NORTH-HOLLAND

A New Statement About the Theorem Determining the Region of Eigenvalues of Stochastic Matrices

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ABSTRACT

Let M_n denote the set of points in the complex plane that are eigenvalues of n-dimensional stochastic matrices. The set M_n is completely determined by the Karpelevich theorem, the statement of which, however, is lengthy and intricate. The paper shortens the presentation of the theorem. © 1997 Elsevier Science Inc.

1. THE KARPELEVICH THEOREM

The following well-known theorem completely determines the set M_n, the set of eigenvalues of n-dimensional stochastic matrices.

THEOREM 1 (Karpelevich [1, 3]). *The region M_n is symmetric relative to the real axis, is included in the unit disc $|z| \leqslant 1$, and intersects the circle $|z| = 1$ at points $e^{2\pi i a/b}$, where a and b run over the relatively prime integers satisfying $0 \leqslant a \leqslant b \leqslant n$. The boundary of M_n consists of these points and of curvilinear arcs connecting them in circular order. Each of*

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these arcs is given by one of the following parametric equations:

$$\lambda^q(\lambda^p - t)^r = (1-t)^r, \tag{I}$$

$$(\lambda^b - t)^d = (1-t)^d \lambda^q, \tag{II}$$

where the real parameter t runs over the interval $0 \leqslant t \leqslant 1$, *and* b, d, p, q, r *are nonnegative integers defined as follows.*

Let the endpoints of an arc be $e^{2\pi i a'/b'}$ *and* $e^{2\pi i a''/b''}$ $(a'/b' < a''/b'')$. *There are two cases:*

$$b''\left[\frac{n}{b''}\right] \geqslant b'\left[\frac{n}{b'}\right], \tag{a}$$

$$b''\left[\frac{n}{b''}\right] \leqslant b'\left[\frac{n}{b'}\right]. \tag{b}$$

If an arc satisfies (a), *then the complex conjugate, counterclockwise arc satisfies* (b). *Thus, due to the symmetry of* M_n, *it will suffice to describe arcs satisfying* (a).

Let $r_1 = b''$, $r_2 = a''$, $r_3, \ldots,$ $r_m = 1$, $r_{m+1} = 0$ *be the nonnegative remainder series of Euclid's algorithm, by which the greatest common divisor of* b'' *and* a'' *is obtained. If* $[n/b''] = 1$ *and m is even, then the counterclockwise arc from* $e^{2\pi i a'/b'}$ to $e^{2\pi i a''/b''}$ *is given by the equation* (I), *where* $r = r_{m-1}$ *and the integers p and q are defined by the relations:*

$$a''p \equiv 1 \pmod{b''} \qquad (0 < p \leqslant b''),$$

$$a''q \equiv -r \pmod{b''} \qquad (0 \leqslant q < b'').$$

Otherwise the counterclockwise arc from $e^{2\pi i a'/b'}$ to $e^{2\pi i a''/b''}$ *is given by the equation* (II), *where* $d = [n/b'']$, $b = b''$, *and the integer q is defined by*

$$a''q \equiv -1 \pmod{b''} \qquad (0 \leqslant q < b'').$$

2. MAIN RESULT

The next is the shortened statement of the Karpelevich theorem.

THEOREM 2. *The region* M_n *is symmetric relative to the real axis, is included in the unit disc* $|z| \leqslant 1$, *and intersects the circle* $|z| = 1$ *at points*

$e^{2\pi ia/b}$, where a and b run over the relatively prime integers satisfying $0 \leqslant a \leqslant b \leqslant n$. The boundary of M_n consists of these points and of curvilinear arcs connecting them in circular order.

Let the endpoints of an arc be $e^{2\pi ia_1/b_1}$ and $e^{2\pi ia_2/b_2}$ $(b_1 \leqslant b_2)$. *Each of these arcs is given by the following parametric equation:*

$$\lambda^{b_2}(\lambda^{b_1} - s)^{[n/b_1]} = (1 - s)^{[n/b_1]}\lambda^{b_1[n/b_1]},$$

where the real parameter s runs over the interval $0 \leqslant s \leqslant 1$.

In order to prove the equivalence between the statements of Theorems 1 and 2 we need the following two lemmas.

LEMMA 1. *The sequence of all reduced nonnegative fractions with denominators not exceeding n, listed in order of their size, is called the Farey sequence of order n. Two reduced nonnegative fractions a'/b'* and *a''/b'' are consecutive in the Farey sequence of order n if and only if $a''b' - a'b'' = 1$* and *$b' + b'' > n$ hold.*

The proof can be found, for instance, in [4].

LEMMA 2. *Let a and b be relatively prime integers satisfying $0 < a < b$,* and *$r_1 = b$, $r_2 = a$, $r_3, \ldots, r_m = 1$, $r_{m+1} = 0$ be the nonnegative remainder series of Euclid's algorithm. Let c be the integer defined by the relation $ac \equiv 1 \pmod{b}$ $(0 < c < b)$. Then*

$$\left[\frac{b}{c}\right] = \begin{cases} r_{m-1}\ (>1) & \text{if } m \text{ is even}, \\ 1 & \text{if } m \text{ is odd}. \end{cases}$$

Proof. Expand b/a into the continued fraction

$$\frac{b}{a} = q_1 + \cfrac{1}{q_2 + \cfrac{1}{\cdots \cfrac{}{q_{m-2} + \cfrac{1}{q_{m-1}}}}},$$

and denote $b/a = /q_1, q_2, \ldots, q_{m-2}, q_{m-1}/$, where $q_1, q_2, \ldots, q_{m-1} = r_{m-1}$ are the positive quotient series of Euclid's algorithm. It is known in general that if m is even, then the continued fraction arranged in reverse, $/q_{m-1}, q_{m-2}, \ldots, q_2, q_1/$, is equal to the reduced fraction b/c, where c satisfies $ac \equiv 1 \pmod b$ and $0 < c < b$ (see, for example, [2]). This means that $[b/c] = r_{m-1} > 1$ if m is even. Similarly, using the identity

$$/q_1, q_2, \ldots, q_{m-2}, q_{m-1}/ = /q_1, q_2, \ldots, q_{m-2}, q_{m-1} - 1, 1/$$

in the case m is odd, we obtain

$$\frac{b}{c} = /1, q_{m-1} - 1, q_{m-2}, \ldots, q_2, q_1/.$$

Thus $[b/c] = 1$. ∎

Proof of the equivalence between the two statements. Let a'/b' and a''/b'' be consecutive fractions in the Farey sequence of order n. By using Lemma 1, $[n/b']$ or $[n/b'']$ is equal to 1. Since M_n is symmetric and all coefficients of the polynomials appearing in Theorems 1 and 2 are real, it will suffice to consider the case $b''[n/b''] \geqslant b'[n/b']$ and to prove coincidence of the equations given by use of the theorems.

Case 1. Suppose further $[n/b''] > [n/b']$. Then $[n/b'] = 1$ and $b' > n/2 \geqslant b''$ hold. The parametric equation given by use of Theorem 2 is

$$\lambda^{b'}(\lambda^{b''} - s)^{[n/b'']} = (1 - s)^{[n/b'']}\lambda^{b''[n/b'']}.$$

Since $[n/b''] \neq 1$, the parametric equation obtained by use of Theorem 1 is of the type (II). Using Lemma 1 we have

$$a''\left(b''\left[\frac{n}{b''}\right] - b'\right) \equiv -1 \pmod{b''} \qquad \left(0 \leqslant b''\left[\frac{n}{b''}\right] - b' \leqslant n - b' < b''\right).$$

This shows that the parametric equation is

$$(\lambda^{b''} - t)^{[n/b'']} = (1 - t)^{[n/b'']}\lambda^{b''[n/b'']-b'}.$$

Case 2. Suppose $[n/b''] < [n/b']$. Then $[n/b''] = 1$ and $b'' > n/2 \geqslant b'$ hold. The parametric equation given by use of Theorem 2 is

$$\lambda^{b''}(\lambda^{b'} - s)^{[n/b']} = (1 - s)^{[n/b']}\lambda^{b'[n/b']}.$$

Since a'' and b'' $(0 < a'' \leqslant b'')$ are relatively prime and $b'' > 1$, we have $0 < a'' < b''$. From Lemma 1 we have

$$a''b' \equiv 1 \pmod{b''} \qquad (0 < b' < b'').$$

Let $r_1 = b''$, $r_2 = a''$, $r_3, \ldots, r_m = 1$, $r_{m+1} = 0$ be the nonnegative remainder series of Euclid's algorithm. Lemma 2 tells us that

$$\left[\frac{b''}{b'}\right] = \begin{cases} r_{m-1}\,(>1) & \text{if } m \text{ is even,} \\ 1 & \text{if } m \text{ is odd.} \end{cases}$$

Since

$$\frac{n}{b'} \geqslant \frac{b''}{b'} \geqslant \left[\frac{n}{b'}\right] > 1,$$

we have

$$\left[\frac{b''}{b'}\right] = \left[\frac{n}{b'}\right] > 1.$$

This means m is even and $[n/b'] = r_{m-1}$. Thus, the parametric equation is of the type (I) and $r = [n/b']$. Using Lemma 1 we have

$$a''\left(b'' - b'\left[\frac{n}{b'}\right]\right) \equiv -\left[\frac{n}{b'}\right] \pmod{b''} \qquad \left(0 \leqslant b'' - b'\left[\frac{n}{b'}\right] < b''\right).$$

Thus we obtain the parametric equation

$$\lambda^{b'' - b'[n/b']}\left(\lambda^{b'} - t\right)^{[n/b']} = (1 - t)^{[n/b']}.$$

Case 3. Suppose $[n/b''] = [n/b']$ $(= 1)$. In this case $b'' \geqslant b'$ holds. If $b'' = b'$, then using Lemma 1 we have $a'' = b'' = b' = 1$, $a' = 0$, and $n = 1$. This case is trivial. Hence, assume $b'' > b'$. The parametric equation by use of Theorem 2 is

$$\lambda^{b''}\left(\lambda^{b'} - s\right) = (1 - s)\lambda^{b'}.$$

From Lemma 1,

$$a''b' \equiv 1 \pmod{b''} \qquad (0 < b' < b'')$$

holds. Let $r_1 = b''$, $r_2 = a''$, $r_3, \ldots, r_m = 1$, $r_{m+1} = 0$ be the nonnegative remainder series of Euclid's algorithm. Since

$$\frac{n}{b'} \geqslant \frac{b''}{b'} \geqslant \left[\frac{n}{b'}\right] = 1$$

holds, we have

$$\left[\frac{b''}{b'}\right] = \left[\frac{n}{b'}\right] = 1.$$

This means that m is odd and the parametric equation obtained by use of Theorem 1 is of the type (II). It is easy to see

$$a''(b'' - b') \equiv -1 \pmod{b''} \qquad (0 < b'' - b' < b'').$$

This means that the parametric equation is

$$\lambda^{b''} - t = (1 - t)\lambda^{b'' - b'}. \qquad \blacksquare$$

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