A Penalty Linear Programming Method
Using Reduced-Gradient Basis-Exchange Techniques*

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Dedicated to Alston S. Householder
on the occasion of his seventy-fifth birthday.

Submitted by G. W. Stewart

ABSTRACT

We offer a variant of the piecewise-linear penalty-function approach to linear programming which was proposed by Conn [5]. Our variant makes use of computational techniques which are more closely related to those in existing computer codes for linear programming and which can be more readily adapted for large sparse problems than were the techniques described by Conn. An experimental code for small dense problems has been prepared and some experience with it is reported.

1. INTRODUCTION

Recently A. R. Conn [5] has proposed a new method of linear programming which makes use of a piecewise differentiable penalty function and active-set projected-gradient techniques. While the penalty-function approach to linear programming appears to hold promise, the techniques used in [5] may hinder this approach from getting the trial it deserves. Firstly, most of the existing software for the linear programming problem is based upon reduced-gradient basis-exchange techniques and cannot be easily modified to explore the ideas in [5]. Secondly, the use of projections, as implemented by orthogonal transformations or by some variant of the classical projector $N(N^TN)^{-1}N^T$, is not readily adapted to the solution of general large sparse problems.

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In this report we describe how Conn's penalty approach can be reformulated to make use of the familiar simplex-method mechanisms of basis matrices and reduced gradients while continuing to offer the flexibility which comes from not having to maintain feasibility.

Our presentation will be expository, closely following the background set by [2, 3, 5]. Accordingly, we will be brief whenever information can be obtained from these earlier works.

2. NOTATION AND TERMINOLOGY

We have tried, where possible, to conform to the notation and the terminology used by Hadley [6].

Consider the problem

\[
\text{minimize } c^T x \\
\text{subject to } Ax = b, \quad (2.1)
\]

\[f \leq x \leq r.
\]

It will be assumed that \(f \leq r\) and, as a convenience, that components "\(-\infty\)" and "\(+\infty\)" will be permitted in \(f\) and \(r\), respectively (corresponding to variables which are unrestricted either below or above or both). Further, we will require that \(A\) have full row rank,

\[A \in \mathbb{R}^{m \times n}, \quad \text{rank}(A) = m.
\]

Take any \(x\) such that \(Ax = b\). [We make no requirement that \(x\) be feasible for (2.1), that is, that \(x\) satisfy the upper- and lower-bound restrictions.] Consider the partitions of \(A\), \(x\), and \(c\) indicated below:

\[
Ax = \begin{bmatrix} B & N \end{bmatrix} \begin{bmatrix} x_B \\ \vdots \\ x_N \end{bmatrix},
\]

\[
c^T x = \begin{bmatrix} c_B^T & c_N^T \end{bmatrix} \begin{bmatrix} x_B \\ \vdots \\ x_N \end{bmatrix},
\]

where \(B\) is \(m \times m\). In the terminology of the simplex method, \(B\) is a \textit{basis matrix} if it is nonsingular; its columns are a \textit{basis} for \(A\). The components of \(x_B\) are the \textit{basic variables} associated with \(B\). The components of \(x_N\) are the
nonbasic variables, and the columns of $N$ are the nonbasic columns. Since $Ax = (AP)(P^{-1}x)$ and $c^T x = (PC)^T(P^{-1}x)$ for any permutation matrix $P$, we can regard any $m$ linearly independent columns of the matrix $A$ as constituting the matrix $B$ shown in (2.2). For simplicity of notation, we shall ignore whatever permutations are needed to bring (2.1) to a form consistent with (2.2) whenever we select a new basis $B$.

The discussion is much simplified if we make a nondegeneracy assumption:

\begin{equation}
\text{No basic variable is at a bound.}
\end{equation}

That is, $x_i \neq f_i$ and $x_i \neq r_i$ for any $i$ associated with a basic column of $A$. As in the simplex method, this assumption can be dispensed with, but to do so requires a much more complicated presentation than we would like to give.

3. DEVELOPMENT

Let $\mu > 0$ be picked, for the moment arbitrarily, and consider the piecewise linear function

\begin{equation}
\phi(x) = \mu c^T x - \sum_i \min(x_i - f_i, 0) - \sum_i \min(r_i - x_i, 0),
\end{equation}

where the primes indicate that terms for which $f_i$ is $-\infty$ or $r_i$ is $+\infty$ are not included in the summation. As in [5], it is true that minimizing (3.1) for a sufficiently small value of $\mu$ will provide an optimal $x$ for (2.1) (or an indication of unboundedness or infeasibility).

We will develop a minimization process which begins at any arbitrary $x$ satisfying $Ax = b$ (but not necessarily satisfying $f < x < r$) and which generates a finite sequence of corrected points $x$, all of which also satisfy the above equations. The outline to be followed is:

1. Find $p \in \mathbb{R}^n$ satisfying $Ap = 0$ such that $\phi(x + \alpha p) < \phi(x)$ for all $\alpha > 0$ small enough.
2. Choose $\alpha > 0$ so as to minimize $\phi(x + \alpha p)$ as a function of $\alpha$.
3. Replace $x$ by $x + \alpha p$.

These steps are repeated until a terminating condition (unboundedness or optimality) is reached. If the resulting $x$ does not solve (2.1) and $\mu$ is still not
negligibly small, then \( \mu \) is reduced, and steps 1–3 are undertaken again. For any sequence of values \( \mu \) approaching zero from above:

(a) either a solution to (2.1) is produced for each value of \( \mu \) in the sequence small enough,
(b) or else the iteration characterized by steps 1–3 will indicate for all values of \( \mu \) that \( \phi \) is unbounded,
(c) or else the iteration will produce points \( x \) which are infeasible for all values of \( \mu \).

Behavior (b) is associated with an unbounded problem (2.1), and behavior (c) indicates that (2.1) is infeasible.

We begin our development by considering the choice of \( p \) in step 1 of (3.2). The condition

\[ Ap = 0 \]  

(3.3)

will guarantee that \( Ax = b \) for all \( x \) which we generate, provided that we start with an \( x \) which satisfies these equations. To satisfy (3.3) we can make use of the partition given by (2.2). Specifically,

\[ p = \begin{bmatrix} p_B \\ \cdots \\ p_N \end{bmatrix}, \]  

(3.4)

so that (3.3) can be satisfied if

\[ p_B = -B^{-1}Np_N. \]  

(3.5)

We wish to obtain a \( p \) such that

\[ \phi(x + \alpha p) < \phi(x) \quad \text{for all } \alpha > 0 \text{ small enough.} \]  

(3.6)

Such a vector \( p \) is a descent direction for \( \phi \) at the point \( x \). We note that

\[ \phi(x + \alpha p) = \phi(x) + \alpha g^T p - \alpha \left[ \sum_i \min(p_i, 0) + \sum_i \min(-p_i, 0) \right], \]  

(3.7)

where

\[ g = \mu c - \sum_i s_i e_i + \sum_i \bar{s}_i e_i. \]  

(3.8)
We have used double primes to indicate summations which include neither indices for which \( f_i, r_j \) are infinite nor indices for which \( x_i, x_j \) are at bound. Stars, on the other hand, indicate summations which are taken over only those indices for which components \( x_i, x_j \) are at bound. (Our nondegeneracy assumption ensures that the starred summations only involve indices associated with the nonbasic portion, \( N \), of the matrix \( A \).) Finally, \( e_i, e_j \in \mathbb{R}^n \) are unit coordinate vectors, and

\[
\begin{align*}
    s_i & = \begin{cases} 
    +1 & \text{if } x_i < f_i, \\
    0 & \text{otherwise},
    \end{cases} \\
    -s_j & = \begin{cases} 
    +1 & \text{if } x_j > r_j, \\
    0 & \text{otherwise}.
    \end{cases}
\end{align*}
\tag{3.9}
\]

From (3.7) we see that the condition (3.6) will be satisfied if and only if

\[
    g^T p - \left( \sum_i^* \min(p_i, 0) + \sum_j^* \min(-p_j, 0) \right) < 0.
\tag{3.10}
\]

Further, consistent with the partitioning of (2.2) and (3.4) and using (3.5), we may write

\[
    g^T p = \begin{bmatrix} g^T_B & g^T_N \end{bmatrix} \begin{bmatrix} p_B \\ p_N \end{bmatrix}
    = \begin{bmatrix} g^T_B & g^T_N \end{bmatrix} \begin{bmatrix} -B^{-1}p_N \\ p_N \end{bmatrix}
    = (g^T_N - g^T_B B^{-1}N) p_N = (g^T_N - \pi^T N) p_N.
\tag{3.11}
\]

Note that (3.11) will permit us to rewrite (3.10) so that it only involves components of \( p_N \).

We have followed the practice of the simplex method by writing \( g^T B^{-1} N \) as \( \pi^T N \), where \( \pi = B^{-1} g_B \). The vector \( g^T_N - \pi^T N \) serves as the counterpart in our discussion to the reduced cost vector (or reduced gradient) appearing in the simplex method. We will take over the usual simplex-method terminology and refer to the components of \( g^T_N - \pi^T N \) as the reduced costs associated with \( x \).

As in the simplex method, it is suitable to restrict the consideration of \( p_N \) to vectors having only one nonzero component. Let \( \sigma \) be an index associated with a nonbasic column of \( A \). Then:

(1) \( x_\sigma \) is not at a bound, and the corresponding reduced cost is nonzero;
(2) $x_\sigma = f_\sigma$, and the corresponding reduced cost is negative;
(3) $x_\sigma = r_\sigma$, and the corresponding reduced cost is positive;
(4) $x_\sigma = f_\sigma$, and the corresponding reduced cost is greater than +1:
(5) $x_\sigma = r_\sigma$, and the corresponding reduced cost is less than −1;
(6) none of the above.

If any one of (1)–(3) holds for any index $\sigma$, then $x$ is not optimal for $\phi$ (and conversely) and a vector $p_N$ with a single nonzero component corresponding to $\sigma$ can be chosen so that

$$
\sum_i * \min(p_i, 0) + \sum_i * \min(-p_i, 0) = 0 \tag{3.13}
$$

and so that

$$
g^T p = (g_N^T - \pi^T N) p_N < 0. \tag{3.14}
$$

This defines a $p$ satisfying (3.10); i.e. (3.6). Choices of $p_N$ for cases (4), (5) can be made to satisfy (3.10), too, but we will ignore them. This is because case (6) implies that

$$(\text{reduced cost})_\sigma = 0 \quad \text{when} \quad x_\sigma \text{ is not at a bound},
0 < (\text{reduced cost})_\sigma < +1 \quad \text{when} \quad x_\sigma = f_\sigma,
-1 < (\text{reduced cost})_\sigma < 0 \quad \text{when} \quad x_\sigma = r_\sigma
$$

for all indices $\sigma$ associated with $N$. However, if $f < x < r$, then $x$ is feasible for (2.1), and the reduced cost vector becomes $c_\Sigma^T - C_\Sigma^T B^{-1} N$ just as in the simplex method. But the optimality conditions for (2.1), according to the simplex method, are

$$(\text{reduced cost})_\sigma = 0 \quad \text{when} \quad x_\sigma \text{ is not at a bound},
0 < (\text{reduced cost})_\sigma < +1 \quad \text{when} \quad x_\sigma = f_\sigma,
(\text{reduced cost})_\sigma < 0 \quad \text{when} \quad x_\sigma = r_\sigma.$$
The extra conditions,

\[
\begin{align*}
\text{(reduced cost)}_a & \leq +1 \quad \text{when } x_a = f_a, \\
\text{(reduced cost)}_a & \geq -1 \quad \text{when } x_a = r_a,
\end{align*}
\]

come from our use of the function \( \phi \) of (3.1) to represent the linear programming problem (2.1). If we ignore these conditions, in effect ignoring (4) and (5) of (3.12), then we may not be able to find a descent direction for \( \phi \) at \( x \) when, in fact, one does exist. But this will be the case only when (1)–(3) are not satisfied at all nonbasic indices. If \( x \) is within its bounds, this is equivalent to stating that \( x \) is optimal for the problem (2.1). If \( x \) is not within its bounds, we may choose to interpret the situation as an indication that \( \mu \) is still too large. The option to reduce \( \mu \) rather than consider (4) or (5) is one which presents itself, and we will choose to take it. Our goal is to arrive at a solution to (2.1) as quickly as possible, not merely to minimize \( \phi \) for a fixed choice of \( \mu \).

Assume that some (nonbasic) index \( \sigma \) can be found which satisfies one of the conditions (1), (2) or (3) of (3.12). Then

\[
\text{let } p_\sigma = -\text{sgn}[(\text{reduced cost})_\sigma]; \text{ let all other components of } p_n \text{ be zero; let } p_B \text{ be given by (3.5). (3.15)}
\]

It is easily checked that (3.13) and (3.14), and hence (3.6), are satisfied. This completes the discussion on step 1 of (3.2).

To carry out step 2 of (3.2), we note that (3.7) can be written as

\[
\phi(x + \alpha p) = \phi(x) + \alpha g^{(0)} r p,
\]

where \( g^{(0)} = g \) as given in (3.8). This representation of \( \phi(x + \alpha p) \) as a function of \( \alpha \) will be valid until \( \alpha \) is increased to a value for which some component of \( x + \alpha p \) strikes a bound. Note, however, that only the basic components together with the \( \sigma \)th component, which is nonbasic, of \( x + \alpha p \) are subject to change as \( \alpha \) is increased. The values of \( \alpha \) at which these components equal their upper or lower bounds are given by the ratios

\[
\frac{f_k - x_k}{p_k} \quad \text{and} \quad \frac{r_l - x_l}{p_l}
\]

for \( k, l = \sigma \) or \( k, l \) corresponding to a basic variable, but not including indices for which \( f_k \) is \(-\infty\) or \( r_l \) is \(+\infty\).
Only the nonnegative ratios (3.17) are of importance. Let
\[ \alpha^{(1)} \leq \alpha^{(2)} \leq \cdots \leq \alpha^{(l)} \]
be these (at most \(2m+2\)) ratios. The representation (3.16) is valid for \(0 < \alpha < \alpha^{(1)}\).

Suppose \(\alpha\) is increased beyond \(\alpha^{(1)}\). For \(\alpha^{(1)} < \alpha < \alpha^{(2)}\), \(\phi(x + \alpha p)\) will have the representation
\[
\phi(x + \alpha p) = \phi(x) + \alpha^{(1)} g^{(0)} T p + (\alpha - \alpha^{(1)}) g^{(1)} T p,
\]
where \(g^{(1)}\) differs from \(g^{(0)}\) to account for the component of \(x\) (associated with \(\alpha^{(1)}\)) which has reached and passed a bound. The change from \(g^{(0)}\) to \(g^{(1)}\) occurs in only one component of \(g^{(0)}\) and is easily determined. We need only inspect the component of \(x + \alpha p\) which crosses a bound as \(\alpha\) crosses the value \(\alpha^{(1)}\) and reset the associated component of \(g^{(0)}\) consistent with (3.8) and (3.9). [Ties, i.e., several ratios (3.17) with a common value, can be handled as if the ratios were different by random infinitesimal amounts.]

After \(g^{(1)}\) is obtained, we may note whether \(p\) is still a descent direction; i.e., \(g^{(1)} T p < 0\). If this is so, \(\alpha\) may be increased further. A loop results:

\[
\begin{align*}
\nu & := 0; \quad t := g^{(0)} T p; \\
\text{while } & t < 0 \text{ and } \nu < \lambda \text{ do} \\
& \begin{align*}
\nu & := \nu + 1; \\
t & := g^{(\nu)} T p \\
\end{align*}
\end{align*}
\]

This loop clearly accommodates even the case in which no ratio (3.17) is nonnegative (i.e. \(\lambda = 0\)). The loop terminates with \(\nu = \lambda\) and \(g^{(\lambda)} T p < 0\), which indicates unboundedness, or it terminates at some stage \(0 < \nu < \lambda\) with \(g^{(\lambda)} T p > 0\). The corresponding ratio \(\alpha^{(\nu)}\) is associated with an index \(\rho = k\) or \(\rho = l\) from (3.17). The component \(x_{\rho} + \alpha p_{\rho}\) has hit an upper or a lower bound at \(\alpha = \alpha^{(\nu)}\) depending upon which ratio of (3.17) provided the value \(\alpha^{(\nu)}\). As with the simplex method, the assumption of nondegeneracy (2.3) will ensure that the ratio \(\alpha^{(\nu)}\) is distinct from all other ratios of (3.17) and that \(\rho\) is uniquely determined. This concludes the discussion for step 2 of (3.2).

According to step 3, \(x\) is to be replaced by \(x + \alpha p\) (\(\alpha = \alpha^{(\nu)}\)) for the next execution of steps 1–3. If \(\rho = \sigma\), then we may continue to use \(B\) as a basis. If \(\rho \neq \sigma\), then we must drop column \(A_{\rho}\) from the basis and replace it with \(A_{\sigma}\). This basis exchange step, then, exactly corresponds to the basis exchange which takes place in the simplex method.
We end the discussion of (3.2) by rewriting the proposed minimization process in a more detailed and structured fashion:

\[
\begin{align*}
\{ & \text{Choose any basis } B \text{ for } A \}; \\
\{ & \text{Determine any } x_N, \text{ and let } x_B = B^{-1}(b - Nx_N) \}; \\
\{ & \text{Determine } g \text{ as in (3.8)} \}; \\
\text{repeat} & \\
\{ & \text{Solve } B^T \pi = g_B \text{ for } \pi \}; \\
\{ & \text{Compute } g_N^T - \pi^TN \}; \\
\{ & \text{Find a nonbasic index } \sigma \text{ from (3.12) (1)-(3)} \}; \\
\text{if} & \{ \text{no } \sigma \text{ found} \} \text{ then} \\
& \begin{cases} \\
\{ & \text{if } \{ x \text{ not within bounds} \} \text{ then } \{ \text{decrease } \mu \} ; \\
& \{ \text{break loop} \} \\
\end{cases} \\
\{ & \text{Obtain } p_{\sigma} \text{ from (3.15)} \}; \\
\{ & \text{Solve } Bp_B = -\text{sgn}(p_{\sigma})A_{\sigma} \text{ for } p_B \}; \\
\{ & \text{Compute the ratios given in (3.17)} \}; \\
\{ & \text{Carry out (3.18) to find the index } \rho \text{ at some stage } \nu \}; \\
\text{if} & \{ \text{no } \rho \text{ found} \} \text{ then} \\
& \begin{cases} \\
\{ & \text{decrease } \mu \} ; \text{ break loop } \} \text{ end; } \\
\end{cases} \\
\{ & \text{Set } x_B := x_B + \alpha^{(\rho)}p_B \text{ and } x_{\sigma} := x_{\sigma} + \alpha^{(\nu)}p_{\sigma} \}; \\
\{ & \text{Adjust } g \text{ according to (3.8) to be correct at } x \}; \\
\text{if} & \rho \neq \sigma \text{ then } \{ \text{replace } A_{\rho} \text{ in } B \text{ by } A_{\sigma} \} \\
\text{until} & \{ x \text{ is optimal for (2.1)} \} \text{ or } \{ \mu \text{ is too small} \}.
\end{align*}
\]

4. PROBLEM FORMULATION AND PROBLEM CHANGE

In order to begin the minimization process, it is necessary to pick an initial basis. Unlike the simplex method, however, the feasibility of a basis is not important, and this removes the necessity of executing a so-called phase-1 section. If \(m\) linearly independent columns of \(A\) are known, then they may be chosen to provide \(B\), and any initial \(x\) consistent with \(B\) may be used. This allows an arbitrary choice for the nonbasic variables \(x_N\); hence any \textit{a priori} knowledge about the problem may be used in determining these variables. In particular we observe that, in postoptimality analysis, one has
frequently just solved a problem of the form (2.1) with slightly changed \( A, b, c, f, \) or \( r \). The solution (and possibly the basis) from this prior problem is likely to offer an excellent start for the current problem.

Without any knowledge about which columns of \( A \) are linearly independent, we may take a hint from the simplex method's phase-1 process to get started. By introducing \( m \) artificial variables, \( z \), we may convert (2.1) into the equivalent problem

\[
\begin{align*}
\text{minimize} & \quad \begin{bmatrix} c^T & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \\
\text{subject to} & \quad \begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = b, \\
& \quad f \leq x \leq r, \quad 0 \leq z \leq 0.
\end{align*}
\]  

(4.1)

It is easily seen that (4.1) will be unbounded or infeasible if (2.1) is unbounded or infeasible, respectively. If (2.1) is neither unbounded nor infeasible, then the optimal points \( x \) for (2.1) correspond to optimal \( x \) for (4.1). The minimization process (3.19) can be applied to (4.1) using the components of \( z \) as the initial basic variables. (The resulting minimization method will not be identical with phase 1 of the simplex method.)

Between no knowledge of a starting basis and complete knowledge there is a realm in which a starting basis consisting of some columns of \( A \) and some artificial columns may be chosen. The examples below suggest a few of the possibilities.

Suppose the following linear programming problem is given:

\[
\begin{align*}
\text{minimize} & \quad 3x_1 - 4x_2 + x_3 - 2x_4 \\
\text{subject to} & \quad 2x_1 + x_2 + 2x_3 + x_4 = 10, \\
& \quad x_3 + 2x_4 < 10, \\
& \quad x_1 - x_2 + x_4 \geq -5, \\
& \quad 5 < 2x_1 + 3x_2 + x_3 + x_4 < 20, \\
& \quad x_1 \geq 0, x_2 > 0, x_3 > 0, x_4 > 0.
\end{align*}
\]

To this we may add slack variables \( x_5, \ldots, x_8 \) to obtain a problem of the form (2.1):

\[
\begin{align*}
\text{minimize} & \quad 3x_1 - 4x_2 + x_3 - 2x_4 \\
\text{subject to} & \quad 2x_1 + x_2 + 2x_3 + x_4 = 10, \\
& \quad x_3 + 2x_4 + x_5 = 10, \\
& \quad x_1 - x_2 + x_4 - x_6 = -5, \\
& \quad 2x_1 + 3x_2 + x_3 + x_4 - x_7 = 5, \\
& \quad 2x_1 + 3x_2 + x_3 + x_4 + x_8 = 20, \\
& \quad 0 \leq x_i < +\infty, \quad i = 1, \ldots, 8.
\end{align*}
\]
EXAMPLE 1. It may be noted that columns 4, 5, 6, 7, 8 of the constraint matrix in (4.2) will serve as a basis, and this can be used to begin the minimization.

EXAMPLE 2. It may be noted merely that columns 5, 6, 7, 8 would serve as a partial basis. A single artificial variable can be added to (4.2), for example to the first equation, to obtain

\[ 2x_1 + x_2 + 2x_3 + x_4 + 0x_5 + 0x_6 + 0x_7 + 0x_8 + z_1 = 10, \]
\[ 0 \leq z_1 \leq 0. \]

The resulting columns 5, 6, 7, 8, 9 can be used as a first basis.

EXAMPLE 3. The internal structure of the constraint matrix may be ignored entirely, and artificial variables \( z_1, z_2, z_3, z_4, z_5 \) can be added to (4.2) to obtain a starting basis [see (4.1)].

EXAMPLE 4. Using 4, 5, 6, 7, 8 as a starting basis as in Example 1, the method given by (3.19) terminates with an optimal \( x \) given by

\[ x_1 = 0, \quad x_2 = 5, \quad x_3 = 0, \quad x_4 = 5 \]
\[ x_5 = 0, \quad x_6 = 5, \quad x_7 = 15, \quad x_8 = 0. \]

This point \( x \) is associated with the basis consisting of columns 2, 4, 5, 6, 7. Any changes can be made to \( c, b, f, r \) and to columns 1, 3 or 8 of \( A \), and the minimization process can be reinitiated with the basis given above. If no change has been made to \( b \), then the \( x \)-values above may be reused. Otherwise \( x_1, x_3, \) and \( x_8 \) may be retained and \( x_2, x_4, x_5, x_6, x_7 \) recomputed.

5. EXPERIENCE, VARIATIONS, RELATED WORK

Some preliminary trials have been carried out to compare the proposed method (3.19) started at an all-artificial basis (4.1) with the standard phase-1–phase-2 simplex method. The results have been quite encouraging. We have generated a number of random problems having known solutions by using the ideas laid out in [7]. Briefly, given \( m < n \), choose \( m \) integers \( i_1, \ldots, i_m \) randomly and without replacement from the first \( n \) positive integers. These serve to define the indices of an optimal basis. Define \( x \) consistently with these indices by selecting \( x_{i_1}, \ldots, x_{i_m} \) to be random positive numbers and by letting \( x_j = 0 \) for all other \( j \). Generate random numbers as
entries for the matrix $A$. Define $g$ as an optimal reduced-cost vector consistent with $x$ by selecting $g_i$ to be a random positive number for each $i \in \{1, \ldots, n\} - \{i_1, \ldots, i_m\}$ and letting $g_{i_1}, \ldots, g_{i_m}$ be zero.

If we select any random values for $\pi_1, \ldots, \pi_m$ and compute $b, c$ according to

$$b = Ax,$$
$$c = A^T \pi + g,$$

then $x$ solves the problem

$$\text{minimize} \quad c^T x$$
subject to $Ax = b, \quad x \geq 0,$

and $\pi$ will be the dual solution.

In constructing our trial problems, we used the uniform random-number generator \texttt{GGUBF} which is included in [8]. The pertinent \texttt{FORTRAN} code was:

```fortran
DO 20 I = 1,M
  10 CONTINUE
    J = IFIX(GGUBF(ISEED)*FLOAT(N)) + 1
    IF (X(J) .NE. ZERO) GO TO 10
    X(J) = (GGUBF(ISEED) + HALF)*XSPRED
  20 CONTINUE

DO 40 J = 1,N
  DO 30 I = 1,M
    A(I,J) = (GGUBF(ISEED) - HALF)*ASPRED
  30 CONTINUE
  40 CONTINUE

DO 50 I = 1,M
  B(I) = SDOT(N, A(I,1),1, X,1)
  PI(I) = (GGUBF(ISEED) - HALF)*PSPRED
  50 CONTINUE

DO 60 I = 1,N
  C(I) = SDOT(M, A(1,I),1, PI,1)
  IF (X(I) .NE. ZERO) GO TO 60
    C(I) = C(I) + (GGUBF(ISEED) + HALF)*GSPRED
  60 CONTINUE
```
It is assumed that the array $X$ has been initialized to zero, that $\text{IAR}$ gives the value of the row-dimension of the array $A$, that $\text{HALF} = 0.5$ and $\text{ZERO} = 0.0$, and that $\text{SDOT}$ is an inner-product routine:

$$\text{SDOT}(K, V, IV, W, IW) = V(1) \cdot W(1) + V(1 + IV) \cdot W(1 + IW) + \cdots + V(1 + (K - 1) \cdot IV) \cdot W(1 + (K - 1) \cdot IW).$$

For the results quoted in Tables 1 and 2, $\text{XSPRED}$, $\text{ASPRED}$, $\text{PSPRED}$ and $\text{GSPRED}$ were all taken to be $10.0$. (If these parameters are varied, aspects of problem conditioning and computational accuracy can be investigated.) The phase-1–phase-2 method used for comparison was $\text{ZX3LP}$ of [8], which is a straightforward implementation of the revised simplex method. The penalty parameter $\mu$ used for all of our trials was $0.025$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>Penalty LP</th>
<th>Phase-1–Phase-2</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>7.2</td>
<td>6.6</td>
</tr>
<tr>
<td>10</td>
<td>15.8</td>
<td>15.8</td>
</tr>
<tr>
<td>15</td>
<td>25.8</td>
<td>25.4</td>
</tr>
<tr>
<td>20</td>
<td>32.4</td>
<td>34.0</td>
</tr>
<tr>
<td>30</td>
<td>49.6</td>
<td>65.8</td>
</tr>
<tr>
<td>40</td>
<td>71.8</td>
<td>83.2</td>
</tr>
<tr>
<td>45</td>
<td>86.0</td>
<td>112.2</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>$m$</th>
<th>Penalty LP</th>
<th>Phase-1–Phase-2</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>9.4</td>
<td>8.8</td>
</tr>
<tr>
<td>10</td>
<td>21.0</td>
<td>19.2</td>
</tr>
<tr>
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<td>33.2</td>
<td>34.8</td>
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<tr>
<td>20</td>
<td>42.0</td>
<td>47.8</td>
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<tr>
<td>30</td>
<td>64.8</td>
<td>83.0</td>
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<tr>
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<td>107.2</td>
<td>125.6</td>
</tr>
<tr>
<td>45</td>
<td>121.6</td>
<td>147.0</td>
</tr>
</tbody>
</table>

We have only quoted numbers of iterations in our comparisons. As far as timing was concerned, our code ran 40% faster than $\text{ZX3LP}$ (averaged over the tests), but this could merely be an artifact of a difference in programming styles. We would much rather point out that our proposed techniques
use fundamentally the same computational tools as does the classical simplex method. There is no reason why an iteration of our proposed algorithm should need to take materially longer to carry out than an iteration of the simplex method. Given that this is true, we find it very promising that our techniques seem to require fewer and fewer iterations to attain an optimum, relative to the simplex method, as larger and larger problems are solved.

Again, we point out that our method was started at an all-artificial basis in the above tests. In this regard note that the choice of where to start a linear programming method often strongly influences how quickly the method reaches optimum. The proposed method is exceedingly flexible in this regard, whereas the standard simplex method is quite rigid. Wherever information is available about a good starting basis, whether feasible or infeasible, the proposed method can easily take advantage of it. Hence this method may be particularly suited to situations in which many postoptimality studies are carried out and previous optima can be used as starting points.

As an illustration of the flexibility we have in starting the proposed method, let us consider the problem given by (4.2). The simplex method might conventionally introduce artificial variables in rows 1 and 4 to obtain the phase-1 problem

\[
\begin{align*}
\text{minimize} & \quad x_9 + x_{10} \\
\text{subject to} & \quad 2x_1 + x_2 + 2x_3 + x_4 + x_9 = 10, \\
& \quad x_3 + 2x_4 + x_5 = 10, \\
& \quad x_1 - x_2 + x_4 - x_6 = -5, \\
& \quad 2x_1 + 3x_2 + x_3 + x_4 - x_7 + x_{10} = 5, \\
& \quad 2x_1 + 3x_2 + x_3 + x_4 + x_8 = 20, \\
& \quad 0 \leq x_i, \quad i = 1, \ldots, 10.
\end{align*}
\]

If the starting basis involves indices (5, 6, 8, 9, 10), then phase 1 proceeds through the bases

1. (5, 6, 8, 9, 10),
2. (2, 5, 6, 8, 9),
3. (2, 3, 5, 6, 8),

and phase 2 proceeds through the bases

4. (2, 3, 5, 6, 7),
5. (2, 3, 4, 5, 7),
6. (2, 3, 4, 7, 8).

A total of 6 steps have been required to reach an optimum.

If the proposed algorithm is started with the same basis, involving the same two artificial variables (but with the appropriate objective function

\[3x_1 - 4x_2 + x_3 - 2x_4 + 0x_5 + 0x_6 + 0x_7 + 0x_8 + 0x_9 + 0x_{10},\]
then the progress to an optimum is given by

1. (5, 6, 8, 9, 10),
2. (5, 6, 8, 9, 2),
3. (5, 6, 8, 2, 3),
4. (5, 8, 2, 3, 7),
5. (5, 2, 3, 7, 4),
6. (5, 2, 7, 4, 6).

(Note: there are several optimal bases for this problem.)

If the proposed algorithm is started at the (infeasible) basis involving the indices (4, 5, 6, 7, 8) as given in (4.2), then only 4 steps are required:

1. (4, 5, 6, 7, 8),
2. (4, 6, 7, 8, 1),
3. (4, 6, 7, 1, 2),
4. (4, 6, 7, 2, 5).

In both cases \( \mu = 0.1 \) was taken, and the algorithm had no occasion to change this value.

An extensive series of tests is currently being undertaken by R. J. Hanson et al. at Sandia Laboratories. They will be studying this and several other linear programming codes for small, dense problems.

The ideas presented here are being explored for their applicability to the network simplex method by Rainer von Saleski as part of his Ph.D. thesis for the Department of Mathematical Sciences at The Johns Hopkins University.

Finally, Brown and Graves [4] have reported significant advantages of a similar method over conventional techniques in solving large problems having some 0-1 variables.

REFERENCES


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