study are defined in terms of almost split sequences. It is therefore essential to show that these sequences behave nicely under the crossed product construction. To do this, it is convenient, in view of the functorial description of such sequences, to generalize the crossed product construction to groups acting on categories, in particular on categories of type f.p. $((\text{mod} \Lambda)^{op}, \text{Ab})$, also denoted mod(mod $\Lambda$), of finitely presented contravariant functors from mod $\Lambda$ to abelian groups. With this application in mind, we generalize some results from rings to the categorical setting.

Our categorical description is also useful for establishing a connection with the important theory of coverings. Namely, for finite groups the Galois coverings correspond in almost all characteristics of the field $k$ to the skew group algebra construction.

It is of interest to know to which extent the operation of going from $\Lambda$ to $\Lambda \rtimes G$ can be reversed with a similar construction. From the point of view of knowing that properties are preserved by this construction, it is sufficient to decide whether we can get back in a finite number of steps. We prove that if $\Lambda$ is an algebra over an algebraically closed field and $G$ is a solvable group, then we can get from $\Lambda G$ to $\Lambda$ in a finite number of steps by skew group algebra constructions, changing, if necessary, the rings up to Morita equivalence. In [26] there are more general results along these lines.

Most of the results about skew group algebras in this chapter are also valid for crossed product algebras.

If the 2-cocycles $\gamma$ and $\gamma'$ in $H^2(G, U(R))$ differ by a coboundary, then the
resulting crossed product algebras are isomorphic, but we give examples to show that the converse is not necessarily true. We give examples of crossed product algebras \( A * G \) which are not skew group algebras of \( A \) by any group, but we do not know if we get essentially new classes of algebras this way.

Also when only interested in studying skew group algebras, the crossed product algebras come up naturally in connection with the following problem. If every element of \( G \) acts as an inner automorphism, is then \( AG \) isomorphic to the ordinary group algebra \( A(G) \)? We prove that this is the case when \( G \) is cyclic, but not in general.

We now describe the content of the paper section by section. In Section 1 we give some basic properties shared by \( A \) and \( AG \) or \( A * G \). Section 2 is devoted to discussing examples, in particular giving methods for constructing \( AG \) from \( A \). We here establish the connection between hereditary algebras of Dynkin class \( A_{2n-3} \) and \( D_n \), and the corresponding connection between selfinjective algebras of finite type.

In Section 3 we give our generalization to the categorical setting and prove that almost split sequences go to direct sums of almost split sequences. In Section 4 we apply these results to showing that properties like being of finite type with no oriented cycles of irreducible maps or being a tilted algebra of finite type are preserved, and that preprojective partitions correspond.

In Section 5 we prove the result about getting back for solvable groups and give the connection between skew group algebras and Galois coverings. We study the case when all elements of \( G \) act as inner automorphisms, both for algebras and categories.

We will denote the category of finitely generated \( A \)-modules by \( \text{mod } A \), and \( \text{ind } A \) will denote the isomorphism classes of indecomposable objects in \( \text{mod } A \). \( \text{Hom}_A(A,B) \) will often be shortened to \( (A,B) \). We write \( A | B \) for \( A \) being a summand of \( B \). \(|G|\) will denote the order of \( G \), and \( G = g^{Z/nZ} \) will often denote the cyclic group of order \( n \) with generator \( g \). Instead of \( 1_e \) we write the identity of \( AG \) as \( 1 \).

We assume that the reader is familiar with the theory of almost split sequences and irreducible maps (see [29]).

1. Basic Results

Let \( A \) be an artin algebra, \( G \) a finite group whose order is invertible in \( A \) and \( G \rightarrow \text{Aut } A \) a group homomorphism. In this section we prove some basic properties, some of which are known, which hold for \( A \) if and only if they hold for the skew group algebra \( AG \), and which are closely related to the types of questions studied in representation theory.

Before we go on we point out some easy examples of skew group rings
AG. The ordinary group rings are such examples, and if \( G = H \ltimes N \) is a semidirect product and \( k \) is a field, then \( kG \simeq (kN)H \), where the action of \( H \) on \( kN \) is induced by the conjugation action of \( H \) on \( N \). If \( A = k \times \cdots \times k \), a product of \( n \) copies of the field \( k \), \( G = \langle g \rangle \) cyclic of order \( n \), with the action of \( G \) on \( A \) given by \( g(r_1, \ldots, r_n) = (r_n, r_1, \ldots, r_{n-1}) \), then \( AG = M_n(k) \), the full \( n \times n \) matrix ring over \( k \).

To get a better understanding for which properties of the skew group algebra construction are being used, and to make clearer how things might be generalized, we consider an arbitrary ring monomorphism \( i: A \to \Gamma \), where \( A \) and \( \Gamma \) both are artin algebras, and we consider the induced functors \( F = \Gamma \otimes_A: \text{mod } A \to \text{mod } \Gamma \) and the restriction functor \( H: \text{mod } \Gamma \to \text{mod } A \). We consider three important properties of this setup, and show that they hold for \( i: A \to AG \) when \( i(\lambda) = \lambda e \), \( e \) the identity of \( G \).

1.1. The sets of properties we consider are the following.

(A) (i) \( A \) is a twosided summand of \( \Gamma \).

(ii) The product map \( \Gamma \otimes_A \Gamma \to \Gamma \) is a split epimorphism as twosided \( \Gamma \)-modules.

(B) \( (H, F) \) is an adjoint pair of functors.

(C) \( \Gamma r = r\Gamma = \text{rad } \Gamma \), where \( r \) denotes the radical of \( A \).

We point out that it is easy to see that \( A \) is a twosided summand of \( \Gamma \) if and only if the natural morphism \( I : \text{mod } A \to \text{mod } \Gamma \) is a split monomorphism of functions, where \( I : \text{mod } A \to \text{mod } A \) is the identity functor, and that (A)(ii) holds if and only if the natural morphism \( FH \to J \) is a split epimorphism of functions, where \( J : \text{mod } \Gamma \to \text{mod } \Gamma \) is the identity functor. \( F \) is always a left adjoint of \( H \), and (B) states that \( F \) is also a right adjoint of \( H \).

We have the following result for skew group rings, whose proof we include for the sake of completeness.

**Theorem 1.1.** If \( A \) is an artin algebra, \( G \) a finite group whose order \( n \) is invertible in \( A \) and \( i: A \to AG \) the natural ring monomorphism, then (A), (B), (C) hold.

The following lemma, whose proof is straightforward and which is essentially Maschke, is important in proving this theorem.

**Lemma 1.2.** Let \( X, Y \) be in \( \text{mod } AG \) and \( t: X \to Y \) a \( \Lambda \)-homomorphism. Then \( \tilde{t}: X \to Y \) defined by \( \tilde{t}(x) = \sum_{g \in G} g^{-1}t(gx) \) is a \( AG \)-homomorphism.

**Proof of Theorem 1.1.** For \( Y \) in \( \text{mod } A \) and \( X \) in \( \text{mod } AG \) the natural maps given by \( (F, H) \) being an adjoint pair of functors are \( i_Y: Y \to AG \otimes_A Y \), where \( i_Y(y) = 1 \otimes y \) for \( y \in Y \), and \( q_X: AG \otimes_A X \to X \) given by
An element \( y \in \mathcal{A}G \otimes \Lambda \) \( X \) can be written in the form \( y = \sum_{g \in G} g^{-1} \otimes y_g \), and in a unique way since the \( g^{-1} \) are a basis for \( \mathcal{A}G \) as a right \( \Lambda \)-module. A \( \Lambda \)-splitting for \( i_y \) is given by \( p_y: \mathcal{A}G \otimes \Lambda \rightarrow Y \), where \( p_y(\sum_{g \in G} g^{-1} \otimes y_g) = y_e \). A \( \Lambda \)-splitting for \( q_x \) is given by \( 1/nx \). This proves (A)(i) and (A)(ii).

To prove (B), we want to find an isomorphism \( \text{Hom}_\Lambda(X, Y) \rightarrow \text{Hom}_{\mathcal{A}G}(X, \mathcal{A}G \otimes \Lambda Y) \), which is functorial in \( X \) and \( Y \). For \( s \in \text{Hom}_\Lambda(X, Y) \), define \( \rho(s) = \tilde{\psi}s \), and for \( f \in \text{Hom}_{\mathcal{A}G}(X, \mathcal{A}G \otimes \Lambda Y) \), define \( \psi(f) = p_yf \). We observe that if \( y = \sum_{g \in G} g^{-1} \otimes y_g \) is in \( \mathcal{A}G \otimes \Lambda Y \), then for \( h \in G \), \( p_y(hy) = p_y(\sum_{g \in G} h g^{-1} \otimes y_g) \). Hence we have \( y = \sum_{g \in G} g^{-1} \otimes p_y(gy) \). For \( s \in \text{Hom}_\Lambda(X, Y) \), we have \( \psi(s)(x) = p_y(\sum_{g \in G} g^{-1} \otimes s(gx)) = s(x) \) for \( x \) in \( X \). For \( f \in \text{Hom}_{\mathcal{A}G}(X, \mathcal{A}G \otimes \Lambda Y) \), we have \( \psi(f)(x) = \sum_{g \in G} g^{-1} \otimes p_y(gf(x)) = f(x) \) by the above formula, when \( x \in X \). This shows that \( \psi \) and \( \rho \) both are identities, so that we have our desired isomorphism, which is clearly functorial in \( X \) and \( Y \).

(C)(i) We want to show that \( r(\mathcal{A}G) = \text{rad}(\mathcal{A}G) \). \( r \) is a \( G \)-invariant submodule of \( \mathcal{A} \), so that \( G \) operates on \( A' = A/r \), which is semisimple. Then \( A'G \) is also semisimple, since by (A) every \( X \) in mod \( A'G \) is a summand of some \( A'G \otimes \Lambda \) \( X \) where \( X \) is a projective \( A' \)-module, and hence all objects in mod \( A'G \) are projective. Consider the surjection \( \phi: \mathcal{A}G \rightarrow A'G \) induced by the surjection \( \psi: \Lambda \rightarrow A' \), defined by \( \psi(\sum_{g \in G} r_g g) = \sum_{g \in G} \psi(r_g) g \). Then \( \text{Ker } \phi = \{ \sum_{g \in G} r_g g; r_g \in r \} = r(\mathcal{A}G) \). Since \( A'G \) is semisimple, \( r(\mathcal{A}G) = \text{Ker } \phi = \text{rad}(\mathcal{A}G) \). To finish the proof we want to show that \( r(\mathcal{A}G) \) is nilpotent. Since each \( g \in G \) gives an automorphism of \( r \), we have \( (A\Gamma)^r = r(\mathcal{A}G) \). By induction we then get that \( r^i(\mathcal{A}G) = (r(\mathcal{A}G))^i \), which shows that \( r(\mathcal{A}G) \) is nilpotent.

We have already seen that if \( G = H \ltimes N \), a semidirect product, then \( \mathcal{A}G = (\mathcal{A}N)H \), hence \( i: \mathcal{A}N \rightarrow \mathcal{A}G \) fits into the above setup in this case. It is also not hard to see that it can be proved similarly to the above that if \( H \) is an arbitrary subgroup of \( G \), then (A) and (B) hold for \( AH \rightarrow \mathcal{A}G \), but not necessarily (C). In particular, (C) is not a consequence of (A) or (B).

In connection with stable equivalence there occur pairs of functors \( (F, H) \) having properties closely related to (A), (B), (C). For example, if \( r^2 = 0 \) for \( A \) and \( A \) is a trivial extension \( A/r \ltimes r \), then the natural ring map \( A \rightarrow \Gamma = (A/\mathcal{A}] \otimes \Lambda/\mathcal{A}] \) induces an equivalence \( \text{mod } A \rightarrow \text{mod } \Gamma \) between the categories modulo projective [4]. It is not hard to see that (A) and (C) hold, by, for example, using the study of this ring map from [4]. Since \((0 \otimes \Lambda/\mathcal{A}] \) is a left \( \Gamma \)-summand of \( \Gamma \) which is not projective as \( \Lambda \)-module, (B) does not hold.

We shall get back to more relationships between the conditions in the next section.

1.2. We give some preliminary properties of \( i: A \rightarrow \mathcal{A}G \), which are important from the point of view of representation theory. To make clearer
what the various results depend on, and to see more easily how they might be generalized, we prove the results as consequences of the appropriate (A), (B), (C). Not to make things too difficult to follow, we do not, however, give the most general conditions possible in our statements.

**Theorem 1.3.** Let $i: A \rightarrow \Gamma$ be a ring monomorphism between two algebras $A$ and $\Gamma$.

(a) If (A) holds, then $A$ is of finite representation type if and only if $\Gamma$ is.

(b) If (B) holds, then $A$ is 1-Gorenstein if and only if $\Gamma$ is.

(c) Assume that (A) and (B) hold. Then we have the following.
   (i) $\text{gl.dim.} A = \text{gl.dim.} \Gamma$.
   (ii) $\text{dom.dim.} A = \text{dom.dim.} \Gamma$.
   (iii) $A$ is selfinjective if and only if $\Gamma$ is.
   (iv) $A$ is an Auslander algebra if and only if $\Gamma$ is.

(d) If (C) holds, we have the following.
   (i) $r^i \Gamma = \Gamma r^i = (\text{rad } \Gamma)^i$ for all $i \geq 1$.
   (ii) $r \Gamma \cap A = r$.
   (iii) If $X$ is simple in mod $A$, then $\Gamma \otimes_A X$ is semisimple in mod $\Gamma$.
   (iv) If $Y$ is simple in mod $\Gamma$, then $Y$ is semisimple in mod $A$.

(e) If (A) and (C) hold, then
   (i) $r^i I \cap A = r^i$.
   (ii) $\mathcal{L}(A) = \mathcal{L}(\Gamma)$, where $\mathcal{L}$ denotes Loewy length.
   (iii) $\mathcal{L}(X) = \mathcal{L}(\Gamma \otimes_A X)$ for $X$ in mod $A$.
   (iv) $\mathcal{L}(Y) = \mathcal{L}(\lambda Y)$ for $Y$ in mod $\Gamma$.

(f) If (B) and (C) hold, then we have for $X$ in mod $A$.
   (i) $\Gamma \otimes_A r^i X/r^{i+1} X \simeq (\text{rad } \Gamma)^i FX/(\text{rad } \Gamma)^{i+1} FX$.
   (ii) $\Gamma \otimes_A \text{soc}^i X/\text{soc}^{i+1} X \simeq \text{soc}^{i+1} FX/\text{soc}^i FX$.

(g) If (A), (B) and (C) hold, then $A$ is Nakayama if and only if $\Gamma$ is Nakayama.

**Proof:** (a) is obvious. Since $(F, H)$ and $(H, F)$ are adjoint pairs, $F$ and $H$ are both exact, and hence both preserve projectives and injectives. Let $I(A)$ denote the injective envelope of $A$. Then $F(I(A)) = \Gamma$ is a submodule of $F(I(A))$ which is projective injective if $I(A)$ is. Conversely if $I(\Gamma)$ is a projective $\Gamma$-module, $H(I(\Gamma)) = \lambda I(\Gamma)$ is a projective injective $\lambda$-module containing $A$, so that $I(A)$ is projective. Since $A$ is said to be 1-Gorenstein if $I(A)$ is projective, we have proved (b).
Since $F$ and $H$ preserve projectives, we have $\text{pd}_A X \geq \text{pd}_I (\Gamma \otimes_A X) \geq \text{pd}_A (\Gamma \otimes_A X) \geq \text{pd}_A X$ for $X$ in mod $A$, using (A)(i). Since by (A)(ii) every $Y$ in mod $I$ is a summand of some $\Gamma \otimes_A X$, we conclude that $\text{gl.dim}.A = \text{gl.dim}.I$.

We recall that the dominant dimension $\text{dom.dim}.A \geq i$ if in a minimal injective resolution $0 \to A \to I_0 \to I_1 \to \cdots \to I_i \to \cdots$, $I_j$ is projective for $j < i$. Using that $\text{A}A$ is a summand of $\text{A}I$ by (A)(i), and $F$ and $H$ preserve projectives and injectives, we get $\text{dom.dim}.A = \text{dom.dim}.I$, and similarly that $A$ is selfinjective if and only if $I$ is. Recalling that $A$ is an Auslander algebra if $\text{dom.dim}.A \geq 2$ and $\text{gl.dim}.A \leq 2$, this finishes the proof of (c).

Since $rI = \text{rad } I = I$, we have $r^2 I = rI = rI = (rI)^2 = (rI)^2$ and so on. $rI \cap A = r$ since obviously $r \subseteq rI \cap A \subseteq A$, and $rI \cap A$ is nilpotent.

Let $X$ be simple in mod $A$. Then $(rI)(\Gamma \otimes_A X) = I \otimes_A X = \Gamma \otimes_A X = 0$, so $\Gamma \otimes_A X$ is semisimple. Let $Y$ be simple in mod $I$. Then $rY \subseteq (rI \cap A)Y \subseteq (rI)Y = 0$, so $Y$ is semisimple in mod $A$, which proves (d).

(e) Since $I = A \otimes X$ in mod $A$, we have $r^j I = r^j A \otimes Y$, so that $r^j I \cap A = r^j$. It follows that $A^j(A) = A^j(A)$. If $r^j X = 0$, then $(rI)^j (\Gamma \otimes_A X) = \Gamma r^j \otimes_A X = \Gamma \otimes_A r^j X = 0$. If $Y$ is in mod $I$ and $(rI)^j Y = 0$, then $r^j Y = 0$. Using that $X$ is a summand of $\text{A}(\Gamma \otimes_A X)$ and $Y$ is a summand of $\Gamma \otimes_A Y$, the rest follows.

(f) The exact sequence $0 \to r^{j+1} X \to r^j X \to r^j X/r^{j+1} X \to 0$ gives rise to the exact sequence $0 \to \Gamma \otimes_A r^{j+1} X \to \Gamma \otimes_A r^j X \to \Gamma \otimes_A r^j X/r^{j+1} X \to 0$. Since by (d) $r^j = (\text{rad } I)^j$, we have that $0 \to (rI)^{j+1} (\Gamma \otimes_A X) \to (rI)^j (\Gamma \otimes_A X) \to (rI)^j (r^j X/r^{j+1} X) \to 0$ is exact.

To prove (ii) we consider the commutative diagram

\[
\begin{array}{ccc}
\text{mod } A & \xleftarrow{F} & \text{mod } I \\
\downarrow D & & \downarrow D \\
\text{mod } A^{op} & \xleftarrow{F'} & \text{mod } I^{op}
\end{array}
\]

where $F'$ and $H'$ are the functors associated with $i: A^{op} \to I^{op}$ (see [12, p. 120]). We then use (i) and the fact that $D(r^j X/r^{j+1} X) \simeq \text{soc }^{j+1} DX/\text{soc }^j DX$.

(g) It is well known that $A$ is Nakayama if and only if $A/r^2$ is Nakayama, and if $A$ is an algebra with $r^2 = 0$, then it is easy to see that $A$ is Nakayama if and only if $I$ is 1-Gorenstein. Since we assume (B), $(H, F)$ is an adjoint pair of functors. $i: A \to I$ induces, by (e), a monomorphism $i': A/r^2 \to I/r^2 I$ and an adjoint pair of functors $(H', F')$. By (B) we have that $A/r^2$ is 1-Gorenstein if and only if $I/r^2 I$ is, that is, if and only if $I/(\text{rad } I)^2$ is. Hence $A$ is Nakayama if and only if $I$ is Nakayama.
We mention that knowing that Auslander algebras are preserved by our construction is useful in representation theory. For describing algebras of finite type is equivalent to describing the Auslander algebras [11, and this method has been used in [31]. The well-known correspondence is given by sending an algebra $A$ of finite type to $\text{End}_A(X)^{op}$, where $X$ is the direct sum of one copy of all indecomposable $A$-modules. This point will be illustrated later.

1.3. The conditions (A), (B), (C), and hence the consequences given in the above theorem, are satisfied more generally by the natural inclusion map $i: A \to A \ast G$, when $A$ is an artin algebra, $G$ a finite group acting on $A$, whose order is invertible in $A$, and $A \ast G$ is a crossed product algebra, as defined in the Introduction.

Note that we assume in this paper that the cocycles have their values in $U(R)$, the units of the center of $A$, so that the action of $G$ on $A$ is given by a group homomorphism $G \to \text{Aut}(A)$. If $\gamma(g, h) = 1$ for all $g, h$ in $G$ we have the special case of skew group algebras. Also if $\gamma: G \times G \to U(R)$ is a coboundary, that is, of the form $\gamma(g, h) = \delta(g) g(\delta(h)) \delta(gh)^{-1}$ for some map $\delta: G \to U(R)$, where $g$ and $h$ are in $G$, and hence represents the zero element in $H^2(G, U(R))$, we shall see in Section 2 that $A \ast G \simeq AG$. Also we shall see there that if $\gamma$ is not a coboundary, then $A \ast G$ and $AG$ may or may not be isomorphic.

The proof that $i: A \to A \ast G$ satisfies properties (A), (B), (C) is similar to the proof for skew group rings, and follows the same outline. We only point out some things which are a little more complicated.

**Theorem 1.4.** The natural ring monomorphism $i: A \to A \ast G$ satisfies (A), (B), (C), when $|G|$ is invertible in $A$.

**Proof.** There is the following generalization of Lemma 1.2.

**Lemma 1.5.** Let $X, Y$ be in $\text{mod} \ A \ast G$, and $t: X \to Y$ a $A$-map. Then $\tilde{t}: X \to Y$ defined by $\tilde{t}(x) = \sum_{g \in G} g^{-1}t(\bar{g}x)$ is a $A \ast G$-map.

**Proof.** We observe that $\bar{g}^{-1} = \gamma(g^{-1}, g)^{-1}g^{-1}$. We have

$$t(\bar{h}x) = \sum_{g \in G} g^{-1}t(\bar{g}hx) = \sum_{g \in G} g^{-1}\gamma(g, h) t(\bar{gh}x)$$

$$= \sum_{g \in G} \gamma(gh^{-1}, h) t(\bar{x})$$

$$= \sum_{g \in G} \gamma(hg^{-1}, gh^{-1})^{-1} \gamma(gh^{-1}, h) \bar{h}^{-1}t(\bar{x})$$

$$= \sum_{g \in G} \gamma(hg^{-1}, g)^{-1} \bar{h}^{-1}t(\bar{x}),$$
using the triple \( \{h g^{-1}, g h^{-1}, h\} \).

\[
\hat{t}(x) = \sum_{g \in G} h g^{-1}(\hat{g} x) = \sum_{g \in G} \hat{h} g^{-1},(g^{-1}, g^{-1} g^{-1})(\hat{g} x)
\]

\[
= \sum_{g \in G} h(g^{-1}, g^{-1}) \gamma(h, g^{-1}) h g^{-1}(\hat{g} x)
\]

\[
= \sum_{g \in G} \gamma(h g^{-1}, g^{-1} h g^{-1})(\hat{g} x),
\]

using \( \{h, g^{-1}, g\} \). This finishes the proof, since \( \tilde{t}(\lambda x) = \lambda \tilde{t}(x) \) follows from \( g^{-1}(g(\lambda)) = \lambda \).

To prove (B), we now write an element \( y \) in \( A \ast G \otimes \Lambda Y \) as \( y = \sum_{g \in G} g^{-1} \otimes y_g \), and get by considering \( p_Y(\hat{h} y) \) where \( p_Y : \Lambda \ast G \otimes \Lambda Y \to U \) is defined by \( p_Y(y) = y_e \) that \( y = \sum_{g \in G} g^{-1} \otimes p_Y(\hat{g} y) \). The rest follows as before.

1.4. Let \( A \) be given by the quiver

\[
\begin{array}{ccc}
\varepsilon_1 & \alpha & \beta \\
\varepsilon_2 & & \\
\varepsilon_3 & & \\
\end{array}
\]

over some field \( k \), and let \( G = \langle g \rangle \) be a group of order 2, where the action of \( g \) on \( A \) is induced by \( g(\varepsilon_1) = \varepsilon_2, g(\varepsilon_2) = \varepsilon_1, g(\varepsilon_3) = \varepsilon_3, g(\alpha) = \beta, g(\beta) = \alpha \).

Assuming that \( \text{char } k \neq 2 \), we can easily compute that \( AG \) is isomorphic to the matrix ring

\[
\Gamma = \begin{pmatrix}
k & k & 0 & 0 \\
k & k & 0 & 0 \\
k & k & k & 0 \\
k & k & 0 & k \\
\end{pmatrix},
\]

by arranging a \( k \)-basis in the following way:

\[
\begin{pmatrix}
\varepsilon_1 & \varepsilon_1 g & 0 & 0 \\
\varepsilon_2 g & \varepsilon_2 & 0 & 0 \\
1/2(\alpha + \beta g) & 1/2(\beta + \alpha g) & 1/2(\varepsilon_3 + \varepsilon_3 g) & 0 \\
1/2(\alpha - \beta g) & 1/2(-\beta + \alpha g) & 0 & 1/2(\varepsilon_3 - \varepsilon_3 g) \\
\end{pmatrix}.
\]

The induced natural map \( \iota : A \to \Gamma \) is given by \( \iota(\varepsilon_1) = E_{11}, \iota(\varepsilon_2) = E_{22}, \iota(\varepsilon_3) = E_{33} + E_{44}, \iota(\alpha) = E_{31} + E_{41}, \iota(\beta) = E_{32} - E_{42} \), where \( E_{ij} \) is the element of \( \Gamma \) whose \((i,j)\) entry is 1 and the others 0.

If the characteristic of \( k \) is 2, we can still consider the ring map \( \iota : A \to \Gamma \),
with $\Gamma$ and $i$ defined as above. With respect to all the homological properties we have considered, the properties of $\Gamma$ are the same regardless of the characteristic of $k$. But the close connection between $A$ and $\Gamma$ in the characteristic 2 case cannot be gotten from Theorem 1.3, by considering $i: A \to \Gamma$. For it turns out that none of the properties (A), (B), (C) hold in this case. rad $\Gamma$ is clearly

$$\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
k & k & 0 & 0 \\
k & k & 0 & 0
\end{pmatrix}.$$

$$\Gamma r = \begin{pmatrix}
* & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
r & 0 & 0 & 0 \\
0 & s & 0 & 0
\end{pmatrix} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
a & b & 0 & 0 \\
a & (-b) & 0 & 0
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
ra & rb & 0 & 0 \\
sa & (-sb) & 0 & 0
\end{pmatrix}; \ a, b, r, s \text{ in } k$$

$$= \text{rad } \Gamma.$$

$$r\Gamma = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
a & b & 0 & 0 \\
a & (-b) & 0 & 0
\end{pmatrix} \begin{pmatrix}
c & d & 0 & 0 \\
e & f & 0 & 0 \\
* & * & & &
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
ac + be & ad + bf & 0 & 0 \\
ac - be & ad - bf & 0 & 0
\end{pmatrix}; \ a, b, c, d, e, f \text{ in } k.$$

$$\ne \text{ rad } \Gamma \quad \text{when char } k = 2, \text{ since then } ac - be = ac + be.$$

We now show that (A)(i) does not hold. Assume to the contrary that there is some $X_\lambda$ such that $\lambda I_A = \lambda A \downarrow \lambda X_\lambda$. Let $x \neq 0$ in $X$. $x = \lambda_1 E_{11} + \lambda_2 E_{22} + \lambda_3 E_{33} + \lambda_4 E_{44} + \lambda_{12} E_{12} + \lambda_{11} E_{11} + \lambda_{21} E_{21} + \lambda_{31} E_{31} + \lambda_{32} E_{32} + \lambda_{41} E_{41} + \lambda_{42} E_{42}$. $E_{11} \times E_{11} = \lambda_1 E_{11} \in X$, so that $\lambda_{11} = 0$, since $E_{11} \in A$. Similarly
\( \lambda_{32} \) is 0. \( E_{22} \times E_{11} = E_{21} = \lambda_{12} E_{12} \in X \) and \( E_{11} \times E_{22} = \lambda_{12} E_{12} \in X \). Since \( E_{21} \not\in A \), there is some \( x \in X \) with \( \lambda_{21} \neq 0 \), so that \( E_{21} \in X \), and similarly \( E_{12} \in X \). \( (E_{32} - E_{42}) E_{21} = E_{31} - E_{41} \in X \) since \( E_{32} - E_{42} \in A \). This is a contradiction when \( \text{char } k = 2 \), since then \( E_{31} - E_{41} = E_{31} + E_{41} \in A \).

One can also show that \( (A)(ii) \) and \( (B) \) do not hold. We note that if we replace \( i: A \rightarrow \Gamma \) by the map \( j: A \rightarrow \Gamma \), where the only difference is that \( j(\beta) = E_{32} + E_{42} \), the properties \( (A) \), \( (B) \) and \( (C) \) also fail, with the same proof as above for \( (C) \) and \( (A)(i) \).

If for \( y \in k \) we change the ring map \( i: A \rightarrow \Gamma \) to the map \( i': A \rightarrow \Gamma \) where the only change is that \( i'(\beta) = E_{32} + y E_{42} \), we ask if there is some \( y \) such that \( (A) \), \( (B) \), \( (C) \) hold when \( k \) is a field of characteristic 2 with more than two elements. For example, we see that if \( y \neq 1 \), by using the above calculations, that \( r\Gamma = \text{rad } \Gamma = \Gamma r \), so that \( (C) \) holds. But assume that \( \lambda_{33} E_{31} + \lambda_{44} E_{41} \in X \). As before we get \( (E_{32} + y E_{42}) E_{21} = E_{31} + y E_{41} \in X \), and \( (E_{31} + E_{41}) E_{42} = E_{32} + E_{42} \in X \). There must be some \( z \) in \( X \) such that the coefficients \( \lambda_{33} \) of \( E_{33} \) and \( \lambda_{44} \) of \( E_{44} \) are not both zero. \( z \cdot (E_{31} + E_{41}) = \lambda_{33} E_{31} + \lambda_{44} E_{41} \in X \), and \( z \cdot (E_{32} + y E_{42}) = \lambda_{33} E_{32} + \lambda_{44} y E_{42} \in X \). This implies, since \( E_{31}, E_{32} \) are not in \( X \), that \( \lambda_{44} = y \lambda_{33} \) and \( \lambda_{44} y = \lambda_{33} \). Hence \( y^2 = 1 \), so that \( y = \pm 1 \). Combining with the above we see that it is necessary for \( (A)(i) \) to be satisfied for \( i': A \rightarrow \Gamma \) that \( y = -1 \), the case coming from skew group algebras for characteristic not 2, and consequently for no choice of \( y \in k \) when the characteristic of \( k \) is 2.

In concrete examples as above one will usually find a ring map \( i: A \rightarrow \Gamma \), coming from a skew group algebra \( \Gamma \cong AG \) when the order of \( G \) is invertible in \( A \), which is also defined otherwise. And usually \( \Gamma \) will have similar properties as \( AG \) also in the exceptional characteristics. It would be nice to have a way of describing the algebras arising in this fashion, and to have a theory including also these exceptional cases. Since our conditions \( (A) \), \( (B) \), \( (C) \) are not always satisfied, one would need to find a theory under weaker conditions to be able to include such a situation.

1.5. Since we have seen that \( A \) and \( AG \) or \( A \ast G \) share many properties, it is of interest to have methods for constructing \( AG \) from \( A \) and \( G \). Section 2 is devoted to discussing this. For illustration we indicate here how this can be done for some semisimple algebras. Since we have seen that in general \( G \) induces an action on \( A/r \) and \((A/r)G \cong AG/\text{rad}(AG)\), knowing what happens to semisimple algebras also gives information in general. The following more general result on reduction to indecomposable algebras, which was essentially pointed out to us by Farkas and Snider, gives some information about skew group rings of semisimple rings.

**Proposition 1.6.** Let \( A_1, \ldots, A_t \) be isomorphic indecomposable algebras and \( G \) a finite group acting on the product algebra \( A = A_1 \times \cdots \times A_t \),
containing elements \( g_i = e, \ldots, g_t \) such that \( g_i(A_i) = A_i \). Let \( H = \{ g \in G; g(A_i) = A_i \} \). Then \( AG \) is Morita equivalent to \( A_i H \).

**Proof.** We write \( 1 = e_1 + \cdots + e_t \), where \( e_i \) is the identity of \( A_i \). We observe that \( e_i A_i e_i = A_i H \). \( A_i H \subset e_i A_i e_i \) since \( e_i \) is the identity of \( A_i H \). Consider conversely \( e_i \lambda g e_i = (e_i \lambda) g(e_i) \lambda e_i \lambda \in A_i \), so if \( e_i \lambda g(e_i) \lambda \neq 0 \), then \( g(e_i) \lambda \in H \), that is \( g \in H \), so we are done.

Consider now the left \( AG \)-module isomorphism \( f: AG \to AG \), given by \( f(x) = xg \), \( f \) takes \( A_i g e_i \) to \( A_i g e_i g_i = A_i g e_i \). Since \( AG = \bigoplus A_i g e_i \), \( A_i g e_i \) contains a copy of each indecomposable projective \( AG \)-module. This shows that \( AG \) and \( A_i H \) are Morita equivalent.

As a consequence we get information on semisimple rings, like, for example, the following.

**Example 1.7.** Let \( A \) be the product of \( t \) copies of the algebraically closed field \( k \).

(a) If \( G \) is a finite cyclic group whose order is invertible in \( A \) with the generator acting as a cyclic permutation of the \( t \) copies of \( k \), then \( AG \) is Morita equivalent to the product of \( m \) copies of \( k \), where \( m \) is the order of the subgroup leaving the first copy of \( k \) fixed.

(b) If the symmetric group \( S_t \) acts by permuting the \( t \) copies of \( k \), then \( AS_t \) is Morita equivalent to \( kS_{t-1} \).

The above example, and any other description of skew group rings of semisimple rings, also gives some information on how \( AG \otimes A S \) decomposes as a \( AG \)-module, when \( S \) is a semisimple or simple \( A \)-module. More generally, it is interesting to study how \( AG \otimes A X \), or \( A \ast G \otimes A X \), decomposes when \( X \) is indecomposable in \( \mod A \), and how \( Y \) decomposes when \( Y \) is indecomposable in \( \mod A \ast G \). Before we give a result of this nature, we introduce the following notation. For \( X \) in \( \mod A \) and \( g \) in \( G \), we denote by \( {}^g X \) the \( A \)-module whose underlying set and additive structure is the same as for \( X \), and where \( g \cdot x \) is defined to be \( g^{-1}(\lambda)x \). The subset \( \tilde{g} \otimes X = \{ \tilde{g} \otimes x; x \in X \} \) of \( A \ast G \otimes A X \) has a natural \( A \)-module structure given by \( \lambda(\tilde{g} \otimes x) = \tilde{g} \otimes x = \tilde{g}^{-1}(\lambda) \otimes x = \tilde{g} \otimes g^{-1}(\lambda)x \), so that \( {}^g X \) and \( \tilde{g} \otimes X \) are isomorphic \( A \)-modules.

**Proposition 1.8.** Let \( A \) and \( A \ast G \) be as usual and \( F: \mod A \to \mod A \ast G \) and \( H: \mod A \ast G \to \mod A \) the associated functors. If \( X \) and \( Y \) are ind \( A \), we have the following.

(a) \( HF(X) \simeq eX \amalg {}^e X \amalg \cdots \amalg {}^e X \), where \( G = \{ e, g_2, \ldots, g_n \} \).

(b) \( FX \simeq FY \Leftrightarrow X \simeq {}^g Y \) for some \( g \in G \).

(c) The number of summands in a decomposition of \( FX \) into a direct
sum of indecomposable modules is at most the order \( m \) of \( H \), where \( H = \{ g \in G; g^X \simeq X \} \).

Assume now that \( A \) is a \( k \)-algebra where \( k \) is algebraically closed.

(d) If \( G \) is cyclic of order \( n \) and \( X \simeq g^X \) for all \( g \in G \), then \( F_X \) has exactly \( n \) summands.

(e) If \( H = \{ g \in G; g^X \simeq X \} \) is cyclic of order \( m \), then \( F_X \) has exactly \( m \) summands.

**Proof:**

(a) \( A \ast G \otimes_A X = g_1 \otimes X \ast \cdots \ast g_n \otimes X \simeq g_1 X \ast \cdots \ast g_n X \) as \( A \)-modules by the above remark.

(b) It is easy to see that \( \phi: A \ast G \otimes_A X \to A \ast G \otimes_A X \) given by \( \phi(h \otimes x) = h \otimes g \otimes x \) is a \( A \ast G \)-isomorphism.

Assume that \( A \ast G \otimes_A X \simeq G \otimes_A Y \). Then \( Y \) is a summand of \( A \ast G \otimes_A X \simeq g_1 X \ast \cdots \ast g_n X \) as \( A \)-modules, so that \( Y \simeq g^X \) for some \( g \in G \).

(c) Consider \( A \ast G \otimes_A X = Y_1 \ast \cdots \ast Y_r \), with the \( Y_i \) indecomposable. As a \( A \)-module we have \( Y_i = g_i X \ast \cdots \ast g_i X \). Since \( Y_i \) is a \( A \ast G \)-module, we have an isomorphism \( \phi: Y_i \to g^X Y_i \), given by \( \phi(y) = g^{-1} y \) for \( y \in Y_i \). Hence \( Y_i \simeq g^X Y_i \ast \cdots \ast g^X Y_i \). This shows that each \( Y_i \) has a summand from each isomorphism class of the \( g^X \), that is, \( Y_i \) has at least \( n/m \) summands of indecomposable \( A \)-modules. The claim follows from this.

(d) By [16] it is possible to choose an isomorphism \( \sigma: X \to g^X \) such that \( g^{n-1}(\sigma) \cdots g(\sigma) \sigma = id_X \). We want to show that the identity in \( \text{End}_{A \ast G}(A \ast G \otimes_A X) \) can be written as a sum of \( n \) nonzero orthogonal idempotents. Considering the natural isomorphism \( \text{End}_{A \ast G}(A \ast G \otimes_A X) \simeq \text{Hom}_A(X, A \ast G \otimes_A X) \), we choose the elements corresponding to \( e_i = 1/n (id_X, \zeta^i \sigma, \zeta^2 g(\sigma) \sigma, \ldots, \zeta^{n-1} g^{n-2}(\sigma) \cdots g(\sigma) \sigma) \) in \( \text{Hom}_A(X, A \ast G \otimes_A X) \simeq \text{Hom}_A(X, X \otimes g^X \otimes \cdots \otimes g^{n-1}X) \), where \( \zeta \) is a primitive \( n \)-th root of unity. This gives the desired idempotents. For further details, compare with 3.1.

An alternative proof for \( AG \) can be given using results which we establish in Sections 3 and 5. We can then reduce the problem to assuming that \( X \) is a simple module, and further to the ring decomposition of \( M_n(k)G \), for some action of \( G \) on \( M_n(k) \). Now it is known that every \( k \)-automorphism of \( M_n(k) \) is inner [33] and in Section 5 we prove that if \( G \) is cyclic, we can assume that the action of \( G \) is trivial. Hence \( M_n(k)G \) is Morita equivalent to \( kG \), which decomposes into a product of \( n \) copies of \( k \).

(e) Consider \( A \subset A \ast H \subset A \ast G \). Since \( A \ast H \otimes_A X \) has \( m \) summands by (d), \( A \ast G \otimes_A X \) has at least \( m \) summands. On the other hand, \( A \ast G \otimes_A X \) has at most \( m \) summands by (c), so we are done.

We end this section by pointing out that \( \text{mod} A \) can be recovered from
mod $A \ast G$, in the following known result \cite{14}, whose proof we sketch for the sake of completeness.

**Proposition 1.9.** The functor $F: \text{mod} A \rightarrow \text{mod} A \ast G$ induces an equivalence between $\text{mod} A$ and the category $\text{gr}(\text{mod} A \ast G)$ of finitely generated $A \ast G$-modules $G$-graded over $A$, with degree $e$ maps.

**Proof:** For $X$ in $\text{mod} A$, $FX$ is clearly $G$-graded. Conversely, $Y = \bigoplus_{g \in G} Y_g$ in $\text{gr}(\text{mod} A \ast G)$ is isomorphic to $FY_e$ since $Y_g = AgY_e$. Finally, any degree $e$ map $f: A \ast G \otimes_A X \rightarrow A \ast G \otimes_A X'$ has the form $A \otimes f'$ for a unique $A$-homomorphism $f': X \rightarrow X'$.

2. **Examples**

Throughout this chapter we assume that $A$ is a finite-dimensional algebra over an algebraically closed field $k$. As usual we only consider finite groups $G$ whose order is invertible in $k$, and the actions of $G$ on $A$ will always be trivial on $k = k1_A \subseteq A$.

The examples of skew group algebras studied in this section will illustrate that for seemingly unrelated basic algebras $A$ and $A'$, $A'$ can still be Morita equivalent to $AG$. On the other hand, we will obtain a list of properties not preserved under the transition from $A$ to $AG$. If $A$ is basic and $G$ is cyclic we described the quiver of the basic algebra Morita equivalent to $AG$ in terms of the quiver of $A$. We also give examples of crossed product algebras.

2.1. We begin with an algebra $A$ and a group $G$ such that $AG$ is already determined by the behavior of the simple $A$-modules under $AG \otimes_A$. Let $A$ be the basic hereditary algebra given by the quiver

```
\begin{tikzpicture}
  \node (1) at (0,0) {$\alpha_1$};
  \node (2) at (1,0) {$\alpha_2$};
  \node (3) at (2,0) {$\alpha_{n-1}$};
  \node (4) at (3,0) {$n$};
  \node (5) at (4,0) {$\alpha'_1$};
  \node (6) at (5,0) {$\alpha'_2$};
  \node (7) at (6,0) {$\alpha'_{n-1}$};
  \node (8) at (7,0) {$n'$};

  \draw[->] (1) -- (2);
  \draw[->] (2) -- (3);
  \draw[->] (3) -- (4);
  \draw[->] (4) -- (5);
  \draw[->] (5) -- (6);
  \draw[->] (6) -- (7);
  \draw[->] (7) -- (8);
\end{tikzpicture}
```

Let $g$ be the automorphism of $A$ switching $i$ and $i'$ for $i = 2,..., n$ and $\alpha_i$ and $\alpha'_i$ for all $i$, and let $G = g^{Z/2Z}$. By 1.2, the semisimple part $AG/\text{rad}(AG)$ of $AG$ is isomorphic to $(A/r)G$, where the action of $G$ on $A/r$ is induced by the action on $A$. We therefore obtain that
\[ AG \otimes_A S_i \cong T_i \amalg T'_i, \]
\[ AG \otimes_A S_i = AG \otimes_A S'_i = T_i \quad \text{for } i = 2, \ldots, n. \]

where \( S_i \) and \( S'_i \) denote the simple \( A \)-modules corresponding to the vertices \( i \) and \( i' \), respectively, and \( T_1, T'_1 \) and \( T_i \) are the pairwise nonisomorphic simple \( AG \)-modules.

Recall that the functor
\[ F = AG \otimes_A : \mod A \to \mod AG \]
is exact and preserves projectives as well as their radical series and socle series. Hence the projective cover \( P_1 \) of \( S_1 \) yields a projective \( AG \)-module with radical series, in order from top to bottom
\[ (T_1, T'_1, T_2, \ldots, T_n) \]
and for \( i \geq 2 \), the projective covers \( P_i \) and \( P'_i \) of \( S_i \) and \( S'_i \) both yield a projective over \( AG \), with radical series \( (T'_i, T_i, \ldots, T_n) \).

We conclude that \( AG \otimes P_1 \) is the direct sum of two uniserial projectives with radical series \( (T_1, T_2, \ldots, T_n) \) and \( (T'_1, T'_2, \ldots, T_n) \), and that \( AG \otimes P_i \) and \( AG \otimes P'_i \) are uniserial for \( i > 2 \). The quiver of \( AG \) has the form
\[ \begin{array}{c}
1 \\
2 \\
\vdots \\
n
\end{array}
\]
and, since \( AG \) is hereditary (1.2), it determines \( AG \) up to Morita equivalence.

We could have argued the same way if we had reversed some of the pairs of arrows \( \{a_i, a'_i\} \), so that we obtain the following result. For a hereditary algebra \( A \) of type \( A_{2n-1} \) for which there exists an automorphism \( g \) of order 2 on the quiver, with only one fixed vertex, the skew group algebra \( AG \) is hereditary of type \( D_{n+1} \), where \( G = g \mathbb{Z}/2\mathbb{Z} \). Notice that neither the maximal number \( a(A) \) of summands of the middle term is an almost split sequence nor the maximal number \( \beta(A) \) of nonprojective such summands needs to be preserved when passing from \( A \) to \( AG \). In fact, as soon as \( n \geq 3 \), we have \( a(A) = \beta(A) = 2 \) and \( a(AG) = \beta(AG) = 3 \) in our example.

2.2. The method used in 2.1 in order to find the basic algebra \((AG)_b\)
Morita equivalent to \( AG \) only works if for each projective indecomposable \( A \)-module \( P \) the radical and socle series of \( AG \otimes_A P \) already determine all
indecomposable summands of $AG \otimes_A P$ up to isomorphism. It is therefore rather limited, and there are examples where more information is needed in order to obtain $(AG)_b$ (see 2.5). Our aim is to describe $(AG)_b$ by quiver and relations if $A$ is basic and $G$ cyclic, which will give us a lot of examples.

First let $A$ be any algebra, not necessarily basic, and let $G$ operate on $A$. Since $G$ preserves the powers of the radical, there is an induced action of $G$ on $A/r$ via algebra automorphisms and on $r/r^2$ via $A/r - A/r$-bimodule automorphisms. The kernel of the algebra homomorphism

$$
\pi: AG \to (A/r)G
$$

which is given by $\pi(\lambda g) = \hat{\lambda} g$ where $\hat{\lambda}$ denotes the image of $\lambda$ in $A/r$ is the ideal $rAG = \text{rad } AG$, so that we have an isomorphism

$$
AG/\text{rad } AG \cong (A/r)G.
$$

Using the exactness of $AG \otimes_A$ we obtain an isomorphism

$$(AG \otimes_A r)/(AG \otimes_A r^2) \to AG \otimes_A r/r^2$$

of $AG - A$-bimodules. This yields an isomorphism of $AG - AG$-bimodules

$$
\text{rad } AG/\text{rad}^2 AG \to AG \otimes_A r/r^2
$$

if we define a right $AG$-module structure on $AG \otimes_A r/r^2$ by

$$(\lambda g \otimes \bar{a})h = \lambda gh \otimes h^{-1}(\bar{a})$$

for $g, h \in G$, $\lambda \in A$, $\mu \in r$, $\bar{a} \in r/r^2$. Finally, we obtain an isomorphism of $(A/r)G - (A/r)G$-bimodules

$$
\text{rad } AG/\text{rad}^2 AG \to (A/r)G \otimes_{A/r} r/r^2.
$$

As a consequence, the quiver of $AG$ is determined by the semisimple algebra $(A/r)G$ and the $(A/r)G - (A/r)G$-bimodule $(A/r)G \otimes_{A/r} r/r^2$.

Now let $A$ be basic with quiver $Q$. The action of $G$ on $A/r$ and on the $A/r - A/r$-bimodule $r/r^2$ induces an action of $G$ on the quiver algebra $kQ$ such that all vector spaces of paths of fixed length are preserved. We have the following.

**Proposition 2.1.** Let $A$ be basic with quiver $Q$, and let $G$ operate on $A$ and on $kQ$ as described. Then there is a $G$-equivariant algebra epimorphism $\phi: kQ \to A$.

**Proof.** The argument is based on an averaging process à la Maschke. First we find a $G$-equivariant section $\sigma$ for the algebra epimorphism $\pi: A \to$
Let $A$ have Loewy length $s > 1$. By induction, we may suppose that we are given a $G$-equivariant section $\sigma_2$ for the epimorphism $\pi_2: A/r^{s-1} \to A/r$. Let $\pi_1: A \to A/r^{s-1}$, and choose a section $\sigma_1$ for $\pi = \pi_2 \pi_1$ such that $\pi_1 \sigma_1 = \sigma_2$.

\[
\begin{array}{c}
A \\ \pi_1 \downarrow \\
\xrightarrow{\pi_2} A/r^{s-1} \\
\sigma_2 \\
\xleftarrow{\pi_1} A/r
\end{array}
\]

Such a section $\sigma_1$ always exists, since any complete set of primitive orthogonal idempotents of $A/r^{s-1}$ can be lifted to $A$, but of course $\sigma_1$ is not $G$-equivariant in general. The map $\sigma: A/r \to A$ obtained by averaging $\sigma_1$ over $G$ is $G$-equivariant by definition and satisfies $\pi \sigma = 1$. We will show that it is an algebra homomorphism in addition.

For $g \in G$ and $x \in A/r$, $g(\sigma_1 x)$ and $\sigma_1 (gx)$ have the same image under $\pi_1$, since both $\pi_1$ and $\sigma_2$ are $G$-equivariant, and hence we have

\[ g(\sigma_1 x) = \sigma_1 (gx) \mid r(g, x) \]  

for some $r(g, x)$ is $r^{s-1}$. Since $\sigma_1$ is an algebra homomorphism, we obtain

\[ r(g, xy) = \sigma_1 (gx) r(g, y) + r(g, x) \sigma_1 (gy) \]  

for $x, y \in A/r$ and $g \in G$. Notice that the product $r(g, y) r(g, x)$ vanishes as it lies in $r^{s-2} \subset r^s$. For $x$ in $A/r$, we set

\[ \sigma(x) = \frac{1}{n} \sum_{g \in G} g(\sigma_1(g^{-1}x)), \]

where $n$ is the order of $G$. Using ($*$), we obtain

\[ \sigma(x) = \sigma_1 x + \frac{1}{n} \sum_{g \in G} r(g, g^{-1}x), \]

and by ($**$) we conclude that $\sigma$ is in fact an algebra homomorphism.

Using $\sigma$ we identify $A/r$ with a $G$-stable subalgebra of $A$. In a second step, we find a $G$-equivariant section $\sigma'$ for the canonical map $\pi': r \to r/r^2$ of $A/r - A/r$-bimodules. Starting from any section $\sigma'_1$, we now obtain a $G$-equivariant one right away by setting

\[ \sigma'(x) = \frac{1}{n} \sum_{g \in G} g(\sigma'_1(g^{-1}x)) \]

for all $x$ in $r/r^2$. 

\[ \sigma(x) = \sigma_1 x + \frac{1}{n} \sum_{g \in G} r(g, g^{-1}x), \]  

and by ($**$) we conclude that $\sigma$ is in fact an algebra homomorphism.
We define $\phi: kQ \to A$ using the sections $\sigma$ and $\sigma'$ for the vertices and arrows, respectively. By construction, $\phi$ is $G$-equivariant. This completes the proof.

If $A$ is basic and $G$ operates on $A$, we can therefore view $A$ as $kQ/I$ in such a way that the action of $G$ on $A$ is induced by an action of $G$ on $kQ$ which leaves $I$ stable and preserves the natural grading on $kQ$ by the length of paths. Then $AG$ is isomorphic to $(kQ)G/I((kQ)G)$. We define $(kQ)G$ in the usual way even though $kQ$ may be infinite dimensional. If $r_1, \ldots, r_s$ generate the ideal $I$, then $I((kQ)G)$ is generated by the $r_jg$ for $j = 1, \ldots, s$ and $g$ in $G$.

The action of $G$ on $kQ$ is particularly simple if $Q$ contains no multiple arrows. Each $g \in G$ permutes the vertices of $Q$ and maps each arrow $\alpha: \varepsilon \to \varepsilon'$ onto a scalar multiple of the only arrow from $g(\varepsilon)$ to $g(\varepsilon')$.

In case $G = \mathbb{Z}/n\mathbb{Z}$ is cyclic, we obtain a precise description of the action of $g$ even if $Q$ does contain multiple arrows. Let $\varepsilon$ and $\varepsilon'$ be two vertices, and let the size of the $G$-orbit through $\varepsilon$ and $\varepsilon'$ be $s$ and $s'$, respectively. For each $\alpha \neq 0$ in the vector space $V$ spanned by all arrows of $Q$ from $\varepsilon$ to $\varepsilon'$, $g^i(\alpha)$ belongs to $V$ if and only if $i$ is a multiple of $[s, s']$, the least common multiple of $s$ and $s'$. The automorphism $g^{i[s,s']}$ of $V$ has finite order, not divisible by the characteristic of $k$, and is therefore diagonal with respect to a suitable basis $a_1, \ldots, a_s$ of $V$. We choose $g^i(a_1), \ldots, g^i(a_s)$ as a basis for $g^i(V)$ for $i = 0, \ldots, [s, s'] - 1$. Since the vector space spanned by all arrows of $Q$ which do not belong to $\sum g^i(V)$ is $G$-stable, we can repeat the argument. Eventually, we find a basis such that the image under $g$ of every arrow is a multiple, necessarily a root of unity, of another arrow.

2.3. We want to describe the quiver $Q_G$ of $AG$ in case $A$ is basic and $G = \mathbb{Z}/n\mathbb{Z}$ is cyclic. Let $A = kQ/Z$ be as in 2.2, and assume the basis of the vector space spanned by the arrows of the quiver $Q$ of $A$ is chosen as above. We fix a primitive $n$th root of unity $\zeta$.

In order to determine the vertices of $Q_G$, we have to compute $AG/\text{rad } AG \simeq (A/\text{rad } A)G$, and it suffices to concentrate on one $G$-orbit of vertices of $Q$ at the time. We let $A_0 = \prod_{i=0}^{s-1} k^g^i(\varepsilon)$, where $s$ is the size of the $G$-orbit of the vertex $\varepsilon$. For each $\mu$ with $0 \leq \mu < m = n/s$, we let $M_\mu$ denote a copy of the algebra of $s \times s$-matrices, and we define a $k$-linear map

$$\phi: \prod_{\mu = 0}^{m-1} M_\mu \to A_0 G$$

by setting

$$\phi(E_{i,j}^{(\mu)}) = \frac{1}{m} \sum_{i=0}^{m-1} \zeta^{\mu i} g^{-i} g^{j + si}$$
for \( l, j = 0, \ldots, s - 1 \), where \( E_{ij}^{(a)} \) is the elementary matrix of \( M_a \) which has a 1 in row \( l \) and column \( j \), where rows and columns range from 0 to \( s - 1 \). Using that
\[
\frac{1}{m} \sum_{i=0}^{m-1} \zeta^{si\lambda} = \begin{cases} 1 & \text{if } \lambda \equiv m \pmod{n}, \\ 0 & \text{otherwise}, \end{cases}
\]
it is not hard to check that \( \phi \) is an algebra homomorphism, and \( \phi \) is surjective since for each integer \( h \) and each \( j, l = 0, \ldots, s - 1 \)
\[
g^{-l}\zeta^{j+sh} = \sum_{\mu=0}^{m-1} \zeta^{-\mu sh} \phi(E_{ij}^{(\mu)})
\]
lies in the image. As both algebras have dimension \( ms^2 = ns \), \( \phi \) is an isomorphism. For the vertices of \( Q_G \), we choose the \( \phi(E_{00}^{(\mu)}); \quad \mu = 0, \ldots, m - 1, \)
obtained from a set \( \mathcal{F} \) of representatives of the \( G \)-orbits of vertices of \( Q \); i.e., one diagonal elementary matrix from each matrix ring.

Given our choice of a basis, the vector space spanned by all arrows of \( Q \) decomposes as a direct sum of \( G \)-modules spanned by the \( G \)-orbit of one arrow. We may therefore compute the contribution of one \( G \)-orbit of arrows of \( Q \) to the set of arrows of \( Q_G \) at the time.

Let \( \varepsilon \) and \( \varepsilon' \) be in \( \mathcal{F} \), and let the size of the \( G \)-orbit of \( \varepsilon \) and \( \varepsilon' \) be \( s \) and \( s' \), respectively. Assume \( Q \) contains an arrow \( \alpha \colon \varepsilon \to g^{-t}(\varepsilon') \) for some integer \( t \) with \( 0 \leq t < (s, s') \), the greatest common divisor of \( s \) and \( s' \). Then
\[
g^{[s,s']}(\alpha) = \zeta^{[s,s']t}\alpha
\]
for some integer \( a \). We may assume that \( A \) is given by the following quiver which consists of the vertices \( \varepsilon, g(\varepsilon), \ldots, g^{s-1}(\varepsilon), \varepsilon', g(\varepsilon'), \ldots, g^{s'-1}(\varepsilon') \) and the arrows \( \alpha, g(\alpha), \ldots, g^{[s,s']-1}(\alpha) \), for example,

\[
\begin{align*}
\varepsilon &= g^3(\varepsilon) \\
g(\varepsilon) &\quad g(\alpha) \\
g^2(\varepsilon) &\quad g^2(\varepsilon')
\end{align*}
\]
\[
\begin{align*}
\varepsilon' &= g^3(\varepsilon') \\
g(\varepsilon') &\quad g(\alpha) \\
g^2(\varepsilon') &\quad g^2(\varepsilon)
\end{align*}
\]

We let
\[
\eta_\mu = \phi(E_{00}^{(\mu)}), \quad 0 \leq \mu < m = n/s,
\]
\[
\eta_{\mu'} = \phi(E_{00}^{(\mu')}), \quad 0 \leq \mu' < m' = n/s',
\]
be the vertices of $Q_G$ obtained from $G\varepsilon$ and $G\varepsilon'$, respectively. We claim that 

$$\eta_{\mu'} \phi(E_{0t}^{(u')} \alpha \eta_{\mu}) \neq 0$$

if and only if 

$$\mu \equiv \mu' + a \mod (m, m').$$

If the claim is true, we find $mm'/\langle m, m' \rangle$ different arrows in $Q_G$ and hence 

$$ss' \frac{mm'}{(m, m')} = \frac{n^2}{(m, m')} = n[s, s']$$

linearly independent elements in $\text{rad} \ A G$. The factor $ss'$ is motivated by the sizes $s$ and $s'$ of the matrix rings to which $\eta_{\mu}$ and $\eta_{\mu'}$ belong. But the dimension of $\text{rad} \ A G$ is $n[s, s']$.

Let us compute $\eta_{\mu'} \phi(E_{0t}^{(u')} \alpha \eta_{\mu})$, which we denote by $A$. By definition, we obtain

$$A = \frac{1}{mm'} \sum_{r=0}^{m-1} \sum_{t'=0}^{m'-1} \zeta^{r[s, s'] + s't'} \zeta^{s't'} g^{t + s't'}(a) g^{s + s't'}.$$

We write 

$$si = r[s, s'] + sj$$

for $0 \leq r < \frac{n}{[s, s']} = (m, m')$ and $0 \leq j < \frac{s'}{s}$, 

$$s'i' = r'[s, s'] + s'j'$$

for $0 \leq r' < (m, m')$ and $0 \leq j' < \frac{s'}{s'}$.

This yields 

$$g^{t + s't'}(a) = g^{t + r[s, s'] + s'j'}(a) = \zeta^{t[s, s'] + r + s'j'}(a),$$

and if we set $R = r + r'$, we get $A = BC$, where 

$$B = \frac{1}{mm'} \left( \sum_{r'=0}^{m'-1} \zeta^{r'[s, s'](-u + \mu' + a)} \right)$$

and

$$C = \sum_{k=0}^{(m, m') - 1} \sum_{j, j'} \zeta^{R[s, s'] + s + s'j' + \mu} g^{t + s'j'}(a) g^{R[s, s'] + sj + s'j' + t}.$$
The integers \( R[s, s'] + sj + s'j' + t \) occurring as exponents of \( g \) in the sum form a system of representatives of \( Z/nZ \). Therefore the element \( A \) is different from zero if and only if the factor \( B = \sum_{r=0}^{(m, m')} r \zeta^{r[s, s'j](-a + u' + a)} \) does not vanish, which is the case precisely when \( \mu \equiv \mu' + a \) modulo \( (m, m') \).

Summing up, we have obtained the following description of \( Q_G \): Each vertex of \( Q \) in \( E \) whose \( G \)-orbit has size \( s \) gives rise to \( n/s \) vertices of \( Q_G \). Let \( \varepsilon \) and \( \varepsilon' \) lie in \( E \) and have orbits of size \( s \) and \( s' \), respectively, and let \( \alpha : \varepsilon \to g^{-1}(\varepsilon') \) be an arrow of \( Q \). If \( \eta_{0}, \ldots, \eta_{m-1} \) and \( \eta'_{0}, \ldots, \eta'_{m'-1} \) are the vertices of \( Q_G \) arising from \( G\varepsilon \) and \( G\varepsilon' \), there is an arrow from \( \eta_{\mu} \) to \( \eta'_{\mu'} \) in \( Q_G \) if and only if \( \mu \equiv \mu' + a \mod (m, m') \), where \( a \) is defined by

\[
g^{[s, s']}(\alpha) = r_{[\varepsilon, \varepsilon']}^{a} a.
\]

Applied to concrete examples, this construction is less cumbersome than it appears.

2.4. We compute \( AG \) for some basic algebras \( A \) and some cyclic groups \( G \). In all our examples, we could have reversed the orientation for some \( G \)-orbits of arrows. Notice also that there is an action on the basic algebra \( A' = (AG)^{b} \) Morita equivalent to \( AG \) such that \( A'G \) is again Morita equivalent to \( A \). We will study this phenomenon in more detail in Section 5.

(a) \[
A = \prod_{i=1}^{n} k\varepsilon_{i}, \quad G = g^{Z/nZ}, \quad g(\varepsilon_{i}) = \varepsilon_{i+1}.
\]

In this case, \( AG \) is semisimple and Morita equivalent to \( A \) itself.

In the examples (b)–(e), we take \( A \) hereditary, \( G = g^{Z/2Z} \), and the action of \( g \) on \( A \) is induced by a reflection on the quiver. We will draw the symmetry axis of \( g \) as a dotted line. Since \( AG \) is hereditary, it is determined by its quiver. We note that the examples (b)–(h) could also be gotten using the method given in 2.1.

(b) (Compare with 2.1.)

\[\begin{array}{c}
1 \\
2 \\
\vdots \\
n
\end{array}
\begin{array}{c}
1' \\
2' \\
\vdots \\
n'
\end{array}
\]

\( A \) (\( AG \))
(f) $A$ is given by the quiver of (c) and the relations $r^2 = 0$, and $g$ acts as in (c). Then $AG$ is described by the quiver of $(AG)_b$ in (c) and the relations
For the definition of simply connected we refer to [10].

(g) \( A \) is hereditary, given by

\[
\begin{array}{c}
\downarrow \\
\downarrow \\
\end{array}
\]

and \( G = g^{\mathbb{Z}/3\mathbb{Z}} \) acts on the quiver nontrivially. The quiver of \( AG \) is

\[
\begin{array}{c}
\downarrow \\
\downarrow \\
\downarrow \\
\end{array}
\]

These examples show that tame hereditary algebras of different types can be obtained from each other by skew group algebra constructions. In particular, the four subspace quiver and the double arrow are linked this way.

(h)

\[
\begin{array}{c}
\downarrow \\
\downarrow \\
\end{array}
\]

Finally, we consider two examples where the action of \( G \) on \( A \) is not induced by an action on the quiver of \( A \). In both cases \( AG \) is basic, and the \( k \)-category given by the same quiver and relations as \( AG \) is a Galois covering of the one given by the quiver and relations defining \( A \). We will examine the connections with coverings in Section 5.

(i) \( A \) is given by

\[
\begin{array}{c}
\circ \\
\downarrow \\
\end{array}
\alpha,
\begin{array}{c}
\downarrow \\
\alpha' = 0 \text{ for some } r > 1,
\end{array}
\]

and \( G = g^{\mathbb{Z}/n\mathbb{Z}} \) acts by \( g(\alpha) = \zeta \alpha \), where \( \zeta \) is a primitive \( n \)th root of unity. Then \( AG \) is given by
and all relations $\alpha' = 0$.

(j) $A$ is given by

and $G = g^{\mathbb{Z}/n\mathbb{Z}}$ acts by $g(\alpha) = \alpha$, $g(\beta) = \zeta \beta$, where $\zeta$ is a primitive $n$th root of unity. Then $\Lambda G$ is hereditary, given by

2.5. In 2.1 we have seen that hereditary algebras of type $D_n$ and $A_{2n-3}$ whose quiver admits a nontrivial automorphism $g$ of order 2 correspond to each other bijectively up to Morita equivalence under $A \to \Lambda G$, where $G = g^{\mathbb{Z}/2\mathbb{Z}}$, provided the characteristic of $k$ is different from 2. In this section we will see that selfinjective algebras of finite type behave in an analogous way. We begin with an example.

Let $A$ be given by the quiver

and the relations

$$\alpha_1' \alpha_2' - \alpha_3 \alpha_2, \alpha_1 \alpha_3 \alpha_2 - \alpha_5 \alpha_4, \alpha_4 \alpha_1', \alpha_2 \alpha_5, \alpha_2' \alpha_5, \alpha_1' \alpha_1 \alpha_3, \alpha_2 \alpha_1 \alpha_3'.$$

By definition, $A$ belongs to the set of 2-cornered selfinjective algebras of tree class $D_5$ defined in [111]. The cyclic group $G = g^{\mathbb{Z}/2\mathbb{Z}}$ acts on $A$ by switching
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\(e_3\) and \(e'_3\), \(a_3\) and \(a'_3\), \(a_2\) and \(a'_2\), and by fixing all the other vertices and arrows. We compute the quiver \(Q_G\) of \(AG\) as in 2.3, and we obtain

\[Q_G = \begin{array}{c}
\eta_1^+ = \frac{1}{2}(e_i \pm e_i g) \quad \text{for} \quad i = 1, 2, 4, \\
\eta_3 = e_3, \\
\beta_1^+ = a_i \pm a_i g \quad \text{for} \quad i = 1, 2, 4, 5, \\
\beta_2^+ = a_3 \pm a'_3 g.
\end{array}\]

\(AG\) is selfinjective of finite type (1.2), and, given its quiver, it must be of tree class \(A_7\) [11]. Using the explicit description of the arrows, we can compute the relations directly. We have, for instance,

\[\beta_1^+ \beta_1^+ = (a_3 - a'_3 g)(a_2 + a_2 g) = (a_3 \alpha_2 - a'_3 a'_2)(1 + g) = 0,\]
\[\beta_1^+ \beta_1^+ = (a_4 + a_4 g)(a_1 + a_1 g) = 2a_4 a_1(1 + g) = 0,\]
\[\beta_2^+ \beta_2^+ = 4a_2 a_1(a_3 + a_3' g) = 4a_2 a_1 a_3 - \beta_2^+ \beta_1^+ \beta_2^-,\]

and the remaining relations are

\[\beta_3^+ \beta_2^-, \beta_3^+ \beta_3^+, \beta_2^+ \beta_2^-, \beta_4^+ \beta_1^-, \beta_1^+ \beta_3^+, \beta_2^+ \beta_2^-, \beta_3^+ \beta_3^+, \beta_4^+ \beta_4^- - 2\beta_3^+ \beta_4^+, \beta_1^+ \beta_3^- \beta_2^- - 2\beta_3^+ \beta_4^- .\]

So the stable part of the Auslander–Reiten quiver of \(AG\) is \(ZA_7/t^{12}\) [31].

If we let \(g\) act on the same algebra \(A\) by switching \(e_3\) and \(e'_3\), \(a_2\) and \(a'_2\), \(a_3\) and \(a'_3\) and by sending \(a_1\) and \(a_4\) to their negatives while fixing all other vertices and arrows, the quiver of \(AG\) becomes
where \( \eta_i^\pm \) and \( \beta_i^\pm \) are defined as before. The relations are

\[
\beta_1^+\beta_1^- + \beta_2^+\beta_2^- + \beta_3^+\beta_3^- + \beta_4^+\beta_4^- + \beta_5^+\beta_5^- - 2\beta_4^+\beta_4^- = 0,
\]

In this case, \( AG \) is still selfinjective of tree class \( A \), but it is a "Moebius strip"; i.e., the stable part of its Auslander–Reiten quiver is \( ZA_5/(\tau^7\phi)^2 \), where \( \phi \) is the reflection of \( ZA_4 \) which fixes the vertices on the central line \( [30] \). So \( AG \) need not inherit the property of \( A \) of being weakly symmetric or symmetric.

Consider the algebra \( A' \) which is given by the same quiver \( Q \) and the relations

\[
\alpha_1\alpha_2' - \alpha_3\alpha_4', \alpha_1\alpha_3\alpha_2 - \alpha_5\alpha_4, \alpha_4\alpha_1, \alpha_2\alpha_5, \alpha_2'\alpha_3, \alpha_2\alpha_1\alpha_3, \alpha_2'\alpha_1\alpha_3',
\]
i.e., the selfinjective algebra of class \( D \), whose Auslander–Reiten quiver is given by the same configuration as \( I_A \), but where the fundamental group is generated by \( \tau^7\psi \), where \( \psi \) is the automorphism of \( ZD_5 \) of order 2 \( [30] \). The cyclic group \( G = \mathbb{Z}/2^2 \) operates on \( A' \) nontrivially in two ways as it does on \( A \), and repeating the computations above for \( A' \), we see that \( (A'G)_b \) and \( (AG)_b \) are isomorphic for either operation. Hence \( AG \) and \( A'G \) can be isomorphic for nonisomorphic basic algebras \( A \) and \( A' \).

Now let \( A \) be any two-cornered algebra. We use the notions and results of \([11, 7.1]\). The quiver \( Q \) of \( A \) is obtained by replacing an arrow \( \alpha: x \to y \) in a Brauer quiver \( P \) with \( n - 2 \) vertices by

The automorphism \( g \) of \( Q \) which exchanges \( z \) and \( z' \), \( \gamma_1 \) and \( \delta_1 \), \( \gamma_2 \) and \( \delta_2 \), and fixes all other arrows and vertices induces an automorphism of \( kQ \) under which the relations defining \( A \) are stable. The quiver \( Q_G \) of \( AG \) contains two vertices \( p^+ \) and \( p^- \) for each vertex \( p \neq z, z' \) of \( Q \) and one vertex \( z \) for \( \{z, z'\} \).
Each arrow $\alpha$ (or $\beta$) from $p$ to $q$ in $Q$ gives rise to a pair of arrows $\alpha^+ = \alpha + \alpha g: p^+ \to q^+$, $\alpha^- = \alpha - \alpha g: p^- \to q^-$ (or $\beta^+: p^+ \to q^+$, $\beta^-: p^- \to q^-$). The pair $\gamma^\alpha$, $\delta^\alpha$ of arrows of $Q$ yields $\gamma^\alpha_1 = \gamma_1^\alpha + \gamma^\alpha_1 g: x^+ \to z$ and $\gamma^\alpha_1 = \gamma_1^\alpha - \gamma^\alpha_1 g: x^- \to z$, and from the pair $\gamma^\beta$, $\delta^\beta$ we obtain the two arrows $\gamma^\beta_2 = \gamma_2^\beta + \delta^\beta g: z \to y^+$ and $\gamma^\beta_2 = \gamma_2^\beta - \delta^\beta g: z \to y^-$ of $Q_G$. Clearly, $Q_G$ is a Brauer quiver with $2n - 3$ vertices, obtained from $P$ by first replacing the arrow $\alpha: x \to y$ by the path $x \to z \to y$ and then "doubling" all vertices except $z$ as well as all arrows.

We claim that $AG$ is Morita equivalent to the algebra given by $Q_G$ and the relations of [11, 6.1]; i.e., that $AG$ is weakly symmetric of multiplicity one. Since we already know that $AG$ is selfinjective of finite type, it suffices to exhibit for each vertex $u$ of $Q_G$ a nontrivial path $\sigma: u \to u$ in $Q_G$ which is not zero in $(AG)_b$, but which yields zero if composed with any arrow. If $u$ has the form $p^+$ (or $p^-$) for some vertex $p$ of $P$, we let $b_1 \cdots b_1: p \to p$ be the shortest nontrivial composition of $\beta$-arrows from $p$ to $p$, and we choose $v = \beta^+ b_1 \cdots b_1 (1 + g)$ (or $v = \beta^- b_1 \cdots b_1$). For $u = z$, we take $v = \gamma_1^+ \cdots \gamma_1^+ \delta 1$, where $\alpha_\alpha \cdots \alpha_\alpha: y \to x$ is the shortest composition of $\alpha$-arrows from $y$ to $x$.

Conversely, we start from a Brauer quiver $Q$ with $2n - 3$ vertices which admits an automorphism $g$ that exchanges $\alpha$- and $\beta$-arrows [31, 3]. Then $G = g^{2/2z}$ operates on the algebra $A$ of class $A_{2n - 3}$ which is given by $Q$ and the relations of [31, 6.1]. Similar arguments as above show that $AG$ is Morita equivalent to the two-cornered algebra of class $D_n$ whose quiver is obtained from $Q/G$ by "doubling" the orbit of the fixed point under $g$ and the arrows stopping and starting there. The process $A \to AG$ induces a bijection from the set of two-cornered algebras of class $D_n$ to the set of selfinjective algebras of class $A_{2n - 3}$ which admit an orientation reversing automorphism. Notice that to use our method we need to assume that the characteristic of $k$ is different from 2. It is not, however, for these algebras of class $D_n$ that characteristic 2 plays a special role in the classification of selfinjective algebras [32].

There are three nonisomorphic algebras of class $D_4$ whose quiver is

```
Q = \begin{array}{c}
1 \\
\downarrow \\
0 \\
\downarrow \\
2 \\
\downarrow \\
3
\end{array}
```

depending on the relations. As above for the two-cornered algebras, there are actions of the cyclic groups of order 2 and 3 which take any isomorphism class to any other under $A \to (AG)_b$. 

2.6. Let \( G = G_1 \rtimes G_2 \) be a semidirect product acting on some basic algebra \( A \). Then we know that \( AG = (AG_1) G_2 \), where \( G_2 \) acts on \( AG_1 \) by

\[
g_2(\lambda g_1) = g_2(\lambda) g_2 g_1 g_2^{-1}.
\]

If \( G_1 \) and \( G_2 \) are both cyclic, we know by 2.3 how to compute \( (AG_1)_b \), and if \( AG_1 \) is basic, we can go on and compute \( (AG)_b \). Actually, we can still determine \( (AG)_b \) if \( AG_1 \) is not basic but only contains an idempotent \( e \) which is fixed by \( G_2 \) and such that \( e(AG_1)e \) is basic and Morita equivalent to \( AG_1 \), because of the following lemma.

**Lemma 2.2.** Let \( G \) act on \( A \), and let \( e \) be an idempotent of \( A \) fixed by \( G \). Then \( (eAe)_G \) is isomorphic to \( e(AG)e \). If \( eAe \) is Morita equivalent to \( A \), then \( (eAe)_G \) is Morita equivalent to \( AG \).

**Proof.** The first assertion is clear. As for the second, \( P = Ae \) contains a representative of each isomorphism class of indecomposable projective \( A \)-modules, and hence \( AG \otimes_A P = (AG)e \) has the same property with respect to indecomposable projective \( AG \)-modules. We conclude that \( AG \) is Morita equivalent to \( \text{End}_{AG}(AG)e^{op} = e(AG)e \).

We illustrate this by considering, for instance, the dihedral group \( D_n \) of order \( 2n \). It is the semidirect product of the normal cyclic subgroup \( G = g\mathbb{Z}/n\mathbb{Z} \), the group of rotations, together with the group \( G_2 = h\mathbb{Z}/2\mathbb{Z} \) generated by a reflection, where \( hgh = g^{-1} \). Let \( A \) be given by the quiver

\[
Q = \begin{array}{c}
\varepsilon_2 \\
\varepsilon_1 \\
\varepsilon_0 \\
\varepsilon_n \\
\end{array}
\]

\( D_n \) operates on the regular \( n \)-hedron with center \( \varepsilon_0 \) and the vertices \( \varepsilon_1, \ldots, \varepsilon_n \) in the usual way; i.e., \( g \) is the anticlockwise rotation by \( 2\pi/n \) and \( h \) the reflection fixing \( \varepsilon_n \). We take the induced action on \( Q \). Then \( (AG_1)_b \) is given by

\[
Q' = \begin{array}{c}
\eta_2 \\
\eta_1 \\
\eta_0 \\
\eta_n \\
\eta_{n-1} \\
\end{array}
\]
where

$$\eta_i = \frac{1}{2} \sum_{j=0}^{n-1} \zeta^{ij} e_0 g^j \quad \text{for } i = 0, \ldots, n - 1,$$

$$\eta_n = e_n,$$

$$\beta_i = \sum_{j=0}^{n-1} \zeta^{ij} e_n g^j \quad \text{for } i = 0, \ldots, n - 1.$$

Here $\zeta$ is a primitive $n$th root of unity. Note that

$$h(\eta_0) = \eta_0,$$

$$h(\eta_i) = \eta_{n-i}, \quad i = 1, \ldots, n - 1,$$

$$h(\eta_n) = \eta_n.$$

Thus $AG$ is Morita equivalent to $(kQ')G_2$, whose quiver is

if $n$ is odd and

if $n$ is even.

2.7. Obtaining examples of crossed products is a little more complicated. In particular, if $G$ is a cyclic group, $A$ a connected algebra and $\gamma$ any 2-cocycle on $G$ with values in $U(A) \cap Z(A)$, the units in the center of
A, then $A \ast \gamma G$ is isomorphic to the skew group algebra $AG$. For we show in 5.3 that there exists a map $\delta: G \to U(A) \cap Z(A)$ such that

$$\gamma(g, h) = \delta(g) \delta(h) \delta(gh)^{-1}$$

for any two elements of $G$. The isomorphism

$$A \ast \gamma G \to AG$$

is given by

$$\lambda \to \lambda \quad \text{for} \quad \lambda \in A,$$

$$g \to \delta(g)g \quad \text{for} \quad g \in G.$$ 

The smallest group $G$ with $H^2(G, k^*) \neq 1$ is $G = g^{\mathbb{Z}/2\mathbb{Z}} \times h^{\mathbb{Z}/2\mathbb{Z}}$. It is easy to see that the map $\gamma: G \times G \to k^*$ given by

$$\gamma(g, h) = \gamma(gh, g) = \gamma(h, gh) = i,$$

$$\gamma(h, g) = \gamma(g, gh) = \gamma(gh, h) = -i,$$

and by the value 1 on the remaining pairs of groups elements, is a cocycle, but not a coboundary. We will give some examples of algebras $A \ast G$, using the group $G$ and the cocycle $\gamma$ above.

(a) Let $A = k$ with $G$ acting trivially. The map

$$\phi: M_2(k) \to A \ast G$$

given by

$$\phi(E_{11}) = \frac{1}{2}(e + e\overline{g}), \quad \phi(E_{12}) = \frac{1}{2}(e\overline{h} + ieg\overline{h}),$$

$$\phi(E_{21}) = \frac{1}{2}(e\overline{h} - ieg\overline{h}), \quad \phi(E_{22}) = \frac{1}{2}(e - e\overline{g})$$

is easily seen to be an isomorphism of algebras, where $e$ is the identity of $k$ and $E_{ij}$ the elementary matrix with a 1 is the $i$th row and $j$th column.

(b) Choose $A = k\varepsilon \times k\eta$, and let $G$ operate by $g(\varepsilon) = \varepsilon$, $g(\eta) = \eta$, $h(\varepsilon) = \eta$, $h(\eta) = \varepsilon$. Then we obtain an isomorphism

$$M^{(1)}_2(k) \times M^{(2)}_2(k) \overset{\phi}{\to} A \ast G$$

by

$$\phi(E^{(1)}_{11}) = \frac{1}{2}(\varepsilon + e\overline{g}), \quad \phi(E^{(1)}_{11}) = \frac{1}{2}(e\overline{h} + ieg\overline{h}),$$

$$\phi(E^{(1)}_{12}) = \frac{1}{2}(\eta\overline{h} - inger\overline{h}), \quad \phi(E^{(1)}_{22}) = \frac{1}{2}(\eta - \eta g),$$

$$\phi(E^{(2)}_{11}) = \frac{1}{2}(\eta + \eta g), \quad \phi(E^{(2)}_{12}) = \frac{1}{2}(\eta\overline{h} + ieg\overline{h}),$$

$$\phi(E^{(2)}_{21}) = \frac{1}{2}(e\overline{h} - ieg\overline{h}), \quad \phi(E^{(2)}_{22}) = \frac{1}{2}(e - e\overline{g}).$$
So the result is the same as for the skew group algebra $A^G$ (with the same action), although $\gamma$ is not a coboundary.

(c) Take $A = \prod_{i=1}^{4} k e_i$, and let $G$ act as the Klein 4-group permuting the four idempotents, e.g., $g = (e_1 e_2)(e_3 e_4)$ and $h = (e_1 e_3)(e_2 e_4)$. Then $A * G$ is a $4 \times 4$ matrix ring, and we define the isomorphism $\phi$ by putting $\phi(F_{ij})$ in the $i$th row and $j$th column of a $4 \times 4$ matrix:

\[
\begin{pmatrix}
  e_1 & e_1 g & e_1 h & e_1 gh \\
  e_2 g & e_2 & ie_2 gh & -ie_2 h \\
  e_3 h & -ie_3 gh & e_3 & ie_3 g \\
  ie_4 h & ie_4 g & -ie_4 & e_4
\end{pmatrix}
\]

(d) Finally, let $A$ be given by the quiver

\[
\begin{array}{c}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\varepsilon_4
\end{array}
\begin{array}{c}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4
\end{array}
\]

and take the action of $G$ on $A$ induced by the action on $\{e_1, \ldots, e_4\}$ of (c). Then the quiver of $A * G$ is

\[
e_1 \xrightarrow{\alpha_1 + \alpha_2} e_1 \quad \text{with} \quad \frac{1}{2}(\eta + \eta g).
\]

3. Skew Group Categories

Many properties of artin algebras which are important in representation theory are expressed in terms of almost split sequences. Hence it is of interest to investigate how almost split sequences behave with respect to constructing skew group algebras and crossed product algebras.

To do this, it is convenient to generalize the constructions and some of the results of Section 1 from mod $A$ to the categories mod(mod $A$) and mod((mod $A$)$^o$) of finitely presented contravariant and covariant functors from mod $A$ to abelian groups. Starting with an additive category $C$ with a finite group $G$ acting, we define the skew group category $CG$. We shall assume that any $f$ in any Hom$_C(X, Y)$ is uniquely divisible by $|G|$, which we
for short refer to as $|G|$ being invertible in $C$. We investigate some connections between $C$ and $CG$ which are of interest in themselves. The categorical point of view is useful also when dealing only with module categories, in making more apparent what is going on. Also in connection with stable equivalence it is useful to be able to make our constructions directly with the stable category mod $A$. The categorical description is further convenient for seeing the close connection with finite coverings, to be discussed in Section 5.

3.1. Let $C$ be a preadditive category and $G$ a finite group of order $n$, and assume that we have a group homomorphism from $G$ to the group of equivalences from the category $C$ to itself. We define the category $C\{G\}$ as follows. The objects are $n$-tuples $\{X\} = \{g_iX, 1 \leq i \leq n\}$. Here $e = g_1, g_2, \ldots, g_n$ are the distinct elements of $G$, and $g_iX$ denotes the image of $X$ under the equivalence given by $g_i$. A morphism $f: \{X\} \rightarrow \{Y\}$ is an $n \times n$ matrix $(f_{g_i,g_j})$ where $f_{g_i,g_j}: X \rightarrow g_jY$ are arbitrary morphisms in $C$, and $f_{g_i,g_j}: g_iX \rightarrow g_jY = g_j(f_{g_i,g_j}^{-1}g_jX)$. Composition is given by multiplying matrices. Alternatively, a morphism is a matrix $(f_{g_i,g_j})$ which is left fixed under the action of each $g$ in $G$.

The category $C\{G\}$ is again a preadditive category, which is additive if $C$ is. Even if we assume that idempotents split in $C$, this is not necessarily the case for $C\{G\}$, as we shall soon see. We denote by $CG$ the category we get by making idempotents split in $C\{G\}$ (see [22]), and call $CG$ the skew group category of $C$ by $G$.

Assume now that $C$ is an additive category and that idempotents split in $C$. We also assume that $|G|$ is invertible in $C$, which is the case, for example, if $C$ is a category over a field $k$ with $|G|$ invertible in $k$. We have the following natural functors between $C$ and $C\{G\}$, and $C$ and $CG$. We define $F: C \rightarrow C\{G\} \subset CG$ by $F(X) = \{X\}$ for $X$ in $C$. If $h: X \rightarrow Y$ is a morphism in $C$, we define $F(h) = (f_{g_i,g_j})$ where $f_{g_i,g_j} = g_j(h)$ and $f_{g_i,g_j} = 0$ for $i \neq j$.

$H: C\{G\} \rightarrow C$ is defined by $H(\{g_iX, 1 \leq i \leq n\}) = \bigcup_{i=1}^{n} g_iX$. If $S: \{X\} \rightarrow \{Y\}$ is a morphism in $C\{G\}$, then $H(f): \bigcup_{i=1}^{n} g_iX \rightarrow \bigcup_{i=1}^{n} g_iY$ is induced by the matrix $(f_{g_i,g_j})$ in the natural way. $H$ induces a functor $H: CG \rightarrow C$.

Let $A$ be an artin algebra and $G$ a finite group of order $n$ with a group homomorphism $G \rightarrow \text{Aut}(A)$. For $g \in \text{Aut}(A)$ we consider the induced equivalence from mod $A$ to mod $A$ given by sending $X$ to $gX$ for $X$ in mod $A$. Let $X$ and $Y$ be in mod $A$. By adjointness we have $\text{Hom}_{\text{mod }A}(A g \otimes \Lambda X, A g \otimes \Lambda Y) \simeq \text{Hom}_{\Lambda}(X, A g \otimes \Lambda Y) \simeq \text{Hom}_{\Lambda}(X, e \otimes Y \oplus \cdots \oplus g_n \otimes Y)$. We have seen that we have natural isomorphisms $g_i \otimes Y \simeq g_iY$. By considering what the induced $A$-map $A g \otimes \Lambda X \rightarrow A g \otimes \Lambda Y$ looks like, it is easy to see that sending $A g \otimes \Lambda X$ to $\{g_iX, 1 \leq i \leq n\}$ induces an equivalence of categories between the full subcategory of mod $A$ whose objects are of the form $A g \otimes \Lambda X$ for $X$ in mod $A$, and $(\text{mod }A)\{G\}$. If $n$ is invertible in $A$, there is an
induced equivalence between \( \text{mod } A G \) and \( (\text{mod } A )G \). Since \( A G \otimes_A X \) is not necessarily an indecomposable \( A G \)-module even when \( X \) is an indecomposable \( A \)-module, we see that idempotents do not necessarily split in \( (\text{mod } A )G \).

By restriction, if \( p(A) \) denotes the category of finitely generated projective \( A \)-modules, \( p(A)G \) and \( p(AG) \) are equivalent categories.

3.2. As for algebras, we isolate some of the essential properties of \( C \cong^F_H CG \). So let \( C \) and \( D \) be additive categories and \( F: C \to D \) and \( H: D \to C \) additive functors. We denote by \( I: C \to C \) and \( J: D \to D \) the identity functors, and consider the following properties.

(A) (i) There is a split monomorphism of functors \( I \to HF \).

(ii) There is a split epimorphism of functors \( FH \to J \).

(B) \( (F, H) \) and \( (H, F) \) are adjoint pairs of functors.

We postpone the discussion of an analogue of property (C), which is somewhat more complicated, to the next section, and show here that (A) and (B) hold for our skew group categories.

**Theorem 3.2.** Let \( C \) be an additive category where idempotents split, \( G \) a finite group, with a group homomorphism \( G \to \text{Aut}(C) \). Assume that \( |G| \) is invertible in \( C \). Then (A) and (B) hold for \( C \cong^F_H CG \).

**Proof.** For \( X \) in \( C \), consider the monomorphism \( I(X) = X \to HF(X) = eX \oplus \cdots \oplus s^nX \) given by sending \( X \) to the first coordinate. Defining the map \( HF(X) \to X \) to be the projection to the first summand, we get a splitting which is easily seen to be functorial.

Let \( Y = \{eX, \ldots, s^nX\} \) be in \( C[G] \). Consider the natural map \( FH(Y) = \{e(X \oplus \cdots \oplus s^nX), \ldots, s^n(X \oplus \cdots \oplus s^nX)\} \to Y = \{eX, \ldots, s^nX\} \) given by letting \( e(X \oplus \cdots \oplus s^nX) \to s_iX \) be the \( i \)th projection map. This gives a morphism of functors \( FH \to J \). To get a splitting, define \( t_Y: Y \to FH(Y) \), where \( t_{s_i}e: eX \to s_i(eX \oplus \cdots \oplus s^nX) \) is given by sending \( eX \) to \( s_iX \), \( s_i \) by the identity. Letting \( s = 1/n \), \( s: J \to FH \) is our desired morphism of functors. Finally, we extend from \( C[G] \) to \( CG \). This finishes the proof of (A).

Let \( X \) be in \( C \) and \( Z = \{eY, \ldots, s^nY\} \) in \( C[G] \). Since by definition a morphism \( f = (f_{e,Y}, s^nY): \{eX, \ldots, s^nX\} \to \{eY, \ldots, s^nY\} \) in \( C[G] \) is uniquely determined by the \( f_{s_i}e: X \to s_iY \), we have a natural isomorphism \( (X, eY \oplus \cdots \oplus s^nY) \to (eX, \ldots, s^nY) \), that is, \( (X, HZ) \to (FX, Z) \). Since also the \( f_{s_i}e: X \to Y \) determine \( f \) uniquely, we have a natural isomorphism \( (HZ, X) \to (Z, FX) \). This shows that \( (F, H) \) and \( (H, F) \) are adjoint pairs of functors between \( C \) and \( C[G] \), hence also between \( C \) and \( CG \).
We have the following analogue of part of Theorem 1.3, which by Theorem 3.2 specializes to skew group categories.

**Proposition 3.3.** Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories with enough projectives and $F: \mathcal{C} \to \mathcal{D}$, $H: \mathcal{D} \to \mathcal{C}$ additive functors satisfying (A) and (B). Then the following hold.

(a) $\text{gl.dim.} \mathcal{C} = \text{gl.dim.} \mathcal{D}$.

(b) $\mathcal{C}$ is 1-Gorenstein if and only if $\mathcal{D}$ is.

(c) $\text{dom.dim.} \mathcal{C} = \text{dom.dim.} \mathcal{D}$.

(d) The projectives coincide with the injectives in $\mathcal{C}$ if and only if they do in $\mathcal{D}$.

**Proof.** Again $F$ and $H$ clearly are exact and preserve projectives and injectives, and the proof follows from this. Here we say that $\mathcal{C}$ is 1-Gorenstein if for each projective $P$ in $\mathcal{C}$ there is a monomorphism $P \to I$ with $I$ projective injective, and $\text{dom.dim.} \mathcal{C} \geq i$ if for each projective $P$ there is an exact sequence $0 \to P \to I_0 \to I_1 \cdots I_{i-1}$ with the $I_j$ projective injective.

3.3. Even if we are only interested in studying the categories $\text{mod } A$ for an artin algebra $A$, the study of almost split sequences leads us naturally to investigating the category $\text{mod}(\text{mod } A)$ of finitely presented contravariant functors from $\text{mod } A$ to abelian groups. For an almost split sequence $0 \to X \to Y \to Z \to 0$ induces a simple object $F$ determined by the exact sequence of functors $0 \to (, X) \to (, Y) \to (, Z) \to F \to 0$. Hence the question of what happens to almost split sequences under the skew group algebra construction is closely connected with the question of what happens to simple objects, in another category, and this is closely related to property (C). The following commutativity results is useful here.

**Proposition 3.4.** Let $\mathcal{C}$ be an additive category where idempotents split, $G$ a finite group acting, with the previous assumptions. With the action of $G$ on $\text{mod } \mathcal{C}$, induced by sending $(, C)$ to $(, ^G C)$, $(\text{mod } \mathcal{C})G$ and $\text{mod}(\mathcal{C}G)$ are equivalent categories.

**Proof.** $F: \text{mod } \mathcal{C} \to (\text{mod } \mathcal{C})G$ and $H: (\text{mod } \mathcal{C})G \to \text{mod } \mathcal{C}$ preserve epimorphisms and projectives (where projective is defined via the lifting property) using that $(F, H)$ and $(H, F)$ are adjoint pairs of functors. Since $\mathcal{C}$ is equivalent to the projectives in $\text{mod } \mathcal{C}$, we then get that $\mathcal{C}G$ is equivalent to the projectives in $\text{mod } \mathcal{C}$. So the full subcategories of projectives in $\text{mod } \mathcal{C}G$ and $(\text{mod } \mathcal{C})G$ are equivalent. To obtain the desired equivalence between $\text{mod } \mathcal{C}G$ and $(\text{mod } \mathcal{C})G$, it suffices to show that like $\text{mod } \mathcal{C}G$, also $(\text{mod } \mathcal{C})G$ has projective presentations and cokernels. It is easy to see that we have projective presentations of the objects in $(\text{mod } \mathcal{C})G$. Consider a map.
f: FA → FB, with A and B in modC. Since mod C has cokernels, we have an exact sequence \( H(FA) \to H(FB) \to X \to 0 \), hence an exact sequence \( FH(FA) \to FH(FB) \to FX \to 0 \). By (A)(ii) we have a commutative diagram

\[
\begin{array}{ccc}
FH(FA) & \xrightarrow{FH(f)} & FH(FB) \to FX \to 0 \\
\downarrow s & & \downarrow t \\
FA & \xrightarrow{f} & FB \\
\downarrow u & & \downarrow v \\
FH(FA) & \xrightarrow{FH(f)} & FH(FB) \to FX \to 0
\end{array}
\]

such that \( us \) and \( vt \) are idempotents and \( su = 1_{FA} \) and \( tv = 1_{FB} \). There is then induced an idempotent \( e: FX \to FX \). Since idempotents split in \((\text{mod } C)_G\), \( \text{Im } e \) exists. It is not hard to see that we have an exact sequence \( FA \to f FB \to \text{Im } e \to 0 \), so that \( f: FA \to FB \) has a cokernel in \((\text{mod } C)_G\). It now follows that \((\text{mod } C)_G\) and \(\text{mod}(CG)\) are equivalent categories.

In dealing with an analogue of property (C), we shall make more assumptions on our additive category. We want a setting containing the categories \(\text{mod}(\text{mod } A)\) for an artin algebra \( A \). It will be convenient to assume that we have \text{mod } C for some dualizing \( R \)-variety \( C \), where \( R \) is a commutative artin ring, that is, each \( \text{Hom}_C(X, Y) \) is a finitely generated \( R \)-module, and \((C^{\text{op}}, \text{mod } R) \to (C, \text{mod } R)\) induces a duality between the categories of finitely presented functors \text{mod } C and \text{mod } C^{\text{op}} [5]. Here \((C^{\text{op}}, \text{mod } R)\) denotes the category of additive contravariant functors from \( C \) to \text{mod } R, and \((C, \text{mod } R)\) the category of additive covariant functors from \( C \) to \text{mod } R. \text{mod } C \) is known to be a dualizing \( R \)-variety, and an abelian category having projective covers [5].

We consider the following condition for the setup \(\text{mod } C \simeq_{\text{hu}} \text{mod } D\), for dualizing \( R \)-varieties \( C \) and \( D \).

\( \text{(C) } F \) and \( H \) preserve semisimple objects and projective covers.

It will be useful to consider the following related property for \( C \simeq_{\text{hu}} D \), which clearly holds in the setting \( C \simeq_{\text{hu}} CG \), for an additive category \( C \) with idempotents splitting and \( |G| \) invertible.

\( \text{(C') } \) There is a finite group \( G = \{e, g_2, ..., g_n\} \) acting on \( C \) such that for \( C \) in \( C \) \( HF(C) = \bigsqcup_{g \in G} gC \) and if \( f: X \to Y \) is in \( C \), there is a decomposition \( HF(f) = (f, g_2(f), ..., g_n(f)) : eX \sqcup ... \sqcup eX \to eY \sqcup ... \sqcup eY \).

The importance of \( \text{(C') } \) is that it shares with \( \text{(A) } \) and \( \text{(B) } \) the following fact, which is not too hard to prove.

**Proposition 3.5.** Let \( C \) and \( D \) be additive categories with functors \( F: C \to D \) and \( H: D \to C \). If one of the properties \( \text{(A), (B), (C') } \) holds, then it
holds for \( \text{mod } C \) and \( \text{mod } D \), with the induced functors \( F: \text{mod } C \to \text{mod } D \) and \( H: \text{mod } D \to \text{mod } C \), induced by \( F((, C)) = (, FC) \) and \( H((, D)) = (, HD) \), for \( C \) in \( \mathbf{C} \) and \( D \) in \( \mathbf{D} \).

We now prove property (C) for our skew group categories, by proving the following result.

**Theorem 3.6.** Let \( C \) and \( D \) be dualizing \( R \)-varieties with functors \( F: \text{mod } C \to \text{mod } D \) and \( H: \text{mod } D \to \text{mod } C \) satisfying (A), (B) and (C). Then (C) also holds.

**Proof.** We first prove that \( F \) preserves semisimple objects and projective covers. We claim that if \( Y \) is indecomposable in \( \text{mod } D \) and \( HY \) is semisimple, then \( Y \) is simple. For assume that \( Y \) is not simple. Then there is some nonsplit exact sequence \( 0 \to X \to Y \to Z \to 0 \). Since \( HY \) is semisimple, \( 0 \to HX \to HY \to HZ \to 0 \) splits. Consider

\[
0 \to FH(X) \to FH(Y) \to FH(Z) \to 0
\]

where the upper sequence then splits, and by (A)(ii) \( FH(Y) \to Y \) is a split epimorphism which is functorial in \( Y \). \( Y \to Z \) would then also be a split epimorphism, which contradicts our assumptions.

If \( Y \) is simple in \( \text{mod } C \), then \( HF(Y) \) is semisimple. For we know that \( HF(Y) = eY \sqcup \cdots \sqcup e^n Y \), and all the \( e^n Y \) are simple since \( g_i: C \to C \) is an equivalence of categories. Combining with the above we have that \( Y \) simple implies \( FY \) semisimple.

We next claim that \( F((, X)/r(, X)) = (, FX)/r(, FX) \) for \( X \) indecomposable in \( C \). Here \( r(, X) \) denotes the radical of \( (, X) \), that is, the intersection of the maximal subfunctors. Since \( F \) and \( H \) take projectives to projectives, they induce functors between \( \mathbf{C} \) and \( \mathbf{D} \) which we also denote by \( F \) and \( H \). Applying \( F \) to \( (, X) \to (, X)/r(, X) \to Y \to 0 \), we get that \( (, FX) \to F((, X)/r(, X)) \to 0 \) is exact, so that \( (, Y) \to F((, X)/r(, X)) \to 0 \) is a projective cover, where \( Y \) is a summand of \( FX \). Applying \( H \) gives that \( (, HY) \to HF((, X)/r(, X)) \to 0 \) is exact. Since \( HF(Y) = eY \sqcup \cdots \sqcup e^n Y \), its projective cover is \( (, eX \sqcup \cdots \sqcup e^n X) \). Since \( HY \) is a summand of \( HFX = eX \sqcup \cdots \sqcup e^n X \), we must then have \( HY = HFX \), so that \( Y = FX \), and the claim follows.

We now show that \( H \) preserves semisimple objects and projective covers. Let \( \tilde{Y} \) be simple in \( \text{mod } D \). Write \( \tilde{Y} = (, Y)/r(, Y) \), and choose by (A)(ii) \( X | HY \) in \( \mathbf{C} \) such that \( Y | FX \). By the above we have that \( F((, X)/r(, X)) = (, FX)/r(, FX) \), and \( \tilde{Y} = (, Y)/r(, Y) \) is a summand of \( (, FX)/r(, FX) \).
We have $H((,FX))/r((,FX)) = (,X)/r((,X)) \amalg \cdots \amalg \bigotimes (,X)/r\otimes (,X) = (,HF(X))/r((,HF(X)))$, which is semisimple. This shows that $H\tilde{Y}$ is semisimple, and that $H$ preserves projective covers.

This finishes the proof of the theorem.

We have the following immediate consequence, using that $(\text{mod } C)G$ is equivalent to $\text{mod}(CG)$ by Proposition 3.4, and hence is a dualizing $R$-variety if $CG$ is.

**Corollary 3.7.** If $C$ and $CG$ are dualizing $R$-varieties, with the usual assumptions on $G$, then $\text{mod } C \xrightarrow{H} (\text{mod } C)G$ satisfies (A), (B), (C), ($\tilde{C}$).

We can now get information on almost split sequences. We are mostly interested in $\text{mod } A$ in this paper, but we give the proof more generally for dualizing $R$-varieties, since it might be interesting in other situations. For example, all the categories occurring as covering of categories given by algebras are dualizing $R$-varieties [29].

**Theorem 3.8.** Let $C$ be a category with a finite group $G$ acting such that $|G|$ is invertible, and assume that $C$ and $CG$ are dualizing $R$-varieties. Let the functors $F$ and $H$ between $\text{mod } C$ and $(\text{mod } C)G$ be as before.

(a) If $0 \to X \to Y \to Z \to 0$ is an almost split sequence in $\text{mod } C$ (or $(\text{mod } C)G$), then $0 \to FX \to FY \to FZ \to 0$ (or $0 \to HX \to HY \to HZ \to 0$) is a direct sum of almost split sequences in $(\text{mod } C)G$ (or $\text{mod } C$).

(b) If $X \to Y$ is a minimal left or right almost split map in $\text{mod } C$ (or $(\text{mod } C)G$), then $FX \to FY$ (or $HX \to HY$) is a direct sum of minimal left or right almost split maps in $(\text{mod } C)G$ (or $\text{mod } C$).

**Proof.** To see this, we use the functorial description of almost split sequences and minimal right and left almost split maps. For if $Y$ is indecomposable, $f: X \to Y$ in $\text{mod } C$ is minimal right almost split if and only if $(,X) \to (,Y) \to \text{Coker}(,f) \to 0$ is a minimal projective presentation in $\text{mod}(\text{mod } C)$ with $\text{Coker}(,f)$ simple [5]. And if $X$ is indecomposable, $f: X \to Y$ in $\text{mod } C$ is minimal left almost split if and only if $(Y, \to X, \to \text{Coker}(,f)\to 0$ is a minimal projective presentation in $\text{mod}(\text{mod } C^{op})$, with $\text{Coker}(,f)$ simple.

We now use that $\text{mod } C \xrightarrow{H} (\text{mod } C)G$ satisfies (A), (B), ($\tilde{C}$), hence so does the induced diagram $\text{mod}(\text{mod } C) \xrightarrow{H} \text{mod}(\text{mod } C)G$. Hence (C) holds, that is, semisimple objects and projective covers are preserved.

3.4. Considering the property ($\tilde{C}$) was convenient for clarifying the structure of our proofs. But using ($\tilde{C}$) in itself at the end of the previous proof was not so essential, since by Proposition 3.4, $\text{mod}(\text{mod } C)G$ and $(\text{mod}(\text{mod } C))G$ are equivalent categories. If we want to get a similar relationship between almost split sequences over $A$ and $A \ast G$, however, it is
important to consider (C). For (C) is clearly satisfied for \( \mod A \cong H \mod A \ast G \), under the usual assumptions on \( G \), and hence the proof of Theorem 3.8 goes as above.

An alternative way of treating the relationship between \( A \) and \( A \ast G \) would be to try to generalize the notion of skew group categories to crossed product categories, a construction which would also be of interest in its own right. We indicate how this can be done for the case we consider in this paper, namely, the case when the 2-cocycle \( \gamma \) defined on \( G \times G \) has its values in the units of the center of \( A \). For the units of the center have a categorical interpretation. So let \( C \) be an additive category with a finite group \( G \) acting such that \( |G| \) is invertible. Let \( \gamma \) be a map from \( G \times G \) to the natural automorphisms of the identity functor for \( C \), satisfying the following properties, like those for rings:

1. \( \gamma(gg', g'') = \gamma(g, g'g'') \gamma(g, g'g'') \) for \( g, g', g'' \) in \( G \).

2. \( \gamma(e, g) = 1 = \gamma(g, e) \) for \( g \in G \), \( e \) identity of \( G \).

The only difference from the definition of the skew group category is the following.

A morphism \( f: \{X\} \to \{Y\} \) is now an \( n \times n \) matrix \((f_{g_i, 0}, f_{g_i, e})\), where as before \( f_{g_i, 0}, f_{g_i, e}: \mathcal{C}X \to \mathcal{C}Y \) are arbitrary morphisms in \( C \) and now \( f_{g_i, 0}, f_{g_i, e}: \mathcal{C}X \to \mathcal{C}Y \) is \( \gamma(g_j, g_j^{-1}g_i) g_j(f_{g_i, 0}, f_{g_i, e}) \). Again composition is defined by multiplying matrices. It is easy to check that the composed matrix is of the desired form and that composition is associative.

As for skew group categories, we have that if \( A \) is an algebra, \( \gamma \) a 2-cocycle with values in the units in the center of \( A \), and \( \gamma \) also denotes the induced 2-cocycle for \( \mod A \), then the categories \( \mod(A \ast G) \) and \( \mod(A) \ast G \) are equivalent.

3.5. The categorical description is also useful in connection with stable equivalence, as it is then possible to apply the skew group category construction directly to the stable category \( \mod A \). We can then construct new pairs of algebras which are stably equivalent.

Let \( A \) and \( A' \) be stably equivalent algebras, and assume that there are automorphisms \( \sigma: A \to A \) and \( \sigma': A' \to A' \) of order \( n \) such that the following diagram commutes.

\[
\begin{array}{ccc}
\mod A & \xrightarrow{g} & \mod A \\
\downarrow{\alpha} & & \downarrow{\alpha} \\
\mod A' & \xrightarrow{g'} & \mod A'
\end{array}
\]

Here \( \sigma \) and \( \sigma' \) denote the induced equivalences of categories, and \( \alpha: \mod A \to \mod A' \) is an equivalence. Let \( G = \langle g \rangle \) be the cyclic group of order \( n \), acting
on mod $A$ by sending $g$ to $g$. From the commutative diagram it follows that $(\text{mod } A)G$ and $(\text{mod } A')G$ are equivalent categories. Further, we have the following.

**Lemma 3.9.** $(\text{mod } A)G$ and $(\text{mod } A)G$ are equivalent categories.

*Proof.* For $\{X\}$ and $\{Y\}$ in $(\text{mod } A)G$ we need to show that a map $f = (f_{g_i}, g_j): \{X\} \to \{Y\}$ factors through a projective object in $(\text{mod } A)G$ if and only if each $f_{g_i}$ factors through a projective object in mod $\Lambda$. This follows easily, using that $\{X\}$ is projective if and only if $X$ is projective in mod $\Lambda$, if and only if all $g_iX$ are projective in mod $\Lambda$. We first get an equivalence between $(\text{mod } A)\{G\}$ and $(\text{mod } A')\{G\}$. Then we use that idempotents in $\text{End}\{X\}$ can be lifted to idempotents in $\text{End}\{X\}$.

As a consequence we can now deduce that $(\text{mod } A)G$ and $(\text{mod } A')G$ are equivalent categories. That is, we have shown that $AG$ and $A'G$ are stably equivalent algebras.

We point out that there are examples of selfinjective algebras $A$ with a group of automorphisms acting in two different ways so that the resulting skew group algebras are not stably equivalent. Hence the commutativity of the above diagram is essential for our conclusions.

3.6. In the case $\mathbf{C} = \text{mod } A$ and $\mathbf{D} = \text{mod } \Gamma$ we deal in this chapter with the setting $\text{mod } A \rightleftharpoons \text{mod } \Gamma'$, with $(F, H)$ an adjoint pair of functors. In Section 1 we started with a ring monomorphism $i: \Lambda \to \Gamma$ and considered the induced pair of functors $(F, H)$. To tie things together, we point out when a pair of adjoint functors $(F, H)$ is induced by a ring map.

**Proposition 3.10.** For $\text{mod } A \rightleftharpoons \text{mod } \Gamma$, where $(F, H)$ is an adjoint pair of functors, there is a commutative diagram

$$
\begin{array}{ccc}
\text{mod } A & \xrightarrow{F} & \text{mod } \Gamma \\
\downarrow{K} & & \downarrow{L} \\
\text{mod } A' & \xleftarrow{F'} & \text{mod } \Gamma'
\end{array}
$$

with $K, L$ equivalences of categories, such that $F'$ is induced by a ring map $f: \Lambda' \to \Gamma'$, if and only if

(i) $F$ preserves projectives.

(ii) If $Q$ is projective in mod $\Gamma$, there is some projective $P$ in mod $\Lambda$ with $Q \mid FP$.

*Proof.* Since $(F, H)$ is an adjoint pair, there is a bimodule $rE_A$ such that $F = rE_A \otimes$ and $H = \text{Hom}_r(rE_A, )$. We see that (i) and (ii) are equivalent to
$rE$ being a projective generator. If $F$ is induced by a ring map $f:A \to \Gamma$, then $rE_A = r\Gamma_A$, hence a projective $\Gamma$-generator.

Assume now that $rE_A$ is a finitely generated projective $\Gamma$-generator, and consider the composition $\mathrm{mod} A \to rE_A \otimes \mathrm{mod} \Gamma \to (rE)_{\gamma} \otimes \mathrm{mod} \Omega$, where $\Omega = \text{End}_{\Gamma}(rE)^{op}$. $(rE)^{\ast} \otimes \text{Hom}_{\Gamma}(rE, \gamma)$ is a Morita equivalence. Now $(rE)^{\ast} \otimes \Gamma rE_A \simeq \text{Hom}_{\Gamma}(rE_A, rE_A) = \Omega$ as $\Omega - \Lambda$-bimodule, so that the composite of the above functors is $\gamma_{\Omega A} \otimes$, which is given by the ring map $A \to \Omega$ inducing the structure of $\Omega$ as right $\Lambda$-module. This finishes the proof.

We also mention that in the presence of (A)(i), the ring map must be a monomorphism.

We also comment on the relationships between condition (C) in this section and the corresponding one in Section 1. We observe that if we deal with module categories, $F$ preserves semisimple objects if and only if $F(A/rA) = r/r$ is semisimple, that is, if and only if $r\Gamma \subseteq \text{rad} \Gamma$. $F$ preserves projective covers if and only if $F(A) = \Gamma \to F(A/rA) = r/r$ is a projective cover, which happens if and only if $r\Gamma \subseteq r$. Hence $F$ preserves both semisimple objects and projective covers if and only if $r\Gamma = \text{rad} I$, and in this case $r\Gamma \cap A = r$, so that $H$ automatically preserves semisimple objects and projective covers. To get a condition specializing to $r\Gamma = \text{rad} \Gamma = r$ for module categories, we could have added the dual condition that our functors also preserve injective envelopes, but this was not necessary for the results we wanted to prove. For we have a commutative diagram

\[
\begin{array}{ccc}
\text{mod} A & \xrightarrow{F} & \text{mod} \Gamma \\
\downarrow{D} & & \downarrow{D'} \\
\text{mod} A^{op} & \xrightarrow{F'} & \text{mod} \Gamma^{op}
\end{array}
\]

since $D'(\Gamma \otimes A X) \simeq DX \otimes_A \Gamma$, where $D$ and $D'$ are the ordinary dualities.

4. Relationships between $A$ and $AG$

In this section we apply some of the results of the previous section to investigate the relationship between $A$ and $AG$ or $A \ast G$ with respect to ring theoretic properties defined in terms of almost split sequences and irreducible maps. We prove that there are no cycles of irreducible maps for $A$ if and only if the same is true for $A \ast G$, under the usual assumptions on $G$. We also investigate the relationship between the preprojective partitions for $A$ and $A \ast G$, and we discuss the property of being a tilted algebra. In the last section we discuss the property of being an Auslander algebra. We assume throughout that $A$ is an artin algebra, and $G$ acts on $A$ with $|G|$ invertible in $A$. 
4.1. We have seen in 3.3 and 3.4 that the functors $F: \text{mod } \mathcal{A} \to \text{mod } \mathcal{A} \ast G$ and $H: \text{mod } \mathcal{A} \ast G \to \text{mod } \mathcal{A}$ carry almost split sequences to a direct sum of almost split sequences and minimal left or right almost split maps to a direct sum of minimal left or right almost split maps.

As a consequence we get a close connection between irreducible maps in \(\text{mod } \mathcal{A}\) and \(\text{mod } \mathcal{A} \ast G\) (or \(\text{mod } \mathcal{A} \ast G\)). For \(X\) in \(\text{ind } \mathcal{A}\), we denote by \([X]\) the set of indecomposable \(\mathcal{A}\)-modules in the \(G\)-orbit of \(X\). For \(Z\) in \(\text{ind } \mathcal{A} \ast G\), choose \(X\) in \(\text{ind } \mathcal{A}\) such that \(Z\) is a summand of \(\mathcal{A} \ast G \otimes_{\mathcal{A}} X\). Denote by \([Z]\) the set of nonisomorphic indecomposable summands of \(\mathcal{A} \ast G \otimes_{\mathcal{A}} X\). When \(X\) and \(Z\) are related as above, we write \([X] = [Z]\). We then have \([X] = [Z] \Leftrightarrow Z \mid FX \Rightarrow ^gX \mid HZ\) for some \(g \in G\). This gives partitions of \(\text{ind } \mathcal{A}\) and \(\text{of } \text{ind } \mathcal{A} \ast G\).

It is convenient to make the following definitions. If there is an irreducible map from an object in \([X]\) to an object in \([Y]\), we say that there is an irreducible map \([X] \rightarrow [Y]\). Then we have the following relationship between \(\mathcal{A}\) and \(\mathcal{A} \ast G\).

**Lemma 4.1.** With the usual assumptions and notation, the following are equivalent, where \(X\) and \(Y\) are in \(\text{ind } \mathcal{A}\).

(a) There is an irreducible map \([X] \rightarrow [Y]\).
(b) Given \(X'\) in \([X]\) there is some irreducible map \(X' \rightarrow Y'\) with \(Y'\) in \([Y]\).
(c) Given \(Y'\) in \([Y]\) there is some irreducible map \(X' \rightarrow Y'\) with \(X'\) in \([X]\).
(d) There is an irreducible map \([\tilde{X}] \rightarrow [\tilde{Y}]\).
(e) Given \(Z\) in \([\tilde{X}]\) there is some irreducible map \(Z \rightarrow U\) with \(U\) in \([\tilde{Y}]\).
(f) Given \(U\) in \([\tilde{Y}]\) there is some irreducible map \(Z \rightarrow U\) with \(Z\) in \([\tilde{X}]\).

**Proof:** If \(X \rightarrow Y\) is irreducible with \(X\) and \(Y\) in \(\text{ind } \mathcal{A}\), consider the minimal right almost split map \(X \sqcup X' \rightarrow Y\). \(FX \sqcup FX' \rightarrow FY\) is then minimal right almost split in \(\text{mod } \mathcal{A} \ast G\). Given \(Z\) in \([\tilde{X}]\), we have \(Z \mid FX\), so there is an irreducible map \(Z \rightarrow U\) with \(U \mid FY\), that is, \(U\) in \([\tilde{Y}]\).

If \(Z \rightarrow U\) is irreducible in \(\text{mod } \mathcal{A} \ast G\), consider similarly the minimal right almost split map \(Z \sqcup Z' \rightarrow U\), which gives rise to a minimal right almost split map \(HZ \sqcup HZ' \rightarrow HU\). If \(X\) is arbitrary with \(Z\) in \([\tilde{X}]\), then \(X \mid HZ\), so there is an irreducible map \(X \rightarrow Y\) with \(Y \mid HU\), that is, \(U\) in \([\tilde{Y}]\). From this and dual arguments, the lemma follows.

We make similar definitions for \(DTr\). Let \(X\) be nonprojective in \(\text{ind } \mathcal{A}\) and consider an almost split sequence \(0 \rightarrow DTrX \rightarrow E \rightarrow X \rightarrow 0\). Then for \(g\) in \(G\) we have an almost split sequence \(0 \rightarrow \delta(DTrX) \rightarrow \delta E \rightarrow \delta X \rightarrow 0\). This shows
that $DTr(x) \simeq x(DTr)$, so that $[x] = [x']$ implies $[DTrX] = [DTrX']$. We then define $DTr[X] = [DTrX]$. If $Y$ is in $[DTrX]$, there is clearly some $X'$ in $[X]$ such that $Y \simeq DTrX'$. We also have the following result.

**Lemma 4.2.** For $X$ nonprojective in $\text{ind } A$ we have $DTr[X] = [DTrX]$.

**Proof.** Let $Z$ be in $[X]$. $Z$ is then not projective, and we consider the almost split sequences $0 \to DTrZ \to U \to Z \to 0$ and $0 \to DTrX \to E \to X \to 0$. We have a direct sum of almost split sequences $0 \to F(DTrX) \to FX \to 0$, so that $F(DTrX) \simeq DTr(FX)$. The indecomposable summands of $F(DTrX)$ are those of $[DTrX]$, and $DTrZ$ is in $[DTrX]$.

We recall that there is a decomposition of $\text{ind } A$, for an artin algebra $A$, into components, with respect to the equivalence relation generated by $X$ and $Y$ being related if there is an irreducible map $X \to Y$. If $\mathcal{C}$ is such a component, which then corresponds to a connected component of the Auslander–Reiten quiver, $\mathcal{C}$ is clearly also, where the objects in $\mathcal{C}$ are those of the form $x$ for $x$ in $\mathcal{C}$. And the correspondence sending $[x]$ to $[\Xi]$ induces a correspondence between finite sets of components for $A$ and $\text{mod } A \ast G$. $[\mathcal{C}]$ denotes the set of components of the form $\mathcal{C}$, and $[\mathcal{C}]$ consists of the components in $\text{ind } A \ast G$ having an object from some $[\Xi]$ with $X$ in $\mathcal{C}$. We also say that a component $\mathcal{C}$ has the $DTr$-property if for each $X$ in $\mathcal{C}$, $DTr^iX$ is projective for some $i \geq 0$.

On the basis of the above preliminary statements, we have the following connection between $A$ and $A \ast G$.

**Theorem 4.3.** Let $A$ be an artin algebra and $A \ast G$ such that $|G|$ is invertible in $A$.

(a) There are no oriented cycles of irreducible maps in some component $\mathcal{C}$ of $\text{ind } A$ if and only if the same is true for the components $\mathcal{C} \in [\mathcal{C}]$ in $\text{ind } A \ast G$.

(b) There are no oriented cycles of irreducible maps between indecomposable modules for $\text{mod } A$ if and only if the same is true for $\text{mod } A \ast G$.

(c) A component $\mathcal{C}$ in $\text{ind } A$ has the $DTr$-property if and only if each component $\mathcal{D} \in [\mathcal{C}]$ has the $DTr$-property.

(d) $A$ has the $DTr$-property if and only if $A \ast G$ does.

(e) $\mathcal{C}$ is a preprojective component of $\text{ind } A$, that is, $\mathcal{C}$ has no oriented cycles of irreducible maps and $\mathcal{C}$ has the $DTr$-property [20], if and only if each $\mathcal{D}$ in $[\mathcal{C}]$ is a preprojective component of $\text{ind } A \ast G$.

**Proof.** (a) Assume there is a chain of irreducible maps $X_1 \to X_2 \to \cdots \to X_n \to X_1$ in $\mathcal{C}$, and no such chain in any $\mathcal{D} \in [\mathcal{C}]$. There is then a chain of irreducible maps $[X_1] \to [X_2] \to \cdots \to [X_n] \to [X_1]$, hence a chain of irreducible maps $Z_1 \to Z_2 \to \cdots \to Z_n \to Z_1$, within some $\mathcal{D}$ in $[\mathcal{C}]$, where $Z_1$
is in $[\tilde{\mathcal{X}}]$, $Z'$ in $[\tilde{\mathcal{X}}]$. By Lemma 4.1, we continue this chain, and since $[Z_1]$ has only a finite number of objects, we get an oriented cycle of irreducible maps in $\mathcal{S}$. The proof of the converse is the same.

(c) Let $X$ be in $\mathcal{S}$. $X$ is projective if and only if each $Z$ in $[\tilde{X}]$ is projective. Since $\mathcal{DTr}[X] = DTr[\tilde{X}]$, $DTrX$ is projective if and only if $DTrZ$ is projective for $Z$ in $[\tilde{X}]$, and we are done.

(b) and (e) follow directly from (a) and (c).

(e) follows from (a) and (b).

A similar idea can be used to prove the following related result.

**Proposition 4.4.** With the usual notation, $A$ is a factor of an hereditary algebra if and only if $A \ast G$ is.

**Proof.** Let $f: Z \to Z'$ be a nonzero map in $\text{mod} A \ast G$, where $Z$ and $Z'$ are indecomposable. Choose $X$ and $X'$ in $\text{ind} A$ such that $Z \mid FX$ and $Z' \mid FX'$, and extend $f$ to a map $\tilde{f}: FX \to FX'$. By adjointness, $\text{Hom}_A(X, HFX')$ is then not zero, so we get a nonzero map $t: X \to X''$, with $X''$ in $[X']$. Similarly, given a nonzero map $s: X \to X'$ in $\text{mod} A$, with $X$ and $X'$ in $\text{ind} A$, we choose $Z$ and $Z'$ in $\text{ind} A \ast G$ such that $X \mid HZ$ and $X' \mid HZ'$. We then get some nonzero map $Z'' \to Z'$ with $Z''$ in $[Z]$. To finish the proof we use that $F$ and $H$ preserve projectives and that an artin algebra $\Gamma$ is not a factor of an hereditary artin algebra if and only if there is some chain of nonzero maps $Q_1 \to Q_2 \to \cdots \to Q_n \to Q_1$ between indecomposable projective $\Gamma$-modules.

**4.2.** Another interesting concept closely related to irreducible maps is the concept of preprojective partition and preprojective module, introduced in [6]. This is a decomposition $\text{ind} A = \bigcup_{i=0}^\infty \mathcal{P}_i \cup \mathcal{P}_\infty$, into disjoint sets $\mathcal{P}_i$, $0 \leq i \leq \infty$, where $\mathcal{P}_n$ is finite for $n < \infty$, and $\mathcal{P}_n$ for $n < \infty$ is minimal with respect to the property that each indecomposable module not in $\bigcup_{i=0}^{n-1} \mathcal{P}_i$ is a factor module of a finite direct sum of modules from $\mathcal{P}_n$. $p(A)$ is defined to be the number of nonempty layers in the preprojective partition. We have the following connection between $A$ and $A \ast G$.

**Theorem 4.5.** With the previous notation, with $\{\mathcal{P}_i\}$ the preprojective partition for $A$ and $\{Q_i\}$ the preprojective partition for $A \ast G$, the following are equivalent for an $X$ in $\text{ind} A$.

(a) $X$ is in $\mathcal{P}_i$.
(b) All $X'$ in $[X]$ are in $\mathcal{P}_i$.
(c) Some $Z$ in $[\tilde{X}]$ is in $Q_i$.
(d) All $Z$ in $[\tilde{X}]$ are in $Q_i$. 


In particular, \( p(A) = p(A \ast G) \).

**Proof.** To prove this, it is convenient to use a functorial description of the \( \mathcal{S}_i \), as given in [2]. For a finitely presented covariant functor \( K \), denote by \( \text{Supp} \ K \) the set of \( X \) in \( \text{ind} \ A \) such that \( (X, )/r(X, ) \) is a composition factor of \( K \). Define \( K_1 \) to be the minimal subfunctor of \( K \) such that \( \text{Supp} \ K/K_1 \) is contained in \( \text{Supp} \ K/rK \). Assuming \( K_m \) is defined, define \( K_{m+1} \) to be the minimal subfunctor of \( K_m \) with \( \text{Supp} \ K/K_{m+1} \) contained in \( \text{Supp} \ K/rK_m \). For \( K = (\Lambda, ) \) we have \( \text{Supp}(K_m/rK_m) = \mathcal{S}_m [2] \).

Let \( L \) be a finitely presented covariant functor on \( \text{mod} \ A \). If \( ^sL \simeq L \) for all \( g \) in \( G \), then clearly \( ^s(L/rL) \simeq L/rL \) for all \( g \) in \( G \). \( \text{Supp} \ L \) is closed under the action of \( G \), hence is some union \( \bigcup_{i \in I} [X_i] \) for \( X_i \) in \( \text{ind} \ A \). Since \( F \) preserves projective covers, we have \( \text{Supp} \ FL = \bigcup_{i \in I} [\tilde{X}_i] \) and clearly \( \text{Supp} \ HF(L) = \bigcup_{i \in I} [X_i] \).

Since \( F \) preserves projective covers and semisimple objects, we also have \( F(L/rL) \simeq FL/r(FL) \).

Let \( K = (P, ) = K_0 \) where \( P \) is the direct sum of one copy of each indecomposable projective \( \Lambda \)-module. Then \( ^sK \simeq K \) for all \( g \) in \( G \). Since we have that \( F(K_m/rK_m) \simeq FK_m/r(FK_m) \), it is sufficient to prove that \( K_m \simeq ^sK_m \) for all \( g \in G \) and that \( F(K_m) = (FK)_m \) for all \( m \geq 0 \), where \( (FK)_0 = FK \). We have \( r(^sK) = ^s(rK) \), and since \( K \simeq ^sK \), we then have \( \text{Supp} \ K/rK = \text{Supp} \ ^sK/r(^sK) \).

It follows that \( (^sK)_1 = (FK)_1 \), so that \( ^sK_1 \simeq K_1 \) since \( K \simeq K \). Then \( K_1/rK_1 = \text{Supp} \ ^sK_1/r(^sK)_1 \) and \( \text{Supp} \ K/rK_1 = \text{Supp} \ ^sK/r(^sK)_1 \), so that also \( ^sK_2 = (FK)_2 \), and hence \( K_2 \simeq ^sK_2 \). Continuing this way we get \( ^sK_m = (FK)_m \) and \( K_m \simeq ^sK_m \) for all \( m \geq 0 \) and all \( g \in G \).

Assume we have proved that \( F(K_m) = (FK)_m \) for some \( n \geq 0 \). Since \( F \) preserves projective covers and semisimple objects, we have \( F(K_m/rK_m) \simeq F(K_m)/rF(K_m) \). Since \( K_m \simeq ^sK_m \), we then have \( \text{Supp} F(K_m)/rF(K_m) = [\tilde{X}_1] \cup \cdots \cup [\tilde{X}_n] \), if \( \text{Supp} K_m/rK_m = [X_1] \cup \cdots \cup [X_n] \). Using that \( ^sK_m = (FK)_m \), we get \( ^s(K_m/K_{m+1}) \simeq K_m/K_{m+1} \), so that \( \text{Supp} (K_m/K_{m+1}) \) is stable under \( G \). It follows that \( (FK)_{m+1} \subset F(K_{m+1}) \subset rF(K_m) \subset F(K_m) \subset FK \), so that \( H((FK)_{m+1}) \subset H(k(K_{m+1}) \subset H(K) \). \( H((FK)_{m+1}) \subset (FK)_{m+1} \cup \cdots \cup (FK)_n \subset FK \), where \( G = \{ g_1, \ldots, g_n \} \). If \( (FK)_{m+1} \subset F(K_{m+1}) \) was a proper inclusion, then so is \( H((FK)_{m+1}) \subset (FK)_{m+1} \cup \cdots \cup (FK)_n \).

For consider the diagram with projective covers

\[
\begin{array}{ccc}
(X, ) & \rightarrow & (Y, ) \\
\downarrow & & \downarrow \\
(FK)_{m+1} \subset F(K_{m+1}) & \downarrow & \downarrow \\
0 & = & 0
\end{array}
\]
which gives rise to the commutative diagram with projective covers

\[
\begin{array}{ccc}
(IX, \pi) & \rightarrow & (HY, \pi) \\
\downarrow & & \downarrow \\
H((FK)_{m+1}) & \subset & HF(K_{m+1}) \\
0 & & 0
\end{array}
\]

Since \((X, \pi) \rightarrow (Y, \pi)\) is not an isomorphism, \(Y \rightarrow X\) is not an isomorphism. Hence \(HY \rightarrow HX\) is not an isomorphism since \(H\) is a restriction functor. Then some \(Z\) in \(\text{Supp} K_m/\pi K_m\) would be in \(\text{Supp} \, eK_{m+1}/\pi eK_{m+1} \cup \ldots \cup eK_{m+1}/\pi eK_{m+1}\), which would contradict the minimality of \((\xi K)_{m+1}\), for some \(i\). Hence we conclude that \(F(K_{m+1}) = (FK)_{m+1}\).

4.3. We now discuss how tilted algebras behave with respect to the construction of crossed product algebras. We recall from [20] that an artin algebra \(A\) is a tilted algebra if there is a hereditary algebra \(C\) and a tilting module \(T\) such that \(A = \text{End}_C(T)^{op}\). Here \(T\) is a tilting module over a hereditary algebra if \(\text{Ext}^1_C(T, T) = 0\) and the number of nonisomorphic summands of \(T\) is equal to the number of nonisomorphic simple \(C\)-modules.

In [9] a section \(\mathcal{E}\) of the Auslander-Reiten quiver \(\Gamma_A\) of an algebra \(A\) is a subquiver such that the point set \(\mathcal{E}_0\) is a set of representatives of the \(DTr\)-orbits of \(\text{ind} A\) and the set of arrows \(\mathcal{E}_1\) is a set of representatives of the \(\sigma\)-orbits in \(\Gamma_A\). We recall that for an arrow \(\alpha: x \rightarrow y\), corresponding to an irreducible map \(X \rightarrow Y\) between the corresponding indecomposable modules, with \(Y\) not projective, \(\sigma \alpha\) is defined to be the arrow \(\tau y \rightarrow x\), where \(\tau y\) corresponds to \(DTrY\). The corresponding concept to a section is that of a complete slice in [20]. We shall use the characterization that an artin algebra \(A\) of finite type is a tilted algebra if and only if \(\Gamma_A\) has no oriented cycles and has a section [20, 9].

**Theorem 4.6.** Let \(A\) be an artin algebra of finite type and \(G\) a finite group acting on \(A\), with \(|G|\) invertible in \(A\), \(\gamma: G \times G \rightarrow U(A) \cap Z(A)\) as usual. Then \(A\) is a tilted algebra if and only if \(A * G\) is.

This result was first proved by de la Peña for skew group algebras, when \(G\) is a solvable group (private communication). In his proof he showed that if \(A\) is a tilted algebra of finite type with \(G\) acting on \(A\), then \(A\) has a section \(\mathcal{E}\) stable under the action of \(G\). We need here the following more general result.

**Lemma 4.7.** Assume that \(S\) is a tilted algebra of finite type, where \(S\) is \(\sigma\)
or $A \ast G$ above. Then $\Gamma_S$ contains a section $\mathcal{F}$, such that if $X$ is in $\mathcal{F}$ then each $X'$ in $[X]$ is in $\mathcal{F}$.

Proof. We know that $\Gamma_S$ has no oriented cycles of irreducible maps. Let $\mathcal{U}$ be a section for $I_S$, and let $P$ be projective in ind $S$. Let $P_1, \ldots, P_r$ be the nonisomorphic objects in $[P]$. Since $\mathcal{U}$ contains exactly one module from each $DTr$-orbit, there is a unique $n_i$ such that $TrD^nP_i$ is in $\mathcal{U}$. Choose $n = n_p$ as the smallest of the $n_i$ for $i = 1, \ldots, r$, and replace $X_i = TrD^nP_i$ by $X'_i = TrD^nP_i$ for $i = 1, \ldots, r$. We repeat this process for every $[Q]$, where $Q$ is projective in ind $S$, by defining $n_0$ in an analogous way. Then each $X$ in $\mathcal{U}$ is replaced by some $X'$, and we let $\mathcal{F}$ be the full subquiver of $\Gamma_S$ whose point set corresponds to these $X'$. Then $\mathcal{F}$ clearly contains exactly one representative from each $DTr$-orbit. Let now $X + Y$ be an irreducible map where $X$ and $Y$ are in $\mathcal{U}$. To show that $\mathcal{F}$ is also a section, we want to show that there is an irreducible map $X' + Y'$ or an irreducible map $Y' + X'$.

We have that $X' = DTr^iX$ for some $i \geq 0$. By definition of the $X'$ there is some $X_i$ in $[X']$ with $X_i$ in $\mathcal{U}$. Since $TrD^iX_i$ is then in $[X]$, there is some irreducible map $TrD^iX_i \to Y_i$, with $Y_i$ in $[Y]$. Since $\mathcal{U}$ is a section, there is some irreducible map $X_1 \to Y_2$ or $Y_2 \to X_1$, with $Y_2$ in $\mathcal{U}$, where $Y_2 = DTr^iY_i$ or $Y_2 = DTr^{i+1}Y_1$. This shows that $Y' = DTr^iY$, with $j > i$. Assume $j > i + 1$, and choose $Y_j$ in $[Y]$ with $DTr^jY_j \in \mathcal{U}$. Then there is an $X_j$ in $[X]$ and an irreducible map $X_3 \to Y_3$. Since $\mathcal{U}$ is a section there would then be an irreducible map $DTr^iX_3 \to DTr^iY_3$ or $DTr^iY_3 \to DTr^jX_3$, with $DTr^jX_3$ or $DTr^jX_3$ in $\mathcal{U}$. Since $X' = DTr^iX$ and $j - 1 > i$, this is a contradiction.

Proof of Theorem 4.6. Let $\mathcal{F}$ be a section for $A$, with the property of Lemma 4.8. We define a subquiver $\mathcal{F}'$ of $\Gamma_A \ast G$ as follows. Consider the indecomposable $A \ast G$-modules which are in $[X]$ for some $X$ in $\mathcal{F}$, with an arrow each time there is some irreducible map. By our previous observations on $DTr$, we clearly have exactly one representative from each $DTr$-orbit. Since for a tilted algebra of finite type there are no oriented cycles of irreducible maps, there are also no oriented cycles of irreducible maps for $A \ast G$ by Theorem 4.3.

Let $Z \to U$ be an irreducible map in ind $A \ast G$. Then $DTr^iZ$ is in $\mathcal{F}'$ for some $i$. Choose $X$ in $\mathcal{F}$ such that $[X] = [DTr^iZ]$. Then $[TrD^iX] = Z$, and hence there is an irreducible map $TrD^iX \to Y$, where $[Y] = [U]$. Since $X$ is in $\mathcal{F}'$ there is then either an irreducible map $X \to DTr^iY$ or $DTr^{i+1}Y \to X$, in $\mathcal{F}'$.

Since $DTr^iU$ is in $[DTr^iY]$ and $DTr^{i+1}U$ in $[DTr^{i+1}Y]$, we have that $DTr^iZ \to DTr^iU$ or $DTr^{i+1}U \to DTr^iZ$ is in $\mathcal{F}'$. This shows that $\mathcal{F}'$ is a section.

The proof of the converse is analogous.

4.4. Using the results developed in this section, we get an alter-
native way of seeing that some selfinjective algebras of finite type of Dynkin class $A_{2n-3}$ are connected with selfinjective algebras of finite type of Dynkin class $D_n$, using the skew group algebra construction.

Let $A$ be a basic selfinjective $k$-algebra of finite representation type over an algebraically closed field $k$, of Dynkin class $A_{2n-3}$. Assume that there is an automorphism $\sigma$ of order 2 on the Auslander–Reiten quiver $I_A$, taking a section $\mathcal{S}$ of the stable part into itself. Then $\sigma$ induces an automorphism of order 2 on the Auslander algebra $I_A$ and on the ordinary quiver of $A$ (and hence on $A$), since $A$ is a standard algebra (see [31, 10]). For each $X$ in $\mathcal{S}$ we consider the indecomposable summands of $AG \otimes_A X$, where $G = \langle \sigma \rangle$ is of order 2. From our previous results we have that these indecomposables form a section $\mathcal{S}'$ of the stable part of the Auslander–Reiten quiver $I_{AG}$. Using the connection between irreducible maps in $\text{mod } A$ and $\text{mod } AG$, we see that $\mathcal{S}'$ is of type $D_n$, and we are done.

4.5. As a consequence of our results in Section 3 we have the following: If $A$ is an artin algebra of finite representation type with $G$ acting on $A$ (as usual), and $I_A$ and $I_{AG}$ are Auslander algebras for $A$ and $AG$, then $(\text{mod } I_A)G$ and $\text{mod } I_{AG}$ are equivalent categories. This provides some insight into the hard problem of describing the artin algebras $A$ of finite type whose Auslander algebra $I_A$ is of finite type, since it says that $A$ and $AG$ behave the same way with respect to this property. For example, the algebras given by the quivers

$$\begin{array}{c}
\circ \\
\circ
\end{array}$$

and

$$\begin{array}{c}
\circ \\
\circ
\end{array}$$

behave the same way. Here the Auslander algebra is of finite type in both cases.

5. MISCELLANEOUS

In this final section we discuss various topics, in particular, the possibility of recovering $A$ from $AG$ (5.1), connections between skew group constructions and coverings (5.2) and some results on $AG$ if $G$ acts by inner automorphisms. For background on coverings we refer to [10, 16]. We assume that $A$ is an algebra over an algebraically closed field $k$ and that the action of $G$ on $k \cdot 1_A$ is trivial and $|G|$ is invertible in $k$.

5.1. Our first result explains a phenomenon we found for all our examples of skew group algebras formed with cyclic groups $G$ in Section 2.
We could always define an action of $G$ on $A_G$ such that $(A_G)G$ was Morita equivalent to $A$.

Denote by $X$ the group of characters on $G$; i.e., the group homomorphisms $\chi: G \to k^* = k\setminus\{0\}$. Then $X$ operates on $A_G$, via $\chi(\lambda g) = \chi(g) \lambda g$.

**Proposition 5.1.** The map

$$\phi: (A_G)_X \to \text{End}_{A_G} A_G$$

given by

$$\phi(\lambda g\chi)(\mu h) = \chi(h) \lambda g\mu h$$

is an algebra isomorphism, where $G'$ denotes the commutator subgroup of $G$ and where $A_G$ is considered as right $A_G'$-module.

**Proof.** In order to see that $\phi(\lambda g\chi)$ lies in $\text{End}_{A_G} A_G$, notice that $\chi(h) = 1$ for $h \in G'$. It is easy to check that $\phi$ is an algebra homomorphism. In order to see that $\phi$ is injective, we choose

$$\sum_{\chi, g} \lambda_{g, \chi} g\chi$$

in the kernel of $\phi$. For each $h$ in $G$, we have

$$\phi\left(\sum_{\chi, g} \lambda_{g, \chi} g\chi\right)(h) = 0 = \sum_{\chi, g} \chi(h) \lambda_{g, \chi} g h,$$

and hence $\sum_{\chi} \chi(h) \lambda_{g, \chi} = 0$ for all $g$ and $h$. The value $\chi(h)$ depends only on the coset $G' h = h$, and the matrix $(\chi(h))$ with $\chi \in X$ and $h \in G/G'$ is invertible. As a consequence, $\lambda_{g, \chi} = 0$ for all $g$ and $\chi$.

The order $|X|$ of $X$ equals $|G|/|G'|$, and hence $(A_G)_X$ has dimension $\dim A(\dim G/\dim G')$. On the other hand, $A_G$ is isomorphic to $(A_G')^{\dim G/\dim G'}$ as a right $A_G'$-module, and therefore the dimension of $\text{End}_{A_G} A_G$ equals

$$\frac{|G|^2}{|G'|^2} \dim A G' = \frac{|G|^2}{|G'|^2} (\dim A) |G'|.$$

Thus $\phi$ is an isomorphism.

**Corollary 5.2.** With the notations above, $(A_G)_X$ is Morita equivalent to $A_G'$.

**Proof.** The right $A_G'$-module $A_G$ is clearly a projective generator of $\text{mod } A_G'$.

If $G$ is abelian, the corollary says that $(A_G)G$ is Morita equivalent to $A$. 

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**SKEW GROUP ALGEBRAS**

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for some action of $G \simeq X$ on $\mathcal{AG}$. If $G$ is solvable, we let $G^{(i+1)}$ be the commutator subgroup of $G^{(i)}$, and $X^{(i)}$ the character group of $G^{(i)}$, and we set $A_{i} = \mathcal{AG}^{(i)}$, where $G^{(i)}$ acts as a subgroup of $G$. Then $(\mathcal{AG}^{(i)})X^{(i)}$ is Morita equivalent to $\mathcal{AG}^{(i+1)}$ for all $i$, where $X^{(i)}$ acts as described above.

Hence we have the following result.

**Proposition 5.3.** If $\mathcal{AG}$ is a skew group algebra with $G$ a solvable group, we can get from $\mathcal{AG}$ to $\mathcal{A}$ by using a finite number of skew group algebra constructions, combined with Morita equivalences.

Since skew group algebras $\mathcal{AG}$ are easier to study when $G$ is cyclic, it is of interest to know when we can reduce our considerations to the cyclic case, that is, when $\mathcal{AG}$ can be constructed from $\mathcal{A}$ by applying a finite number of skew group algebra constructions with cyclic groups (combined with Morita equivalences). As a direct consequence of the above we have the following, which is of theoretical interest.

**Proposition 5.4.** If $G$ is a solvable group, then $\mathcal{AG}$ can be obtained from $\mathcal{A}$ using a finite number of skew group algebra constructions with cyclic groups, combined with Morita equivalences.

**Proof.** The above gives that we get from $\mathcal{AG}$ to $\mathcal{A}$ by using abelian groups, namely, the $X^{(i)}$, hence we also get by using the above, from $\mathcal{A}$ to $\mathcal{AG}$ by using abelian groups. Then we use that $\mathcal{A}(G_{1} \times G_{2}) \simeq (\mathcal{AG})_{1}G_{2}$, so that the abelian case is reduced to the cyclic case.

**5.2.** Let $\mathcal{A}$ be a category given by a quiver $Q$ and relations $I$; i.e., $\mathcal{A} = kQ/I$, where $kQ$ is the path category and $I$ an ideal of $kQ$ which is contained in the square of the radical of $kQ$ \[10\]. Suppose a finite group $G$ operates freely on $\mathcal{A}$, that is, no element $\neq e$ fixes an object.

In this situation, Gabriel \[16\] studied the category $\mathcal{A}/G$ whose objects are the $G$-orbits of objects of $\mathcal{A}$ and whose morphisms are the families $(f_{xy} \in \mathcal{A}(x, y))$ which are fixed under $G$, where $x$ and $y$ range over two $G$-orbits. He showed that the natural functor $\mathcal{A} \to \mathcal{A}/G$ is a covering functor in the sense defined in \[10\].

We can also consider the skew group category $\mathcal{AG}$ as defined in 3.1. Since idempotents split in $\mathcal{A}/G$, it is clear from the definitions of $\mathcal{AG}$ and $\mathcal{A}/G$ that these two categories are isomorphic.

Using our results on functorial splittings (3.2), we find some of the statements of \[16\] for the special case of a finite group $G$.

**Proposition 5.5.** (a) If $\mathcal{A}/G$ is locally representation finite, then the same is true for $\mathcal{A}$.
If the characteristic of $k$ does not divide the order of $G$, then the converse holds.

Note that we did not need any assumptions on the order of $G$ to obtain the splitting of the natural transformation $I \rightarrow HF$ from the identity functor on $\text{mod} \ A$ to $HF$.

5.3. Let $G$ act on $A$ in two different ways. We are interested in knowing when the two skew group algebras obtained from the two actions are isomorphic. In particular, if $G$ acts on $A$ via inner automorphisms, is then $AG$ isomorphic to the ordinary group algebra which we denote by $A_0G$?

Assume $G$ acts via inner automorphisms on $A$, and let $u_g \in U(A)$ be a unit such that $g(\lambda) = u_g \lambda u_g^{-1}$ for $g \in G$. Then $g$ determines $u_g$ uniquely up to multiplication with factors in the center $Z(A)$ of $A$. In particular, we have

$$u_{gh} = \gamma(g, h) u_g u_h$$

for all $g$ and $h$ with $\gamma(g, h) \in Z(A) \cap U(A)$. It is easy to check that $\gamma$ is a cocycle. In this situation we obtain:

**Lemma 5.6.** $AG$ is isomorphic to $A \rtimes \gamma G$, where $AG$ is formed with the given action of $G$ on $A$ and $A \rtimes \gamma G$ with trivial action and the cocycle $\gamma$.

**Proof.** The isomorphism

$$\phi: AG \rightarrow A \rtimes \gamma G$$

is given by setting

$$\phi(\lambda) = \lambda \quad \text{for} \quad \lambda \in A,$$

$$\phi(g) = u_g \tilde{g}.$$

So studying $AG$ when $G$ acts by inner automorphisms is reduced to studying crossed products $A \rtimes \gamma G$, where $G$ acts trivially on $A$. The cocycles of $G$ (with respect to any operation of $G$ on $A$) with values in the units of the center of $A$ form a group, with composition given by multiplication of the values in $U(A) \cap Z(A)$. A coboundary is a cocycle $\delta^+$ given by

$$\delta^+(g, h) = \delta(g) g(\delta(h)) \delta(gh)^{-1},$$

for some map $\delta: G \rightarrow U(A) \cap Z(A)$, and the coboundaries form a normal subgroup of the cocycles. The isomorphism class of $A \rtimes \gamma G$ depends only on the class of $\gamma$ in the factor group $H^2(G, Z(A) \cap U(A))$.

**Lemma 5.7.** Let $\gamma$ and $\gamma' = \gamma \delta^+$ be two cocycles having the same image in $H^2(G, C(A) \cap U(A))$. Then $A \rtimes \gamma G$ and $A \rtimes \gamma' G$ are isomorphic.
Proof. Define
\[ \phi: A \ast \gamma G \to A \ast \gamma G \]
by setting
\[ \phi(\lambda) = \lambda \quad \text{for} \quad \lambda \in A, \]
\[ \phi(\bar{g}) = \delta(g) \bar{g}. \]

Here the elements of \( A \ast \gamma G \) are denoted by \( \sum_{k \in G} \lambda_k \bar{g}. \)

Remark. We have seen in 2.6 that the converse of this statement is false.

We are now ready to see that \( AG \) need not be isomorphic to the ordinary group algebra \( A_0 G \) even if \( G \) acts on \( A \) by inner automorphisms. We let \( A \) be the algebra of \( 2 \times 2 \)-matrices over \( k \) and \( G = g^{\mathbb{Z}/2\mathbb{Z}} \times h^{\mathbb{Z}/2\mathbb{Z}} \). We let \( g, h \) and \( gh \) act by conjugation with

\[
\begin{align*}
    u_g &= \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \\
u_h &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
u_{gh} &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{align*}
\]

The cocycle \( \gamma \) defined as above takes the values
\[
\gamma(g, h) = \gamma(h, gh) = \gamma(gh, g) = i,
\]
\[
\gamma(h, g) = \gamma(g, gh) = \gamma(gh, h) = -i,
\]
and 1 otherwise. By Lemma 5.1, \( AG \) is isomorphic to \( A \ast \gamma G \), which is in turn isomorphic to the algebra of \( 4 \times 4 \)-matrices over \( k \). The image of the elementary \( 4 \times 4 \)-matrix \( E_{ij} \) under this isomorphism is the entry \( (i, j) \) in the following picture, where \( e_{11}, e_{12}, \ldots \) denote the elementary \( 2 \times 2 \)-matrices.

\[
\frac{1}{2} \begin{pmatrix}
    e_{11}(1 + \bar{g}) & e_{11}(1 + \bar{g})\bar{h} & e_{12}(1 + \bar{g}) & e_{12}(1 + \bar{g})\bar{h} \\
    e_{11}(1 - \bar{g})\bar{h} & e_{11}(1 - \bar{g}) & e_{12}(1 - \bar{g})\bar{h} & e_{12}(1 - \bar{g}) \\
    e_{21}(1 + \bar{g}) & e_{21}(1 + \bar{g})\bar{h} & e_{22}(1 + \bar{g}) & e_{22}(1 + \bar{g})\bar{h} \\
    e_{21}(1 - \bar{g})\bar{h} & e_{21}(1 - \bar{g}) & e_{22}(1 - \bar{g})\bar{h} & e_{22}(1 - \bar{g})
\end{pmatrix}
\]

On the other hand, it is easy to see that the ordinary group algebra \( A_0 G \) is isomorphic to 4 copies of the \( 2 \times 2 \) matrix algebra.

We have the following positive result:

**Proposition 5.8.** If \( G = g^{\mathbb{Z}/n\mathbb{Z}} \) is cyclic, then \( H^2(G, U(R)) = 1 \) with respect to any action of \( G \) on \( R \), where \( R \) is a local commutative algebra.
Since for an indecomposable algebra $A$, $Z(A)$ is commutative local, this implies that when the values of $\gamma$ are units in the center of $A$, $A \ast \gamma G$ is isomorphic to the skew group algebra $\mathcal{A}G$ for any action of $G$ when $G$ is cyclic. If $G$ acts on $A$ by inner automorphisms, then $\mathcal{A}G$ is isomorphic to the ordinary group algebra $A \ast_0 G$.

**Proof:** It suffices to show that any cocycle with values in $U(R)$ is a coboundary.

Let $\gamma$ be a cocycle. First we claim that

$$\lambda = \gamma(g, g) \gamma(g^2, g) \cdots \gamma(g^{n-1}, g)$$

is fixed under $g$. Indeed, expressing the cocycle condition for the triple $(g, g^i, g)$, we find that

$$\gamma(g, g^i) \gamma(g^{i+1}, g) = g(\gamma(g^i, g)) \gamma(g, g^{i+1}),$$

and the claim follows by taking the product over all these equations for $i = 0, \ldots, n - 1$. Since the characteristic of $k$ does not divide $n$, $n$th roots exist for units in the commutative local algebra $R^G$, and we let $\mu$ be an $n$th root of $\lambda$. Set

$$\delta(g^i) = \mu^i \gamma(g, g)^{-1} \gamma(g^2, g)^{-1} \cdots \gamma(g^{i-1}, g)^{-1},$$

for $i \in \mathbb{Z}$. Then

$$\delta(1) = \delta(g^n) = 1.$$

We check that

$$\gamma(g^i, g^j) = \delta(g^i) g^j (\delta(g^i)) \delta(g^{i+j})^{-1}$$

by induction on $j$. For $j = 1$, we have

$$\delta(g^i) g^j (\delta(g^i)) \delta(g^{i+1})^{-1} = \mu^i \gamma(g, g)^{-1} \cdots \gamma(g^{i-1}, g)^{-1} g^j(\mu) \mu^{-i-1} \gamma(g^i, g) \cdots \gamma(g^i, g) = \gamma(g^i, g),$$

since $g^i(\mu) = \mu$. For $j > 1$, we use the cocycle condition for the triple $(g^i, g, g^{i-1})$, which gives

$$\gamma(g^i, g) \gamma(g^{i+1}, g^{i-1}) = g^i(\gamma(g, g^{i-1})) \gamma(g^i, g^i).$$
and we compute

\[
\gamma(g^i, g^j) = \gamma(g^i, g) \gamma(g^{i+1}, g^{j-1}) g^i(\gamma(g, g^j))^{-1}
\]

\[
= \frac{\delta(g^i) g^i(\delta(g))}{\delta(g^{i+1})} \cdot \frac{\delta(g^{i+1}) g^{i+1}(\delta(g^{j-1}))}{\delta(g^{i+j})}
\]

\[
\cdot \frac{g^i(\delta(g^j))}{g^i(\delta(g)) g^{i+1}(\delta(g^{j-1}))}
\]

\[
= \frac{\delta(g^i) g^i(\delta(g^j))}{\delta(g^{i+j})}.
\]

This finishes the proof.

Alternatively, we could have used that \( H^2(G, k^*) = 1 \) for \( G \) cyclic [8] and that \( H^2(G, k^*) \approx H^2(G, U(R)) \) by using the proof of [3, Theorem 6.5].

There are other cases where \( AG \) is isomorphic to \( A_0 G \) when \( G \) acts by inner automorphisms. For dihedral groups of order \( 2n \) with \( n \) odd this is a consequence of the following result.

**Proposition 5.9.** Let \( G = D_n \) be the dihedral group of order \( 2n \) with \( n \) odd. Then \( H^2(G, U(R)) = 1 \) for every commutative local algebra \( R \), with respect to the trivial action.

**Proof.** We write \( G = g^{\mathbb{Z}/n\mathbb{Z}} \rtimes h^{\mathbb{Z}/2\mathbb{Z}} \) with \( hgh = g^{-1} \). By Proposition 5.8, we may assume that any cocycle \( \gamma \) (with respect to the trivial action of \( G \)) satisfies

\[
\gamma(g^i, g^j) = 1 \quad \text{and} \quad \gamma(g^ih, g^jh) = 1
\]

for all \( i \) and \( j \). Computing the cocycle condition for the triples

\[
(g^i, g^ih, g^{i+j}h) \quad \text{and} \quad (g^{i+j}h, g^i, g^jh)
\]

we obtain

\[
\alpha_{i,j} = \gamma(g^i, g^j) = \gamma(g^ih, g^{i+j}h) = \gamma(g^{i+j}h, g^i).
\]

Considering \((g^i, g^j, h)\), we obtain

\[
\alpha_{i,j} = \alpha_{j,0} \alpha_{i,j},
\]

and finally the triple \((g^{i-1}, gh, g)\) yields

\[
\alpha_{i-1,1} \alpha_{1,i-1} = \alpha_{1,0} \alpha_{i-1,0} = \alpha_{i,0} \alpha_{1,0} \alpha_{i-1,0}^{-1}
\]
hence
\[ \alpha_{i,0}^2 = \alpha_{i-1,0}^2 \alpha_{1,0}^2 = \alpha_{1,0}^{2i}. \]

Thus \( \gamma \) is determined by the \( \alpha_{1,0} \) with this condition, and the only restriction we have found so far is that
\[ \alpha_{n,0}^2 = 1 = \alpha_{1,0}^{2n}. \]

The map \( \delta: G \to U(R) \) with \( \delta^+ = \gamma \) which we want to find has to satisfy
\[ \delta(g^i) = \delta(g)^i, \quad \delta(g)^n = 1, \quad \delta(g^ih)^2 = 1. \]

If \( \alpha_{n,0}^i = 1 \), we set \( \delta(g) = \alpha_{1,0} \) and otherwise \( \delta(g) = -\alpha_{1,0} \). Remember that \( n \) is odd. We let \( \delta(h) = 1 = \delta(gh) \) and \( \delta(g^ih) = \pm 1 \) according as \( \alpha_{i,0} = \pm \delta(g)^i \) for \( i \geq 1 \).

**Remarks.**
(a) For \( n \) even, \( H^2(G, U(R)) \) is cyclic of order 2. The proof above suggests how to find a candidate for a nontrivial cocycle: Choose a primitive \( 2n \)th root of unity for \( \alpha_{1,0} \). We used this cocycle in Chapter 2 for \( n = 2 \).

(b) We do not know whether there exists a group \( G \) acting on a basic algebra \( A \) by inner automorphisms such that \( AG \) is not isomorphic to \( A_0G \). Using the results of Section 2 or [3, Theorem 6.5] we can show that it suffices to consider cocycles with values in \( k^* \), and it is easy to see that the nontrivial cocycle for \( D_{2n} \), for instance, cannot arise from an action by inner automorphisms, by reducing modulo the radical.

5.4. We now generalize the results of Section 5.3 to crossed product categories. An automorphism \( g \) of a preadditive category \( C \) is said to be inner if the functor \( g \) is isomorphic to the identity functor of \( C \). Then we have for each object \( X \) of \( C \) an isomorphism \( u_g(X): X \to ^gX \) such that for each \( f: X \to Y \) in \( C \) the following diagram commutes:

\[
\begin{array}{ccc}
^gX & \xrightarrow{gf} & ^gY \\
\uparrow u_{g}(X) & & \uparrow u_{g}(Y) \\
X & \xrightarrow{f} & Y
\end{array}
\]

The natural transformation \( u_g \) is uniquely determined by \( g \) up to composition with an element in the units of the center \( Z(C) \) of \( C \); i.e., an automorphism of the identity functor of \( C \).
Let the finite group $G$ act on $C$ by inner automorphisms, and denote the natural transformation representing $g$ by $u_g$. Then we have

$$u_g(hX) u_h(X) = \gamma(g, h)(ghX) u_{gh}(X) = u_{gh}(X) \gamma(g, h)(X)$$

for all objects $X$ and all $g, h \in G$, where $\gamma$ is a 2-cocycle on $G$ with values in the units of $Z(C)$. As for algebras we have the following.

**Lemma 5.10.** $CG$ is equivalent to $C \rtimes_{\gamma} G$, where $CG$ is formed with the given action of $G$ on $C$ and $C \rtimes_{\gamma} G$ with trivial action and the cocycle $\gamma$.

**Proof:** We exhibit the equivalence

$$F: CG \to C \rtimes_{\gamma} G.$$

Let $G = \{g_1 = e, g_2, \ldots, g_n\}$ and let

$$\alpha = (\alpha_{g,h}, e): \{\xi_1 X, \ldots, \xi_n X\} \to \{\xi_1 Y, \ldots, \xi_n Y\}$$

be a morphism in $CG$. We set $F\alpha = \beta$, where

$$\beta = (\beta_{g,h}, e): \{\xi_1 X, \ldots, \xi_n X\} \to \{\xi_1 Y, \ldots, \xi_n Y\}$$

is given by

$$\beta_{g,e} = u_g(Y)^{-1} \alpha_{g,e}: X \to Y = \xi Y$$

and

$$\beta_{xh,e} = \gamma(g, h) \beta_{h,e}: \xi X \to \xi Y = \xi Y.$$

By definition, $\beta$ is a morphism in $C \rtimes_{\gamma} G$, where $G$ acts trivially on $C$ (see 3.4). We have to show that

$$F\alpha' F\alpha = F(\alpha' \alpha)$$

where

$$\alpha = (\alpha_{g,h}, e): \{\xi_1 X, \ldots, \xi_n X\} \to \{\xi_1 Y, \ldots, \xi_n Y\},$$

$$\alpha' = (\alpha'_{g,h}, e): \{\xi_1 Y, \ldots, \xi_n Y\} \to \{\xi_1 Z, \ldots, g^n Z\}.$$

Set $\alpha'' = \alpha' \alpha$, $\beta = F\alpha$, $\beta' = F\alpha'$, and $\beta'' = F\alpha''$. It suffices to see that

$$\beta''_{e h, e} = \beta'_{e h, e} \beta_{e, e}$$

for all $g$ and $h$. By the definition of morphisms in $CG$, we have

$$\alpha''_{e h, e} = g(\alpha'_{h,e}) = u_g(hZ) u_{h,e} u_g(Y)^{-1},$$
and hence
\[ \beta_{gh, e} = \gamma(g, h) \beta_{h, e} - \gamma(g, h) u_h(Z)^{-1} \alpha_{h, e} \]
\[ = \gamma(g, h) u_h(Z)^{-1} u_g(hZ)^{-1} \alpha_{gh, e} u_g(Y) \]
\[ = u_{gh}(Z)^{-1} \alpha_{gh, e} u_g(Y). \]
This implies
\[ \beta_{gh, e} = (u_{gh}(Z)^{-1} \alpha_{gh, e} u_g(Y))(u_g(Y)^{-1} \alpha_{g, e}) \]
\[ = u_{gh}(Z)^{-1} \alpha_{gh, e} a_{g, e} = \beta_{gh, e}. \]
Clearly \( F \) is dense and fully faithful.

A map \( \delta \) from \( G \) to the units of \( Z(C) \) defines a coboundary \( \delta^+ \), which is a 2-cocycle, given by
\[ \delta^+(g, h) = \delta(g) g(\delta(h)) \delta(gh)^{-1}. \]
Again, \( C \ast \gamma G \) depends only on the class of \( \gamma \) in \( H^2(G, C(C)^*) \), up to equivalence, where \( C(C)^* \) denotes the units in \( C(C) \).

**Lemma 5.11.** Let \( \gamma \) and \( \gamma' = \gamma \delta^+ \) be two 2-cocycles on \( G \) with values in \( Z(C)^* \). Then \( C \ast \gamma G \) and \( C \ast \gamma' G \) are equivalent categories.

**Proof.** We define the equivalence
\[ F: C \ast \gamma G \to C \ast G \]
as follows. For a morphism
\[ \alpha = (a_{s, t}): \{ s_1X, \ldots, s_nX \} \to \{ s_1Y, \ldots, s_nY \} \]
in \( C \ast \gamma G \), we set
\[ F\alpha = \beta = (\beta_{s, t}): \{ s_1X, \ldots, s_nX \} \to \{ s_1Y, \ldots, s_nY \} \]
with
\[ \beta_{s, t} = \delta(g) a_{s, t} \]
and
\[ \beta_{gh, e} = \gamma(g, h) \beta_{h, e}. \]
It is not hard to check that \( F \) is compatible with composition, and it is an equivalence.

Combining these two lemmas with 5.3, we obtain:
COROLLARY 5.12. If $G$ is either a dihedral group $D_n$ with $n$ odd or a cyclic group, acting by inner automorphisms on a preadditive category $C$ whose center $Z(C)$ is a finite-dimensional local algebra, then $CG$ is equivalent to the category $C,G$ formed with $G$ acting trivially.

The condition on the center is satisfied for $C = \text{mod} A$ and $C = \text{mod}(\text{mod} A)$, where $A$ is an indecomposable algebra, because of the following result:

**Lemma 5.13.** Let $C$ be a dualizing $k$-variety. If $Z(C)$ is a local finite-dimensional $k$-algebra, then the same is true for $Z(\text{mod} C)$.

**Proof:** We claim that the algebra homomorphism

$$\phi: C(\text{mod} C) \to C(C)$$

given by $(\phi a)(D) = a((, D))$ for $D$ in $C$ is injective. Indeed, $\phi a = 0$ implies that $a$ vanishes on all projective functors, hence on all finitely presented functors.

If $a(F)$ is an isomorphism for some $F$ in $\text{mod} C$, then $a(P)$ is an isomorphism for the projective cover $P$ of $F$. Since $Z(C)$ is local, $a(Q)$ is an isomorphism for all projective functors $Q$, and hence $a$ is a unit.

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