Regular embeddings of complete multipartite graphs

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Abstract

In this paper, we classify all regular embeddings of the complete multipartite graphs $K_{p,\ldots,p}$ for a prime $p$ into orientable surfaces. Also, the same work is done for the regular embeddings of the lexicographical product of any connected arc-transitive graph of prime order $q$ with the complement of the complete graph of prime order $p$, where $q$ and $p$ are not necessarily distinct. Lots of regular maps found in this paper are Cayley maps.

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1. Introduction

A map is a 2-cell embedding of a graph into a closed surface. The embedded graph is called the underlying graph of the map. Throughout this paper we only consider maps on orientable surfaces. By an automorphism of a map $\mathcal{M}$, we mean an automorphism of the underlying graph $\mathcal{G}$ which can be extended to an orientation preserving self-homeomorphism of the surface. The set of automorphisms forms a group, called the automorphism group $\text{Aut}(\mathcal{M})$ of the map $\mathcal{M}$. Moreover, it is well known [9] that $\text{Aut}(\mathcal{M})$ acts semi-regularly on the arcs of $\mathcal{G}$. If it acts regularly, the map as well as the corresponding embedding is called regular.
The classification of regular embeddings of a given simple graph was initiated by Heffter [6] who constructed regular embeddings of the complete graph $K_p$, $p$ a prime. The classification for the complete graphs has been completed by the works of Biggs [1] and James and Jones [8]. The proof uses a characterization of sharply doubly transitive permutation groups. It follows that such embeddings can exist only if $n = p^k$ is a prime power and the automorphism groups of the maps are the groups of affine transformations of the field $\mathbb{F}_{p^k}$. Besides the complete graphs, regular maps of the cocktail-party graphs $K_{n,n} \sim nK_2$ are classified in [11]. As far as other families of simple graphs are concerned, only partial results are known. For instance, by using lifting techniques, some regular embeddings of the $n$-dimensional cube $Q_n$ and of the complete bipartite graph $K_{n,n}$ are constructed in [12]. Each solution of the congruence $e^t \equiv 1 \pmod{n}$ gives rise to such a regular embedding of $Q_n$, and also of $K_{n,n}$. It is conjectured in [12] that there are no other regular embeddings of $Q_n$ and also $K_{n,n}$. Recently, in [13] the conjecture was verified affirmatively for the graph $K_{p,p}$, where $p$ is a prime.

The difficulty of the classification problem is emphasized by the fact that for the complete $n$-partite graphs $K_{m,m,\ldots,m}$, no general constructions of regular embeddings are known, provided that $n > 3$ and $m > 1$. In Section 5, we give a classification for the case when $m$ is prime (see Theorem 5.2). As a consequence of Theorem 5.2, we get the following: for a given prime $p$, there exists a regular embedding of the complete $n$-partite graph $K_{p,p,\ldots,p}$ ($n \geq 2$) if and only if $n$ is not two, three or $p$. This is quite different from the situation which appears for the regular embeddings of complete graphs. Note that $K_n$ can be viewed as the complete $n$-partite graph $K_{1,1,\ldots,1}$. Recall that the lexicographical product $G[H]$ of two graphs $G$ and $H$ is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ and edge set $E(G[H]) = \{(u_1,v_1)(u_2,v_2) \mid u_1 = u_2 \text{ and } v_1v_2 \in E(H); \text{ or } u_1u_2 \in E(G)\}$. To prove Theorem 5.2, in Section 4 we also classify all the regular embeddings of the lexicographical product of any connected graph of prime order $q$ with the complement of the complete graph of prime order $p$, where $q$ and $p$ are not necessarily distinct (see Theorem 4.4). The key step in proving the main classification theorem is based on the analysis of the structure of the automorphism group of regular maps in question. It essentially depends on the classification of the maximal subgroups of the two-dimensional projective linear groups over the finite field of characteristic $p$ (see Proposition 3.2).

Throughout this paper, we denote by $\mathbb{Z}_n$ and $\mathbb{D}_n$ the cyclic and the dihedral groups of order $n$, respectively, and by $\mathbb{F}_p$ the finite field of $p$ elements, and by $N : H$ a semidirect product of the group $N$ with the group $H$, by $\mathbb{Z}_n^2$ the direct product $\mathbb{Z}_n \times \mathbb{Z}_n$, and by $G^{(k)}$ the group $\{g^k \mid g \in G\}$ for a group $G$ and for a positive integer $k$, and by $S^n$ the multiplication group of a given ring $S$.

2. Regular maps from groups

All graphs considered in this section will be simple, finite, and connected. Consequently, groups associated with our graphs and maps will be finite. Every edge of a graph gives rise to a pair of opposite arcs. Let $\tilde{G} = (V, D)$ be a simple graph with vertex set $V = V(\tilde{G})$ and arc set $D = D(\tilde{G})$. By $S_V$ and $S_D$ we denote the symmetric groups on the vertex set and on the arc set, respectively. The involution $L$ in $S_D$ interchanging the two arcs underlying
The subgroup \( \text{Mon} \) of \( \mathcal{M} \) of regular \( \mathcal{M} \) by \( \mathcal{M} \) (see [10]). Let \( \mathcal{G} = \mathcal{G}(G; R, L) \) be a finite group and \( H \) a proper subgroup of \( G \) with the free core, that is, \( \cap_{g \in G} H^g := \cap_{g \in G} g^{-1} H g = 1 \). Let \( B = B^{-1} \). From now on, we use \( \mathcal{G} = \mathcal{G}(G; H, B) \) to denote the **coset graph** with \( V(\mathcal{G}) = \{ H g | g \in G \} \) and \( D(\mathcal{G}) = \{ (H g, H b g) | b \in B, g \in G \} \). Note that \( \mathcal{G} \) acts faithfully and arc-transitively on the coset graph by right multiplication. In what follows, we often identify the group \( G \) with the corresponding group of right multiplications.

**Definition 2.1.** Let \( G = \langle r, \ell \rangle \) be a finite two-generator group with \( \ell^2 = 1 \) and \( (r) \cap (\ell)^\mathbf{c} = 1 \). By an algebraic map \( \mathcal{M}(G; r, \ell) = (\mathcal{G}; R, L) \), we mean the map whose underlying graph is the coset graph \( \mathcal{G} = \mathcal{G}(G; \langle r \rangle, \langle r \rangle \ell \langle r \rangle) \) and rotation \( R \) is determined by \( e^R = e^{r_1^{-1} r g_1} = (\langle r \rangle g_1, \langle r \rangle g_2 g_1^{-1} r g_1) \) for any arc \( e = (\langle r \rangle g_1, \langle r \rangle g_2) \) in \( D(\mathcal{G}) \).

The following two propositions are well known (see for instance [9]).

**Proposition 2.2.** Let \( \mathcal{G}' = (\mathcal{G}', R', L') \) be a k-valent regular map with a simple underlying graph \( \mathcal{G}' \), a rotation \( R' \) and an arc-reversing involution \( L' \), and let \( G = \text{Aut}(\mathcal{M'}) \). Let \( \{ u, v \} \) be a fixed edge of \( \mathcal{G}' \). Then, there exist two automorphisms \( r \in G_u \) and \( \ell \in G_{\{u,v\}} \) such that \( G_u = \langle r \rangle \cong \mathbb{Z}_k \), \( G_{\{u,v\}} = \langle \ell \rangle \cong \mathbb{Z}_2 \), \( G = \langle r, \ell \rangle \), \( \ell \) interchanges \( u \) and \( v \), and \( \mathcal{M}' \) is isomorphic to an algebraic map \( \mathcal{M}(G; r, \ell) \).

**Proposition 2.3.** Let \( G \) be an abstract group with two pairs of generators \( \langle r_1, \ell_1 \rangle \) and \( \langle r_2, \ell_2 \rangle \) satisfying the condition in Definition 2.1. Then, \( \mathcal{M}(G; r_1, \ell_1) \cong \mathcal{M}(G; r_2, \ell_2) \) if and only if there exists an element \( \sigma \in \text{Aut}(G) \) such that \( r_1^\sigma = r_2 \) and \( \ell_1^\sigma = \ell_2 \).
By Propositions 2.2 and 2.3, one can transfer the classification problem of regular embeddings of a given graph into a purely group theoretical problem. Proposition 2.2 says that each regular map $M$ is isomorphic to an algebraic map $M(G; r, \ell)$, where $G \cong \text{Aut}(M)$. Also, note that two isomorphic maps have isomorphic automorphism groups. Therefore, one may find all the regular maps of a given underlying arc-transitive graph $G$ with valency $s$ in the following two steps:

1. Find the representatives $G$ (as abstract groups) of the isomorphic classes of arc-regular subgroups of $\text{Aut}(G)$ with cyclic vertex-stabilizers.
2. For each group $G$ given in (1), determine all the algebraic regular maps $M(G; r, \ell)$ with underlying graphs isomorphic to $G$ or, equivalently, determine the representatives of the orbits of $\text{Aut}(G)$ on the set of generating pairs $(r, \ell)$ of $G$ such that $|r| = n, |\ell| = 2$ and $G(G; \langle r \rangle, \langle r \ell r \rangle) \cong G$.

As a demonstration of this approach, one can determine all the regular embeddings of any arc-transitive graph of an odd prime order $q$ and valency $s$, where $s$ must be even. It is shown in [3] that $s \mid (q - 1)$ and for any such pair $(q, s)$ there exists only one arc-transitive graph up to isomorphism, say $G(q, s)$, with order $q$ and valency $s$. Let $t$ be an element of order $s$ in $\mathbb{F}_q^*$ and let

$$G = \langle a, b \mid a^q = b^s = 1, a^b = a^t \rangle.$$  \hfill (2.1)

Then $G \cong \mathbb{Z}_q : \mathbb{Z}_s$, a Frobenius group, and the group $G$ is independent of the choice of the element $t$ up to group isomorphism. As a consequence we get the following statement classifying regular embeddings of graphs with $q$ vertices.

**Theorem 2.4.** For any arc-transitive graph $G$ of odd prime order $q$ and valency $s$, there exist exactly $\phi(s)$ nonisomorphic regular maps with underlying graph $G$, where $\phi$ is the Euler function. Moreover, such regular maps can be represented by the algebraic regular maps $M(G; b^e, b^s a)$ with $e \in \mathbb{Z}_q^*$, where $G$ is the group defined in the Eq. (2.1).

For example, the complete graph $K_p$, $p$ a prime, has $\phi(p - 1)$ nonisomorphic regular embeddings into orientable surfaces, as shown by Biggs in [1].

### 3. Group theoretical results

In this section, we review some group theoretical results which will be used later.

**Proposition 3.1** ([15, Theorem 6.11]). Let $H$ be a subgroup of a group $G$. Then $C_G(H)$ is a normal subgroup of $N_G(H)$ and the quotient $N_G(H)/C_G(H)$ is isomorphic to a subgroup of $\text{Aut}(H)$.

**Proposition 3.2** ([4]). Let $p$ be an odd prime. Then:

1. The maximal subgroups of the projective special linear group $\text{PSL}(2, p)$ are:
   - One class of subgroups isomorphic to $\mathbb{Z}_p : \mathbb{Z}_{p-1}$; one class isomorphic to $\mathbb{D}_{p-1}$, when $p \geq 13$; one class isomorphic to $\mathbb{D}_{p+1}$, when $p \neq 7$; two classes isomorphic to $A_5$, when $p \equiv \pm 1(\text{mod } 10)$; two classes isomorphic to $S_4$, when $p \equiv \pm 1(\text{mod } 8)$;
and one class isomorphic to $A_4$, when $p = 5$ or $p \equiv 3, 13, 27, 37 \pmod{40}$ and $p \geq 5$.

(2) The maximal subgroups of the projective general linear group $\text{PGL}(2, p)$ are:

One class of subgroups isomorphic to $\mathbb{Z}_p : \mathbb{Z}_{p-1}$; one class isomorphic to $\mathbb{D}_{2(p-1)}$, when $p \equiv 7$; one class isomorphic to $\mathbb{D}_{2(p+1)}$; one class isomorphic to $S_4$, when $p = 5$ or $p \equiv 3, 13, 27, 37 \pmod{40}$ and $p \geq 5$; and one subgroup $\text{PSL}(2, p)$.

Lemma 3.3. Let $p$ and $q$ be primes and $q \geq 3$, and let $s$ be an even divisor of $q - 1$. Let $H$ be a Frobenius subgroup of the general linear group $\text{GL}(2, p)$ isomorphic to $\mathbb{Z}_q : \mathbb{Z}_s$.

Then $H$ is isomorphic to one of the groups $H_k$, $1 \leq k \leq 4$, defined as follows:

1. $p = 2, q = 3$ and $s = 2$. $H_1 = \langle x, y \rangle \cong \mathbb{D}_6$, where

$$x = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$  

2. $q \mid (p - 1), p \geq 5$ and $s = 2$. $H_2 = \langle x, y \rangle \cong \mathbb{D}_{2q}$, where

$$x = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$ 

for an element $t$ of order $q$ in $\mathbb{F}_p^*.$

3. $q \mid (p + 1), p \geq 5$ and $s = 2$. $H_3 = \langle x, y \rangle \cong \mathbb{D}_{2q}$, where

$$x = \begin{pmatrix} e & f \theta \\ f & e \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$ 

where $\mathbb{F}_p^* = \langle \theta \rangle$, $\mathbb{F}_{p^2} = \mathbb{F}_p(\alpha), \alpha^2 = \theta$ and $e + f \alpha$ is a fixed element of order $q$ in $\mathbb{F}_{p^2}^*.$

4. $q = p \geq 3$ and $s \mid (p - 1)$. $H_4 = \langle x, y \rangle \cong \mathbb{Z}_p : \mathbb{Z}_s$, where

$$x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $t$ is a fixed element of order $s$ in $\mathbb{F}_p^*$.

Moreover, $\text{GL}(2, p)$ has only one conjugacy class of subgroups isomorphic to $H_k$ for each $1 \leq k \leq 4$.

Proof. Let $A = \text{GL}(2, p), Z = Z(A)$ the center of $A$, and $\overline{A} = \text{PGL}(2, p) = A/Z$. Let $H$ be a Frobenius subgroup of $A$ isomorphic to $\mathbb{Z}_q : \mathbb{Z}_s$, where $q \geq 3$ is a prime and $s$ is an even divisor of $q - 1$. Noting that $|\text{GL}(2, p)| = p(p - 1)^2(p + 1), (p - 1, p + 1) = 2$ for $p \geq 3$, and $q \geq 3$, one can have the following four cases: (i) $p, q = (2, 3)$; (ii) $q \mid (p - 1)$ and $p \geq 5$; (iii) $q \mid (p + 1)$ and $p \geq 5$; and (iv) $p = q \geq 3$. We discuss each case separately.

(i) If $p = 2, q = 3$ and $s = 2$, then $H = A = \text{GL}(2, 2) \cong \mathbb{D}_6$ and we can set $H = H_1$ as in the statement (1).

In what follows, we assume $p \geq 3$. Since $H \cap Z = 1$, we have $H \cong (HZ)/Z$ and so $\text{PGL}(2, p)$ contains a non-Abelian proper subgroup, say $\overline{H}$, isomorphic to $\mathbb{Z}_q : \mathbb{Z}_s$. Thus $\overline{H}$ must be contained in a maximal subgroup of $\overline{A}$. A direct checking of the maximal subgroups of $\overline{A}$ listed in Proposition 3.2 shows that $H \cong \mathbb{Z}_q : \mathbb{Z}_2 \cong D_{2q}$, where $s = 2, p \geq 5,$
and either \( q \mid (p - 1) \) or \( q \mid (p + 1) \); or \( H \cong \mathbb{Z}_p : \mathbb{Z}_a \) where \( p = q \geq 3 \) and \( s \mid (p - 1) \). We shall deal with these case by case below.

(ii) \( q \mid (p - 1) \) and \( p \geq 5 \). Let \( D \) be the diagonal subgroup of \( A \). Let \( y \) be the matrix defined in the statement (2). Then the normalizer \( N_A(D) \) of \( D \) in \( A \) is \( D : \langle y \rangle \) [7, II. Theorem 8.3]. Since \( A \) contains only one class of dihedral subgroups of order \( 2h \) for any \( h \mid (p - 1) \), each such subgroup in \( A \) must be conjugate to one of them contained in \( N_A(D) \). Let \( x \) be the matrix in (2). Then it is easy to check that \( \langle x \rangle \) is the unique noncentral normal cyclic subgroup of order \( q \) in \( N_A(D) \), noting that \( q \) is a prime. Hence we may conclude \( H = H_2 \) as in the statement (2).

(iii) \( q \mid (p + 1) \) and \( p \geq 5 \). Let \( \mathbb{F}_p^\times = \langle \theta \rangle \) and let \( \mathbb{F}_{p^2} = \mathbb{F}(\alpha) \) where \( \alpha^2 = \theta \). Assume \( e_1 + f_1 \theta \) is a fixed element of order \( p^2 - 1 \) in \( \mathbb{F}_{p^2} \). Then it is well known that \( A \) has a cyclic subgroup \( S \) of order \( p^2 - 1 \) which is called the Singer–Zyklus subgroup generated by the matrix \( \left( \begin{array}{cc} e_1 & f_1 \\ f_1 \theta & e_1 \end{array} \right) \). Let \( x \) be an element of order \( q \) in \( S \). Then \( N_A(\langle x \rangle) = \langle x \rangle : \langle y \rangle \), where \( x \) and \( y \) are chosen as in the statement (3) [7, III. Theorem 8.4]. Since \( A \) has only one conjugacy class of the cyclic subgroups of order \( q \) and \( (q, p - 1) = 1 \), the group \( A \) should have the same property. Therefore \( A \) has only one conjugacy class of dihedral subgroups of order \( 2q \), and then one may conclude that \( H = H_3 \), as in the statement (3).

(iv) \( p = q \geq 3 \) and \( s \mid (p - 1) \). Since all the subgroups \( P \) of order \( p \) are conjugate in \( A \) and \( N_A(P) \cong \mathbb{Z}_p : \mathbb{Z}_{p-1} \), all the subgroups isomorphic to \( \mathbb{Z}_p : \mathbb{Z}_{p-1} \) are conjugate in \( A \) too. However, for any \( s \mid (p - 1) \), each subgroup isomorphic to \( H \cong \mathbb{Z}_p : \mathbb{Z}_s \) is uniquely contained in one \( \mathbb{Z}_p : \mathbb{Z}_{p-1} \). Therefore all such subgroups \( H \) are conjugate in \( A \), and then one can get \( H = H_4 \) as in the statement (4). \( \square \)

Lemma 3.4. Let \( p \) be an odd prime and \( s \) an even divisor of \( p - 1 \). Let \( Q \) be a non-Abelian group of order \( p^3 \) with exponent \( p^2 \). Let \( G = \langle y \rangle \) with \( [\langle y \rangle] = s \), and let the centralizer \( C_G(Q) \) of \( Q \) in \( G \) be equal to the center \( Z(Q) \) of \( Q \). Then \( G \) has the following presentation:

\[
G = \langle x, a, y \mid x^{p^2} = a^p = y^s = 1, x^a = x^{1+p}, x^y = x^t, a^y = a \rangle,
\]

where \( t \) is a fixed element of order \( s \) in \( \mathbb{Z}_{p^2} \).

Proof. Since \( C_G(Q) = Z(Q) \), the conjugacy action of \( y \) induces an automorphism \( \varphi \) of order \( s \) on the group \( Q \). By the assumptions, \( G \) is generated by two elements, say \( x \) and \( a \), such that \( |x| = p^2 \) and \( |a| = p \). Then \( Z(Q) = \langle x^p \rangle \). We are going to show that one can choose the generators \( x \) and \( a \) so that the relations in Eq. (3.1) hold.

First, we claim that \( \varphi \) fixes setwise a cyclic subgroup of order \( p^2 \) of \( Q \). To do this, we identify \( Q/Z \) with the two-dimensional vector space \( V = V(2, p) \) over \( \mathbb{F}_p \). Then the induced action of \( \varphi \) on \( Q/Z = \langle xZ, aZ \rangle \) gives an action on the vector space \( V \) as well as on the projective space \( \text{PG}(V) \) of \( V \). Then \( \varphi \) has a presentation, say \( \varphi \), in the general linear group \( \text{GL}(2, p) \). If \( \varphi \) is contained in the center \( Z(\text{GL}(2, p)) \) of \( \text{GL}(2, p) \), then \( \varphi \) fixes \( xZ \) and so \( y \) fixes \( \langle x \rangle \) in its conjugacy action. Assume \( \varphi \notin Z(\text{GL}(2, p)) \). Then \( \varphi Z(\text{GL}(2, p)) \) is an element in \( \text{PGL}(2, p) \) whose order is a divisor of \( s \). Since \( s \mid (p - 1) \), each element of order \( s \) in \( \text{PGL}(2, p) \) fixes at least two points in \( \text{PG}(V) \) (see [7, II. Theorem 8.3]).
Thus $\varphi$ must fix a point except for $\langle aZ \rangle$. Consequently, $\varphi$ also fixes a cyclic subgroup of order $p^2$ of $Q$.

Similarly, by considering the induced action of $\varphi$ on $\langle x^p, a \rangle$, one can induce that $\varphi$ must fix a noncentral subgroup of order $p$ of $Q$.

By the above arguments, one may assume that $x^y = x^t$ and $a^y = a^k$ for some $t \in \mathbb{Z}_{p^2}$ and $k \in \mathbb{F}_p^*$. Assume $a^{-1}xa = x^i$ for some integer $i$. Since $y$ preserves this relation, we have $a^{-k}x^t a^k = x^{it}$, which implies that $t^{k-1} = 1$ in $\mathbb{Z}_{p^2}$ and so $k = 1$. Hence $y$ fixes $a$. Replacing $a$ and $y$ by their appropriate powers, one can assume that $i = 1 + p$ and $t$ is an element of order $s$ in the $\mathbb{Z}_{p^2}$.

Accordingly, one can get a presentation for $G$ as shown in Eq. (3.1).

Lemma 3.5. Let $q$ be an odd prime and $s$ an even divisor of $q - 1$. Let $\mathcal{H}$ be either a graph isomorphic to $G(q, s)$ for $s < q - 1$ mentioned in Section 2 or the complete graph $K_n$ (including the case $G(q, q - 1)$). Let $\mathcal{G} = \mathcal{H}(\overline{K}_m)$ and let $G$ be a subgroup of $\text{Aut}(\mathcal{G})$ acting arc-regularly and having a cyclic point-stabilizer. Let $K$ be the normal subgroup of $G$ fixing each block of $G$ setwise in its action on $V(\mathcal{G})$. Then $|K| = m^2$ and $G/K$ acts on the quotient (block) graph $\mathcal{H}$ arc-regularly with cyclic point-stabilizers.

Proof. Clearly, $G/K$ induces an arc-transitive action on the quotient graph $\mathcal{H}$. Since $|K||G/K| = |G| = |D(\mathcal{G})| = m^2|D(\mathcal{H})|$, we have $|K| \leq m^2$. We show that $|K| = m^2$.

If $\mathcal{H} \cong G(q, s)$ for $s < q - 1$, then each arc-transitive subgroup of $\text{Aut}(\mathcal{H})$ must be arc-regular. Hence $|G/K| = qs$, which implies $|K| = m^2$, noting that $|G| = sqm^2$. Assume $\mathcal{H} \cong K_n$. Then $|G| = m^2n(n - 1)$. Take $u \in V(\mathcal{G})$. We know that $(G_u)^{(n-1)}$ fixes each block, which means $(G_u)^{(n-1)} \leq K$ (see Section 1 for the notation $(G_u)^{(n-1)}$). Take an arc $(u, v) \in D(\mathcal{G})$. Since $G$ is arc-regular and $G_u$ is cyclic, we have $G_u \cap G_v = 1$ and so $(G_u)^{(n-1)} \cap (G_v)^{(n-1)} = 1$. Therefore $|K| \geq |(G_u)^{(n-1)}, (G_v)^{(n-1)}| \geq m^2$. Thus $|K| = m^2$.

The arc-regularity of the action of $G/K$ on the quotient graph $\mathcal{H}$ follows. In addition, for any $u \in V(\mathcal{G})$ and the block $B$ containing $u$, we have that $(G/K)_B \leq G_uK/K$ is cyclic.

4. Regular embeddings of $G(q, s)[\overline{K}_p]$

Let $p$ and $q$ be primes with $q \geq 3$, and $s$ an even divisor of $q - 1$. Let $G(q, s)$ be the unique arc-transitive graph of order $q$ and valency $s$ defined as in Section 2. Let

$$G(p, q, s) = G(q, s)[\overline{K}_p]$$

be the lexicographical product of the graph $G(q, s)$ and $\overline{K}_p$. Then the graph $G(p, q, s)$ is arc-transitive of order $pq$ and valency $s$. Moreover, $\text{Aut}(G(p, q, s)) \cong S_p \wr \text{Aut}(G(q, s))$, the wreath product of the symmetric group $S_p$ and $\text{Aut}(G(q, s))$, and $|\text{Aut}(G(p, q, s))| = (p!)^s|\text{Aut}(G(q, s))|$ (see [14]). In this section, we shall classify all the regular maps with underlying graph $G(p, q, s)$ following the approach discussed in Section 2. First, we determine the representatives $G$ (as abstract groups) of the isomorphism classes of arc-regular subgroups of $\text{Aut}(G(p, q, s))$ whose point-stabilizers are cyclic. Next, for each of
those groups $G$, we determine all nonisomorphic algebraic regular maps $\mathcal{M}(G, r, \ell)$, in which the corresponding coset graphs are isomorphic to $\mathcal{G}(p, q, s)$.

In advance, we introduce two groups $G_1$ and $G_2$.

Let $H$ be one of the subgroups $H_k = \langle x, y \rangle$, $1 \leq k \leq 4$, of $GL(2, p)$ described in Lemma 3.3. Set $\alpha = (1, 1) \in V(2, p)$ if $k = 2$, or $\alpha = (1, 0) \in V(2, p)$ if $k = 1, 3$ or 4; and set $\beta = (0, 1) \in V(2, p)$ for $1 \leq k \leq 4$. Let $a$ and $b$ be the translations in the affine geometry $AG(2, p)$ by $\alpha$ and $\beta$ respectively, and set $P = \langle a, b \rangle \cong \mathbb{Z}_p^2$. Now, let $G_1$ be the subgroup of the affine group $AGL(2, p)$ defined as follows:

$$G_1 = P : H = \langle a, b : (x, y) \rangle. \quad (4.2)$$

Let $p$ be an odd prime, $s$ an even divisor of $p - 1$, and let $t$ be a fixed element of order $s$ in $\mathbb{Z}_p^*$. Let $G_2$ be the group defined by the Eq. (3.1) with those $p, s, t$. That is,

$$G_2 = \langle x, a, y | x^p = a^p = y^s = 1, x^a = x^{1+p}, x^y = x^t, a^y = a \rangle. \quad (4.3)$$

**Lemma 4.1.** Let $\mathcal{G}(p, q, s) = \mathcal{G}(q, s)[\mathcal{K}_p]$ be the graph as in the Eq. (4.1), and let $G$ be an arc-regular subgroup of $\text{Aut}(\mathcal{G}(p, q, s))$ with cyclic point-stabilizer. Then $G$ is isomorphic either to $G_1$ defined in (4.2) or to $G_2$ defined in (4.3).

**Proof.** Let $\mathcal{G} = \mathcal{G}(p, q, s)$. Note that $\text{Aut}(\mathcal{G}) = K : E$, where $K \cong S_p \times \cdots \times S_p$ ($q$ times) and $E \cong \text{Aut}(\mathcal{G}(q, s))$, which is isomorphic to a Frobenius group $\mathbb{Z}_q : \mathbb{Z}_s$ if $s < q - 1$ or to $S_q$ if $s = q - 1$. Denote by $B = \{B_0, B_1, \ldots, B_{q-1}\}$ the set of imprimitive blocks of $\text{Aut}(\mathcal{G})$, forming the orbits of $K$ on $V(\mathcal{G})$. Denote by $\overline{G}$ the quotient (block) graph of $G$ induced by $K$. Then $\overline{G} \cong \mathcal{G}(q, s)$. Moreover, the Sylow $p$-group of $K$ is isomorphic to $\mathbb{Z}_p^q$.

Let $G$ be a subgroup of $\text{Aut}(\mathcal{G})$ acting arc-regularly and having a cyclic point-stabilizer. Then $[G] = sq^2$ and $[G_v] = sp$ for any $v \in V(\mathcal{G})$. Let $P = K \cap G$, that is the kernel of $G$ on $B$. Then by Lemma 3.5, $P \cong \mathbb{Z}_p^2$ and $G/P \cong \mathbb{Z}_q : \mathbb{Z}_s$ acts arc-regularly on the quotient $\overline{G}$ with a cyclic point-stabilizer. More precisely, $P = \langle P_{uv}, P_{vu} \rangle$ for any $v_i \in B_i$ and $v_j \in B_j$ where $i \neq j$. In fact, for any $v \in V(\mathcal{G})$, $P_v$ fixes precisely $p$ points in $V(\mathcal{G})$, noting that $[N_G(P_v)]/[N_{G_v}(P_v)] = (p^2s)/(ps) = p$ [16, Theorem 3.5]. Hence $P_{uv}$ is transitive on the block $B_j$, and consequently, $P_{uv} \neq P_{vu}$.

Since $C_G(P)/P$ is normal in $G/P \cong \mathbb{Z}_q : \mathbb{Z}_s$, $C_G(P)/P$ is isomorphic to the identity, to $\mathbb{Z}_q$, or to $G/P$. In the last two cases, one can derive that $C_G(P)$ is transitive on $V(\mathcal{G})$, contradicting $P_v \neq 1$ for any $v \in V(\mathcal{G})$. Therefore we have $C_G(P) = P$.

By Proposition 3.1 we have $\mathbb{Z}_q : \mathbb{Z}_s \cong G/P \cong G/C_G(P)$, which is isomorphic to a subgroup of $\text{Aut}(\mathbb{Z}_p^q) \cong GL(2, p)$. Consequently, $GL(2, p)$ contains a Frobenius subgroup $H$ isomorphic to $\mathbb{Z}_q : \mathbb{Z}_s$. Using Lemma 3.3, we get the four cases for $H$, which will be discussed separately as follows.

**Case 1.** $p = 2, q = 3$, and $s = 2$. Then $|G| = 24$ and $G$ has the a normal subgroup $P \cong \mathbb{Z}_2^2$ such $C_G(P) = P$. By [7, II. Theorem 8.17], $G \cong S_4$. Hence we may conclude that $G \cong G_1 = P : H_1$ defined by (4.2).

**Case 2.** $q | (p - 1)$, $p \geq 5$, and $s = 2$. By using the fact that $G$ contains a normal subgroup $P$ satisfying $(|P|, |G : P|) = 1$ and the Schur–Zassenhaus theorem [15, II. Theorem 8.10], one can show that there exists a subgroup $H$ such that $G = P : H$ and $H \cong G/P \cong \mathbb{D}_{2q}$.

Since $C_G(P) = P$, $G$ is isomorphic to a subgroup of the affine group
AGL(2, p) containing the translation normal subgroup $P \cong \mathbb{Z}_p^2$. By Lemma 3.3, GL(2, p) has only one conjugacy class of subgroups isomorphic to $H$. Hence AGL(2, p) has only one conjugacy class of the subgroups isomorphic to $G$. Therefore we may conclude that $G \cong G_1 = P : H_s$ defined by (4.2).

Case 3. $q \mid (p + 1)$, $p \geq 5$, and $s = 2$. Using an argument similar to that of Case 2, one can conclude that $G \cong G_1 = P : H_3$ defined by (4.2).

Case 4. $p = q \geq 3$ and $s = (p - 1)$. Then $|G| = p^3$. Let $Q$ be the Sylow $p$-subgroup of $G$. Then $Q$ is normal in $G$ and contains $P$. From $C_G(P) = P$, we know that $Q$ is non-Abelian. It is well known that $Q$ has two nonisomorphic types depending on $\text{Exp} Q = p$ or $\text{Exp} Q = p^2$.

Assume $\text{Exp} Q = p$. Then $P$ has a complement of order $p$ in $Q$. Since $P$ is an Abelian normal subgroup in $G$ and $(|P|, |G : Q|) = 1$, it follows from the Gaschütz theorem [7, I. Theorem 17.4] that $P$ has a complement, say $H$, in $G$. Then $H \cong G/P \cong \mathbb{Z}_p : \mathbb{Z}_s$. As in Case 2, one can show that $G$ can be identified with a subgroup of AGL(2, p) containing the translation normal subgroup $P \cong \mathbb{Z}_p^2$. Consequently, one can conclude that $G \cong G_1 = P : H_4$ defined by (4.2).

Assume $\text{Exp} Q = p^2$. Since $G_v \cong \mathbb{Z}_{sp}$, one can assume that $G_v = \langle a \rangle \times \langle y \rangle$ where $a \in Q \setminus Z(Q)$ and $y \in G \setminus Q$ is of order $s$. Then $G = Q : \langle y \rangle$. By Lemma 3.4, we obtain that, up to isomorphism, the unique group $G = G_2$ defined in (4.3).

In what follows, we determine all the algebraic regular maps $M(G_1; r, \ell)$ with the corresponding coset graphs isomorphic to $G(p, q, s)$. We shall do this separately for $G_1$ in Lemma 4.2 and for $G_2$ in Lemma 4.3.

**Lemma 4.2.** Let the graph $G(p, q, s)$ and the group $G_1$ be defined as in (4.1) and (4.2), respectively. Then any regular map $M(G_1; r, \ell)$ with the corresponding coset graph isomorphic to $G(p, q, s)$ is isomorphic to one of the following regular maps:

$$M_1(p, q, s, i, j) := M(G_1; ay^j, x^iy^j),$$

where $p$ and $q$ are primes with $q \geq 3$, $s$ is an even divisor of $q - 1$, if $p \neq q$ then $s = 2$, $i \in \mathbb{F}_q^*/(\mathbb{F}_q^*)^{(\frac{q-1}{2})}$, and $j \in \mathbb{Z}_s^*$. Moreover, the maps in (4.4) are uniquely determined by the given parameters $p$, $q$, $s$, $i$, and $j$.

**Proof.** The lemma is proved by the following two steps.

**Step 1.** Show that for any admissible choice of the parameters, the map given by (4.4) is a well-defined regular map whose underlying coset graph is isomorphic to $G(p, q, s)$.

Let $r_j = ay^j$ and $\ell_i = x^i y^j$ for any $i \in \mathbb{F}_q^*$, where $i$ and $j$ satisfy the condition. By the definition of $G_1$, we know that $ay = y a$ and $(s, p) = 1$, which implies $(r_j) = \langle a, y^j \rangle \cong \mathbb{Z}_{sp}$. Moreover, $(r_j, \ell_i) = \langle a, y^j, \ell_i \rangle = \langle a, H \rangle = G_1$. Note that $(r_j) \cap (r_j) = 1$. By Definition 2.1, the maps in (4.4) are well defined.

Let us consider the coset graph $G = G(G_1; r_j, (r_j)\ell_i(r_j))$. The normal subgroup $P$ of $G_1$ (defined above) induces an imprimitive complete block system on $V(G)$. Assume that the block $B_0$ contains the point $u = (r_j)$. Then $\langle a \rangle$ in $(G_1)_u$ is transitive on the each block except for $B_0$. Since $G_1/P = \langle x P, y P \rangle \cong \mathbb{Z}_q : \mathbb{Z}_s$ acts arc-regularly on the quotient
(block) graph \( \mathcal{Y} \), we obtain that \( \mathcal{Y} \) is isomorphic to \( \mathcal{G}(q, s) \). Thus \( \mathcal{G} \cong \mathcal{G}(p, q, s) \) and we are done.

**Step 2.** Given \( p, q, \) and \( s \), determine all the nonisomorphic regular maps \( \mathcal{M}(G_1; r, \ell) \) in which the corresponding coset graph is isomorphic to \( \mathcal{G}(p, q, s) \).

Let \( r_j \) and \( \ell_i \) be as in Step 1. Then it suffices to prove that the pairs \((r_j, \ell_i)\) are precisely all the representatives of orbits of \( \text{Aut}(G_1) \) on the set of the generating element pairs \((r, \ell)\) in \( G_1 \) such that \(|r| = sp, |\ell| = 2\) and the corresponding coset graph is isomorphic to \( \mathcal{G}(p, q, s) \). We divide our discussion into four cases according to the four cases for \( H \), noting that \( H = H_k \) for \( k = 1, 2, 3, \) or \( 4 \).

1. **Suppose \( k = 1 \).** Then \( p = 2, q = 3, \) and \( s = 2 \). Then \( G_1 \cong S_4 \). We only need to determine all the possible pairs \((r, \ell)\) in \( S_4 \). Clearly, \( r \in S_4 \setminus A_4 \) and \( S_4 \) has only one conjugacy class of such elements. Thus we may assume \( r = (1234) \). Then there are four involutions: \( S = \{(12), (23), (34), (41)\} \) generating \( S_4 \) with \( r \). Clearly, the conjugacy action of \( (r) \) is transitive on \( S \). Hence we may assume \( \ell = (12) \). In other words, we have only one possibility for the choice of pairs \((r, \ell)\) up to the action of \( \text{Aut}(S_4) \). Accordingly, we may take \( r = ay \) and \( \ell = xy \), as in (4.4).

2. **Suppose \( k = 2 \).** Then \( q \mid (p - 1), p \geq 5, \) and \( s = 2 \). Since \(|G_1| = 2qp^2\), \( G_1 \) has only one conjugacy class of involutions. Since \( C_{G_1}(y) = \langle ay \rangle \cong \mathbb{Z}_{2p}, G_1 \) has only one cyclic subgroup of order \( 2p \) containing \( y \), and consequently \( G_1 \) has only one conjugacy class of cyclic subgroups of order \( 2p \). Therefore we may assume \( r \in \langle ay \rangle \). Furthermore, since the conjugacy action of \( Z(\text{GL}(2, p)) \) on \( \text{AGL}(2, p) \) fixes \((x, y)\) pointwise and maps \( a \) to \( a^h \) for any \( h \in \mathbb{F}_p^* \), one can take \( r = ay \).

Now, we determine the possible involutions \( \ell \) generating \( G_1 \) with \( r = ay \). Since \( C_{G_1}(y) = \langle r \rangle \), such involutions \( \ell \) have the form \( y^{b^i} \) for each \( h \in \mathbb{F}_q \) and \( i \in \mathbb{F}_q^* \). Note that each such involution \( \ell \) is contained uniquely in \( (G_1)_v \) for some vertex \( v \) in \( B_0 \), where \( B_0 \) is the block containing \( (r) \). Since \( (a) \) is transitive on each block except for \( B_0 \), the conjugacy action of \( (a) \) is transitive on the set of involutions \( \{ y^{b^i} \mid h \in \mathbb{F}_p \} \) for each \( i \in \mathbb{F}_q^* \). Hence we may assume \( \ell = y^{x^{-2}} = x^iy \) for any \( i \in \mathbb{F}_q^* \). Noting that \((x^iy)^y = x^{-i}y \), we may let \( i \in \mathbb{F}_q^*/(\mathbb{F}_q^*)^{(q - 1)/2} = \{1, 2, \ldots, (q - 1)/2\} \).

Finally, we prove that any automorphism in \( \langle \text{Aut}(G_1) \rangle \), cannot map \( x^iy \) to \( x^jy \) for any \( i \neq j \) in \( \{1, 2, \ldots, (q - 1)/2\} \) and \( q \geq 5 \) as well. Therefore we have precisely \((q - 1)/2\) choices for \( \ell \) as well as for \((r, \ell)\), as claimed in (4.4). In fact, suppose that \( \sigma \) is such an automorphism in \( \langle \text{Aut}(G_1) \rangle \). Then it satisfies

\[
a^\sigma = a, \quad b^\sigma = a^mb^n, \quad y^\sigma = y \quad \text{and} \quad x^\sigma = x^{ji^{-1}},
\]

where \( n \in \mathbb{F}_p^* \). From (4.2) and Lemma 3.3(2), we have that \( a^x = a^{r^{-i}}b^{t^{-i}}b^i \) and \( b^x = b^i \). Since \( \sigma \) preserves the relation \( b^x = b^i \), we have \( (a^mb^n)^{xji^{-1}} = (a^mb^n)^y \).

Solving it, we get \( t^{ji^{-1}}m = tm \) and \( (m + n)t^{ji^{-1}} = t(m + n) \). Noting that \( t^{ji^{-1}} \neq t \), we get \( m = n = 0 \), a contradiction.

3. **Suppose \( k = 3 \).** Then \( q \mid (p + 1), p \geq 5, \) and \( s = 2 \). Arguments similar to those in (2) can be applied (we omit the details).
(4) Suppose \( k = 4 \). Then \( q = p \geq 3 \) and \( s \mid (p - 1) \). Now, \( |G_1| = sp^3 \). Then \( G_1 \) has one conjugacy class of cyclic subgroups of order \( s \). Since \( C_{G_1}((y)) = (ya) \cong \mathbb{Z}_{ps} \), \( G_1 \) has one conjugacy class of cyclic subgroups of order \( sp \). Therefore we may assume \( r \in \langle ay \rangle \). Furthermore, since the conjugacy action of \( \mathbb{Z}_{2}(\mathbb{GL}(2, p)) \) on \( A\mathbb{GL}(2, p) \) fixes \( \langle x, y \rangle \) pointwise and takes \( a \) to \( a^h \) for any \( h \) in \( \mathbb{F}_p^* \). We may assume \( r = ay^j \), where \( j \in \mathbb{Z}_p^* \). We claim that there exists no automorphism of \( G_1 \) mapping \( ay^j \) to \( ay^{j'} \) for some \( j \neq j' \) in \( \mathbb{Z}_p^* \). For the contrary, suppose that \( \sigma \) is such an automorphism. Then it satisfies

\[
a^\sigma = a, \quad b^\sigma = a^mb^n \quad \text{and} \quad y^\sigma = y^{j'^{-1}},
\]

where \( n \in \mathbb{F}_p^* \). From (4.2) and Lemma 3.3(4) we have \( y^{-1}by = b' \). Since \( \sigma \) preserves this relation, we have \( y^{-j'^{-1}}(a^mb^n)y^{j'^{-1}} = (a^mb^n)y' \). Solving the equality we get \( m = 0 \) and either \( n = 0 \) or \( j' - 1 = 0 \), a contradiction.

By Lemma 3.3(4) \( x^\gamma = x^t \) for some element \( t \) of order \( s \) in \( \mathbb{F}_p^* \). Similarly, as in (2), one may assume that \( \ell \in \langle x^i \rangle \) for some element \( i \) of order \( s \) in \( \mathbb{F}_p^*/(\mathbb{F}_p^*)^\ell \). Using arguments similar to those in (2) one can prove that there exists no element in \( \langle \text{Aut}(G_1) \rangle \), which maps \( y^\gamma x^i \) to \( y^\gamma x^j \), for any \( i \neq j \) in \( \mathbb{F}_p^*/(\mathbb{F}_p^*)^\ell \).

By the above arguments, there exist exactly \( ((p - 1)\phi(s))/s \) choices for the generating pairs \( (r, \ell) \), as claimed in (4.4). \( \square \)

**Lemma 4.3.** Let the graph \( G(p, p, s) \) and the group \( G_2 \) be defined in (4.1) and (4.3) respectively. Then any regular map \( \mathcal{M}(G_2; r, \ell) \) with the corresponding coset graph isomorphic to \( G(p, p, s) \) is isomorphic to one of the following regular maps:

\[
\mathcal{M}_2(p, s, j) := \mathcal{M}(G_2; (ay)^j, xy^j), \quad (4.5)
\]

\( j \in \mathbb{Z}_{ps}^* \). Moreover, the maps in (4.5) are uniquely determined by the given parameters \( p, s, \) and \( j \).

**Proof.** We split the proof into the following two steps.

**Step 1.** Show that the maps defined in (4.5) are well-defined regular maps with underlying coset graph isomorphic to \( G(p, p, s) \).

The proof can be done by replacing \( P \) with \( \langle x \rangle \) in Step 1 in the proof of Lemma 4.2 and following the arguments used there. We shall omit the details.

**Step 2.** Given \( p \) and \( s \), determine all nonisomorphic regular maps \( \mathcal{M}(G_2; r, \ell) \) in which the corresponding coset graphs are isomorphic to \( G(p, p, s) \).

Let \( Q = \langle x, a \rangle \). Since \( G_2 = G \) (as \( (s, p) = 1 \), \( G_2 \) has only one conjugacy class of subgroups of order \( s \) of the form \( (y)^i \) as well as one class of involutions of the form \( (y^i)^x \) for \( i \in \mathbb{Z}_{sp}^* \). Since \( C_G(y) = (ay) \cong \mathbb{Z}_{sp} \), \( G \) has only one conjugacy class of cyclic subgroups of order \( sp \). Thus we may assume \( r \in \langle ay \rangle \).

Now, we deal with the involutions \( \ell \) such that \( G_2 = \langle r, \ell \rangle \). Clearly, such an \( \ell \) is of the form \( (y^i)^x = x^iy^i \), where \( i \in \mathbb{Z}_{ps}^* \). Since the conjugacy action of \( \langle ay \rangle \) is transitive on the set of these involutions, we may assume \( \ell = xy^j \). Let \( \sigma \) be an element in \( \text{Aut}(G_2) \)
5. Regular embeddings of the complete multipartite graph

Let \( \ell \) be a prime. Since \( \sigma \) preserves the relations \( x^a = x^{1+p} \) and \( y^a = y^i \), we have \( x^a = x^{1+p} \) and \( y^a = y^i \). This implies \( j \equiv 1 \pmod{p} \) and \( j \equiv 1 \pmod{s} \), and so \( \sigma \) is the identity. Therefore up to the action of \( \text{Aut}(G_2) \), we have precisely \( \phi(p,s) \) choices for \( r \) as well as for \( (r, \ell) \), as claimed in (4.5).

Combining Lemmas 4.1–4.3, we obtain one of the main results of this paper.

**Theorem 4.4.** Let \( G(p, q, s) = G(q, s) \mathcal{K}_p \) be the graph defined by (4.1) for any two primes \( p, q \) and an even divisor \( s \) of \( q - 1 \). Let \( \mathcal{M} \) be a regular map with underlying graph \( G(p, q, s) \). Then one of the following two statements holds:

1. \( p \neq q, s = 2, q \mid (p \pm 1) \), and \( \mathcal{M} \cong \mathcal{M}_1(p, q, 2, i, 1) \) for some \( i \in \{1, 2, \ldots, (q - 1)/2\} \);
2. \( p = q \geq 3 \) and \( \mathcal{M} \) is isomorphic either to \( \mathcal{M}_1(p, p, s, i, j) \) for some \( i \in \mathbb{Z}^*_p (\mathbb{Z}^*_p)^{\pm 1} \) and \( j \in \mathbb{Z}^*_s \), or to \( \mathcal{M}_2(p, s, j) \) for some \( j \in \mathbb{Z}^*_p \).

**Corollary 4.5.** Let \( G(p, q, s) = G(q, s) \mathcal{K}_p \) be the graph defined by (4.1) for any two primes \( p, q \) and an even divisor \( s \) of \( q - 1 \). Then

1. there exist \( (q - 1)/2 \) nonisomorphic regular embeddings of \( G(p, q, 2) \) if \( q \mid (p \pm 1) \),
2. there exist \( \phi(s)(p - 1)(1 + 1/s) \) nonisomorphic regular embeddings of \( G(p, p, s) \),
3. there are no regular embeddings of \( G(p, q, s) \) for all other triples of parameters \( (p, q, s) \).

**Remark 4.6.** Given a Cayley graph \( G = G(G, S) \), one can choose a cyclic permutation \( \rho \) of the generators in \( S \). At each vertex \( v \) of \( G \) the permutation \( \rho \) induces a local rotation of arcs emanating at \( v \). In such a way, an embedding of \( G \) into an orientable surface together with an associated map \( \mathcal{M} \) is defined. Such maps are called **Cayley maps** (see [2, Definition 5.3.3]). The automorphism group of the Cayley map \( \mathcal{M} \) contains a subgroup isomorphic to the group \( G \) which acts regularly on vertices of \( \mathcal{M} \). The converse is also true: a given map \( \mathcal{M}' \) is isomorphic to a Cayley map if and only if \( \text{Aut}(\mathcal{M}') \) contains a vertex-regular subgroup [2, Theorem 5.3.4]. One can check that all the regular maps \( \mathcal{M}_1(p, q, s, i, j) \) and \( \mathcal{M}_2(p, q, j) \) are Cayley maps, except for the case when \( q \mid (p + 1) \) and the map is isomorphic to \( \mathcal{M}_1(p, q, 2, i, 1) \) for some \( i \).

5. Regular embeddings of the complete multipartite graph \( K_n[\mathcal{K}_p] \)

In this section, we use Theorem 4.4 to classify all the regular maps whose underlying graphs are the complete multipartite graph \( K_n[\mathcal{K}_p] \), where \( n \) is an integer and \( p \) is a prime. First, we shall deal with the case \( n = 2 \).

For a prime \( p \), let \( G_3 = ((a) \times (b)) : (y) \cong \mathbb{Z}_p^2 : \mathbb{Z}_2 \) with \( a^y = b \), and let

\[
\mathcal{M}_3(p) := \mathcal{M}(G_3, a, y)
\]  

be an algebraic map with two generators \( a \) and \( y \).
Lemma 5.1. Given a prime $p$, any regular map with underlying graph $K_{p,p}$ is isomorphic to the map $\mathcal{M}_3(p)$ defined in (5.1). Moreover, $\mathcal{M}_3(p)$ is a Cayley map.

Proof. Clearly, the map $\mathcal{M}_3(p)$ is well defined and its underlying graph is isomorphic to $K_{p,p}$. Since the group $G_3$ contains a vertex-regular subgroup $\langle ab \rangle : \langle y \rangle \cong \mathbb{Z}_{2p}$, $\mathcal{M}_3(p)$ is a Cayley map.

Since $\text{Aut}(K_{p,p}) \cong (S_p \times S_p) : \mathbb{Z}_2$ and its arc-regular subgroup $G$ with a cyclic point-stabilizer is uniquely determined up to isomorphism, we may assume $G = G_3$. To determine the generating pairs $(r, \ell)$, we may assume $r \in \langle a \rangle$ without any loss of generality. Since $G_3$ has $p$ involutions and the conjugacy action of $b$ fixes $\langle a \rangle$ setwise and is transitive on the set of involutions, we may assume $\ell = y$. For any $i \in \mathbb{Z}_p^*$, define $\sigma : G \to G$ by $(a^ib^jy^h)^\sigma = a^{iy}b^{kj}y^h$ for any $j, k \in \mathbb{Z}_p$ and $h \in \mathbb{Z}_2$. Then $\sigma$ is an automorphism of the group $G$ fixing $y$ and maps $a$ to $a^i$. Therefore we may fix $r = a$ and, consequently, only one regular map is obtained. □

We are now ready to classify all the regular embeddings of the complete multipartite graph $K_n[\mathbb{K}_p]$.

Theorem 5.2. If a complete multipartite graph $K_n[\mathbb{K}_p]$ has a regular embedding into an orientable surface for a prime $p$ and an integer $n \geq 2$, then $n$ must be prime and one of the following cases holds:

1. $n = 2$ and $\mathcal{M} \cong \mathcal{M}_3(p)$, or
2. $n = 3 \neq p$ and $\mathcal{M} \cong \mathcal{M}_1(p,3,2,1,1)$, or
3. $n = p \geq 3$ and $\mathcal{M} \cong \mathcal{M}_1(p,p,p-1,1,j)$ for some $j \in \mathbb{Z}_{p-1}^*$ or $\mathcal{M} \cong \mathcal{M}_2(p,p-1,j)$ for some $j \in \mathbb{Z}_{p(p-1)}^*$.

Proof. Suppose that $\mathcal{M}$ is a regular map with underlying graph $K_n[\mathbb{K}_p]$. If $n$ is prime then the statement follows from Theorem 4.4 and Lemma 5.1. In what follows, we need only to show that $n$ must be a prime.

It is known in [14] that $\text{Aut}(K_n[\mathbb{K}_p]) = K : E \cong S_p$ wr $S_n$, where $K \cong S_p \times S_p \times \cdots \times S_p$ ($n$ times) and $E \cong S_n$. Denote by $B = \{B_0, B_1, \ldots, B_{n-1}\}$ the set of imprimitive blocks of $\text{Aut}(K_n[\mathbb{K}_p])$, which are the orbits of $K$ on $V(K_n[\mathbb{K}_p])$. As in the proof of Lemma 4.1, let $G$ be a subgroup of $\text{Aut}(K_n[\mathbb{K}_p])$ acting arc-regularly and having a cyclic point-stabilizer. Let $P = K \cap G$, that is the kernel of $G$ on $B$. Then by Lemma 3.5, $P \cong \mathbb{Z}_p^*$ and $G/P$ acts arc-regularly on the quotient graph $K_n$ with a cyclic point-stabilizer. By the classification of regular maps of complete graphs (see [8]), the number of vertices of the quotient is a prime power, say $n = q^d$. Moreover, the automorphism group of the quotient map is isomorphic to $\text{AGL}(1,q^d) \cong \mathbb{Z}_q^d : \mathbb{Z}_{q^d-1}$.

Suppose $d \geq 2$. Since $K_n[\mathbb{K}_p]$ is a complete $q^d$-partite graph and the action of $G$ is arc-regular, each non-trivial element of $G$ cannot fix two points in different blocks. Therefore for any $v_i \in B_i$ and $v_j \in B_j$ with $i \neq j$, $P = \langle P_{v_i}, P_{v_j} \rangle$. Since $C_G(P)/P$ is normal in $G/P \cong \mathbb{Z}_q^d : \mathbb{Z}_{q^d-1}$, we get that either $C_G(P)/P = 1$ or $C_G(P)/P$ contains a subgroup isomorphic to $\mathbb{Z}_q^d$. The latter case implies that $C_G(P)$ is transitive on $V(K_n[\mathbb{K}_p])$, contradicting $P_v \neq 1$ for any $v \in V(K_n[\mathbb{K}_p])$. Therefore $C_G(P) = P$. Using Proposition 3.1, we have that $\text{GL}(2,p)$ contains a subgroup, say $Q$, isomorphic to
\[ \mathbb{Z}_d^q : \mathbb{Z}_{q^d-1} \text{ and } Q \cap Z = 1, \text{ where } Z = \text{GL}(2, p). \] Consequently, PGL(2, p) contains such a subgroup, say \( \overline{Q} \), as well. Using Proposition 3.2, one can check all the possibilities for the existence of \( \overline{Q} \leq \text{PGL}(2, p) \). One concludes that \( q = d = 2 \), because \( d \neq 1 \) by the assumption. This implies that \( Q \) contains a subgroup, say \( O \), isomorphic to \( \mathbb{Z}_2^2 \). Then \( \mathbb{Z}_2 \cong O \cap \text{SL}(2, p) \subset \text{GL}(2, p) \setminus Z \). However, it is well known [15, p. 393] that \( \text{SL}(2, p) \) contains only one involution and this involution is contained in the center of \( \text{GL}(2, p) \), a contradiction. Therefore \( d = 1 \) and \( n = q \) is a prime. □

**Corollary 5.3.** The nonisomorphic regular embeddings of the complete uniform \( n \)-partite graph \( K_n[\overline{K}_p] \) with partition sets of a prime size \( p \) are enumerated as follows:

1. There is only one regular embedding of \( K_{p,p} \).
2. There is only one regular embedding of \( K_{p,p,p} \) for \( p \neq 3 \).
3. There are exactly \( p\phi(p-1) \) regular embeddings of \( K_{p}[\overline{K}_p] \).
4. There are no regular embeddings of \( K_n[\overline{K}_p] \) provided \( n \geq 4 \) and \( n \neq p \).

**Remark.** We do not have an idea of how to classify regular embeddings of complete multipartite graphs \( K_n[\overline{K}_m] \) in general. Actually, the classification problem is not solved even for the complete bipartite graphs \( K_{m,m} = K_2[\overline{K}_m] \), \( m \) an integer. This suggests that the problem is hard. A further step could be to attack the classification of regular embeddings of \( K_n[\overline{K}_p] \) for \( m = 2p, m = p^2 \) or \( m = pq \), where \( p, q \) are primes.

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