Computers and Mathematics with Applications 61 (2011) 578–585

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

# Approximate solutions for nonlinear oscillation of a mass attached to a stretched elastic wire

Seher Durmaz [a], Sezgin Altay Demirbağ [b], Metin Orhan Kaya [a,*]

[a] *Faculty of Aeronautics and Astronautics, Istanbul Technical University, 34469 Maslak, Istanbul, Turkey*

[b] *Faculty of Science and Letts, Istanbul Technical University, 34469 Maslak, Istanbul, Turkey*

ARTICLE INFO

*Article history:*
Received 27 April 2010
Received in revised form 30 November 2010
Accepted 2 December 2010

*Keywords:*
Nonlinear oscillator
He's max–min approach
He's frequency–amplitude method
Parameter-expansion method

ABSTRACT

In this paper, the approximate solutions of the mathematical model of a mass attached to a stretched elastic wire are presented. At the beginning of the study, the equation of motion is derived in a detailed way. He's max–min approach, He's frequency–amplitude method and the parameter-expansion method are implemented to solve the established model. The numerical results are further compared with the approximate analytical solutions for both a small and large amplitude of oscillations, and a very good agreement is observed. The relative errors are computed to illustrate the strength of agreement between the numerical and approximate analytical results.



## 1. Introduction

A considerable amount of study has focused on ways to solve nonlinear problems. In general, the analytical approximation of the solution of a given nonlinear problem is more difficult that the numerical solution approximation.

During the past few decades, several types of methods have been proposed for obtaining approximate solutions of various types of nonlinear differential equations. Among them are the variational iteration method [1–5], the homotopy perturbation method [6–14], the harmonic balance method [15], He's frequency–amplitude formulation [16–23], He's max–min approach [24–28], and the parameter-expansion method [29–34].

The purpose of this paper is to determine the periodic solutions for a nonlinear oscillator of a mass attached to a stretched elastic wire using He's max–min approach, He's frequency–amplitude method, and the parameter-expansion method. In engineering sciences, it is easy to find the maximal and minimal solution thresholds of nonlinear problems. Considering this, the max–min approach was first proposed by He [24], based on ancient Chinese mathematics. After applying the max–min approach, the approximate solution of the model is solved by the frequency–amplitude method, which is also based on old Chinese techniques. Generally, ancient Chinese mathematics has been used to solve not only the oldest nonlinear equations but also complex modern equations such as the Bethe Equation, which describes the behavior of an energetic particle [35–37]. The last technique used to solve the established model belongs to the family of perturbation methods: parameter expansion, also referred to as the *bookkeeping parameter*. All solutions resulting from the three techniques are compared, and periodic solutions are plotted for various values of the parameters of the equation. The relative errors are also computed to illustrate good agreement between the numerical and approximate analytical results.

* Corresponding author. Tel.: +90 (212) 2853110; fax: +90 (212) 2853139.
*E-mail addresses:* durmazseh@itu.edu.tr (S. Durmaz), saltay@itu.edu.tr (S. Altay Demirbağ), kayam@itu.edu.tr (M.O. Kaya).

0898-1221/$ – see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2010.12.003

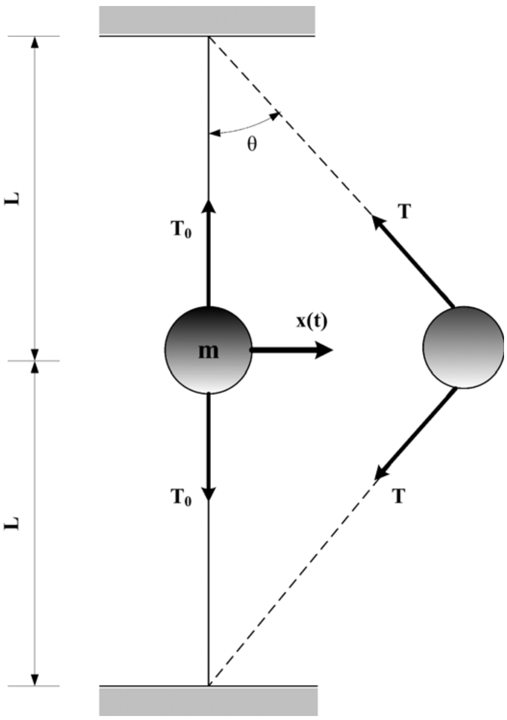



**Fig. 1.** Mass attached to a stretched elastic wire.

## 2. Equation of motion

The governing differential equation of motion and the associated initial conditions for a mass attached to a stretched elastic wire [38], shown in Fig. 1, are:

$$m\ddot{x} = -2T\sin\theta \tag{1a}$$
$$x(0) = X \quad \text{and} \tag{1b}$$
$$\dot{x}(0) = 0 \tag{1c}$$

where $T$ is the tension on the elastic wire after stretching and $X$ is the initial amplitude.

$$T = T_0 + EA\frac{\sqrt{L^2 + x^2} - L^2}{L^2}. \tag{2}$$

With some geometric manipulations according to Fig. 1, Eq. (2) expresses the tension $T$ in terms $T_0$ and $EA$, which denote the initial tension and the axial rigidity of the elastic wire, respectively.

Here, Eq. (1) is rewritten using Eq. (2) to yield Eq. (3)

$$m\ddot{x} + 2EA\frac{x}{L} + \frac{\frac{x}{L}}{\sqrt{1 + \left(\frac{x}{L}\right)^2}}(2T_0 - 2EA) = 0. \tag{3}$$

The following dimensionless quantities (Eqs. (4a) and (4b)) are chosen and substituted into Eq. (3) to derive the nondimensional form of the equation, which is given in Eq. (5)

$$u = \frac{x}{L}, \tag{4a}$$
$$\tau = \frac{t}{\sqrt{\frac{mL}{2EA}}} \tag{4b}$$

$$u'' + u - \left(1 - \frac{T_0}{EA}\right)\frac{u}{\sqrt{1 + u^2}} = 0, \quad 0 \le T_0 \le EA \tag{5}$$

where $(\,)'$ is the derivative with respect to $\tau$.

The coefficient of the third term is introduced with a new parameter $\lambda$, which characterizes the system of values between 0 and 1:

$$u'' + u - \frac{\lambda u}{\sqrt{1+u^2}} = 0, \quad 0 \le \lambda \le 1 \tag{6a}$$

$$u(0) = \frac{X}{L} = a \quad \text{and} \tag{6b}$$

$$u'(0) = 0. \tag{6c}$$

Eq. (6a) is the equation of motion for a mass attached to the center of a stretched elastic wire in dimensionless form. It is, indeed, an example of a conservative nonlinear oscillatory system having an irrational elastic force.

## 3. He's max–min approach

Here, we apply the max–min approach based on He Chengtian's inequality.

We can write Eq. (6a) in the following form

$$u'' + uf(u) = 0 \tag{7}$$

where $f(u)$ is a non-negative function of $u$.

$$f(u) = 1 - \frac{\lambda}{\sqrt{1+u^2}}, \quad 0 \le \lambda \le 1. \tag{8}$$

According to the max–min approach, we use a trial function,

$$u(\tau) = a\cos(\omega\tau), \tag{9}$$

where $\omega$ is assumed to be the frequency to be further determined.

It can be observed that the square of frequency $\omega^2$ is never less than that in the solution of the governing differential equation given by Eq. (6a) with the initial conditions

$$u'' + uf_{\min} = 0 \tag{10}$$

$$u(0) = a \quad \text{and} \tag{11a}$$

$$u'(0) = 0. \tag{11b}$$

Similarly $\omega^2$ never exceeds the solution of Eq. (6a)

$$u'' + uf_{\max} = 0 \tag{12}$$

$$u(0) = a \quad \text{and} \tag{13a}$$

$$u'(0) = 0. \tag{13b}$$

Here $f_{\min}$ and $f_{\max}$ are the minimum and maximum values of the function $f(u)$.

Using Eqs. (10) and (12), we can write

$$\frac{f_{\min}}{1} < \omega^2 < \frac{f_{\max}}{1}. \tag{14}$$

Using the trial function given in Eq. (9), Eq. (14) is rearranged as follows:

$$1 - \lambda < \omega^2 < 1 - \frac{\lambda}{\sqrt{1+a^2}}. \tag{15}$$

He Chengtian actually used the inequality $\frac{a}{b} < x < \frac{d}{c}$, where $a$, $b$, $c$ and $d$ are real numbers; thus, $\frac{a}{b} < \frac{ma+nd}{mb+nc} < \frac{d}{c}$, and $x$ is approximated by

$$x = \frac{ma+nd}{mb+nc}. \tag{16}$$

Introducing $k = \frac{n}{m+n}$ and using Eqs. (15)–(16), we have

$$\omega^2 = 1 + \lambda(k-1) - \frac{\lambda k}{\sqrt{1+a^2}}. \tag{17}$$

Therefore, the solution of Eq. (6a) can be written

$$u(\tau) = A\cos\left(\sqrt{1+\lambda(k-1)-\frac{\lambda k}{\sqrt{1+a^2}}}\tau\right). \tag{18}$$

We can write Eq. (6a) in the following form by using the approximate solution of Eq. (18):

$$u'' + \underbrace{\left(1+\lambda(k-1)-\frac{\lambda k}{\sqrt{1+a^2}}\right)}_{\omega^2} u = \lambda(k-1)u - \frac{\lambda k}{\sqrt{1+a^2}}u + \frac{\lambda}{\sqrt{1+u^2}}u. \tag{19}$$

If Eq. (18) is the exact solution of Eq. (6a), then the right side of Eq. (19) vanishes. On the other hand, if it is only an approximation of the exact solution, the right hand side of Eq. (19) expands into the Fourier series:

$$\begin{aligned}\lambda(k-1)u - \frac{\lambda k}{\sqrt{1+a^2}}u + \frac{\lambda}{\sqrt{1+u^2}}u &= \sum_{j=0}^{\infty} b_{2j+1}\cos[(2j+1)\omega\tau] \\ &= b_1\cos(\omega\tau) + \sum_{j=1}^{\infty} b_{2j+1}\cos[(2j+1)\omega\tau].\end{aligned} \tag{20}$$

We set $b_1$ using Eq. (9):

$$b_1 = \frac{2a\lambda}{T}\int_0^{T/2}\left[(k-1)-\frac{k}{\sqrt{1+a^2}}+\frac{1}{\sqrt{1+a^2\cos^2(\omega\tau)}}\right]\cos^2(\omega\tau)\mathrm{d}\tau = 0, \quad T=\frac{2\pi}{\omega} \tag{21a}$$

$$\int_0^{\pi/2}\frac{(k-1)\sqrt{1+a^2}-k}{\sqrt{1+a^2}}\cos^2(\theta)\mathrm{d}\theta + \int_0^{\pi/2}\frac{\cos^2(\theta)}{\sqrt{1+a^2\cos^2(\theta)}}\mathrm{d}\theta = 0. \tag{21b}$$

Applying a simple transformation to the last term in Eq. (21b), the integral is evaluated in terms of $E(-a^2)$ and $K(-a^2)$, which are the first and second kinds of complete integrals [39].

$$\int_0^1 \frac{x^2}{\sqrt{(1-x^2)(1+k^2x^2)}}\mathrm{d}x = \frac{E(-a^2)-K(-a^2)}{a^2}. \tag{22}$$

Next, $k$ is identified in the following:

$$k = \frac{\sqrt{1+a^2}}{\sqrt{1+a^2}-1}\left[1-\frac{4E(-a^2)-4K(-a^2)}{\pi a^2}\right]. \tag{23}$$

Thus, the solution $u(t)$ is identical to the result obtained by Belendez et al. [7].

$$u(\tau) = a\cos\left[\sqrt{1-\frac{4\lambda}{\pi a^2}[E(-a^2)-K(-a^2)]}\tau\right]. \tag{24}$$

## 4. He's frequency–amplitude method

The main purpose of this section is to present how to apply the frequency–amplitude formulation, proposed recently by He [16].

The method is based on determining the amplitude–frequency relation. To do so, the trial functions $u_1(\tau) = a\cos(\tau)$ and $u_2(\tau) = a\cos(\omega\tau)$, which are the solutions of the following linear equations, are substituted into Eq. (6a) to obtain the residuals.

$$u'' + \omega_1^2 u = 0, \quad \omega_1^2 = 1 \tag{25}$$

$$u'' + \omega_2^2 u = 0, \quad \omega_2^2 = \omega^2. \tag{26}$$

The residuals are

$$R_1(\tau) = -\frac{\lambda a\cos(\tau)}{\sqrt{1+a^2\cos^2(\tau)}} \tag{27}$$

$$R_2(\tau) = -a\omega^2\cos(\omega\tau) + a\cos(\omega\tau) - \frac{\lambda a\cos(\omega\tau)}{\sqrt{1+a^2\cos^2(\omega\tau)}}. \tag{28}$$

$\hat{R}_1$ and $\hat{R}_2$ are introduced as follows:

$$\hat{R}_1 = \frac{4}{T_1}\int_0^{T_1/4} R_1(\tau)\cos(\tau)\mathrm{d}\tau = -\frac{2\lambda}{\pi a}[E(-a^2) - K(-a^2)] \tag{29}$$

$$\hat{R}_2 = \frac{4}{T_2}\int_0^{T_2/4} R_2(\tau)\cos(\omega\tau)\mathrm{d}t = \frac{a}{2}(1-\omega^2) - \frac{2\lambda}{\pi a}[E(-a^2) - K(-a^2)]. \tag{30}$$

Applying He's amplitude frequency method,

$$\omega^2 = \frac{\omega_1^2\hat{R}_2 - \omega_2^2\hat{R}_1}{\hat{R}_2 - \hat{R}_1} = 1 - \frac{4\lambda}{\pi a^2}[E(-a^2) - K(-a^2)]. \tag{31}$$

Therefore, the solution is given in the following equation:

$$u(\tau) = a\cos\left[\sqrt{1 - \frac{4\lambda}{\pi a^2}[E(-a^2) - K(-a^2)]}\tau\right]. \tag{32}$$

## 5. The parameter-expansion method

The parameter-expansion method which was first developed in Ref. [29] is based on perturbation techniques. Eq. (6a) is rewritten to be solved via the parameter-expansion method

$$u'' + 1\cdot u - \lambda\cdot\frac{u}{\sqrt{1+u^2}} = 0. \tag{33}$$

The solution can be expressed as a power series in $p$ such that,

$$u = u_0 + pu_1 + p^2u_2 + \cdots \tag{34}$$

where $p$ is a bookkeeping parameter, $p = 1$. The coefficient of $u$ and $\frac{u}{\sqrt{1+u^2}}$ are expanded respectively into a power series in $p$.

$$1 = \omega^2 + pb_1 + p^2b_2 + \cdots \tag{35}$$

$$\lambda = pe_1 + p^2e_2 + \cdots. \tag{36}$$

Substituting Eqs. (34)–(36) into (33) yields the following form:

$$\begin{aligned}&(u_0'' + pu_1'' + p^2u_2'' + \cdots) + (\omega^2 + pb_1 + p^2b_2 + \cdots)\cdot(u_0 + pu_1 + p^2u_2 + \cdots)\\ &\quad -(pe_1 + p^2e_2 + \cdots)\cdot\frac{(u_0 + pu_1 + p^2u_2 + \cdots)}{\sqrt{1 + (u_0 + pu_1 + p^2u_2 + \cdots)^2}} = 0.\end{aligned} \tag{37}$$

The terms of the same power are collected in the following form:

$$u_0'' + \omega^2 u = 0 \tag{38}$$

$$u_1'' + \omega^2u_1 + b_1u_0 - e_1\frac{u_0}{\sqrt{1+u_0^2}} = 0. \tag{39}$$

The solution of Eq. (38) can be written as follows:

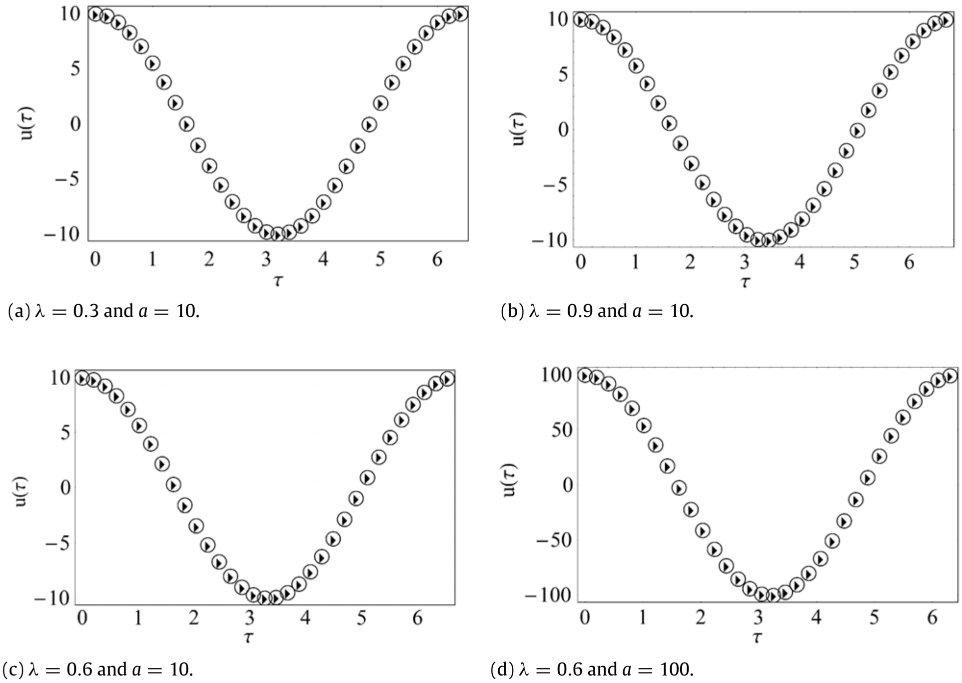
$$u_0(\tau) = a\cos(\omega\tau). \tag{40}$$

Inserting $u_0$ into Eq. (39) results in

$$u_1'' + \omega^2u_1 + b_1a\cos(\omega\tau) - e_1\frac{a\cos(\omega\tau)}{\sqrt{1+a^2\cos^2(\omega\tau)}} = 0. \tag{41}$$

The following Fourier series expansion is valid for the last term of Eq. (41):

$$\begin{aligned}\frac{a\cos(\omega\tau)}{\sqrt{1+a^2\cos^2(\omega\tau)}} &= \sum_{n=0}^{\infty} c_{2n+1}\cos[(2n+1)\omega\tau]\\ &= c_1\cos(\omega\tau) + c_3\cos(3\omega\tau) + \cdots\end{aligned} \tag{42}$$




(a) $\lambda = 0.3$ and $a = 10$.

(b) $\lambda = 0.9$ and $a = 10$.

(c) $\lambda = 0.6$ and $a = 10$.

(d) $\lambda = 0.6$ and $a = 100$.

**Fig. 2.** The comparison of the approximate analytical results [14] with approximate results. ○: Approximate analytical solution; ►: Approximate solutions.

where $c_1$ can be determined by Fourier series,

$$c_1 = \frac{4}{\pi}\int_0^{\pi/2} \frac{a\cos(\theta)}{\sqrt{1+a^2\cos^2(\theta)}}\cos(\theta)\mathrm{d}\theta. \tag{43}$$

By means of Eqs. (41) and (42), we find that

$$u_1'' + \omega^2 u_1 + \cos(\omega\tau)\left[b_1 a - \frac{4e_1}{\pi}\int_0^{\pi/4} \frac{a\cos(\theta)}{\sqrt{1+a^2\cos^2(\theta)}}\cos(\theta)\mathrm{d}\theta\right] - e_1\sum_{n=1}^{\infty} c_{2n+1}\cos[(2n+1)\omega\tau] = 0. \tag{44}$$

The absence of secular terms in $u_1$ requires that

$$b_1 - \frac{4e_1}{\pi}\int_0^{\pi/4} \frac{\cos(\theta)}{\sqrt{1+a^2\cos^2(\theta)}}\cos(\theta)\mathrm{d}\theta = 0. \tag{45}$$

If a first order approximation solution is sufficient, then from Eqs. (35) and (36), we have

$$b_1 = 1-\omega^2 \tag{46}$$

$$e_1 = \lambda. \tag{47}$$

Solving equations from (45)–(47), we obtain

$$\omega^2 = 1 - \frac{4\lambda}{\pi a^2}[E(-a^2) - K(-a^2)]. \tag{48}$$

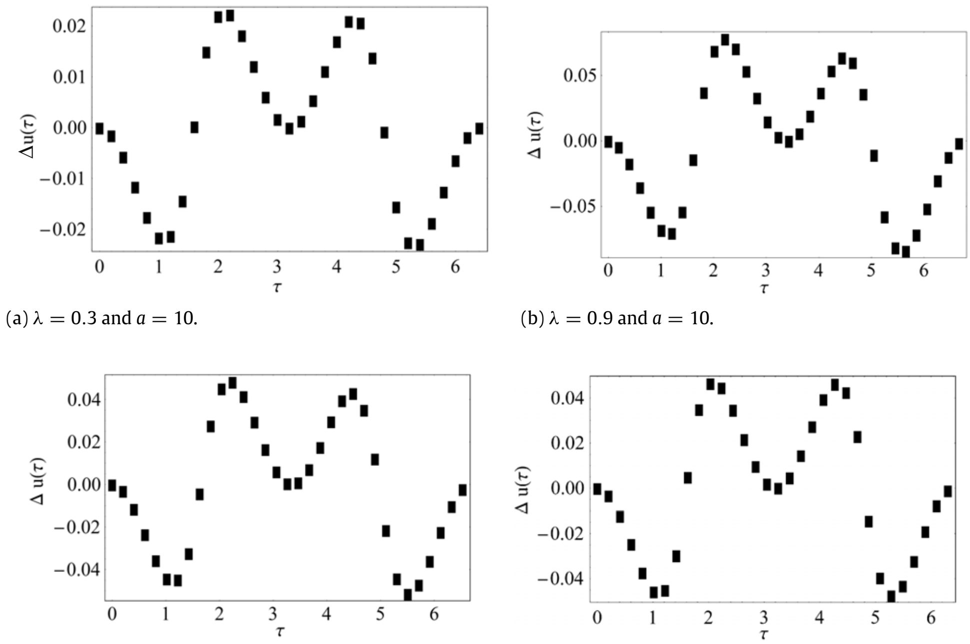
Therefore, the solution is given in the following equation:

$$u(\tau) = a\cos\left[\sqrt{1 - \frac{4\lambda}{\pi a^2}[E(-a^2) - K(-a^2)]}\tau\right]. \tag{49}$$

## 6. Results and discussion

It can be clearly observed that the approximate methods presented here yield a solution of the same form, which is given by Eqs. (24), (32) and (49). All calculations and drawings were carried out using MATHEMATICA®. The approximate solutions obtained here and the approximate analytical solutions are given in Fig. 2.

Because the results are indistinguishable from analytical ones, the difference between the approximate and approximate analytical [15] results is plotted in Fig. 3. The difference is sufficiently small, even for large values of amplitude $a$ and parameter $\lambda$.




(a) $\lambda = 0.3$ and $a = 10$.

(b) $\lambda = 0.9$ and $a = 10$.

(c) $\lambda = 0.6$ and $a = 10$.

(d) $\lambda = 0.6$ and $a = 100$.

**Fig. 3.** The difference between the approximate and approximate analytical results.

## 7. Conclusion

The mathematical model of a nonlinear oscillator described as a mass attached to a stretched elastic wire is established. The solution of the nonlinear oscillator model is efficiently handled by He's max–min approach, He's frequency–amplitude formulation, and the parameter-expansion method. The periodic solutions obtained by the methods presented here have been compared to the approximate analytical results for both small and large amplitude values of oscillation and are found to agree well with both.

## References

[1] J.H. He, Variational iteration method—some recent results and new interpretations, J. Comput. Appl. Math. 207 (2007) 3–17.
[2] A.A. Hemeda, Variational iteration method for solving nonlinear partial differential equation, Chaos Solitons Fractals 39 (2009) 1297–1303.
[3] M. Rafei, D.D. Ganji, H. Daniali, H. Pashaei, The variational iteration method for nonlinear oscillators with discontinuities, J. Sound Vib. 305 (2007) 614–620.
[4] V. Marinca, N. Herisanu, Periodic solutions for some strongly nonlinear oscillations by He's variational iteration method, Comput. Math. Appl. 54 (2007) 1188–1196.
[5] T. Ozis, A. Yıldırım, A study of nonlinear oscillators with $u^{1/3}$ force by He's variational iteration method, J. Sound Vib. 306 (2007) 372–376.
[6] L. Cveticanin, The homotopy-perturbation method application for solving complex-valued differential equations with strong cubic nonlinearity, J. Sound Vib. 285 (2005) 1171–1179.
[7] A. Belendez, T. Belendez, C. Neipp, A. Hernandez, M.L. Alvarez, Approximate solutions of a nonlinear oscillator typified as a mass attached to a stretched elastic wire by the homotopy perturbation method, Chaos Solitons Fractals 39 (2009) 746–764.
[8] A. Belendez, T. Belendez, A. Marquez, C. Neipp, Application of He's homotopy perturbation method to conservative truly nonlinear oscillators, Chaos Solitons Fractals 37 (2008) 770–780.
[9] A. Belendez, C. Pascual, T. Belendez, A. Hernandez, Solution for an anti-symmetric quadratic nonlinear oscillator by modified He's homotopy perturbation method, Nonlinear Anal. RWA 10 (2009) 416–427.
[10] A. Belendez, C. Pascual, M. Ortuño, T. Belendez, S. Gallego, Application of a modified He's homotopy perturbation method to obtain higher-order approximations to a nonlinear oscillator with discontinuities, Nonlinear Anal. RWA 10 (2009) 601–610.
[11] A. Belendez, A. Hernandez, T. Belendez, C. Neipp, A. Marquez, Higher accuracy analytical approximations to a nonlinear oscillator with discontinuity by He's homotopy perturbation method, Phys. Lett. A 372 (2008) 2010–2016.
[12] T. Ozis, A. Yıldırım, A note on He's homotopy perturbation method for van der Pol oscillator with very strong nonlinearity, Chaos Solitons Fractals 34 (2007) 989–991.
[13] A. Belendez, C. Pascual, S. Gallego, M. Ortuño, C. Neipp, Application of a modified He's homotopy perturbation method to obtain higher-order approximations of an $x^{1/3}$ force nonlinear oscillator, Phys. Lett. A 371 (2007) 421–426.
[14] M. Rafei, D.D. Ganji, H.R. Mohammadi Daniali, H. Pashaei, Application of homotopy perturbation method to the RLW and generalized modified Boussinesq equations, Phys. Lett. A 364 (2007) 1–6.
[15] W.P. Sun, B.S. Wu, C.W. Lim, Approximate analytical solutions for oscillation of a mass attached to a stretched elastic wire, J. Sound Vib. 300 (2007) 1042–1047.
[16] J.H. He, Comment on He's frequency formulation for nonlinear oscillators, European J. Phys. 29 (2008) L1–L4.
[17] J.H. He, An improved amplitude–frequency formulation for nonlinear oscillators, Int. J. Nonlinear Sci. Numer. Simul. 9 (2008) 211–212.
[18] J. Fan, He's frequency–amplitude formulation for the Duffing harmonic oscillator, Comput. Math. Appl. 58 (2009) 2473–2476.
[19] H.L. Zhang, Application of He's frequency–amplitude formulation to an $x^{(1/3)}$ force nonlinear oscillator, Int. J. Nonlinear Sci. Numer. Simul. 9 (2008) 297–300.

[20] H.L. Zhang, Application of He's frequency–amplitude formulation to nonlinear oscillator with discontinuity, Comput. Math. Appl. 58 (2009) 2197–2198.
[21] L. Zhao, He's frequency–amplitude formulation for nonlinear oscillators with an irrational force, Comput. Math. Appl. 58 (2009) 2477–2479.
[22] A.E. Ebaid, Oscillations in an $x((2n+2)/(2n+1))$ potential via He's frequency–amplitude formulation, Z. Nat. forsch. A 64 (2009) 877–878.
[23] Z.F. Ren, G.Q. Liu, Y.X. Kang, et al., Application of He's amplitude–frequency formulation to nonlinear oscillators with discontinuities, Phys. Scr. 80 (2009) 045003.
[24] J.H. He, He Chengtian's inequality and its applications, Appl. Math. Comput. 151 (2004) 887–891.
[25] J.H. He, Max–min approach to nonlinear oscillators, Int. J. Nonlinear Sci. Numer. Simul. 9 (2008) 207–210.
[26] Y.Y. Shen, L.F. Mo, The max–min approach to a relativistic equation, Comput. Math. Appl. 58 (2009) 2131–2133.
[27] D.Q. Zeng, Y.Y. Lee, Analysis of strongly nonlinear oscillator using the max–min approach, Int. J. Nonlinear Sci. Numer. Simul. 10 (2009) 1361–1368.
[28] J.H. He, J. Tang, Rebuild of King Fang 40 BC musical scales by He's inequality, Appl. Math. Comput. 168 (2005) 909–914.
[29] J.H. He, Bookkeeping parameter in perturbation methods, Int. J. Nonlinear Sci. Numer. Simul. 2 (2001) 257–264.
[30] J.H. He, Some asymptotic methods for strongly nonlinear equations, Internat. J. Modern Phys. B 20 (2006) 1141–1199.
[31] S.Q. Wang, J.H. He, Nonlinear oscillator with discontinuity by parameter-expansion method, Chaos Solitons Fractals 35 (2008) 688–691.
[32] L. Xu, Application of He's parameter-expansion method to oscillations of a mass attached to a stretched elastic wire, Phys. Lett. A 368 (2007) 259–262.
[33] M.O. Kaya, S. Altay Demirbağ, Application of parameter expansion method to the generalized nonlinear discontinuity equation, Chaos Solitons Fractals 42 (2007) 1967–1973.
[34] D.H. Shou, J.H. He, Application of parameter expanding method to strongly nonlinear oscillator, Int. J. Nonlinear Sci. Numer. Simul. 8 (2007) 121–124.
[35] J.H. He, Application of He Chengtian's interpolation to Bethe equation, Comput. Math. Appl. 58 (2009) 2427–2430.
[36] T. Zhong, Ancient Chinese musical scales: best approximations, but why? Int. J. Nonlinear Sci. Numer. Simul. 10 (2009) 161–166.
[37] J.C. Lan, Z. Yang, Continued fraction method for an ancient Chinese musical equation, Int. J. Nonlinear Sci. Numer. Simul. 10 (2009) 167–169.
[38] R.E. Mickens, Oscillations in Planar Dynamic Systems, World Scientific Publishing, Singapore, 1996.
[39] I.S. Gradshteyn, I.M. Ryzhik, A. Jeffrey, D. Zwillinger, Table of Integrals, Series, and Products, Academic Press, 2000.