Matrix Tensor Notation
Part II. Skew and Curved Coordinates

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Abstract—In Part I, a notation called Matrix-Tensor Notation was introduced for rectilinear orthogonal coordinates.
Part II discusses how the same notation is equally efficient for vectors and 2nd order tensors in skew bases and skew curved coordinates.
Without considering coordinates first, vectors and 2nd order tensors are described in skew bases, covariant and contravariant descriptions being distinguished similar to tensor notation. A consistent interpretation of deformation and strain in terms of skew bases is given.
The use of the transpose symbol complicates the matrix algebra, but integrates it with tensor algebra which makes it possible to interpret customary tensor equations as relations in space with all the advantages that an isomorphism with Euclidean space has. It is demonstrated how different metrics can be assigned arbitrarily to define higher-dimensional orthogonal vectors in abstract higher-dimensional space.
A consistent notation is given to distinguish between subspace and subbase.
To write higher order tensors as matrices, or 2nd order tensors as vectors, partial transpose is introduced, which is a formal stretching operation to write any higher order tensor product operation as matrix-vector product.
The description of the variable bases in generally curved coordinates is given in this notation, together with the corresponding notation for the partial differentiation rules. The vector operations grad, div and curl are discussed and an almost-physical notation for the Christoffel symbols is given. However, the replacement of tensor notation for higher than second order tensors by Matrix Tensor Notation is not feasible.
A version of Matrix Tensor Notation that merges with the customary printed form is presented in an Appendix.
Further applications are deferred to Part III, applying the new notation to curved coordinates in function space, and the equations of mechanics projected on finite-dimensional subspaces.

Keywords—Mechanics, Tensors, Continuum mechanics, Vector spaces.

14. INTRODUCTION
The Matrix-Tensor Notation, Part II, is the continuation of Matrix-Tensor Notation, Part I, [24], applied to skew bases and curved coordinates. For ease of continuation, sections, equations, figures, and references are numbered continuing from Part I.
The same equations of transformations and rotation as for Cartesian base are generalized to include the well-known relations for skew bases, including deformation, in Matrix Tensor Notation.

15. SKEW BASE
15.1. Posed Skew Base
To describe a skew base, it must be assumed that the metric of the space is known, which in Euclidean space is axiomatic, otherwise the term skew makes no sense. If the base is drawn on a
piece of paper, then the paper supplies the implied orthonormal space. Alternatively the angles between all the base vectors may be given. If a base is used to describe the motion of a part of a machine, then the machine supplies the orthonormal space. These are geometric or graphical descriptions.

The graphical description of a skew base is shown in Fig. 8(a), in which only two dimensions are shown, which is sufficient to see all the necessary features. The base with the name \( q \), chosen for its phonetic similarity with the word "skew," is denoted by

\[
\overrightarrow{E}_q = \begin{bmatrix} \overrightarrow{e}_{q1} & \overrightarrow{e}_{q2} & \overrightarrow{e}_{q3} \end{bmatrix}
\]

corresponding to eqn. (9) in Part I. The skew directions that may have been given in a practical situation are described mathematically by the direction vectors \( \overrightarrow{e}_{q1}, \overrightarrow{e}_{q2}, \overrightarrow{e}_{q3} \), which are now used as a base and collected to the mathematical quantity \( \overrightarrow{E}_q \) according to eqn. (15.1). Let us also assume that no coordinates are used, and therefore, the base vectors are simply chosen as unit magnitude. Note that we avoid the term "length" for base vectors because they are assumed to be physically dimensionless direction vectors.

![Figure 8. Skew bases.](image-url)
The skew components of a vector $\vec{v}$ are shown in Fig.8(a), and are described by the vector array

$$
\vec{v}^q = \begin{bmatrix}
v_{q1} \\
v_{q2} \\
v_{q3}
\end{bmatrix},
$$

(15.2)
corresponding to eqn.(10); however, in skew base the superscripts may not be replaced by subscripts as will be shown subsequently. The fundamental equation corresponding to eqn.(11) is

$$
\vec{v} = \vec{E}_q \cdot \vec{v}^q
$$

(15.3)

where $v_{qi}$ are the scalar components, and $\vec{e}_{qi} v_{qi}$ are the vector components (no tensor addition intended). In addition to the definitions of Part I, we now also introduce the notation for vector components

$$
\vec{e}_{qi} v_{qi} = \tilde{v}_{qi}, \quad \vec{e}_{q2} v_{q2} = \tilde{v}_{q2}, \ldots
$$

written in short as

$$
\vec{e}_{qi} v_{qi} = \tilde{v}_{qi}
$$

(15.4)

(for $i = 1, 2, 3$, no tensor summation intended or possible). In customary Cartesian base, we might write them $v_x, v_y, v_z$. While the result of the multiplication in eqn.(15.4) is not consistent with either our notation rules or tensor notation rules, the symbol $\tilde{v}_{qi}$ is rather like a generalization of the scalar component symbol $v_{qi}$. Note also that the vector component can be written as a result of a tensor operation,

$$
\tilde{v}_{qi} = [\vec{e}_{qi} \cdot \vec{e}_{q1} / \vec{e}_{qi} \cdot \vec{v}^q], \quad \vec{v}.
$$

Seeing that the comfortable rules of notation are apparently violated by eqn.(15.4), there are two questions to be answered. One is, why the rules are violated, the other is, do we need these expressions that violate the rules. The answer to the first question is rather an explanation than an answer. Apparently there are some quantities, like the vector component here, that don’t make sense in the general concept of vector and tensor relations. We have the same appearance in tensor notation. Some quantities just cannot be expressed by the tensor rules because they make no sense. An example is the sum of the squares of the diagonals of a tensor, which is not an invariant. Apparently, we must see the vector component also as a variant rather than an invariant, and accept the fact that therefore it cannot be expressed by proper notation rules. The answer to the second question is that we want a symbol to distinguish the scalar component from the vector component, and there may be other cases where such a vector is needed. We might perhaps express the idea better by saying that the product on the LHS of eqn.(15.4) leads to a vector, and write symbolically

$$
\vec{e}_{qi} v_{qi} \Rightarrow \tilde{v}_{qi}.
$$

Once the vector on the RHS is created, it is a vector with a new name, and the rules are only violated because we don’t give this vector a complete new name but a combination of the terms that created it. Once we have passed the stumbling block of eqn.(15.4), we can carry on with the new vector name $\tilde{v}_{qi}$ like any other, accepting of course that a vector name may consist of more than one letter. Then we may even transform the space symbol of the vector component to the base $q$, and obtain the quantity

$$
\tilde{v}_{qi} = \vec{E}_q \cdot \tilde{v}_{qi}.
$$
This is not the scalar component \( v^{q1} \), but rather a vector array \([v^{q1} 0 0]\). This is a special case of a partial vector, introduced in Section 19. But once we come to the names of the individual components of this partial vector, there must be a distinction between name and subscript, e.g., like

\[
\bar{v}_{q1} = \begin{bmatrix} v_{q1}^{q1} \\ v_{q1}^{q2} \\ v_{q1}^{q3} \end{bmatrix}.
\]

Such a notation allows for the scalar components of a vector component in the inverse base,

\[
\bar{v}_{q1}^{q1} = \begin{bmatrix} v_{q1}^{q1} & v_{q1}^{q2} & v_{q1}^{q3} \end{bmatrix}.
\]

The fact that we distinguish between vectors and matrices by small and capital letters, respectively, also helps.

Obviously, we cannot take this method too far, because we can never include the complete chain of events in a name which is produced by these events. Somewhere along the line, we must make a completely new name of a freshly produced variable.

To be able to invert eqn.(15.3), we need the inverse base \( \bar{E}_q \) defined by

\[
\bar{E}_q \cdot \bar{E}_q = \bar{I},
\]

where eqn.(15.5) corresponds to eqn.(15). Let us postpone, for the moment, the question of how to construct or compute this inverse base. But according to symbolic rules, the notation of the product is

\[
\bar{E}_q \cdot \bar{E}_q = \bar{E}_q^q,
\]

which corresponds to eqn.(17). Therefore,

\[
\bar{E}_q^q = \bar{I},
\]

corresponding to eqn.(18), as well as

\[
\bar{E}_q^q \cdot \bar{E}_q = \bar{E}_q
\]

is also true in skew base. This gives the interesting result that the direction vector arrays \( \bar{e}_{q1}^q, \bar{e}_{q2}^q, \bar{e}_{q3}^q \) are also unit column vectors as in Cartesian base.

The inverse of eqn.(15.3) is now

\[
\bar{v}^q = \bar{E}_q^q \cdot \bar{v},
\]

corresponding to eqn.(13).

The inverse base \( \bar{E}_q^q \) that was produced by the definition of eqn.(15.5) is shown in Fig.8(b). The corresponding direction vectors are denoted according to

\[
\bar{E}_q^q = \begin{bmatrix} \bar{e}_{q1}^q \\ \bar{e}_{q2}^q \\ \bar{e}_{q3}^q \end{bmatrix},
\]

corresponding to eqn.(16). It is a skew base physically different from \( \bar{E}_q \), with different directions of the direction vectors which is clear from the fact that the vector \( \bar{e}_{qi}^q \) is normal to all vectors \( \bar{e}_{q(j\neq i)} \). The direction vectors are also not unit magnitude any more, since

\[
\bar{e}_{qi}^q \cdot \bar{e}_{qi} = |\bar{e}_{qi}^q| \cos(\theta_{ij}) = |\bar{e}_{qi}^q| \cdot \cos(\theta_{ij}) = 1.
\]
It is well known that it is not possible to have original and inverse base with both unit base vectors. Nevertheless, we may express the same physical vector \( \mathbf{v} \) in terms of the inverse base

\[
\mathbf{v} = \mathbf{v}_q \cdot \mathbf{E}^q = v_{q1} \mathbf{e}_q^{q1} + v_{q2} \mathbf{e}_q^{q2} + v_{q3} \mathbf{e}_q^{q3},
\]

where eqn.(15.11) corresponds to eqn.(24), with the row vector

\[
\mathbf{v}_q = \begin{bmatrix} v_{q1} & v_{q2} & v_{q3} \end{bmatrix}.
\]

The vector components \( v_{q1} \mathbf{e}_q^{q1}, v_{q2} \mathbf{e}_q^{q2}, v_{q3} \mathbf{e}_q^{q3} \) are shown in Fig.8(b), summing up to the vector \( \mathbf{v} \) according to the parallelogram law of vectors. Note that \( \mathbf{e} \) is used for base vectors even if nonunit, to be consistent with unit tensor symbol \( \mathbf{E} \).

The inverse of eqn.(15.11) is

\[
\mathbf{v} = \mathbf{v}^q \cdot \mathbf{E}_q^q,
\]

(15.13)

corresponding to eqn.(21). We also introduce the symbols for the reciprocal vector components

\[
v_{qi} \mathbf{e}_q^{qi} = v^q_i.
\]

Seeing that the physical base \( \mathbf{E}_q^q \), which was named base \( q \), is different from the physical base \( \mathbf{E}_q^q \), we may only attach the name \( q \) to the base according to the form \( \mathbf{E}_q^q \). To distinguish the two bases with the base symbol \( q \), we call the base posed base \( \mathbf{E}_q^q \) column base \( q \) and the inverse or reciprocal base \( \mathbf{E}_q^q \) the row base \( q \). Because a vector cannot be inverted, we cannot call \( \mathbf{v}_q \) an inverse vector, rather \( \mathbf{v}^q \) and \( \mathbf{v} \) are reciprocal vector arrays, just like \( \mathbf{E}_q^q \) and \( \mathbf{E}_q^q \) are reciprocal bases, and \( \mathbf{v}^q \) is the column vector, \( \mathbf{v} \) the row vector, using the term “vector” here for “vector array.” Both are arrays of the same physical vector, as in tensor notation where “contravariant vector” and “covariant vector” refers to the vector arrays, not to the physical vectors.

In skew nonunit base \( \mathbf{E}_q^q \), we define the quantities \( v_{qi} \) as the scalar components of the vector \( \mathbf{v}_q \). However, from the magnitude relation

\[
|v_{qi} \mathbf{e}_q^{qi}| = v_{qi} |\mathbf{e}_q^{qi}|,
\]

it is clear that the scalar components are not equal to the magnitudes of the vector components, as in Cartesian base. We have therefore an additional problem with definitions of components in skew nonunit base that there is no term for the magnitude of the vector components. If this is a concern, then additional skew bases with unit direction vectors can be introduced, like bases \( h \) and \( g \) in Section 21.2 and 21.3.

It is now clear that the superscripts and subscripts are necessary to distinguish the inequalities in general,

\[
\mathbf{e}_q^{qi} \neq \left( \mathbf{e}_q^{qi} \right)^\top, \quad \text{or} \quad \mathbf{E}_q^q \neq \left( \mathbf{E}_q^q \right)^\top,
\]

(15.15)

\[
v_{qi} \neq v^q_i, \quad \text{or} \quad \mathbf{v}_q \neq \left( \mathbf{v}_q \right)^\top.
\]

(15.16)

We may define a reciprocal base \( \mathbf{E}_p^p \) with the directions as \( \mathbf{E}_q^q \) but with unit base vectors, and the original base \( \mathbf{E}_p^p = \left( \mathbf{E}_q^q \right)^{-1} \) will then have the same directions as \( \mathbf{E}_q^q \) but with nonunit base...
vectors. These two bases are shown in Fig. 8(c) and (d), in which the vector components of the vector $\mathbf{v}$ are also indicated. All the equations in base $q$ are then still the same in base $p$. These are two specific alternatives for particular skew directions. In the end not one of the two alternative bases will have to have unit base vectors. All the equations in the above can be read again for the general case of all nonunit base vectors and found to be valid, as well as the inequalities. From eqns. (15.3) and (15.11) it follows that the scalar product of two vectors is

$$f_q \cdot \mathbf{v} = f_q \cdot E_q \cdot E_q^T \cdot \mathbf{v};$$

using eqn. (15.5). It seems, therefore, that once a necessity has arisen to use skew base, some vectors, like $\mathbf{v}$, should be expressed in original base as columns, and some vectors, like $f_q$, should be expressed in the inverse base as rows. Then the matrix tensor algebra is the same whether in Cartesian or skew base, and whether any one of the bases has unit vectors is then immaterial.

The graphical method of normal projection on the different axis to obtain scalar components $v_q^i$, e.g., [23], is only valid for unit base vectors of the original base; in the general case, there is no such direct method.

The distinction between superscripted and subscripted quantities corresponds to the same distinction in tensor analysis, for the same reasons, where

$$v^q = \text{column vector} \Rightarrow \text{contravariant vector} \quad v^i$$

$$v_q = \text{row vector} \Rightarrow \text{covariant vector} \quad v_{ji}$$

$$E_q = \text{row base} \Rightarrow \text{covariant base} \quad g_j$$

$$E^q = \text{column base} \Rightarrow \text{contravariant base} \quad g^i$$

except that, in Matrix Tensor Notation, indices stand for base names while, in Tensor Notation, they stand for running counters. Here, the meaning of the symbol $\Rightarrow$ is taken from [25]. But the tensor symbols on the RHS are not as complete as the Matrix Tensor symbols on the LHS because they don’t distinguish between the base $q$ and many possible others—one of the very reasons for introducing Matrix Tensor Notation. The terms vector and tensor in tensor analysis correspond to vector array and tensor matrix, respectively, and we will use the default terms vector or tensor in the same sense as in tensor analysis.

Because of the treatment by matrix algebra, the transpose symbol is also required, which is absent in tensor notation.

The notations and rules in skew base are the same as derived for Cartesian base, particularly that the unit tensor symbol $E$ is treated as a unit, even if skew nonunit base vectors are used; the exception is that the transpose and reciprocal bases and vectors are not the same as borne out by the inequalities (15.15) and (15.16), instead of eqns. (22) and (23), Part I, with the consequences that the transpose symbol actually denotes a different array as shown in the next section.

15.2. Transposed Skew Base

In Part I, the transpose base symbol was introduced which we now also use for skew base

$$\left[ \bar{E}_{q} \right]^T = \bar{E}_{q}^T.$$

(15.17)

The bases $\bar{E}_q$ and $\bar{E}_q^T$ are physically the same, the difference is only mathematical to be able to use matrix algebra for all required operations. In eqn. (15.17), the transpose acts only on the base symbol because the transpose of the space symbol is the same space symbol. Similarly,

$$\left[ E_q \right]^T = E_{rq}.$$

(15.18)
Mathematically, the two new reciprocal bases $\mathbf{E}_q^T$ and $\mathbf{E}_q^T$ have been produced. They are also inverses of each other which follows from eqns.\((15.5), (15.17)\) and \((15.18)\)

$$I = [I]^T = \left[ \mathbf{E}_q^T \cdot \mathbf{E}_q^T \right]^T = \mathbf{E}_q^T \cdot \mathbf{E}_q^T. \quad (15.19)$$

The symbol $\mathbf{E}_q^T$ qualifies therefore as a base name in the same way as any other skew base name. But the physical reciprocal base $\mathbf{q}^T$ is the same as the base $\mathbf{q}$. From the inverse of eqn.\((15.19)\), it also follows that

$$\mathbf{E}_q^T \cdot \mathbf{E}_q^T = \mathbf{I}. \quad (15.20)$$

At this stage, let us now turn to the question of finding the inverse base. This depends very much on the way the skew base is given as well as on the way the inverse base is described. In the graphical description of Fig.8, we assume the base $\mathbf{E}_q^T$ given with respect to a space, and all the magnitudes and angles between the base vectors are known, which means that the symmetric matrix

$$\mathbf{E}_q^T \cdot \mathbf{E}_q^T = \mathbf{E}_q^T \cdot \mathbf{E}_q^T = \mathbf{E}_q^T \cdot \mathbf{E}_q^T. \quad (15.21)$$

is known. Consider the two bases shown in Fig.8(a) and (b) as two different physical bases with names $\mathbf{q}$ and $\mathbf{q}^T$, respectively, then one base may be expressed in terms of the other. Let the inverse base be described in terms of the original base vectors as

$$\mathbf{A} = \mathbf{E}_q^T \cdot \mathbf{A},. \quad (15.22)$$

which substituted in eqn.\((15.5)\) becomes

$$\mathbf{A} = \mathbf{E}_q^T \cdot \mathbf{A} \cdot \mathbf{E}_q^T = \mathbf{I}. \quad (15.23)$$

The known matrix $\mathbf{E}_q^T$ can be inverted numerically, for which an appropriate symbol must be found that complies with the previous rules. To do this, we first define, similar to eqn.\((15.21)\), according to our symbolic rules

$$\mathbf{E}_q^T \cdot \mathbf{E}_q^T = \mathbf{E}_q^T \cdot \mathbf{E}_q^T. \quad (15.24)$$

From the definitions of eqn.\((15.19)\), it is clear that eqn.\((15.24)\) is the inverse of eqn.\((15.21)\), and therefore, the symbolic rules for the inverse apply also to the transpose base

$$\left[ \mathbf{E}_q^T \right]^{-1} = \mathbf{E}_q^T. \quad (15.25)$$

Eqn.\((15.23)\) can now be inverted with the result that $\mathbf{A}^T = \mathbf{A} = \mathbf{E}_q^T$; therefore, eqn.\((15.22)\) becomes

$$\left[ \mathbf{E}_q^T \right]^T = \mathbf{E}_q^T = \mathbf{E}_q^T \cdot \mathbf{E}_q^T. \quad (15.26)$$

This is the equation that corresponds to the familiar tensor notation

$$g^j = g_i \cdot g^{ij}.$$ 

The inverse of eqn.\((15.26)\) is

$$\mathbf{E}_q^T = \mathbf{E}_q^T \cdot \mathbf{E}_q^T \quad (15.27)$$

corresponding to the tensor equation
Eqn.(15.26) must be understood to prove that the symbol algebra rules II and IV

\[ g_\tau \cdot g^\tau = g_\tau \cdot g^\tau, \quad E \cdot E = E \]

give the intended physical meaning of base \( E_\tau \) as defined by eqn.(15.18).

The base vector relations for transpose base \( \gamma_q \) can be derived again in the same manner as the posed relations, and found to be equal to the transpose of the previous relations,

\[
E_\tau = \left[ \begin{array}{c} e\gamma_q1 \\ e\gamma_q2 \\ e\gamma_q3 \end{array} \right], \quad \gamma_q = \left[ \begin{array}{c} v\gamma q1 \\ v\gamma q2 \\ v\gamma q3 \end{array} \right], \quad e\gamma_q = E_\tau \cdot \gamma_q, \quad \gamma_q = E_{\gamma_q} \cdot \gamma_q.
\]

From the one system of skew directions, there are 4 different specific bases that are produced, \( E_\gamma \) and \( E_\gamma \) in the skew directions, \( E_\gamma \) and \( E_\gamma \) in the reciprocal skew directions. Besides those, there are the 4 other physically equal but mathematically different bases \( E_{\gamma} \), \( E_{\gamma} \), \( E_{\gamma} \) and \( E_{\gamma} \).

The transpose is a matrix operation that doesn't change the values of the array; therefore, a column \( \gamma_q \) is still a covariant vector and a row \( \gamma_q \) is still a contravariant vector, if both are referred to base \( q \). However we may consider base \( E_{\gamma} \) a different base with the name \( \gamma_q \), and then we find that a column is always a contravariant vector in the indicated base, and a row is always a covariant vector in the indicated base, and therefore, consistency between Tensor Notation and Matrix Tensor Notation is preserved.

Using the definition \( [v]^T = v_q \) eqn.(12), and eqns.(15.18) and (15.31),

\[
[v]^T = v_\gamma, \quad [v_\gamma]^T = v_\gamma, \quad [v_q]^T = v_\gamma.
\]

In eqns.(15.38) and (15.39), the transpose acts only on the base symbol because the transpose of the vector symbol is the same vector symbol.

On the other hand, because the transpose operation doesn't change values, some redundancy of component names, as in matrix algebra, is introduced while the values of the vector array and its transpose are the same, only arranged differently.
\[ \mathbf{v}_{\tau q} = \begin{bmatrix} v_{\tau q1} \\ v_{\tau q2} \\ v_{\tau q3} \end{bmatrix} = \begin{bmatrix} v_{q1} \\ v_{q2} \\ v_{q3} \end{bmatrix} = \mathbf{v}_q^T, \]

\[ \mathbf{v}_{\tau q} = [v_{\tau q1} \ v_{\tau q2} \ v_{\tau q3}] = [v^{q1} \ v^{q2} \ v^{q3}] = \mathbf{v}_q^T . \]

We note also that the transpose operation on a matrix changes both bases, so that

\[ \left[ E_{\tau q} \right]^T = E_{q}^{\tau q} \quad (15.40) \]

which means that \( E_{\tau q} \) must be symmetric, which is indeed the case.

To summarize, we have the following 4 bases:

- \( E_q \): base \( q \) = posed base \( q \) = column base \( q \)
- \( E_q^\tau \): inverse base \( q \) = reciprocal base \( q \) = row base \( q \)
- \( E_{\tau q} \): transpose base \( q \) = reciprocal base \( \tau q \) = row base \( \tau q \)
- \( E_{\tau q}^\tau \): transpose inverse base \( q \) = posed base \( \tau q \) = column base \( \tau q \)

where inverse refers to the mathematical and reciprocal to the physical base.

It is well advised to point out at this stage that all relations for skew base are equally valid for abstract higher-dimensional skew space, even though its creation may be difficult; an example of an infinite higher-dimensional base was given in Section 12.1. We will also use the term “Cartesian” for an abstract orthonormal base.

### 15.3. The Metric

The matrix \( E_{\tau q} \) is the single important numerical quantity in skew base \( q \). Its elements are the elements of the quantity \( g_{ij} \) of tensor analysis, and the elements of its inverse \( E_{\tau q} \) are the elements of the quantity \( g^{ij} \). Flügge [2] calls \( g_{ij} \) the metric tensor and the individual elements \( g_{ij} \) its covariant components. Eisele [23] calls \( g \) the metric tensor, although \( g \) is usually reserved for the determinant \( \det|g_{ij}| \), and also \( g_{ij} \) and \( g^{ij} \) the metric tensors. Chung [26] calls \( g_{ij} \) and \( g^{ij} \) covariant and contravariant metric tensors. All of the above must be interpreted in terms of the customary practice in tensor notation that tensor means tensor-components. It is clear that the metric tensor and the unit tensor are the same physical quantities, but metric tensor contradicts this practice. Malvern [3], perhaps more physically minded, calls \( 1 \) the unit tensor or fundamental tensor or identity tensor, and \( g_{ij} \) the covariant and \( g^{ij} \) the contravariant components, which corresponds with our terminology, where his \( 1 \) is the same as our \( E_q \). In our notation, we call the matrix \( E_{\tau q} \) the metric, corresponding to the posed skew space, meaning that particular representation of the unit tensor. It is a measure for the skewness of the base. So is the inverse matrix \( E_{\tau q} \), which we will call the inverse or reciprocal metric, because it is the same metric applied to the reciprocal skew base \( E_q \). But \( E_{\tau q} \) remain the components of the unit tensor, not of the metric. On the other hand, we may talk of the elements of a matrix, and in this sense of the elements of the metric.

In tensor notation, the metric is always denoted by the letter \( g \). Some authors, therefore, use the consistent letter \( g \) for the base vectors [2,26], deviating from the familiar \( e \) of Cartesian base. Others use the letter \( e \) [12], corresponding to the base vector symbol in Cartesian base, deviating now from the name for the metric. Still others use \( e \) for unit base vectors derived from the unitary base vectors [23,27], but rather inconsistent names for the unitary base vectors. Our notation uses \( e \) for base vectors and consistent the other way round, \( E \) for the metric, with which we emphasize its symbolic action like a unit, not its numeric unit value. Our distinction between
unit and unitary base vectors is carried through to defining also a new metric for each system of unit base vectors, i.e., the metric for base \( p \) is \( \mathbf{E}_p^T \). This method of notation is also applied to bases derived from skew curved coordinates, see Section 21.

For a finite-dimensional base we define a rank of skewness as follows. The generally skew base may contain some direction vectors which are orthogonal to each other. The symmetric matrix from eqn.(15.21) has altogether \( n(n - 1)/2 \) different elements in the upper or lower off-diagonal triangular matrix, which are the scalar products \( \mathbf{e}_q^T \mathbf{e}_i \). These will be zero if any of the different base vectors are orthogonal to each other, and non-zero if they are skew. We define the rank of skewness as the number of non-zero elements in the off-diagonal triangular matrix \( \mathbf{E}_q^T \) to the number \( n(n - 1)/2 \). That is, rank of skewness is given in terms of two numbers. The rank as defined above is independent of the actual value of these numbers, which would rather indicate the degree of skewness.

The matrix \( \mathbf{E}_q^{-T} \) may be interpreted simply as a transformation matrix according to eqn.(15.27). Its columns represent the base vectors of base \( \mathbf{E}_q \) expressed in terms of base \( \mathbf{E}_q^{-T} \). Similarly, the columns of \( \mathbf{E}_q^T \) represent the base vectors of base \( \mathbf{E}_q^{-T} \) expressed in terms of base \( \mathbf{E}_q \). Another interpretation is that it represents the unit tensor in mixed bases \( \tau q \) and \( q \), as can be seen from the tensor transformation \( \mathbf{E}_q^{-T} = \mathbf{E}_q \mathbf{E}_q^{-T} \).

The metric is used to transform contravariant vectors into covariant vectors, which in our notation, means transformation from base \( q \) to base \( \tau q \), and the inverse operation to transform covariant vectors into contravariant vectors, as follows from eqns.(15.30), (15.31), (15.26) and (15.27), as well as the inverse operation

\[
\mathbf{v}^{-T} = \mathbf{E}_q^{-T} \mathbf{v} \quad \text{and} \quad \mathbf{v} = \mathbf{E}_q^T \mathbf{v}^{-T},
\]

as defined above is independent of the actual value of these numbers, which would rather indicate the degree of skewness.

The actual transformation from column vectors to row vectors and vice versa must be done by the additional transpose operation

\[
\mathbf{v}_q = \left[ \mathbf{E}_q^{-T} \mathbf{v}^{-T} \right]^T, \quad \text{and} \quad \mathbf{v}^{-T} = \mathbf{E}_q^T \left[ \mathbf{v}_q \right]^T.
\]

The extra transpose compared to tensor analysis is necessary because with a matrix operation it is not possible to obtain a row vector from a multiplication of a tensor matrix with a column vector, which corresponds to a multiplication of a covariant tensor with a contravariant vector to produce a covariant vector in tensor analysis.

### 15.4. Transformations in Skew Base

Another way that the skew base \( \mathbf{E}_q \) may be defined, is in terms of a known Cartesian base \( \mathbf{E}_s \) by means of a transformation matrix \( \mathbf{E}_q^s \),

\[
\mathbf{E}_q = \mathbf{E}_s \mathbf{E}_q^s.
\]

It can now easily be shown that the following equations are consistent in skew base

\[
\begin{align*}
\mathbf{E}_q &= \mathbf{E}_s \mathbf{E}_q^s, \\
\mathbf{E}_q^s &= [\mathbf{E}_q^{-1}]^s, \\
\mathbf{E}_\tau q &= \mathbf{E}_s \mathbf{E}_\tau q^s,
\end{align*}
\]
\[
\bar{E}_q = \bar{E}_q^{\tau q} \cdot \bar{E}_s^s, \quad (15.47-d)
\]
\[
\bar{E}_s^{\tau q} = \left[ \bar{E}_r^{\tau q} \right]^{-1}, \quad (15.47-e)
\]
\[
\begin{bmatrix} \bar{E}_s^{\tau q} \end{bmatrix}^\top = \bar{E}_s^q, \quad (15.47-f)
\]
\[
\bar{E}_s^s \cdot \bar{E}_s^q = \bar{E}_q^q. \quad (15.47-g)
\]

The vector transformations follow from the definitions by eqns.(11) (Part I), and (15.3), and (24) (Part I) and (15.11)

\[
\bar{u}^q = \bar{E}_s^q \cdot \bar{u}^s, \quad (15.48-a)
\]
\[
\bar{u}^{\tau q} = \bar{E}_s^{\tau q} \cdot \bar{u}^s, \quad (15.48-b)
\]
\[
\bar{v}^q = \bar{v}^s \cdot \bar{E}_q^s, \quad (15.48-c)
\]
\[
\bar{v}^{\tau q} = \bar{v}^s \cdot \bar{E}_q^{\tau q}. \quad (15.48-d)
\]

Consider eqn.(15.46) as a transformation of eqn.(15.1), then it is clear that the columns of the transformation matrix $\bar{E}_q^s$ are the skew base vectors of $\bar{E}_q^s$ measured in Cartesian base $\bar{E}_s$

\[
\bar{E}_q^s = \begin{bmatrix}
\bar{e}_s^s & | & \bar{e}_q^s & | & \bar{e}_q^s & |
\bar{e}_q^1 & | & \bar{e}_q^2 & | & \bar{e}_q^3 & |
\bar{e}_q^1 & | & \bar{e}_q^2 & | & \bar{e}_q^3 & |
\end{bmatrix}, \quad (15.49)
\]

At the same time $\bar{E}_q^s$ is the unit tensor in mixed Cartesian and skew bases.

Similarly, from the base transformation eqn.(15.47-c), it can be seen that the columns of the transformation matrix $\bar{E}_q^{\tau q}$ are the skew base vectors of the reciprocal base $\bar{E}_q^{\tau q}$ measured in Cartesian base $\bar{E}_s$

\[
\bar{E}_q^{\tau q} = \begin{bmatrix}
\bar{e}_\tau^s & | & \bar{e}_\tau^s & | & \bar{e}_\tau^s & |
\bar{e}_\tau^1 & | & \bar{e}_\tau^2 & | & \bar{e}_\tau^3 & |
\bar{e}_\tau^1 & | & \bar{e}_\tau^2 & | & \bar{e}_\tau^3 & |
\end{bmatrix}. \quad (15.50)
\]

### 15.5. Transformations from Skew Base to Skew Base

Applying the same symbolic algebra to any two different nonunit skew bases $\bar{E}_a = [e_a^1 e_a^2 e_a^3]$ and $\bar{E}_b = [e_b^1 e_b^2 e_b^3]$ by means of transformation matrices $\bar{E}_a^a$ and $\bar{E}_b^b$, respectively, from Cartesian base $\bar{E}_s$, the transformation of one into the other is easily accomplished by the relations

\[
\bar{E}_b = \bar{E}_a \cdot \bar{E}_b^a, \quad (15.51-a)
\]
\[
\bar{E}_a = \bar{E}_b \cdot \bar{E}_a^b, \quad (15.51-b)
\]
\[
\bar{E}_\tau^a = \bar{E}_b \cdot \bar{E}_\tau^b, \quad (15.51-c)
\]
\[
\bar{E}_b = \bar{E}_\tau^a \cdot \bar{E}_b^\tau, \quad (15.51-d)
\]

where
Without use of Cartesian base, but more symbolically,

\[ E_a \cdot E_b = E_a b \]  

The vector transformations can easily be shown to be

\[ v_a = E_b a \cdot v b \]  
\[ v a = E_b a \cdot v b \]  
\[ v a = E b a \cdot v b \]  
\[ v a = E b \tau a \cdot v b \]  
\[ v a = E b \tau a \cdot v b \]

From the base transformation equation (15.51-a), it can be seen that the columns of the transformation matrix \( E_a b \) are the skew base vectors of \( E_a \) measured in skew base \( E_b \).

\[
E_a = \begin{bmatrix}
-e a & -a & -a \\
e b & e b & e b \\
\end{bmatrix}
\]

The numerous possibilities of combinations as given above prove that the symbolic algebra of skew bases are just as valid as the ones for Cartesian base, with the addition of a consistent use of the transpose symbol \( \tau \). The symbolic manipulation would of course be useless if the meaning of the combination of symbols is not made clear. We repeat these in summary for generally skew bases:

The reference base \( E_a \) is the collection of the Cartesian base vectors \( e x 1, e x 2, \ldots, e x n \) in \( n \)-dimensional Euclidean or abstract orthonormal space according to eqn.(9) (Part I). \( E_a \) is the collection of the base vectors \( e x 1, e x 2, \ldots, e x n \) which may be generally skew and nonunit, also in \( n \)-dimensional Euclidean or abstract orthonormal space according to eqn.(15.1) and Fig.8(a). \( E_a \) is the collection of reciprocal base vectors \( e x 1, e x 2, \ldots, e x n \) according to eqn.(15.10) and Fig.8(b). The columns of the matrix \( E_a b \) are the scalar components of the base vectors of \( E_a \) in terms of Cartesian base \( E_a \). Similarly, there may be a second generally skew base \( E_b \). The matrix \( E_a b \) may be obtained by the physical product \( E_a \cdot E_b \) or by the matrix product \( E_a \cdot E_a \). The columns of the matrix \( E_a b \) are the scalar components of the base vectors of \( E_b \) in terms of base \( E_a \). The column \( v a \) is the vector array of the scalar components of vector \( v \) in base \( E_a \). The row \( v a \) is the vector array of the scalar components of vector \( v \) in inverse base \( E_a \). In all
cases, the scalar components are not generally equal to the magnitude of the vector components. Base $\overrightarrow{E}_{\tau a}$ is physically the same as the inverse base $\overrightarrow{E}^a$, but written in transposed form. As such, it represents a mathematically different base than $\overrightarrow{E}^a$, and the symbol $\tau a$ is used like any other skew base symbol $q$. $\overrightarrow{v}^a_\tau$ is the same vector array as the row $\overrightarrow{v}^a_a$ but written in transposed form, i.e., as column, and $\overrightarrow{v}^a_\tau$ is the same vector array as the column $\overrightarrow{v}^a_a$ but written in transposed form, i.e., as row. Column vectors are contravariant vectors and row vectors are covariant vectors in the indicated base. Also the columns of the matrix $\overrightarrow{E}^a_{\tau b}$ are the scalar components of base vectors of $\overrightarrow{E}^a_{\tau b}$ in terms of base $\overrightarrow{E}^a_a$. The rows of the matrix $\overrightarrow{E}^a_{\tau b}$ are the scalar components of the base vectors of inverse base $\overrightarrow{E}^a_q$ in terms of inverse base $\overrightarrow{E}^q_a$, but this interpretation is rather distracting from the posed form, in contrast to Cartesian bases. It is recommended to use only the column interpretation as long as possible as the best way to get used to the meaning of the symbols. It is also useful to connect the transformation symbols strictly to the matrix rule to use the superscript as first index and the subscript as second index for the position in the matrix.

The symbolic system as presented here is rather complicated. But it is no more complicated than the entities it represents, which may be easily overlooked in a simpler notation. Man uses several thousand symbols to distinguish between all he wants to communicate. The new symbols are words in the communication from writer to reader, often from writer to the same writer.

15.6. The Transpose Symbol

Now that we have chosen to use the symbol $\tau$ in conjunction with a base name, we don’t have the choice of writing $\overrightarrow{E}^\tau_q$ any more because the name $E^\tau$ between the base symbols would mean that there is a different tensor $\overrightarrow{E}^\tau_q$ = $[\overrightarrow{E}^\tau_q]^\top$ which in our notation is not so.

At this stage, it has to be made clear that the meaning of the transpose symbol $\tau$ next to a base or vector or tensor name in Part I (pp. 86, 87) was changed, and has a different meaning from the last equations in Part I (p. 87) onwards.

The transpose symbol $\tau$ is an addition to tensor algebra to convert it to matrix algebra. The transpose operation $\tau$ converts the tensor into a mathematically different tensor generally, and the symbol $\tau$ becomes part of its name, e.g., $[\overrightarrow{v}^\tau_\tau]^\top = \overrightarrow{v}^\top_\tau$, $[\overrightarrow{K}^\tau]^\top = \overrightarrow{K}^\top_\tau$. On the vector, the $\tau$ is redundant, and therefore, left out, $\overrightarrow{v}^\top = \overrightarrow{v}$. On the tensor, it is not redundant if the tensor is unsymmetric. The transformation rules now require that the tensor name including the $\tau$ remains unchanged so that the tensor matrix and its transpose are distinguished. For example, even in orthonormal base $\overrightarrow{K}^\tau_q$ is different from $\overrightarrow{K}^{\tau_q}$ if the tensor is not symmetric. In tensor notation, no such distinction is possible, because index symbols can be changed arbitrarily; $K_{ij}$ and $K_{ji}$ stand for the same tensor array, being the array for the same physical tensor. Only when both tensor symbols appear in the same tensor expression, e.g., $K_{ij} + K_{ji}$, then the one is the transpose of the other. In Matrix Tensor Notation, the matrix algebra is carried through right to the same physical tensor, indicating the posed or transposed form, once any one form is defined.

The particular reason for the new meaning of $\tau$ is that we don’t always deal with a total transpose of $\overrightarrow{E}^\tau_q$ to $\overrightarrow{E}^{\tau_q}$, but also do a transpose of each base separately, as in $\overrightarrow{E}^{\tau_q}_q$, which would not be possible with the old meaning of $\tau$. The separation of $\tau$ into each base separately allows the required distinction between all possibilities.

15.7. Tensor Matrix in Skew Base

Applying the vector transformations eqns.(15.3) and (15.9) to the definition of eqn.(26), Part I, of a tensor (of second order), a tensor $\overrightarrow{K}$ can be written in skew base as
\( \mathbf{K}_q^q = \mathbf{E}_s^q \cdot \mathbf{K}_s^s \cdot \mathbf{E}_q^q \), \quad (15.55-a)

\( \mathbf{K}_q^q = \mathbf{E}_s^q \cdot \mathbf{K}_s^s \cdot \mathbf{E}_q^q \), \quad (15.55-b)

\( \mathbf{K}_q^q = \mathbf{E}_s^q \cdot \mathbf{K}_s^s \cdot \mathbf{E}_q^q \), \quad (15.55-c)

\( \mathbf{K}_q^q = \mathbf{E}_s^q \cdot \mathbf{K}_s^s \cdot \mathbf{E}_q^q \), \quad (15.55-d)

but the numerical values may be found rather by transformation from the base in which the tensor matrix of \( \mathbf{K} \) was defined, e.g., from Cartesian base

\( \mathbf{K}_q^q = \mathbf{E}_s^q \cdot \mathbf{K}_s^s \cdot \mathbf{E}_q^q \), \quad (15.56-a)

\( \mathbf{K}_q^q = \mathbf{E}_s^q \cdot \mathbf{K}_s^s \cdot \mathbf{E}_q^q \), \quad (15.56-b)

\( \mathbf{K}_q^q = \mathbf{E}_s^q \cdot \mathbf{K}_s^s \cdot \mathbf{E}_q^q \), \quad (15.56-c)

\( \mathbf{K}_q^q = \mathbf{E}_s^q \cdot \mathbf{K}_s^s \cdot \mathbf{E}_q^q \), \quad (15.56-d)

and \( \mathbf{K} \) in mixed bases becomes

\( \mathbf{K}_b^a = \mathbf{E}_s^a \cdot \mathbf{K}_s^b \cdot \mathbf{E}_b^a \), \quad (15.56-e)

\( \mathbf{K}_b^a = \mathbf{E}_s^a \cdot \mathbf{K}_s^b = \mathbf{K}_a^s \cdot \mathbf{E}_b^a \), \quad (15.56-f)

\( \mathbf{K}_b^a = \mathbf{K}_s^b \cdot \mathbf{E}_b^a = \mathbf{E}_s^b \cdot \mathbf{K}_b^a \), \quad (15.56-g)

and numerous other combinations that satisfy the algebra of superscripts and subscripts.

However if \( \mathbf{K} \) is not symmetric, then

\( \mathbf{K}^\tau q = \mathbf{K}^q , \quad (15.57) \)

but rather

\( \begin{align*}
\mathbf{K}_q^q & = \left[ \mathbf{E}_s^q \cdot \mathbf{K}_s^s \cdot \mathbf{E}_q^q \right]^\tau \\
& = \mathbf{E}_s^q \cdot \mathbf{K}_s^s \cdot \mathbf{E}_q^q \\
& = \mathbf{K}_q^q
\end{align*} \)

(15.58)

The rules above show that the transpose symbol on the base and the transpose symbol on the tensor have different meanings in skew base. The base transpose comes about from switching the base. The tensor transpose comes about from transposing the tensor, which actually creates a new tensor. We can never get a transpose on the tensor by base transformations only; there must be a tensor transpose operation involved. To clarify this point, consider the new tensor \( \mathbf{K}_q^\tau \) that may have to be created from a skew tensor \( \mathbf{K}_q^q \), e.g., when the sum of a tensor and its transpose is used to define a new symmetric tensor. The tensor in skew base is denoted by \( \mathbf{K}_q^q \) and the transpose tensor by \( \mathbf{K}_q^\tau \). Using the fact that the transpose is correctly obtained in Cartesian base, the transpose tensor in skew base is

\( \begin{align*}
\mathbf{K}_q^\tau q & = \mathbf{E}_s^q \cdot \mathbf{K}_s^s \cdot \mathbf{E}_q^q \\
& = \mathbf{E}_s^q \cdot \left[ \mathbf{K}_q^q \right]^\tau \cdot \mathbf{E}_q^q \\
& = \mathbf{E}_s^q \cdot \mathbf{K}_s^s \cdot \left[ \mathbf{K}_q^q \right]^\tau \cdot \mathbf{E}_q^q \\
& = \mathbf{E}_s^q \cdot \mathbf{K}_q^\tau q \cdot \mathbf{E}_q^q \\
& = \mathbf{K}_q^\tau q
\end{align*} \)

(15.59)
which cannot be done without the metric. Note that the same result follows from eqn.(15.58) by the transformation

$$E^q_{\tau q} \cdot K_{\tau q}^{-\tau q} \cdot E^q_{\tau q} = K_{\tau q}^{-\tau q}. \quad (15.60)$$

It is clear then that, to obtain the transpose tensor, we cannot just transpose in skew base

$$K_{\tau q}^{-\tau q} \neq [K^{-\tau q}]^T. \quad (15.61)$$

15.8. The Vector Cross Product

The vector cross product is written in space notation in Section 7 (Part I),

$$\vec{\omega} \times \vec{r} = \vec{v} \quad (15.62)$$

using the angular velocity as example. This form is only possible in 3-dimensional space, while in \((n \neq 3)\)-dimensional space the tensor form must be used

$$\vec{W} \cdot \vec{r} = \vec{v}. \quad (15.63)$$

In skew base, the rules of the Cartesian cross product are not valid any more, and we will have to decide what symbolic notation will be useful for the corresponding rules in skew base.

In Cartesian base, the 3 elements of the vector array \(\vec{w}^s\) are obtained from the antisymmetric tensor \(W^q_s\) by the simple operation of extracting the vector components from the 3 different tensor components. It seems that a similar operation in skew base may be possible if only one physical base is used because a vector is expressed in only one base, and therefore, we start with the eqn.(15.63) transformed to skew base \(q\) in the form

$$\vec{W}^q_{\tau q} \cdot \vec{r}^q = \vec{v}^q. \quad (15.64)$$

It turns out that the tensor matrix \(W^q_{\tau q}\) is also antisymmetric, because symmetry and antisymmetry of a tensor are preserved under this form of apparent mixed bases which is shown later in Section 18. If the 3 different tensor elements are extracted and put into a form of a vector \(\vec{w}\) as in Cartesian base, then the vector \(\vec{w}\) is not the transformed vector

$$\vec{W}^q_{\tau q} \cdot \vec{r}^q = \vec{v}^q. \quad (15.65)$$

Eqn.(15.65) can be written in the form

$$\vec{w}^q = \left[\vec{E}^s_{\tau q}\right]^{-1} \cdot \vec{w}^s = \left[\text{Adj}(\vec{E}^s_{\tau q})/\text{det}(\vec{E}^s_{\tau q})\right] \cdot \vec{w}^s. \quad (15.66)$$

By multiplying out the RHS of eqn.(15.65), and also multiplying out the terms of the transformation equation

$$\vec{W}^q_{\tau q} = \vec{E}^q_{\tau q} \cdot \vec{W}^s_{\tau q} \cdot \vec{E}^s_{\tau q} \quad (15.67)$$

and comparing terms, we find that the extracted vector

$$\vec{\alpha} = \text{det}(\vec{E}^s_{\tau q}) \cdot \vec{w}^q = \sqrt{g} \cdot \vec{w}^q, \quad (15.68)$$

where \(g\) is the customary symbol for the determinant of the metric \(E^q_{\tau q}\). If we define the cross multiplication symbol \(\times\) to be the same operation on elements as in Cartesian base, Section 7
(Part I), with the unit direction vectors $\mathbf{e}_q, \mathbf{e}_q^2, \mathbf{e}_q^3$, then the operation of eqn.(15.64) gives the same result as the operation $\mathbf{r} \times \mathbf{r}^q$.

Therefore, we can write the vector cross product in skew base as

$$\sqrt{g} \mathbf{r}^q \times \mathbf{r}^q = \mathbf{v}^q,$$

in which the form of Cartesian base is embedded. This formula corresponds to the tensor notation formula $\epsilon_{ijk}\omega^jr^j = v_k$, where the non-zero elements of $\epsilon_{ijk}$ are $\pm \sqrt{g}$. Eqn.(15.69) is the 3-dimensional equivalent of the $n$-dimensional eqn.(15.64).

If now the cross product of any other 3-dimensional vectors $\mathbf{a} \times \mathbf{b}$ is required in skew base, then without forming the tensor as for eqn.(15.62) by the operation described in Section 7 (Part I), the cross product can be computed directly as in eqn.(15.69). If any of the vectors are in bases posed other than in in eqn.(15.69), then a transformation with the metric must be done on each vector separately to the form of eqn.(15.69).

By the same method, the relation between transposed vectors are found to be

$$\mathbf{W}^q_{\tau_q} \cdot \mathbf{r}^q = \mathbf{v}^q,$$

$$\mathbf{\bar{\beta}} = \det(\mathbf{E}^q_{\tau_q})\omega^q = (1/\sqrt{g})\omega^q,$$

$$\mathbf{\bar{\beta}} \times \mathbf{r}^q = \mathbf{v}^q,$$

$$\mathbf{(1/\sqrt{g})}\omega^q \times \mathbf{r}^q = \mathbf{v}^q,$$

where $\mathbf{\bar{\beta}}$ is the vector from the extracted elements of $\mathbf{W}^q_{\tau_q}$ by the same pattern as before. Because the cross product is so exceptionally defined, we may just as well let the vectors be row vectors on whose elements the same operation is carried out, so that eqn.(15.73) can be written

$$\mathbf{(1/\sqrt{g})}\omega^q \times \mathbf{r}^q = \mathbf{v}^q.$$

15.9. Hinge Rotations

In this section, one of the customary methods to obtain the elements of the transformation matrix $\mathbf{E}^s_r$ between two Cartesian bases is carried out applying the present Matrix Tensor Notation. To obtain the elements of the transformation matrix $\mathbf{E}^s_r$, between two Cartesian bases, a rigid frame is considered coinciding with $\mathbf{E}^s$, which is rotated to the position of the new base $\mathbf{E}^r$. The rotation may or may not represent an actual mechanical rotation of a rigid body.

A rigid body rotation can be described by the well known different versions of Euler angles, less precisely called orientation angles [16], but they are all merely different assignments of axes to two orthogonal hinges by which the rotating body is imagined to be connected to fixed space, and therefore called “hinge angles” [28]. In this section we consider the motion of a rigid body in real hinges, as well as the rotation of the two intermediate hinges, which may be gimbals, or a sequence of two crosses of a Hooke or Cardan joint.

The original and all intermediate positions of the rotated axes are shown in Fig.9, starting from a fixed Cartesian base $\mathbf{E}^s = \mathbf{E}^r(0)$ as shown in Fig.9(a).

The first rotation about the fixed hinge axis, which is $\mathbf{e}_{s1}$, by the angle $\theta_1$ rotates the body to the position $\mathbf{E}^b_b$, shown in Fig.9(b). This rotation is described by the transformation equation

$$\mathbf{E}^b_b = \mathbf{E}^s_s \cdot \mathbf{E}^s_b,$$

where the transformation matrix

$$\mathbf{E}^s_b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_1 & -\sin \theta_1 \\ 0 & \sin \theta_1 & \cos \theta_1 \end{bmatrix},$$

(15.76)
which with some practice is usually read directly from the figure [29-31], but can be facilitated by drawing the orthogonal projections [15]. The next rotation of \( \theta_2 \) about the axis \( e_{b2} \), shown in Fig.9(c), is the rotation of the second hinge in the first and rotates the body from \( \vec{E}_b \) to \( \vec{E}_c \).

The transformation and transformation matrices are, respectively,

\[
\vec{E}_c = \vec{E}_b \cdot \vec{E}_c^b \tag{15.77}
\]

where

\[
\vec{E}_c^b = \begin{bmatrix}
\cos \theta_2 & 0 & \sin \theta_2 \\
0 & 1 & 0 \\
-\sin \theta_2 & 0 & \cos \theta_2
\end{bmatrix} \tag{15.78}
\]

The last rotation of \( \theta_3 \) about the axis \( e_{c3} \), shown in Fig.9(d), is the rotation of the body in the second hinge and rotates the body from \( \vec{E}_c \) to the final position \( \vec{E}_r \). The transformation and transformation matrices are, respectively,

\[
\vec{E}_r = \vec{E}_c \cdot \vec{E}_r^c \tag{15.79}
\]

where

\[
\vec{E}_r^c = \begin{bmatrix}
\cos \theta_3 & -\sin \theta_3 & 0 \\
\sin \theta_3 & \cos \theta_3 & 0 \\
0 & 0 & 1
\end{bmatrix} \tag{15.80}
\]

The final rotated base \( \vec{E}_r \) is shown in Fig.9(e).

![Figure 9. Hinge rotations.](image)

Note that in this notation some intermediate axes have two names. We consider \( e_{s1} \) as space fixed axis, \( e_{b1} \) and \( e_{b2} \) the orthogonal axes of the first hinge, \( e_{c2} \) and \( e_{c3} \) the orthogonal axes of the second hinge and \( e_{r3} \) the body axis. While an axis cannot rotate mathematically in its own axis, physically the different axes in the same directions rotate with different angular velocities. We prefer this mathematical redundancy to avoid having a direction vector with the same name in two different bases, which would destroy the symbolic algebra of base symbols. As it turns
out, it also points to a difference in physical meaning of the same mathematical direction. Each of the 4 mechanical parts has a different base according to

\[ \text{fixed base } = \vec{E}_s, \text{ first hinge } = \vec{E}_b, \text{ second hinge } = \vec{E}_c, \text{ rotating body } = \vec{E}_r. \]

The well-known results of the multiplications of transformation matrices are, now in our notation,

\[ \vec{E}_c^s = \vec{E}_b^s \cdot \vec{E}_c^b = \begin{bmatrix} c2 & 0 & s2 \\ s1s2 & c1 & -s1c2 \\ -c1s2 & s1 & c1c2 \end{bmatrix}, \quad (15.81) \]

\[ \vec{E}_r^s = \vec{E}_c^s \cdot \vec{E}_r^c = \begin{bmatrix} c2c3 & -c2s3 & s2 \\ s1s2c3 + c1s3 & -s1s2s3 + c1c3 & -s1c2 \\ -c1s2c3 + s1s3 & c1s2s3 + s1c3 & c1c2 \end{bmatrix}, \quad (15.82) \]

where the customary abbreviations \( c1 = \cos \theta_1, \ldots, s3 = \sin \theta_3 \) have been used [16], which is also suitable for computer programming in a subroutine. In a main program, the slightly more descriptive abbreviations \( \cos 1 = \cos \theta_1, \ldots, \sin 3 = \sin \theta_3 \) may be better. Reference [29] uses the abbreviations \( \cos 1 = \cos \theta_1, \ldots, \sin 3 = \sin \theta_3 \) which cannot be written in present alphanumerical computer language. The eqn.(15.82) is obviously the last step of the complete transformation equation

\[ \vec{E}_r^s = \vec{E}_b^s \cdot \vec{E}_c^b \cdot \vec{E}_r^c. \quad (15.83) \]

We also note that the detailed notation allows the intermediate transformation matrix \( \vec{E}_r^b = \vec{E}_c^b \cdot \vec{E}_r^c \), or any inverses if they may be required during detailed engineering design of the hinges. In a numerical application the matrix \( \vec{E}_r^b \) does not have to be programmed explicitly from eqn.(15.82), but rather from all the direct intermediate transformations \( \vec{E}_b^s, \vec{E}_c^b, \vec{E}_r^c \), from which all desired combinations of products can be done by numerical matrix multiplication.

To prove that the transformation matrix is orthogonal is rather complicated if the final form of eqn.(15.82) is used, but the individual transformation matrices can easily be shown to be orthogonal by the multiplication of the analytic forms

\[ \left[ \vec{E}_b^s \right]^T \cdot \vec{E}_b^s = I, \quad \left[ \vec{E}_c^b \right]^T \cdot \vec{E}_c^b = I, \quad \left[ \vec{E}_r^c \right]^T \cdot \vec{E}_r^c = I, \]

and therefore, we may obtain the inverses by transposing, which in our notation becomes

\[ \vec{E}_b^b = \left[ \vec{E}_b^s \right]^T, \quad \vec{E}_c^c = \left[ \vec{E}_c^b \right]^T, \quad \vec{E}_r^r = \left[ \vec{E}_r^c \right]^T. \quad (15.84) \]

Then from the product of eqn.(15.83) it follows that

\[ \left[ \vec{E}_r^s \right]^T \cdot \vec{E}_r^s = \left[ \vec{E}_r^c \right]^T \cdot \vec{E}_r^c = \left[ \vec{E}_b^s \right]^T \cdot \vec{E}_b^s \cdot \vec{E}_c^b \cdot \vec{E}_r^c \]

\[ = \vec{E}_r^c \cdot \vec{E}_b^c \cdot \vec{E}_b^s \cdot \vec{E}_c^b \cdot \vec{E}_r^c \]

\[ = \vec{E}_r^r \cdot \vec{E}_c^c \cdot \vec{E}_r^c \]

\[ = \vec{I}. \]

The physical hinge axes which were originally in the position \( \vec{e}_s1, \vec{e}_s2 \) for the first hinge and \( \vec{e}_s2, \vec{e}_s3 \) for the second hinge, in Fig.9(a), are in the final position \( \vec{e}_r1, \vec{e}_r2 \) for the first hinge and
$e_{b2}, e_{r3}$ for the second hinge. We can define a new skew base consisting of these rotated direction vectors $e_{s1}, e_{b2}, e_{r3}$ which can be picked from the respective columns of $E_s^s, E_b^s, E_r^s$. Formally in Matrix Tensor Notation

$$E_q \equiv \begin{bmatrix} e_{q1} & e_{q2} & e_{q3} \end{bmatrix} = \begin{bmatrix} e_{s1} & e_{b2} & e_{r3} \end{bmatrix} = E_s^s \cdot E_q^s,$$

so that we can pick out the columns from the individual transformation matrices to compile

$$E_s^s = \begin{bmatrix} e_{s1} & e_{b2} & e_{r3} \end{bmatrix}.$$

where of course $e_{s1}$ is well defined. This skew base $E_q^s$ is shown in Fig.9(f).

The rigid right angles $\delta_1$ and $\delta_2$ of the hinges as shown in Fig.9(a) remain orthogonal as shown in Fig.9(f), only the angle between axes 1 and 3, $\alpha$ in $E_s^s$, has changed to $\beta$ in $E_q^s$. The skewness of the rotated hinge axes is given by the metric of the skew base

$$E_q^{\tau q} = \begin{bmatrix} 1 & 0 & c\beta \\ 0 & 1 & 0 \\ c\beta & 0 & 1 \end{bmatrix}$$

with $\beta$ the skew angle, which means that the rank of skewness is 1 out of 3.

The angular velocity $\omega^s$ can be found by differentiating the expression for $E_q^s$ in eqn.(15.82),

$$\dot{E}_r^s = \frac{d}{dt} E_r^s,$$

but this is quite cumbersome, especially because we must still multiply by $E_s^s$ to obtain either $\dot{E}_r^s$ or $E_r^r$. It is easier to use the customary method to realize that $\dot{\omega}$ is the sum of the individual rotations $\dot{\delta}_1, \dot{\delta}_2, \dot{\delta}_3$ about their current rotated axes $e_{s1}, e_{b2}, e_{r3}$, respectively, which are the direction vectors of the skew base $E_q^s$. Therefore the angular velocity can be expressed by

$$\dot{\omega}^s = \dot{e}_{s1} \dot{\delta}_1 + \dot{e}_{b2} \dot{\delta}_2 + \dot{e}_{r3} \dot{\delta}_3 \equiv \dot{E}_q^s \cdot \dot{\omega}_q^s,$$

where we have denoted

$$\dot{\omega}_q^s = \begin{bmatrix} \dot{\delta}_1 \\ \dot{\delta}_2 \\ \dot{\delta}_3 \end{bmatrix}.$$

The question is now whether we are allowed to use the notation in eqn.(15.89) above. On the one hand, it must be checked that eqn.(15.88) is a proper transformation equation. This is certainly true because the $\dot{\delta}_i$ are the skew components in the base $E_q^s$ added by the parallelogram law, which is the definition of a transformation equation. On the other hand, we may consider the derivative of $E_q^s$. It can be seen that such a tensor cannot represent a rigid body rotational velocity because $E_q^s$ is deforming all the time. In fact, the change of $E_q^s$ does not contain the component $\dot{\delta}_3$ at all. The only other problem may be to find out whether $\dot{\omega}^s$ from eqn.(15.88) is the same as derived from the derivative of $E_q^s$ according to equations in Section 10 (Part I). By careful observation it is clear that all $\dot{\delta}_i$ are relative angular velocities, which are appropriately denoted according to Section 10.1 by

$$\dot{\delta}_1 = \dot{\omega}_B = \dot{\omega}_B = \dot{\omega}_B^s,$$

$$\dot{\delta}_2 = \dot{\omega}_C^B = \dot{\omega}_C^B = \dot{\omega}_C^B,$$

$$\dot{\delta}_3 = \dot{\omega}_R^C = \dot{\omega}_R^C = \dot{\omega}_R^C.$$
Here the subscripts \(B, C, R\) refer not to points as in Section 10.1 (obviously a point cannot have a rotation), but to the rigid bases \(b, c, r\), where the "relative subscripts" have been made capital letters so that they are not confused with base component symbols. The subscripts of relative angular velocity vectors follow the rules of relative vectors of Section 10.1 (p. 81, Part I)

\[
\omega_{C/B} = \omega_C - \omega_B. \tag{15.91}
\]

Such a consistent method of subscripts of angular velocities is often used \([15,16,22]\), but sometimes incompletely applied, leading to difficulty in remembering the proper meaning of symbols if they cannot be recognized from their notation \([16]\).

From eqns.\(15.90\) follows

\[
\omega^q = \omega^q_R = \omega^q_B + \omega^q_C/B. \tag{15.92}
\]

Using the tensor relations corresponding to the vector eqn.\(15.92\),

\[
\dot{\mathbf{E}}_r^s = \dot{\mathbf{W}}_r^s = \dot{\mathbf{W}}_r^B + \dot{\mathbf{W}}_{r/C/B} + \dot{\mathbf{W}}_{r/R/C},
\]

which compared to the derivative of eqn.\(15.83\)

\[
\dot{\mathbf{E}}_r^s = \dot{\mathbf{E}}_r^b \cdot \dot{\mathbf{E}}_r^c + \dot{\mathbf{E}}_r^b \cdot \dot{\mathbf{W}}_{c/C/B} \cdot \dot{\mathbf{E}}_r^c + \dot{\mathbf{E}}_r^b \cdot \dot{\mathbf{E}}_r^c \cdot \dot{\mathbf{W}}_{r/R/C},
\]

results in a more generalized version of eqn.\(15.93\) (Part I),

\[
\dot{\mathbf{E}}_r^b = \dot{\mathbf{W}}_b^s, \tag{15.94-a}
\]

\[
\dot{\mathbf{E}}_r^c = \dot{\mathbf{W}}_{c/C/B}, \tag{15.94-b}
\]

\[
\dot{\mathbf{E}}_r^r = \dot{\mathbf{W}}_{r/R/C}, \tag{15.94-c}
\]

where the relative subscripted angular velocity tensors correspond to the relative angular velocities

\[
\dot{\mathbf{W}}_{C/B} = \dot{\mathbf{W}}_C - \dot{\mathbf{W}}_B = \mathbf{\omega}_{C/B} = \mathbf{\omega}_{C} - \mathbf{\omega}_{B}. \tag{15.95}
\]

The inverse is also true if we remember to apply the negative sign according to Section 10 (p. 81, Part I)

\[
\dot{\mathbf{E}}_b^s = -\dot{\mathbf{W}}_s^B, \tag{15.96-a}
\]

\[
\dot{\mathbf{E}}_c^s = -\dot{\mathbf{W}}_c^B, \tag{15.96-b}
\]

\[
\dot{\mathbf{E}}_r^s = -\dot{\mathbf{W}}_r^B, \tag{15.96-c}
\]

and using the rules for relative velocities

\[
-\dot{\mathbf{W}}_s^B = -\dot{\mathbf{W}}_{s/S/B} = \dot{\mathbf{W}}_{s/S/B}, \tag{15.97-d}
\]

\[
-\dot{\mathbf{W}}_{b/C/B} = \dot{\mathbf{W}}_{b/C/B}, \tag{15.97-e}
\]

\[
-\dot{\mathbf{W}}_{c/R/C} = \dot{\mathbf{W}}_{c/R/C}. \tag{15.97-f}
\]

To come back to the validity of eqn.\(15.88\), we may transform eqn.\(15.93\) to

\[
\dot{\mathbf{W}}_s^s = \left[\dot{\mathbf{W}}_{s/B} \cdot \dot{\mathbf{E}}_s^b \cdot \dot{\mathbf{E}}_c^c \cdot \dot{\mathbf{E}}_r^r + \dot{\mathbf{E}}_s^b \cdot \dot{\mathbf{W}}_{b/C/B} \cdot \dot{\mathbf{E}}_c^c \cdot \dot{\mathbf{E}}_r^r + \dot{\mathbf{E}}_s^b \cdot \dot{\mathbf{E}}_c^c \cdot \dot{\mathbf{W}}_{r/R/C} \cdot \dot{\mathbf{E}}_r^r\right] \cdot \dot{\mathbf{E}}_s^r
\]

\[
= \dot{\mathbf{W}}_{s/B} \cdot \dot{\mathbf{E}}_s^b \cdot \dot{\mathbf{E}}_c^c \cdot \dot{\mathbf{E}}_r^r + \dot{\mathbf{E}}_s^b \cdot \dot{\mathbf{W}}_{b/C/B} \cdot \dot{\mathbf{E}}_c^c \cdot \dot{\mathbf{E}}_r^r + \dot{\mathbf{E}}_s^b \cdot \dot{\mathbf{E}}_c^c \cdot \dot{\mathbf{W}}_{r/R/C} \cdot \dot{\mathbf{E}}_r^r,
\]

which is eqn.\(15.88\) expressed in tensors.
15.10. Rotation Tensors

The same generalization to relative angular velocity tensors applies to the corresponding rotation tensors, which must be distinguished by subscripts so that

\[ \overrightarrow{R}_{C/B} \equiv \text{rotation from base } \overrightarrow{E}_b \text{ to } \overrightarrow{E}_c \]  

(15.98)

(Perhaps remembered better as “to \( \overrightarrow{E}_c \) from \( \overrightarrow{E}_b \)”) according to the equations

\[ \overrightarrow{E}_b = \overrightarrow{R}_{C/B} \cdot \overrightarrow{E}_b, \]  

\[ \overrightarrow{E}_c = \overrightarrow{R}_{C/B} \cdot \overrightarrow{E}_b, \]  

\[ \overrightarrow{E}_r = \overrightarrow{R}_{R/C} \cdot \overrightarrow{E}_c, \]  

(15.99-a)

(15.99-b)

(15.99-c)

which is the generalization of eqn.(44) (Part I). The forward and backward rotation tensors are related by

\[ \overrightarrow{R}_{C/B} = \left[ \overrightarrow{R}_{B/C} \right]^{-1} = \left[ \overrightarrow{R}_{B/C} \right]^T. \]  

(15.100)

If we use the term “relative” for rotations, obviously in a different mathematical context than for vectors, then corresponding to the sequence of relative angular velocity vectors, \( \overrightarrow{R}_{C/B} \) is the rotation of base \( c \) relative to base \( b \).

Other notations for the relative rotations can be found in [15] and [16].

Applying now the sequence of rotations tensor-rule

\[ \overrightarrow{R}_{sC} = \overrightarrow{R}_{sC/B} \cdot \overrightarrow{R}_{sB} \]

to eqn.(15.99)

\[ \overrightarrow{E}_c^s = \overrightarrow{R}_{sC/B} \cdot \overrightarrow{E}_b^s \]

\[ = \overrightarrow{E}_b^s \cdot \overrightarrow{R}_{sC/B} \cdot \overrightarrow{E}_b \]

\[ = \overrightarrow{E}_b^s \cdot \overrightarrow{R}_{sC/B} \cdot \overrightarrow{E}_b \]

and using the transformation

\[ \overrightarrow{R}_{cC/B} = \overrightarrow{E}_b \cdot \overrightarrow{R}_{cC/B} \cdot \overrightarrow{E}_c \]

\[ = \overrightarrow{E}_b \cdot \overrightarrow{E}_c \cdot \overrightarrow{E}_b = \overrightarrow{E}_c, \]

therefore, applied to all 3 rotations

\[ \overrightarrow{R}_{sB} = \overrightarrow{R}_{bB} = \overrightarrow{E}_b^s, \]  

(15.101-a)

\[ \overrightarrow{R}_{bC/B} = \overrightarrow{R}_{cC/B} = \overrightarrow{E}_c^b, \]  

(15.101-b)

\[ \overrightarrow{R}_{cR/C} = \overrightarrow{R}_{R/C} = \overrightarrow{E}_r^c, \]  

(15.101-c)

which is the generalization of eqn.(45) (Part I). The total rotation matrix is

\[ \overrightarrow{R}_s = \overrightarrow{R}_r = \overrightarrow{R}_{sR/C} \cdot \overrightarrow{R}_{sC/B} \cdot \overrightarrow{R}_{sB} \]

\[ = \overrightarrow{E}_r^s, \]

(15.102)

which is the same product as eqn.(15.83) but expressed in terms of the relative rotation matrices.
Taking derivatives, we obtain the eqns. (15.94) in terms of rotation matrices

\[
\dot{\mathbf{R}}_{sB} = \mathbf{W}_{sB} \cdot \mathbf{R}_{sB} = \mathbf{R}_{bB} \cdot \mathbf{W}_{bB}',
\]

(15.103-a)

\[
\dot{\mathbf{R}}_{cC/B} = \mathbf{W}_{cC/B} \cdot \mathbf{R}_{cC/B} = \mathbf{R}_{cC/B} \cdot \mathbf{W}_{cC/B}',
\]

(15.103-b)

\[
\dot{\mathbf{R}}_{rR/C} = \mathbf{W}_{rR/C} \cdot \mathbf{R}_{rR/C} = \mathbf{R}_{rR/C} \cdot \mathbf{W}_{rR/C}',
\]

(15.103-c)

which are the equations corresponding to the equations of derivatives in Section 10 (p. 81, Part I).

It is just as well to point out the inequality for relative motion

\[
\dot{\mathbf{R}}_{cC/B} \neq \mathbf{W}_{cC/B} \cdot \dot{\mathbf{R}}_{cC/B}.
\]

The rotated bases and angular velocity tensor matrices in this section are all variables with the parameters \(\theta_i(t)\), which are the three basic variables of the motion, representing real physically measurable quantities of the hinge system. If the rotating motion of an unattached rigid body is studied, one may choose to use imaginary hinges as aid to obtain three basic parameters, but that is then a choice not corresponding to the physical situation.

The sequence of the real hinges is mechanically fixed, but the rotated position of the body does not depend on the sequence of rotations that have occurred. In fact if the derivation is been made with the sequence of rotations \(\theta_3, \theta_2, \theta_1\) about the corresponding positions of the hinge axes, the transformation matrix \(\mathbf{E}_r\) will have the same analytic form, and the skew base \(\mathbf{E}_q\) will be the same. But other auxiliary intermediate bases will occur. Contrary to Euler angles, the sequence of hinge rotations does not affect the result [28].

### 15.11. Delta Hinges

In this section, we develop the rotation matrix of a rigid body which is connected to a fixed base by two hinges with skew axes, in Matrix Tensor Notation. A typical application is the use of \(\delta\)-hinges by which helicopter blades are attached to the rotor to produce favourable motion characteristics [32].

The two hinges in their original position before the rotational displacement are shown in Fig.10(a), where direction vector \(\mathbf{e}_{h1}\) is the axis of the first hinge in the fixed base, \(\mathbf{e}_{h2}\) is the axis between the two hinges, and \(\mathbf{e}_{h3}\) is the body axis on the second hinge. The first hinge consists of the axes \(\mathbf{e}_{h1}\) and \(\mathbf{e}_{h2}\), with the rigid angle \(\delta_1\) between the axes. The second hinge consists of the axes \(\mathbf{e}_{h2}\) and \(\mathbf{e}_{h3}\), with the rigid angle \(\delta_2\) between the axes. The rigid body is attached to the 3rd hinge axis. The hinge direction vectors are of unit magnitude. The original position between the outer hinge axes is given by the angle \(\alpha\). In contrast to the hinges in Section 15.9, we assume that none of the angles \(\delta_1, \delta_2, \alpha\), is a right angle.

\[\text{Figure 10. Skew hinge rotations.}\]
A fixed Cartesian base $\vec{E}_s$ is used as reference. Let the reference original position of the hinge and body axes be given by

$$\vec{E}_h = \begin{bmatrix} e_{h1} \ e_{h2} \ e_{h3} \end{bmatrix} = \vec{E}_s \cdot \vec{E}_h^s.$$  \hspace{1cm} (15.104)

We are not going to use the quantity $\vec{E}_h$ as a base, and therefore, $\vec{E}_h^s$ has the meaning of hinge axis vectors measured in Cartesian base rather than a transformation matrix. The values of $\vec{E}_h^s$ are computed according to the way the design is given, perhaps by direction cosines or by Euler angles on the design drawing. The original position of the rigid body is given by the fixed Cartesian base $\vec{E}_s$. Three hinge rotations $\theta_1, \theta_2, \theta_3$, about the three hinge axes, respectively, rotate the rigid body to the new position, described by a rotated base $\vec{E}_r$. The purpose of this analysis is to find the rotation tensor $\vec{R}(\theta_1, \theta_2, \theta_3)$, whose meaning is given by the equation

$$\vec{E}_r = \vec{R} \cdot \vec{E}_s$$  \hspace{1cm} (15.105)

or in base $s$,

$$\vec{E}_r^s = \vec{R}_s^s$$  \hspace{1cm} (15.106)

by means of Matrix Tensor Notation.

In this problem, there are no natural Cartesian bases that can be used for the intermediate rotations, and therefore, rotation tensors seem to be a better approach to this problem analysis than transformation matrices. It is not useful to use the skew base $\vec{E}_h$; in fact, it may be too skew to be a 3-dimensional base. We also want to include the possibility of one skew hinge only, or even more hinges in the sequence. For the purpose of analysis the rotations have been taken in the sequence $\theta_1, \theta_2$ and $\theta_3$. For the corresponding intermediate positions of the rotated body we may still imagine the same auxiliary bases $\vec{E}_b$ and $\vec{E}_c$ as in Fig.10, hence the rotated positions are referred to $B, C$ and $R$ as in Section 15.9, but we must remember that they are not real positions of the rigid body in its path from $\vec{E}_s$ to $\vec{E}_r$, as the real rotations don’t occur in sequence. The corresponding consecutive positions of the hinge axes are shown in Fig.10(b) and (c), and are described by $\vec{E}_q$, considered as a variable such that $\vec{E}_q(S) = \vec{E}_h$, $\vec{E}_q(B)$ after rotation by $\theta_1$, and final position $\vec{E}_q = \vec{E}_q(C)$ after rotation by $\theta_2$. The final position of the hinge axes is not affected by the last rotation $\theta_3$ which rotates the rigid body to position $\vec{E}_r$.

Because the rotation is not about any axes of a Cartesian base, the rotation matrix cannot be compiled easily as in Section 15.9. Instead, the form based on the rotation angle vector

$$\vec{\theta}_1 = \theta_1 \ e_{h1}$$  \hspace{1cm} (15.107)

must be used. This vector is called Argyris vector [28], and its components in any Cartesian base the Argyris angles, which are different from Euler angles. Previously this vector was given different names [33, 34]. It has been pointed out [28], that the quantity $\vec{\theta}$ is a “vector” but not a “rotation.” Rather a “rotation” is a tensor $\vec{R}$, which is a function of the vector $\vec{\theta}$, such that “rotation” $= \vec{R}(\vec{\theta})$.

Different compact forms of the rotation matrix are available [16, 33, 35], of which the expanded form may be written [33, 35],

$$\begin{bmatrix} 1 - s_2(\theta_2^2 + \theta_3^2) & -s_1 \theta_2 + s_2 \theta_2 \theta_3 & s_1 \theta_3 + s_2 \theta_2 \theta_3 \\ s_1 \theta_2 + s_2 \theta_2 \theta_3 & 1 - s_2(\theta_2^2 + \theta_3^2) & -s_1 \theta_3 + s_2 \theta_2 \theta_3 \\ -s_1 \theta_3 + s_2 \theta_2 \theta_3 & s_1 \theta_3 + s_2 \theta_2 \theta_3 & 1 - s_2(\theta_2^2 + \theta_3^2) \end{bmatrix},$$  \hspace{1cm} (15.108)

$$s_1 \equiv \frac{\sin \theta}{\theta}, \quad s_2 \equiv \frac{1 - \cos \theta}{\theta^2}, \quad \theta \equiv |\vec{\theta}|.$$
Eqn. (15.108) serves as a definition of the function \( \overline{R}(\theta) \).

Since \( \overline{E}_s \) was chosen as reference for the description of the rotation tensor, the components of \( \overline{\theta}_1 \) must also be given in this base,

\[
\overline{\theta}_1^s = \begin{bmatrix}
\theta_1 e_{q1}^s \\
\theta_1 e_{q2}^s \\
\theta_1 e_{q3}^s
\end{bmatrix}
\]

so that the first rotation matrix is

\[
\overline{R}^s_{s_B} = \overline{R}(\overline{\theta}_1^s).
\]

With this rotation matrix the rotated position of the next hinge axis, Fig. 10(b), can be found as

\[
\overline{e}_{q2}^s(B) = \overline{R}^s_{s_B} \cdot \overline{e}_{q2}^s(B).
\]

Still in the same base \( \overline{E}_s \) the next rotation angle vector is

\[
\overline{\theta}_2^s = \begin{bmatrix}
\theta_2 e_{q1}^s(B) \\
\theta_2 e_{q2}^s(B) \\
\theta_2 e_{q3}^s(B)
\end{bmatrix}
\]

with which the next rotation matrix is computed

\[
\overline{R}^s_{s_{C/B}} = \overline{R}(\overline{\theta}_2^s).
\]

With this rotation matrix the rotated position of the last hinge axes, Fig. 10(c), can be found as

\[
\overline{e}_{q3}^s(C) = \overline{R}^s_{s_{C/B}} \cdot \overline{e}_{q3}^s(B).
\]

The last rotation angle vector is

\[
\overline{\theta}_3^s = \begin{bmatrix}
\theta_3 e_{q1}^s(C) \\
\theta_3 e_{q2}^s(C) \\
\theta_3 e_{q3}^s(C)
\end{bmatrix}
\]

and the last rotation matrix is then computed

\[
\overline{R}^s_{s_{R/C}} = \overline{R}(\overline{\theta}_3^s).
\]

The total rotation matrix is as in Section 9 (Part I)

\[
\overline{R}^s_{s} = \overline{R}_r = \overline{R}^s_{s_{R/C}} \cdot \overline{R}^s_{s_{C/B}} \cdot \overline{R}^s_{s_B} = \overline{E}^s_r.
\]

In fact, the only difference between the analysis of this section and Section 15.9 is that the analytic form of the relative rotation matrices for orthogonal hinges could be written down easily.
and explicitly, which for the skew hinge directions are more complicated, and we will not attempt here to give the analytic form of $\mathbf{R}_q^s$.

Besides this, all the equations of Section 15.9 are equally true for this section, including eqn.(15.88) expressing the angular velocity in skew base. This is true even if the transformation matrix $\mathbf{E}_q^s$ is singular. Therefore, also for skew hinges, the angular velocity need not be obtained from the derivative of a rotation matrix, and we may never need the analytic form of $\mathbf{R}_q^s$. On the other hand, if the transformation matrix $\mathbf{E}_q^s$ is not singular, we may even transform the rotation matrix to skew base, $\mathbf{R}_q^q = \mathbf{E}_q^q \cdot \mathbf{R}_s^s \cdot \mathbf{E}_q^s$.

In the final position of the hinge axes, Fig.10(c), the angle between the first axis, $\mathbf{e}_{q1}$, and the third, $\mathbf{e}_{q3}$, has changed to $\beta$, but the rigid hinge angles $\delta_1$ and $\delta_2$ are the same as initially. The deformation of the hinge base from $\mathbf{E}_h^q$ to $\mathbf{E}_q^q$ is, therefore, quite limited.

The equations of this section are valid for any skew angle $\alpha$, including 0 and 180 deg, which are the initial positions for a group of orientation angles [16].
16. DIFFERENT FORMS

It is obvious that with our notation concentrating on transformations between bases, that the equations in terms of the transformation matrices are much easier to read/interpret than the equations in terms of rotation matrices, where much of the index algebra is lost. In some cases, it will be more convenient to work with a rotation matrix rather than a base transformation matrix. In the example of Section 15.9, we could easily leave out the bases $\vec{E}_b$ and $\vec{E}_c$ and work with a single rotated base $\vec{E}_r^s$ or a single relative rotation matrix $\vec{R}_s^r$, which change their values after every step. The final hinge directions in $\vec{E}_q^s$ must be extracted from $\vec{E}_r^s$ at the correct rotated step.

We will show now that the different approaches are strictly coupled to different interpretations. The primary transformation equation of Section 6 (Part I)\(^1\) is

$$\vec{E}_r = \vec{E}_s \cdot \vec{E}_r^s .$$

(16.1)

But consider a position vector $\vec{s}$ in base $\vec{E}_s$, expressed as $\vec{s}^s$. The corresponding rotated vector in base $\vec{E}_r$ cannot be transformed with the transformation equation (16.1) of the bases, because the vector hasn’t got a subscript. If we attempted to introduce a subscript like $s_r$ rotates to $s_r$, then there is no way a similar rotation $s_r = s_r \cdot \vec{E}_r^s$ can be arranged. The only possible arrangement, $s_r = \vec{E}_s^r \cdot \vec{s}^s$, is a transformation of the same vector $\vec{s}$. For the physical rotation of the vector in baseless tensor notation we must apply the rotation tensor, and to distinguish the rotated vector from the original we have chosen the new name $\vec{r}$ as explained in Section 9 (Part I). The rotation is then, expressed in either base,

$$\vec{r} = \vec{R} \cdot \vec{s},$$

$$\vec{r} = \vec{R} \cdot \vec{s},$$

$$\vec{r} = \vec{R} \cdot \vec{s},$$

where we have derived that $\vec{R}_s^r = \vec{E}_r^s$. But the simplest way to obtain the rotated vector array is obtained by expressing the fact that the rotated vector has the same components in the rotated base as the original vector has in the original base, as

$$\vec{r} = \vec{s},$$

which by transformation from one of the rotation equations assumes the complicated from

$$\vec{r} = \vec{E}_s^r \cdot \vec{R}_s^r \cdot \vec{s}^s.$$

It turns out that we can consider basically 3 types of equations that lead to the different forms that may occur.

1. Names and Base symbols are compatible, as in

$$\vec{E}_s^a = \vec{E}_a^s \cdot \vec{E}_b^s,$$

$$\vec{s}^b = \vec{E}_a^s \cdot \vec{s}^s .$$

(16.2)

2. Names are different but base symbols are compatible, as in

$$\vec{E}_b^s = \vec{R} \cdot \vec{E}_a^s,$$

$$\vec{s}^b = \vec{R} \cdot \vec{s}^s,$$

$$\vec{r}^s = \vec{R}_s^r \cdot \vec{s}^s .$$

(16.3)

\(^1\)Not given in this form in that section.
3. Base symbols are not compatible, and therefore, names must be different (see Section 2.4 and 8), as in

\[
\vec{R}_a^r = \vec{E}_b^s \cdot \vec{E}_s^a,
\]

\[\vec{r}^r = \vec{s}\] (16.4)

The differences in appearance are connected to the following different interpretations.

In group 1, we consider bases which are different and relate their orientations to fixed space which in turn we express by the fixed Cartesian base \(\vec{E}_s^r\). The equations from group 1 are transformations in Euler coordinates of the same vectors or tensors in different bases. Consequently, we must primarily interpret eqn.\((16.1)\) as \(\vec{E}_b^s\) is the base \(b\) expressed in base \(s\), rather than an equation of rotation. All equations in that group are transformations. The base symbolic algebra of our Matrix Tensor Notation dictates the validity of equations from this group. Their form in space notation is trivial, e.g., the examples in group 1 become

\[
\vec{E} = \vec{E}_s \cdot \vec{E}_s^r,
\]

\[\vec{s} = \vec{E}_r \cdot \vec{s}\].

Equations of this group cannot be expressed in customary Tensor Notation. The rather clumsy distinction between bases (or coordinates), which is not a standard in Tensor Notation, cannot be used to distinguish between more than 2 bases.

In group 2, we consider changes of a system of fixed identity, such as a rigid body or a position vector of a moving particle, where a tensor operates on a vector or a tensor to produce another vector or tensor which must, therefore, also have another name. For position vectors and orientation of bases this is the Lagrange description inasmuch as the result is related to an original position in space. Group 2 expresses laws like rotations or deformations in geometry, or dynamic relations like force and displacement, but not transformations. All laws of mechanics are originally stated in this form. The base symbolic algebra of our Matrix Tensor Notation does not bring any simplification in group 2, they are written in the customary matrix form, for which the space notation with arrows → is merely a facility for handwriting. These are the equations that are expressed in customary Tensor Notation. The usefulness of Matrix Tensor Notation comes in when transformations of these laws occur for which the symbol algebra of group 1 is applied, then the type of equation is of the type of both groups, e.g., as in

\[
\vec{f}^r = \vec{R}_s^r \cdot \vec{w}^s.
\]

Finally in group 3, we have all the incompatible symbol equations where the symbolic algebra of our notation is of no use. They occur because of the equality between quantities that occurs if a transformation can be used to cancel a rotation or perhaps deformation. The link between the 2 groups of equations may be used here to establish numerical magnitude relations. They are derived strictly from a series of equations and the symbolic algebra of base notation cannot be used to verify their validity. These equations are neither transformations nor rotations or other laws, they are equalities.

Whenever incompatible base symbols appear in our notation, the equation is an equality, e.g., as in eqn.\((16.4)\). In these equations, base notation cannot be replaced by space notation, viz.

\[
\vec{r} \neq \vec{s},
\]

\[\vec{R} \neq \vec{E}_s \cdot \vec{E}_r\]

Therefore, these equations can also not be expressed in customary Tensor Notation.
We can apply the distinction to express our intentions, like in the equation $\vec{E}_b = \vec{E}_a \cdot \vec{E}_a^b$, we want to express the transformation of one base in terms of base vectors of the other in the same space, and we regard $\vec{E}_a^b$ primarily as transformation matrix. The fact that it is also a representation of the space tensor $\vec{E}$ in mixed bases is only used in the original derivation of the transformation equation, to establish its correctness. On the other hand, in the equation $\vec{E}_b^a = \vec{R}_a \cdot \vec{E}_a^s$, although never written that way, we consider the rotation of the base $\vec{E}_a^a$ to the new position $\vec{E}_b^a$ in space, although the numbers of $\vec{R}_a^a$ and $\vec{E}_b^a$ are the same.

To make full use of the different possibilities, we use the term “transformation” mathematically, not physically. In tensor analysis it is clearly stated that transformation means to change components of the same tensor, which in our notation is written as change of base, e.g., in $\vec{v}^r = \vec{E}_s^r \cdot \vec{v}^s$, the $\vec{v}^s$ is transformed to $\vec{v}^r$. The vector $\vec{v}$ remains the same, the physical vector is not transformed. If a physical vector is changed, we don’t call it a “transformation” but rather a rotation or deformation (see Section 17). In a base “transformation” $\vec{E}_r^s = \vec{E}_s^r \cdot \vec{E}_s^r$, the notational intention is not to change base vectors of base $s$ into base vectors of base $r$.

Because of the equalities, some equations can be written in a form where our notation is least suitable. The advantages of algebraic base symbol manipulation are then lost, which makes equations more difficult to develop or read.

The following equations contain the same magnitudes but express different intentions

$$
\begin{align*}
\vec{v}^s_r &= \vec{E}_s^r \cdot \vec{v}^r \quad \text{(transformation)} \\
\vec{v}^s_r &= \vec{R}_s^r \cdot \vec{v}^s \quad \text{(rotation)} \\
\vec{v}^s_r &= \vec{E}_s^r \cdot \vec{v}^s \quad \text{(equality introduced)} \\
\vec{v}^s_r &= \vec{R}_s^r \cdot \vec{v}^r \quad \text{(equality misused or another intention)}.
\end{align*}
$$

The first equation follows from the notation, the second expresses a physical law in valid notation, the validity of the third equation must be derived and the fourth equation, although valid, is already confusing. The problem with the equality equations is that its intention cannot be read, it could mean the intention of the first or the second example.

The fact that Matrix Tensor Notation allows many equations to be written in different forms should not be seen as a disadvantage but be exploited to express different intentions or descriptions. The guide that the proper use of the base symbol algebra provides, facilitates developing new or complicated relations as well as interpretations which would otherwise be very difficult to accomplish.
17. DEFORMING BASE

More generally than the rotation of a rigid body, or the deformation of skew hinge axes, a base may deform such that all angles between axes as well as the magnitude of the direction vectors change. Rigid rotation is included in such a general "deformation."

17.1. Deformation from Cartesian Base to Skew Base

To attach a concrete meaning to such a change, we consider a deformation of a particle of a continuum as shown in Fig.11.

\[ \mathbf{E} = Q \mathbf{E}_s \]  \hspace{1cm} (17.1)

analog to the rigid rotation \( \mathbf{R} \) of eqn. (44). The tensor \( Q \) represents a finite deformation, including rigid rotation, where "deformation" is taken in the sense as "deformation gradient" used in continuum mechanics [3,26], denoted there by \( \mathbf{F} \).

The following relations are derived in detail and are completely analogous to the same relations in rotation of a rigid body, which they become if the deformation is rigid, with the correspondences

\[ \text{base } q \rightarrow \text{base } r, \quad \text{Tensor } Q \rightarrow \text{Tensor } R, \quad \text{Tensor } G \rightarrow \text{Tensor } W. \]

From eqn.(17.1),

\[ \mathbf{E}_q \cdot \mathbf{Q}_s^\dagger \cdot \mathbf{E}_s = \mathbf{E}_q \cdot \mathbf{Q}_s^\dagger. \]
Also,
\[ \overrightarrow{E}_q = \overrightarrow{E}_s \cdot \overrightarrow{E}^q_s; \]
therefore,
\[ \overrightarrow{Q}_q^s = \overrightarrow{E}^q_q. \tag{17.3} \]

Transform
\[ \overrightarrow{Q}_q^q = \overrightarrow{E}^q_s \cdot \overrightarrow{Q}_s^q \cdot \overrightarrow{E}^q_q \]
\[ = \overrightarrow{E}^q_s \cdot \frac{\overrightarrow{E}^q_q}{\overrightarrow{E}^q_q}, \tag{17.4} \]
using (17.3),
\[ = \overrightarrow{E}^q_q. \]
therefore,
\[ \overrightarrow{Q}_q^q = \overrightarrow{Q}^q_q. \tag{17.5} \]

It is an axiom from continuum mechanics that for a proper constitutive equation the tensor \( \overrightarrow{Q}_q \) is the same in the deformation equation eqn.(17.1) for any orientation of the base \( \overrightarrow{E}_s \). A conclusion is that a position vector \( \overrightarrow{s} \) in the undeformed continuum is assumed to deform to a new position vector \( \overrightarrow{r} \) in the deformed continuum according to the same deformation eqn.(17.1) of the base vectors
\[ \overrightarrow{r} = \overrightarrow{Q}_q \cdot \overrightarrow{s}. \tag{17.6} \]
This interpretation means that every particle of the body changes its position linearly according to the corresponding transformation of a position vector in the base.

Using the equality of eqn.(17.3) and transformation eqn.(17.2), eqn.(17.6) can be transformed to
\[ \overrightarrow{r} = \overrightarrow{E}^q_q \cdot \overrightarrow{s}, \tag{17.7} \]
which can be interpreted as: A position vector in the undeformed continuum, \( \overrightarrow{s} = \overrightarrow{E}_s \cdot \overrightarrow{s}_q \), is assumed to deform to a new position vector in space \( \overrightarrow{r} = \overrightarrow{E}_s \cdot \overrightarrow{r}_q \), such that its components deform in proportion to the base vectors.

Implying this assumption, the terms deformation and rigid rotation are used synonymous with deformation and rigid rotation of a body or of a base.

The rate of deformation tensor \( \overrightarrow{G}_q \), i.e., rate of displacement gradient, is defined such that
\[ \overrightarrow{E}^q_q = \overrightarrow{G}_q \cdot \overrightarrow{E}_q = \overrightarrow{G}_q, \tag{17.8} \]
analog to the rigid angular velocity \( \overrightarrow{W}_r \) in eqn.(49) (Part I). From eqn.(17.8), noting that space notation within time derivative can be replaced by a Cartesian base,
\[ \overrightarrow{E}^q_q = \overrightarrow{G}_s \cdot \overrightarrow{E}_q^q = \overrightarrow{G}_q^q = \overrightarrow{E}^q_q \cdot \overrightarrow{G}_q \tag{17.9} \]
which is the generalization of eqns.(51) to (53). From (17.3) and (17.9)
\[ \frac{\dot{\overrightarrow{Q}}_q^s = \overrightarrow{E}^q_q, \tag{17.10}} \]
\[ \frac{\dot{\overrightarrow{Q}}_s^q = \overrightarrow{G}_s \cdot \overrightarrow{Q}_q^s, \tag{17.11}} \]
\[ \frac{\dot{\overrightarrow{Q}}_q = \overrightarrow{G} \cdot \overrightarrow{Q}_q. \tag{17.12}} \]
Taking the derivative of (17.4), then substituting (17.9),

\[ \dot{Q}_q^s = \bar{E}_q^s \cdot \bar{G}_q^s \]

and then again (17.4), produces

\[ \dot{Q}_q^s = \bar{Q}_q^q \cdot \bar{G}_q^q. \]  

(17.13)

Note the difference between eqns.(17.13) and (17.11), considering the equality eqn.(17.5). By taking the derivative of the identity

\[ \bar{E}_q^q \cdot \bar{E}_q^s = I \]

and some base transformations, we obtain the result

\[ \bar{E}_q^q = -\bar{G}_q^q. \]  

(17.14)

It is an interesting exercise in the symbolic algebra of bases to prove the same result by taking the derivative of the transformation equation

\[ \bar{Q}_q^q = \bar{E}_q^q \cdot \bar{G}_q^q \cdot \bar{E}_q^s. \]

The corresponding equation to eqn.(17.14) for rigid rotation (p. 81, Part I) was obtained by using the antisymmetry of the Cartesian tensor matrix \( \bar{W}_s \), which is however not true for the rate of deformation tensor,

\[ [\bar{G}_s^q]^T \neq -\bar{G}_s^q. \]

Eqns.(17.11), (17.12), and (17.13) are the generalizations of the rigid body rotation of Section 10. In passing, we also mention that

\[ \dot{E}_s^q = \left[ \dot{E}_q^s \right]^{-1} \neq \left[ \dot{E}_q^q \right]^{-1} \]

because the derivative and inverse cannot be interchanged, just as in the scalar case.

It may be surprising that the equations for deformation are the same as the ones for rigid rotation, yet they represent something far more complicated. The same equations are easier to derive than in tensor notation because of the mixed base symbols \( s \) and \( q \), which at the same time make the physical interpretation clear. We note also that the rotation tensor \( \bar{R} \) and its generalized counterpart, the deformation tensor \( \bar{Q} \), must be strictly remembered as from original to rotated and deformed, respectively, while the corresponding transformation symbols \( \bar{E}_r^s \) and \( \bar{E}_q^s \) contain this information explicitly, as well as lending them uniquely to denote the inverse transformation. On the other hand, \( \bar{R} \) and \( \bar{Q} \) are physical tensors distinct from \( \bar{E} \).

The previous relations have been derived without the use of coordinates, contrary to the usual approach in tensor analysis. Nevertheless they are all defined and can be evaluated if the single quantity \( \bar{E}_q^s \) is given analytically. These symbolic equations often serve the purpose to interpret the physical meaning of the tensor quantities much better than equations derived from coordinates. Coordinates are introduced in Section 17.4.

Caution must be exercised to interpret eqn.(17.8). From that definition it is clear that \( \bar{G} \) is the rate of deformation at the end of the deformation path from \( \bar{E}_s^q \) to \( \bar{E}_q^q \). We may ask whether the rate at the beginning of the path may be given a similar equation, confusing us then with two quantities \( \bar{G} \), where perhaps \( \bar{G}_s^q \) might be equal to a quantity \( \bar{E}_s^q \), while there is a different
transformation $\overrightarrow{G_s} = \overrightarrow{G_q} \cdot \overrightarrow{E_s}$ which is the rate tensor at the end of the deformation. The error
in this comparison is that $\overrightarrow{E_s} = [0]$ always, because the Cartesian base is fixed. The statement
that $\overrightarrow{E_s}$ is deformed to $\overrightarrow{E_q}$ is too simplified and strictly incorrect as soon as rates of change are
concerned. What actually happens is that the base $\overrightarrow{E_q}$ is originally in the position of base $\overrightarrow{E_s}$,
and then moves to a new final position $\overrightarrow{E_q}$. We have the equality $\overrightarrow{E_q}(0) = \overrightarrow{E_s}$, but that is not
an identity. Writing $\overrightarrow{E_q}$, we tacitly assume that this is not $\overrightarrow{E_q}(0)$ but $\overrightarrow{E_q}(t > 0)$. Similarly, the
tensors $\overrightarrow{Q_q}(t)$ and $\overrightarrow{G_q}(t)$ are generally both functions of time. If we want to express $\overrightarrow{Q_q}$ at the time
that $\overrightarrow{E_q}$ is in the position $\overrightarrow{E_s}$, then we can express this intention in a sequence of equations

$$\overrightarrow{G_q} = \overrightarrow{E_q}, \quad \overrightarrow{E_q} = \overrightarrow{E_s}, \quad \overrightarrow{G_q} = \overrightarrow{G_s}.$$  

This sequence can be expressed as

$$\overrightarrow{G_q}(0) = \overrightarrow{E_q}(0),$$  

which represents a particular instant in a more indicative form of eqn.(17.8)

$$\overrightarrow{G_q}(t) = \overrightarrow{E_q}(t).$$  

$\overrightarrow{G_q}$ and $\overrightarrow{G_s}$ are well defined as long as only one value of the parameter $t$ is involved, otherwise
the distinguishing parameter statement (0) or (t), etc., should be appended. The same applies
to $\overrightarrow{Q_q}(0)$ and $\overrightarrow{Q_q}(t)$, and to the special cases when $\overrightarrow{Q_q} = \overrightarrow{R}$ and $\overrightarrow{G_q} = \overrightarrow{W}$.

Finally, we note that the scalar parameter $t$ may be any other scalar parameter, leading to a
so-called kinetic analogy.

### 17.2. Deformation from Skew Base to Skew Base

The most general transformation of skew nonunit base $\overrightarrow{E_a} = [\overrightarrow{e_{a1}} \overrightarrow{e_{a2}} \overrightarrow{e_{a3}}]$ to skew nonunit
base $\overrightarrow{E_b} = [\overrightarrow{e_{b1}} \overrightarrow{e_{b2}} \overrightarrow{e_{b3}}]$ of Section 15.5, is interpreted now as deformation, expressed by

$$\overrightarrow{E_b} = \overrightarrow{Q_a} \cdot \overrightarrow{E_a}, \quad (17.15)$$

from which follows the equality

$$\overrightarrow{Q_a} = \overrightarrow{E_b} \cdot \overrightarrow{E_a}, \quad (17.16)$$

which expresses the deformation tensor $\overrightarrow{Q_a}$ in terms of the two skew bases. Similar to Section 17.1,
it can be shown that

$$\overrightarrow{Q_a} = \overrightarrow{Q_b} = \overrightarrow{E_a}, \quad (17.17)$$

$$\overrightarrow{Q_a} = \overrightarrow{Q_b} = \overrightarrow{I}. \quad (17.18)$$

The deformation tensor $\overrightarrow{Q_a}$ is the same as in Section 17.1, which is easy to see if we regard $\overrightarrow{E_a}$
as 3 different arbitrary vectors in space transformed to the 3 vectors $\overrightarrow{E_b}$ by the deformation
equation eqn.(17.6) of Section 17.1, using the principle of uniform deformation. This means that
the base $\overrightarrow{E_a}$ is regarded as fixed as much as $\overrightarrow{E_b}$.

Similarly, the same rate of deformation tensor $\overrightarrow{G}$ as given by eqn.(17.8), applies

$$\overrightarrow{E_b} = \overrightarrow{G} \cdot \overrightarrow{E_b} = \overrightarrow{G_b}, \quad (17.19)$$
while
\[ \dot{\vec{E}}_a = [0]. \quad (17.20) \]

If it were that \( \vec{E}_a \neq [0] \), then \( Q_\alpha \) and \( G_\alpha \) would only be relative deformation and relative rate of deformation, respectively. This interpretation of \( Q_\alpha \) and \( G_\alpha \) becomes quite clear if their equalities with transformation matrices of eqns. (17.15) and (17.19) are considered.

Transforming \( \vec{E}_b = \vec{E}_a \cdot \vec{E}_b^a \) in eqn. (17.19) and using eqn. (17.20), and transforming, we obtain
\[ \dot{\vec{E}}_b^a = \frac{\partial}{\partial \vec{E}_b^a} \quad (17.21) \]

Now taking the derivative of the identity
\[ \vec{E}_a \cdot \vec{E}_b^a = \vec{I} \]
and some base transformations, we obtain the result
\[ \dot{\vec{E}}_a^b = -\vec{G}_a^b. \quad (17.22) \]

If at this stage the deformation tensor is required in a Cartesian base, where \( \vec{E}_a = \vec{E}_s \cdot \vec{E}_a^s \) and \( \vec{E}_b = \vec{E}_s \cdot \vec{E}_b^s \), the transformation can be shown to result in the relation
\[ \vec{Q}_s^s = \vec{E}_b^s \cdot \vec{E}_s^a. \quad (17.23) \]

This equality is extremely difficult to read, because it doesn't convey the information that \( \vec{Q}_s^s \) deforms base \( a \) to base \( b \), and neither \( a \) or \( b \) to \( s \). To avoid such a misleading equality—if possible—we would choose the simpler way of using only one skew base \( \vec{E}_q^q \).

17.3. Rigid Rotation of Skew Base

A particular case of “deformation” is the rigid body rotation, where a skew base \( \vec{E}_b^b \) is attached to a rigid body originally in a skew base \( \vec{E}_a^a \). The deformation tensor \( Q_\alpha \Rightarrow \) rotation tensor \( R_\alpha \) and rate of deformation tensor \( \vec{G}_\alpha \Rightarrow \) angular velocity tensor \( \vec{W} \). The Cartesian relations (44), (46), (47), and (48) (Part I), become
\[ \vec{E}_b = \vec{R} \cdot \vec{E}_a, \quad (17.24-a) \]
\[ \vec{R}_a = \vec{E}_b^a, \quad (17.24-b) \]
\[ \vec{R}_a^a = \vec{R}_b, \quad (17.24-c) \]
\[ \vec{E}_b \cdot \vec{E}_a^a = \vec{R}_b, \quad (17.24-d) \]

where the rotation is from \( \vec{E}_a^a \) to \( \vec{E}_b^b \) and \( \vec{E}_a^a \) is fixed. In contrast to the Cartesian base, the skew base tensor \( \vec{R}_a^a \) is not orthonormal or orthogonal any more. The reverse rotation tensor can only be obtained by the inverse
\[ \vec{R}_a^{-1} = \left[ \vec{R}_a \right]^{-1} = \vec{E}_a^b. \quad (17.25) \]

Note also that the rotation matrix is not the unit space tensor matrix, eqn. (17.24-b) is not an identity but an equality. The base symbols behave differently than on the transformation matrix,
\[ \vec{R}_b^a = \vec{E}_b^a \cdot \vec{E}_a^a \neq \left[ \vec{R}_a \right]^{-1}, \]
\[ \vec{R}_a^a = \vec{I}. \]
We obtain a similar difficult equality if we transform the rotation tensor to a Cartesian base \( s \)

\[
\mathbf{R}^s = \mathbf{E}^s \cdot \mathbf{E}^a,
\]

which does not convey the information that \( \mathbf{R} \) rotates base \( a \) to base \( b \), and neither \( a \) or \( b \) to \( s \).

The Cartesian relations for the angular velocity tensor, (49) and (50), become

\[
\mathbf{\dot{E}}_b = \mathbf{W} \cdot \mathbf{E}_b \tag{17.26-a}
\]

\[
= \mathbf{W}_b \tag{17.26-b}
\]

\[
= \mathbf{E}_b \cdot \mathbf{W}_b \tag{17.26-c}
\]

\[
\mathbf{\dot{E}}^a_b = \mathbf{W}^{a}_b \tag{17.26-d}
\]

and the corresponding equation for the reciprocal base becomes

\[
\mathbf{\dot{E}}^{b}_b = -\mathbf{E}^{b}_b \cdot \mathbf{W} \tag{17.27-a}
\]

\[
= -\mathbf{W}^b \tag{17.27-b}
\]

\[
= -\mathbf{W}^b_b \cdot \mathbf{E}^b \tag{17.27-c}
\]

\[
\mathbf{\dot{E}}^{b}_a = -\mathbf{W}^b_a \tag{17.27-d}
\]

The transformation of the angular velocity tensor between the previous Cartesian and the present skew base is

\[
\mathbf{W}^b = \mathbf{E}^b \cdot \mathbf{E}^a \cdot \mathbf{E}^r \tag{17.28}
\]

but in contrast to Cartesian base, the skew base tensor matrix \( \mathbf{W}^b_b \) is not antisymmetric.

In terms of rotation tensor, the eqns.(17.26) can be written, corresponding to the similar equations for Cartesian base in Section 10 (Part I),

\[
\mathbf{\dot{R}}^a_b = \mathbf{\dot{R}}^b_a = \mathbf{W}^a_b \cdot \mathbf{E}^a \tag{17.29}
\]

\[
= \mathbf{R}_b \cdot \mathbf{W}_b \tag{17.30}
\]

17.4. Strain

Of the many definitions of strain, we take 2 examples, viz. Linear Strain and Green Strain, applied to large deformations, to demonstrate that Matrix Tensor Notation cannot be used to assist the physical derivation of a law, nor does any derivation in terms of components, and therefore the derivation is best done in space notation, not in a particular base. However, we then employ our notation to make some aspects much clearer as can be read from space notation.

The application of strain is only intended with a relation to the stress tensor

\[
\mathbf{\sigma} = \mathbf{E}^a \cdot \begin{bmatrix}
\sigma_{xx} & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_{yy} & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \sigma_{zz}
\end{bmatrix} \cdot \mathbf{E}^a,
\]

of which there is no other form in mechanics, if the specialities of spin moments are not introduced. And to make the fundamental issue clear, only isotropic material is considered here. The definition of the stress tensor again is contained in the relation of an infinitesimal force \( d\mathbf{f} \) to an infinitesimal surface element \( d\mathbf{A} \) by
\[ \text{eqn. (17.31): } \frac{d\mathbf{T}}{d\mathbf{A}} = \mathbf{T} \cdot d\mathbf{A}, \]

where \( \frac{d\mathbf{T}}{d\mathbf{A}} \) is the traction vector. Also, the stress tensor is always defined as a linear function of the strain tensor, the linear relation being a 4th order tensor; to be exact in the following discussion, we don't consider a type of strain tensor which must be altered by an addition or subtraction of the identity tensor before being proportional to the stress tensor. The linear relation is taken as symmetric, so that from the symmetry argument of the stress tensor the symmetry of the strain tensor is derived, with the same direction of principal axes. Any nonlinear relation between elongation and stress is, therefore, always expressed in the strain tensor; it would be rather absurd to divide the nonlinearity arbitrarily between strain-elongation and stress-strain relation.

It is found then that any nonlinearity in the many existing strain tensors is either introduced to express a nonlinear stress-elongation relation, of which the Hencky logarithmic strain is an example, or it is introduced to make its determination simpler, of which the Green strain is an example.

Many different strains are derived rather theoretically in customary vector and tensor notation \([3,26]\), all following the more original \([36]\), while \([2]\) presents only the customary infinitesimal and Green strains.

For the basic mechanical derivation of strain in terms of deformation or displacement, the different quantities are shown in Fig. 12(a).

The figure shows a particle of infinitesimal size in original undeformed and in final deformed positions. The original position of the unstrained particle in given by the position vector \( s \), which is the Lagrange coordinate for the displaced particle at position \( r(s) \). These are the positions that are usually called in the literature \( X \) for the original and \( x \) for the deformed position, respectively \([36]\). The displacement of the particle is denoted by \( u(s) \), so that

\[ r = s + u. \quad (17.32) \]

Any infinitesimal line \( ds \) of the unstrained block is changed to a line \( dr \) in the deformed block, by the deformation tensor \( \mathbf{Q} \) of Section 17.1, which applies the same deformation to any arbitrary vector as to the base vectors, expressed by the linear relation

\[ dr = \mathbf{Q} \cdot ds, \quad \text{(17.33)} \]

from which it is clear that

\[ \mathbf{Q} = \frac{dr}{ds} \quad \text{(17.34)} \]

in the notation of Section 11 (Part I), which is, therefore, usually called deformation gradient. Such a use means that \( r \) is called deformation, which includes (rigid body) rotation. In passing we note that eqn.(17.34) is a descriptive notation for the deformation tensor \( \mathbf{Q} \). Comparing eqn.(17.33) with (17.6), we can similarly write eqn.(17.7) in terms of differentials

\[ dr = \mathbf{E} \cdot ds, \quad \text{(17.35)} \]

and therefore, express base \( q \) in terms of the derivative

\[ \mathbf{E} = \frac{dr}{ds}. \quad \text{(17.36)} \]

Eqns.(17.35) and (17.36) raise the interesting prospect to give the past position vector \( s \) the name \( q \), with the following system: \( s \) is the Euler position in space, independent of what occupies space, and independent of time; \( q \) is the unstrained position of a particle of a continuum in space; \( r \) is the present and moving position of the particle, \( r(q,t) \), \( r(q,0) = q \).
To describe the actual deformation due to strain, we assume that the element has undergone a rigid body rotation first, where the line $ds$ has rotated to $dl$, apart from a rigid body translation,

\[ dl' = R \cdot ds, \]
\[ \frac{dl'}{ds} = R, \]  

(17.37)  
(17.38)

where $dl'$ lies somewhere in between $dr'$ and $ds$, as yet undetermined. Let us define a linear stretch tensor $\mathbf{V}$ which stretches the rotated vector to the final strained vector by the linear relation

\[ dr' = \mathbf{V} \cdot dl', \]
\[ \mathbf{V} = \frac{dr'}{dl'}. \]  

(17.39)  
(17.40)

The steps of deformation are then divided into parts according to

\[ \overrightarrow{Q} = \mathbf{V} \cdot R = \frac{dr'}{ds} = \frac{dr'}{dl'} \frac{dl'}{ds}. \]

(17.41)

The actual strain displacement is the vector $\overrightarrow{de}$ shown in Fig.12(b),

\[ \overrightarrow{de} = dr' - dl'. \]

(17.42)

The Linear Strain Tensor is now defined by a tensor $\mathbf{S}$ such that

\[ \overrightarrow{de} = \mathbf{S} \cdot dl'; \]

(17.43)

therefore,
Here $\overrightarrow{E}$ is the same as the identity tensor denoted by $I$ in [3], but our notation allows a transformation to mixed base like for every other tensor. From what was said before, the linear strain tensor is symmetric, and therefore, the stretch tensor $\overrightarrow{V}$ must also be symmetric. As far as our derivation is concerned, this symmetry provides the necessary distinction between the line $dl$ and $d\overrightarrow{r}$. The symmetry of $\overrightarrow{V}$, or $\overrightarrow{S}$, can be understood physically by realizing that stretch must have no more and no less than 6 independent parameters, viz. 3 elongations $e_1, e_2, e_3$ of orthogonal principal axes, and the orientation of these principal axes in terms of 3 angles $\beta_1, \beta_2, \beta_3$, from which again a rotation tensor can be constructed. The total deformation must have an additional 3 parameters $\theta_1, \theta_2, \theta_3$ to allow the rigid rotation by means of the rotation tensor $\overrightarrow{R}$. The total deformation has therefore 9 independent parameters, which correspond to the 9 independent elements of $\overrightarrow{Q}$.

With this we finally arrive at the customary point of departure where the left stretch tensor is defined as the result of the decomposition eqn.(17.41), which is possible according to a theorem of matrix algebra [3]. This particular decomposition is nonlinear, the correct result being defined by the requirement that $\overrightarrow{V}$ is positive definite [5].

The important point of the discussion is that Fig.12 shows that the stress tensor acts on the rotated particle, therefore, has the orientation of the strain tensor $\overrightarrow{V} - \overrightarrow{E}$ in space, and it is this orientation that will have to be used in equations of mechanics. In the application of eqn.(17.31) the direction of the area element is the deformed rotated area element, which we denote by $d\overrightarrow{A}_R$ to distinguish it from the same element in its original undeformed position $d\overrightarrow{A}_S$. Whether the stress tensor is defined with magnitude of $d\overrightarrow{A}_S$ or $d\overrightarrow{A}_R$ in eqn.(17.31) is again a problem of constitutive continuum mechanics, which we cannot address here.

The right stretch tensor $\overrightarrow{U}$, derived from $\overrightarrow{Q} = R \cdot \overrightarrow{U}$ [3, 36], cannot be used for a definition of a real strain tensor because it leads to a stress tensor with the wrong orientation. The stress tensor doesn’t act on the undeformed (unrotated) particle.

Although the decomposition of deformation into rotation and stretch is nonlinear, the stretch and strain relations, eqns.(17.39) and (17.43), respectively, are linear. Linear Strain Tensor is, therefore, an appropriate name. It is also exact.

There is no experimental evidence why this linear strain cannot be used for linear elastic materials for all strains below their practical limit of linear elasticity such as metals. Because strains in most engineering applications are small, there wouldn’t be any significant difference between linear and infinitesimal strain if it wasn’t for large global rotations of parts of a structure. Because linear strain is exact, it is not necessary to use a Cauchy-Green strain for intricate large deformations [37], or spaghetti type problems [38].

We regard the linear strain as the natural generalization of the infinitesimal strain which is its differential, as can be seen by taking the differential, which is an infinitesimal quantity, of the deformation tensor

\[
\overrightarrow{dQ} = d \left( \left[ \overrightarrow{S} + \overrightarrow{E} \right] \cdot \overrightarrow{R} \right)
\]

\[
= \overrightarrow{dS} \cdot \overrightarrow{R} + \left[ \overrightarrow{S} + \overrightarrow{E} \right] \cdot d\overrightarrow{R}
\]

\[
= \overrightarrow{dS} \cdot \overrightarrow{R} + \left[ \overrightarrow{S} + \overrightarrow{E} \right] \cdot d\Theta \cdot \overrightarrow{R},
\]

where $\overrightarrow{dQ}$ is the skew symmetric small angle tensor corresponding to $\overrightarrow{V} dt$ of Section 10 (Part I).
Initially at zero deformation, \( \mathbf{S}_0 = [0], \mathbf{R}_0 = \mathbf{E} \), and therefore,

\[
\frac{d\mathbf{u}}{ds} = d\mathbf{S} + d\mathbf{\Theta},
\]

(17.46)

where \( d\mathbf{S} \) and \( d\mathbf{\Theta} \) are separated by taking the symmetric and skew-symmetric parts, written in finite form,

\[
\mathbf{S} = \frac{1}{2} \left[ \frac{d\mathbf{u}}{ds} + \left( \frac{d\mathbf{u}}{ds} \right)^T \right], \quad \mathbf{\Theta} = \frac{1}{2} \left[ \frac{d\mathbf{u}}{ds} - \left( \frac{d\mathbf{u}}{ds} \right)^T \right].
\]

The actual tensor matrix calculations can be done in any base. In Fig.12(b) are shown the three different bases, \( \mathbf{E}_s, \mathbf{E}_r, \mathbf{E}_q \), attached to the particle during the three steps in the sequence of its deformation. The transformation sequence is

\[
\mathbf{E}_q = \mathbf{E}_s \cdot \mathbf{E}_r \cdot \mathbf{E}_q.
\]

(17.47)

Compared with the deformation eqn. (17.41), and

\[
\mathbf{E}_q = \mathbf{V}_q \cdot \mathbf{E}_r, \quad \mathbf{E}_r = \mathbf{R}_r \cdot \mathbf{E}_s,
\]

(17.48)

which can be written in either Cartesian base \( \mathbf{E}_s \) or \( \mathbf{E}_r \), or even in skew base \( \mathbf{E}_q \), eqn.(17.47) leads to the equations

\[
\mathbf{V}_r = \mathbf{V}_q = \mathbf{E}_q,
\]

(17.50)

\[
\mathbf{Q}_s = \mathbf{V}_s \cdot \mathbf{R}_s = \mathbf{R}_s \cdot \mathbf{V}_r,
\]

(17.51)

while

\[
\mathbf{V}_s = \mathbf{E}_s \cdot \mathbf{V}_r \cdot \mathbf{E}_q
\]

(17.52)

\[
= \mathbf{R}_s \cdot \mathbf{V}_r \cdot \mathbf{R}_q.
\]

(17.53)

(17.54)

The decomposition can be carried out in base \( s \) or in base \( r \) according to eqns.(17.51) or (17.52), respectively, but we have to note that

\[
\mathbf{Q}_s \neq \mathbf{Q}_r,
\]

(17.55)

or it can be carried out on the base transformation

\[
\mathbf{E}_q = \mathbf{E}_r \cdot \mathbf{E}_q.
\]

(17.56)

In eqn.(17.51) the right matrix is orthonormal and in eqn.(17.52) the left matrix is orthonormal. Either decomposition can be done, but must be interpreted correctly. From the definition of the right stretch tensor, \( \mathbf{Q}_s = \mathbf{R}_s \cdot \mathbf{U}_s [3] \), compared to eqn.(17.52), and noting that \( \mathbf{R}_s = \mathbf{R}_r \), it is clear that

\[
\mathbf{V}_r = \mathbf{U}_s.
\]

(17.57)
and it is easy to confuse the two if correct record of the bases is not kept. The equations above are part equalities and part transformations. Both will be needed in an application. None of them though can automatically identify the correct stretch tensor which has been found from mechanical analysis.

Green strain, also called Cauchy strain or Cauchy-Green strain, is based on the observation that the true length differences are given independent of rotation by a form

\[
\left( \frac{dr}{r} \right)^2 - \left( \frac{ds}{s} \right)^2 = \frac{dr}{dr} \cdot \frac{dr}{ds} - \frac{ds}{ds} \cdot \frac{ds}{ds}.
\]

Therefore, comparing with \((1 + e)^2 - e^2\), the Green strain tensor is defined as

\[
\left[ \frac{dr}{r} \cdot \frac{dr}{ds} - \frac{ds}{ds} \cdot \frac{ds}{ds} \right]
\]

and on physical grounds, comparing with \((1 + e)^2 - e^2\), the Green strain tensor is defined as

\[
\sum_{G}^{s} \frac{E}{E}_{s} = \frac{1}{2} \left[ \left( \frac{dr}{r} \cdot \frac{dr}{ds} - \frac{ds}{ds} \cdot \frac{ds}{ds} \right) \left( \frac{dr}{r} \cdot \frac{dr}{ds} - \frac{ds}{ds} \cdot \frac{ds}{ds} \right) \right].
\]

given as \(E = \frac{1}{2}[F^T \cdot F - I]\) in the literature [3]. Particularly this is the Green strain tensor in Lagrange coordinates \(s\). Many authors, e.g., [3], make the statement that this tensor “is referred to the undeformed configuration,” which can be, and is, understood to mean that it is written in Lagrange coordinates \(s\), but it can hardly be understood that this is a rotated tensor, and therefore, not the correctly orientated tensor. The question of the orientation with respect to the deformed particle is not mentioned at all. From the vector notation used, it is obvious that eqn.(17.59) can be written in Cartesian base

\[
\sum_{G}^{s} = \frac{1}{2} \left[ \left( \frac{dr}{r} \cdot \frac{dr}{ds} - \frac{ds}{ds} \cdot \frac{ds}{ds} \right) \left( \frac{dr}{r} \cdot \frac{dr}{ds} - \frac{ds}{ds} \cdot \frac{ds}{ds} \right) \right].
\]

where we have used \(\text{E}_{s} = I\) for easy recognition. In [2] and [26], the derivation is given somewhat differently, using base vectors in original and deformed material. Written in our notation, and simplifying by using Cartesian base \(s\) instead of two skew bases, the deformed line \(dr\) is written in Cartesian base as deformed line \(\sum_{G}^{s} \cdot d\bar{s}^s\), and therefore,

\[
dr \cdot \frac{dr}{r} - ds \cdot \frac{ds}{s} = ds \cdot \sum_{G}^{s} \cdot \frac{dr}{r} \cdot \frac{dr}{s} - ds \cdot \frac{ds}{s}.
\]

But using the equality eqn.(17.5)

\[
dr \cdot \frac{dr}{r} - ds \cdot \frac{ds}{s} = ds \cdot \sum_{G}^{s} \cdot \frac{dr}{r} \cdot \frac{dr}{s} - ds \cdot \frac{ds}{s}.
\]

and again using eqn.(17.3), we arrive at the same eqn.(17.59). Eqn.(17.61) is written in the form of the metric \(\text{E}_{q}^{s}\) to which [26] refers. Flügge [2], raises the question in which base the tensor matrix is, since it is produced by two different bases. But he goes on to show that this mixed form is a tensor, and leaves the question of base unanswered as he turns to the infinitesimal strain after that. But it is interesting to dwell for a moment on a statement such as that from [2]. This question was addressed in the end of Section 8 (Part I), where it is shown that it is easy to define
a tensor from mixed bases. In this section, now we see that such a definition still gives no clue to the physical meaning, i.e., application, of such a construction.

Let us write the Green strain in terms of the linear stretch tensor \( \mathbf{V} \) which has been shown to be properly orientated. So will be any derived form \( [\mathbf{V}]^2 \) because it is symmetric and any power will preserve the principal axes. Using the decomposition in eqn.(17.41) in Cartesian base

\[
\frac{d\tau^s}{ds} = \mathbf{V}^s \cdot \mathbf{R}^s,
\]

which substituted in eqn.(17.60) produces

\[
\overline{S}^s_G = \frac{1}{2} \left[ \left[ \mathbf{R}^s \cdot \mathbf{V}^s \right] \cdot \left[ \mathbf{R}^s \cdot \mathbf{R}^s \right] - \mathbf{I} \right] = \frac{1}{2} \left[ \left[ \mathbf{U}^s \cdot \mathbf{V}^s \cdot \mathbf{U}^s \right] \cdot \left[ \mathbf{U}^s \cdot \mathbf{V}^s \cdot \mathbf{U}^s \right] - \mathbf{I} \right] = \frac{1}{2} \left[ \left[ \mathbf{V}^r_\tau \cdot \mathbf{V}^r_\tau \right] - \mathbf{I} \right].
\]

But this turns out to be a tensor orientated as \( \overline{V} \), expressed in rotated base \( \mathbf{E}_r \), e.g., a tensor matrix \( \overline{S}^r_{G} \). That means as \( \overline{S}^s_G \) and \( \overline{S}^r_{G} \) is confused, the strain which should have been orientated with the rotated particle is actually orientated with the original particle, where the stress doesn’t act. A simple calculation of 2-dimensional rotation with angle \( \theta \) and an elongation in the direction of the original x-axis will show this up. Furthermore, the advantage of the Green strain should have been that it doesn’t need the rotation which is so difficult to find from a deformation, but now it appears in this unknown rotated base. The Green strain has been used in many analysis of large deformations, and the probable reason why it hasn’t caused difficulties is that it was always applied after transformation to global rotated coordinates. Then the error is small because of small strains, yet even larger than the infinitesimal strain which at least accounts approximately for the rotation.

Eqn.(17.60) has been stated in the literature [3,26,36], in the form \( \mathbf{E} = \frac{1}{2} [\mathbf{C} - 1] = \frac{1}{2} [\mathbf{U}^2 - 1] \) considering eqn.(17.57), but then \( \mathbf{U} \) has the wrong orientation, as pointed out before.

The Green strain tensor is used in analysis primarily because of its simple analytic structure, and because practically all engineering structures undergo only small strain, the error is as negligible as the error of determining the elastic constants of the material.

From eqn.(17.63) it is clear that the correct (correctly orientated) Green strain tensor is in base \( r \)

\[
\overline{S}^r_{Gr} = \frac{1}{2} \left[ \left[ \mathbf{V}^r_\tau \right] \cdot \left[ \mathbf{V}^r_\tau \right] - \mathbf{I} \right],
\]

and the transformation to base \( s \) is accomplished by

\[
\overline{S}^s_{Gr} = \overline{S}^r_{Gr} \cdot \mathbf{R}^r_s
\]

\[
\overline{S}^s_{Gr} = \frac{1}{2} \left[ \left[ \mathbf{E}^s_s \cdot \mathbf{V}^s \cdot \mathbf{E}^s_s \right] \cdot \left[ \mathbf{E}^s_s \cdot \mathbf{V}^s \cdot \mathbf{E}^s_s \right] - \mathbf{I} \right] = \frac{1}{2} \left[ \left[ \mathbf{E}^s_s \cdot \mathbf{V}^s \cdot \mathbf{E}^s_s \right] \cdot \left[ \mathbf{E}^s_s \cdot \mathbf{V}^s \cdot \mathbf{E}^s_s \right] - \mathbf{I} \right] = \frac{1}{2} \left[ \left[ \mathbf{V}^r_\tau \cdot \mathbf{V}^r_\tau \right] - \mathbf{I} \right].
\]
where the correct orientation is clear from eqn.(17.65).

The deduction that a scalar, eqns.(17.58) or (17.61), if brought into a quadratic form, will produce a constitutive correct tensor in the same base is incorrect. What we can see from eqn.(17.61) is that the base symbols don’t match. That is because an equality has been used, while eqn.(17.60) is the result of the identity eqn.(17.58) in the same base. By correct use of transformations, which in our notation is done by correct use of base symbol algebra, still nothing can be deducted about what the quantity in brackets in eqn.(17.58) is. We have shown by other equalities that it becomes the quantity in brackets in eqn.(17.63), and still the base symbols don’t match. The step taken to put the quantity in brackets, written in base s, equal to a new Green strain, is therefore again an equality, but this time based on the physical reasoning that went into the deduction of the tensor \( \tilde{V} \), starting from Fig.12.

Note that the opposite also happens: the new definition of a Green strain tensor, eqn.(17.66), doesn’t satisfy the quadratic form equation any more,

\[
(ds)^2 - (dr)^2 \neq ds \cdot S_{Gr} \cdot ds.
\]

Also the matrix product in eqn.(17.66) is not the metric of \( \tilde{E}_q \) any more; it is its inverse.

If in eqn.(17.65) for the Green strain, linear strain tensor from eqn.(17.45) is substituted in the stretch tensor, the result is

\[
S_{Gr} = \mathbf{S} + \frac{1}{2} \left[ \mathbf{S}^2 \right].
\]

This quadratic equation has no experimental basis. In fact, some metals show a softening effect, not a hardening. The particular gray cast iron GG-30 has an elongation curve [39], that can be approximated by \( \sigma = E_0(\varepsilon - 40\varepsilon^2) \). The Green strain tensor was often quoted to be an exact tensor, explicitly or implicitly [40,41], simply because the eqn.(17.58) preserves the true elongation in a rotation. The term exact, however, has probably been interpreted incorrectly as if the quadratic law of eqn.(17.68) was a constitutive necessity. In some results of nonlinear elastic analysis [41-43], this assumption probably obscures other nonlinearities.

By eqn.(17.68) a new “exact” Green strain tensor is defined, but it is still no more exact than the linear strain tensor because the quadratic term is only justified if it is negligible.
18. SPECIAL PROPERTIES

Some special properties of vectors and tensors are given by relations that are expressed in Cartesian coordinates.

1. Orthogonality:
   
   If \( \begin{bmatrix} u^q \end{bmatrix}^\mathsf{T} \cdot \begin{bmatrix} u^q \end{bmatrix} = 0 \) then \( u \) is orthogonal to \( u \).

2. Symmetry:

   If \( \begin{bmatrix} K_{q}^q \end{bmatrix}^\mathsf{T} = K_q^q \) then \( K_q^q \) is symmetric.

3. Rigid Rotation:

   If \( \begin{bmatrix} Q_{q}^q \end{bmatrix}^\mathsf{T} \cdot \begin{bmatrix} Q_{q}^q \end{bmatrix} = I \) then the tensor \( Q_q^q \) is orthonormal, which means that the metric, or skewness, of \( E_a \) and \( E_b \) in a transformation \( E_b = Q_q^q E_a \), is the same.

The opposite conclusions are drawn if the equality, = , is replaced by the inequality, \( \neq \).

In a skew base \( E_q \), these three special relations can be written in our notation, assuming we don't have the metric \( E_q^\tau \), as

\[
\begin{align*}
\begin{bmatrix} u^q \end{bmatrix}^\mathsf{T} \cdot \begin{bmatrix} u^q \end{bmatrix} &= 0, \\
\begin{bmatrix} K_{q}^q \end{bmatrix}^\mathsf{T} &= K_q^q, \\
\begin{bmatrix} Q_{q}^q \end{bmatrix}^\mathsf{T} \cdot \begin{bmatrix} Q_{q}^q \end{bmatrix} &= I.
\end{align*}
\]

That is to say if we have certain vectors \( u \) given in posed skew base and certain other vectors \( v \) given in inverse skew base, we can test orthogonality. If a tensor \( K_q^q \) is given in only the one physical skew base \( E_q \), then an apparent mixed form with two different base symbols is produced from which symmetry can be tested. Orthogonality however of vectors, or of a tensor \( Q_q^q \), cannot be tested if given in one base and its inverse, producing a form with only one base symbol, because there is no way that we can get the form \( v^\tau q \) from \( u^q \), or the form \( Q_q^q \) from \( Q_q^q \), respectively, without the metric.

Rather, we can state the opposite conclusions for a skew base \( E_q \) with a metric \( E_q^\tau q \neq I \)

1. Orthogonality:

   If \( \begin{bmatrix} v^q \end{bmatrix}^\mathsf{T} \cdot \begin{bmatrix} u^q \end{bmatrix} = 0 \) then \( v \) is not orthogonal to \( u \).

2. Symmetry:

   If \( \begin{bmatrix} K_{q}^q \end{bmatrix}^\mathsf{T} = K_q^q \) then \( K_q^q \) is not symmetric.

3. Rigid Rotation:

   If \( \begin{bmatrix} Q_{q}^q \end{bmatrix}^\mathsf{T} \cdot \begin{bmatrix} Q_{q}^q \end{bmatrix} = I \) then \( Q_q^q \) is not orthonormal,

   which means that the metric, or skewness, of bases \( E_a \) and \( E_b \) in a transformation \( E_b = Q_q^q E_a \), is not the same.
Having quantities in only one particular posed form in skew base, the special relations can only be written with aid of the metric, which in our notation become

1. Orthogonality:

\[
\text{If } \begin{bmatrix} v^q \end{bmatrix}^T \cdot \overrightarrow{E}_q^{\tau q} \cdot \overrightarrow{u}^q \equiv v_{\tau q} \cdot \overrightarrow{E}_q^{\tau q} \cdot \overrightarrow{u}^q = 0 \text{ then } \overrightarrow{v} \text{ is orthogonal to } \overrightarrow{u}. \tag{18.7}
\]

2. Symmetry:

\[
\text{If } \begin{bmatrix} K^q_q \end{bmatrix}^T \cdot \overrightarrow{E}_q^{\tau q} = \overrightarrow{E}_q^{\tau q} \cdot \overrightarrow{K}_q^q \text{ then } \overrightarrow{K} \text{ is symmetric.} \tag{18.8}
\]

3. Rigid Rotation:

\[
\text{If } \overrightarrow{E}_q^{\tau q} \cdot \begin{bmatrix} Q^q_q \end{bmatrix}^T \cdot \overrightarrow{E}_q^{\tau q} \cdot \overrightarrow{Q}_q^q = I \text{ then } \overrightarrow{Q} \text{ is orthonormal,} \tag{18.9}
\]

which means that the metric, or skewness, of bases \( \overrightarrow{E}_a \) and \( \overrightarrow{E}_b \) in a transformation \( \overrightarrow{E}_b = \overrightarrow{Q} \cdot \overrightarrow{E}_a \), is the same.

The relations are easier to read if they are written in terms of transformation matrix \( \overrightarrow{E}_b^a \) as follows:

In the transformation \( \overrightarrow{v}^b = \overrightarrow{E}_b^a \cdot \overrightarrow{v}^a \), the corresponding base transformation is \( \overrightarrow{E}_b^a = \overrightarrow{E}_b^a \cdot \overrightarrow{E}_b^a \), where the following 4 cases can occur, which can only be tested if at least one of the bases' metric is known:

1. \( \overrightarrow{E}_a \) is Cartesian, \( \overrightarrow{E}_b \) is Cartesian, and the transformation is a rotation, then \( \overrightarrow{E}_b^a \) is orthonormal.

2. \( \overrightarrow{E}_a \) is skew, \( \overrightarrow{E}_b \) is skew, and the transformation is a rotation, then \( \overrightarrow{E}_b^a \) is skew.

3. \( \overrightarrow{E}_a \) is skew, \( \overrightarrow{E}_b \) is skew, and \( \overrightarrow{E}_b^a \) is orthonormal, then the transformation is a (skew) deformation.

4. \( \overrightarrow{E}_a \) is Cartesian, \( \overrightarrow{E}_b \) is skew, the transformation is a (skew) deformation, then \( \overrightarrow{E}_b^a \) is not orthonormal, and the inverse:

If \( \overrightarrow{E}_b^a \) is not orthonormal, then not both bases can be Cartesian. In this case, we would name the Cartesian base \( \overrightarrow{E}_a \), the skew base \( \overrightarrow{E}_q \), and state that the condition for a tensor \( \overrightarrow{Q} \) to be orthonormal, is

\[
\overrightarrow{Q}_q^q \cdot \overrightarrow{E}_q^{\tau q} \cdot \overrightarrow{Q}_q^q = I, \tag{18.10}
\]

which becomes an identity if \( \overrightarrow{Q}_q^q = \overrightarrow{E}_q^{\tau q} \).

It is obvious that the combinations of the transpose symbol presented in this section produce a complicated notation, and the question is whether this is really necessary and useful. In the
following list, the notations are compared in skew base.

<table>
<thead>
<tr>
<th>Matrix Tensor Notation</th>
<th>Tensor Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\overrightarrow{u}^q$, $\overleftarrow{u}^{-q}$</td>
<td>$u^i$</td>
</tr>
<tr>
<td>$\overrightarrow{f}^q$, $\overleftarrow{f}^{-q}$</td>
<td>$f_j$</td>
</tr>
<tr>
<td>$\overrightarrow{K}^q$</td>
<td>$K^j_i$</td>
</tr>
<tr>
<td>$\overleftarrow{K}^{-q}$</td>
<td>$K_{ij}$</td>
</tr>
<tr>
<td>$\overrightarrow{K}^q$</td>
<td>$K^{ij}$</td>
</tr>
<tr>
<td>$\overrightarrow{K}^q + \overleftarrow{K}^{-q}$</td>
<td>$K_{ij} + K_{ji}$</td>
</tr>
<tr>
<td>$\overrightarrow{K}^q + \overleftarrow{K}^{-q}$</td>
<td>$K^{ij} + K^{ji}$</td>
</tr>
<tr>
<td>$\overrightarrow{K}^q + \overleftarrow{K}^{-q}$</td>
<td>(not possible)</td>
</tr>
</tbody>
</table>

There has been an attempt to distinguish the posed and transpose in tensor notation by a dot symbol [2,3], i.e., $K_{i}^{j}$ is the transpose of $K_{j}^{i}$. This is equivalent to our introduction of the transpose-name symbol but it is a matrix transpose, not a tensor transpose, and therefore, works only in some bases, definitely not for $K_{ij}$. Because symmetry is lost in mixed tensors, the transformed transpose of a tensor is not the transpose of the transformed tensor any more. In our notation, the transposed tensor symbol is necessary, after all the transposed tensor is a different physical tensor and deserves a different name.

Note also that the transpose base symbol falls away for Cartesian bases, and the transposed matrix sign falls away for symmetric tensors, whether the tensor matrix is symmetric or not.

In conclusion, it can therefore be said that our complicated symbols take care of all the complicated forms that may occur in tensor calculations that cannot be expressed in customary tensor notation, and that most of them can be avoided by suitable bases. We believe that that it is an advantage that a complicated symbol draws attention to a complicated form.

18.1. Abstract Space

The well-known linear structural dynamic equation of forced motion in the customary matrix form is, e.g., [44,45],

$$K \cdot \overrightarrow{u} + \overrightarrow{M} \cdot \overleftarrow{u} = \overrightarrow{f},$$

(18.11)

where $\overrightarrow{f}$ and $\overrightarrow{u}$ are the force and displacement vectors of dimension $m$, $\overrightarrow{K}$ and $\overrightarrow{M}$ are the symmetric stiffness and mass matrices of dimension $m \times m$, with $m$ degrees of freedom. Assume now an abstract $m$-dimensional Euclidean space with force $\overrightarrow{f}$, displacement vector $\overrightarrow{u}$, stiffness tensor $\overrightarrow{K}$, mass tensor $\overrightarrow{M}$, and in this space assume the base base $m$ in which the structural eqn.(18.11) is written as

$$\overrightarrow{K}^m \cdot \overrightarrow{u}^m + \overrightarrow{M}^m \cdot \overleftarrow{u}^m = \overrightarrow{f}^m.$$  

(18.12)

With this notation, we have carried the analogy of abstract space and real 3-dimensional Euclidean space further than customary, and write eqn.(18.12) in baseless notation, as introduced for functions in Section 12.1,

$$\overrightarrow{K} \cdot \overrightarrow{u} + \overrightarrow{M} \cdot \overleftarrow{u} = \overrightarrow{f}.$$  

(18.13)

One well-known method of analysis is to find the $m$ eigenvectors of the system, which we will write $\overrightarrow{u}_e^m$, and collect into the matrix

$$\begin{bmatrix} \overrightarrow{u}_e^m \end{bmatrix} = \mathbf{U}_e^m$$  

(18.14)
in the form of a base (it should not be confusing that we use the dimension number \( m \) also as base symbol). The most employed method of normalization of the eigenvectors is

\[
\left[ \bar{U}_e^m \right]^T \cdot \bar{M}^m \cdot \bar{U}_e^m = \bar{I}.
\]  

(18.15)

using the property that the eigenvectors are "orthogonal with respect to the mass matrix." Considering the space, it is obvious that the eigenvectors can be used as a base, which we have given the mnemonic name \( e \). As soon as this choice is taken, the symbol \( U \) stands for the unit space tensor. The abstract bases are therefore denoted by \( \bar{U}_m \) and \( \bar{U}_e \).

To write the RHS. of eqn.(18.15) in terms of appropriate base symbols depends on whether the bases are orthonormal. We will investigate whether we are free to choose the metric of the bases. A first choice may be that base \( m \) is Cartesian (in the abstract sense of Section 15.2). Comparing eqn.(18.15) with the transformation condition eqn.(18.10), it is clear that both bases \( m \) and \( e \) cannot be Cartesian. But it appears on physical grounds that base \( m \) is Cartesian because in the same dynamic system the scalar quantity work is found from the equation

\[
dW = \left[ \overrightarrow{f}^m \right]^T \cdot \bar{u}^m,
\]  

(18.16)

which compared with \( dW = \bar{f} \cdot \bar{u} \) shows that \( \left[ \overrightarrow{f}^m \right]^T = \bar{f}_m \). Therefore, base \( e \) is skew, and eqn.(18.15) becomes

\[
\bar{U}_m \cdot \bar{M}_m \cdot \bar{U}_e^m = \bar{I}.
\]  

(18.17)

If \( \bar{U}_e \) hadn't been a base but just another tensor \( \bar{Q}_e \), then the base symbols in eqn.(18.17) would be \( m \) only, and the transpose sign would have become part of the matrix on the LHS., \( \bar{Q}_e^T \). As it is, \( \bar{U}_e^T = \bar{U}_e \). Note also that in this case from the symmetry of the matrices follows the symmetry of the tensors \( \bar{K} \) and \( \bar{M} \). The customary expression that "eigenvector space is orthogonal with respect to the mass matrix" now simply means that the eigenvector base is skew.

Comparison with eqn.(18.10) shows that eqn.(18.17) is not in a form that the metric of the skew base \( \bar{U}_e \) can be recognized. The skewness is given by the metric \( \bar{U}_m \cdot \bar{U}_e^m \), which is connected in a complicated way to the stiffness and mass matrices.

Consider another choice of interpreting eqn.(18.11) as transformation to base \( m \),

\[
\bar{K}_m \cdot \bar{u}^m + \bar{M}_m \cdot \bar{u}^m = \overrightarrow{f}^m.
\]  

(18.18)

Checking with eqn.(18.16) whether this is allowable, we find

\[
\left[ \bar{f}^m \right]^T \cdot \bar{u}^m = \bar{f}_m \cdot \bar{u}^m,
\]  

(18.19)

which is again a legitimate expression, independent of any skewness of base \( \bar{U}_m \). The symmetry of tensors \( \bar{K} \) and \( \bar{M} \) is also still valid according to eqn.(18.5) for apparent mixed bases.

Collect again the eigenvectors to a base \( \bar{U}_m \) according to eqn.(18.14). The eigenvectors, and \( \bar{U}_e \) are still the same as in the first interpretation, but now we attempt to assume the eigenvector base to be orthonormal, so that the consistent transformed eqn.(18.18) becomes

\[
\bar{U}_m \cdot \bar{K}_m \cdot \bar{U}_m \cdot \bar{u}^m + \bar{U}_m \cdot \bar{M}_m \cdot \bar{U}_e \cdot \bar{u}^m = \overrightarrow{f}^m.
\]  

(18.20)
where the vector transformations are
\[ \vec{u}^m = \overline{U}^m \cdot \vec{u}^e, \]
\[ \vec{f}_e = \overline{f}_m \cdot \overline{U}^m, \]
but assuming the base e to be orthonormal,
\[ \begin{bmatrix} \vec{f}_e \end{bmatrix}^T = \overline{f}^e. \]

The columns of \( \overline{U}^m \) are called modes in mechanics, but are specifically displacement modes, in which the displacements \( \vec{u}^m \) are expressed. The fact that the force modes are different is evident from the expansion of force by \( \overline{f}^m = \overline{U}^m \cdot \overline{f}^e \), so that the columns of \( \overline{U}^m \) are the force modes. The equation that the eigenvectors are orthogonal w.r.t. the mass matrix is expressed in this interpretation by the altered form of eqn. (18.17)
\[ \overline{U}^e \cdot \overline{M}^n \cdot \overline{U}^m = \overline{I}, \]
which was still called \( \overline{M}^e_\tau \) in eqn. (18.20). Eqn. (18.23) is analogous to the condition of an orthonormal tensor, where
\[ \overline{U}^e \cdot \overline{U}^m \cdot \overline{U}^m = \overline{I}, \]
which means that in this interpretation the mass matrix \( \overline{M}^n \) is the metric of the skew base m.

If we call base \( \overline{U}^e \) the modal base, then the customary expressions "modal force" and "modal displacement" coincide with the force vector array in modal base and displacement vector array in modal base. The abstract space concept allows the interpretation that both displacement vectors \( \vec{u}^m \) and \( \vec{u}^e \) are merely transformations of the same vector. It is, therefore, allowable to call both \( \vec{u}^m \) and \( \vec{u}^e \) the displacement vector in the default sense of Section 1 (Part I). Similarly, we may call both force vectors \( \vec{f}_m \) and \( \vec{f}_e \) the force vector. In this interpretation, it is interesting to note that the displacement vector is posed in column form, a contravariant vector, while the force vector is posed in row form, a covariant vector. This coincides with another interesting form.

A conservative force acting on a particle may be derived by the gradient of a scalar potential field \( [29-31] \), which is a function of space, \( P(s) \). But according to Section 11 (Part I) the gradient is a row vector
\[ \vec{f} = \text{grad} \cdot P = \frac{dP}{ds}, \]
where \( \vec{s} \) and \( \vec{u} \) are both position vectors, absolute and relative, respectively. The transformation pair of eqn. (18.21) is such that only the transformation matrix \( \overline{U}^m \) is required, independent of the assignment of orthogonal base to either base m or to base e. In fact, we don’t need to assume any one of these bases to be orthonormal, but put eqn. (18.20) into the form
\[ \overline{U}^e \cdot \overline{K}^e \cdot \overline{U}^m \cdot \vec{u}^e + \overline{U}^e \cdot \overline{M}^e \cdot \overline{U}^m \cdot \vec{u}^e = \overline{U}^e \cdot \overline{f}^m \]
\[ \overline{K}^e \cdot \vec{u}^e + \overline{M}^e \cdot \vec{u}^e = \overline{f}^e, \]
so that the posed form, eqn. (18.18), and transformed form, eqn. (18.26), are completely symmetric w.r.t. to base symbols m and e. The symmetry of stiffness and mass tensors is still valid, as well as the work equation (18.19). It appears that it is an arbitrary choice to name one base as Cartesian if one wishes, yet the advantage of isomorphism of 3-dimensional Euclidean space and abstract space is still not lost if generally skew bases are assumed, and adapting the notation accordingly by eqns. (18.18) and (18.26). Particularly noteworthy is that we can still distinguish...
force vectors that are orthogonal to displacement vectors, in both bases, according to the scalar product eqn.(18.1),
\[
\int_e \cdot \bar{u}^e = \int_m \cdot \bar{U}_m^e \cdot \bar{U}_m^e \cdot \bar{u}^m
\]
\[
= \int_m \cdot \bar{u}^m,
\]
using the inverse \( \bar{U}_m^e \) which is still independent of the choice of skewness of bases. If we make no assumption of the skewness of any one base, then according to case No. 3 we will also never be able to determine whether the transformation
\[
\bar{U}_e = Q \cdot \bar{U}_m
\]
is orthonormal, i.e., a rotation or not.

We may also consider the opposite conclusion according to eqns.(18.4) and (18.6), particularly in the special case when \( M_m^m = I \). From the corresponding equation (18.23), we may not conclude that \( \bar{U}_e \) is an orthonormal base. It is a choice to define it so, and then all transformations must be tracked to establish the skewness of other bases.

The abstraction of matrices into multi-dimensional space is carried out much more consistently isomorphic with 3-dimensional Euclidean space by the distinction between tensor and tensor matrix, carrying the space notation \( \rightarrow \) and base notation \( \rightarrow b \) over into abstract space. Consequently, what is usually called "space" in mathematical treatments, is in our concept "base." Similarly, the customary \textbf{boldface} notation for matrices and vectors in many structural dynamics or Finite Element or vibration analysis texts [44,45,48,49] does not coincide with our abstract tensor concept, considering the correspondence of \textbf{boldface} and space notation (see the Appendix). In their matrix notation, the distinction between bases is not possible. In our notation, the abstract space tensor and vector is cannot be expressed numerically. The numerical expression only occurs indirectly through the medium of the base.

The purpose of abstract space is to use the isomorphism with 3-dimensional Euclidean space. The understanding of relations in terms of transformations, orthogonality, symmetry, etc., creates much more confidence in the results. It is a fair conjecture to say that if equations violate a law of corresponding Euclidean space, they are probably wrong. It is stated that "laws of nature must be stated in covariant form," [23], like eqn.(18.27), perhaps this also applies to abstract space of which eqn.(18.27) is an example. But, unfortunately, such a rule doesn't apply to general tensor equations; there is no way we can bring the equation
\[
\bar{R}_m^m \cdot \bar{K}_m^m \cdot \bar{u}^m + \bar{M}_m^m \cdot \bar{u}^m = \bar{f}^m
\]
into such an invariant form with unknown metric of any base.

There is another direct advantage of abstract Euclidean space. By confident knowledge of Euclidean geometry, many relations in abstract space can simply be written down. To be sure, they should be proved, but this is algebraically much easier if guided by Euclidean geometry.

Let us now also state the inverse. The simple arithmetic operations in multiple variables is often written using the index notation of tensor analysis, of which computer language loops are an example. But the tensor analysis is actually based on tensors, not on matrix algebra. From this point of view an abstract space with abstract tensors is therefore always implied.
19. SUBSPACE AND SUBBASE

19.1. Sub and Partial Base

In an $n$-dimensional space any independent $k < n$ vectors can be collected into a subbase, which again lies in a $k$-dimensional subspace. A subspace must be considered as having a particular position in space, like a particular surface in a 3-dimensional space, while a subbase doesn't. Therefore, a subbase may not be sufficient to describe a subspace. A subspace defined by a single subbase is just a particular simple subspace, containing straight lines, which in 3-dimensional full space may be a line or a plane. In general, we call it a straight subspace (rather than linear). Generally a subspace can be curved, and has a different tangent subbase at each point. But a subspace is geometrically fixed in total space independent of any base, even a straight subspace is not a subbase. This distinction between subspace and subbase is seldom clear in texts on linear algebra and vector spaces, e.g., [46,47], or where they are used in Finite Elements [48]. Our notion of subspace and subbase is rather an abstract $n$-dimensional extension of differential geometry. The definition of a subspace is usually given by equations of coordinates, giving its position in full space, it cannot generally be completely described by a subbase. A subbase is only an auxiliary tool that can be used to describe vectors in subspace. Our concept of subspace is the analog of a generally curved line or surface in Euclidean space, and this differs from the linear subspace of vector spaces. In this section, the notation will be developed, which will be used for typical applications in Part III.

Consider a skew base $\vec{E}_b$ in $n$-dimensional space, and its partitioning into the first $k$ and last $l = n - k$ base vectors $\vec{e}_b^1 \cdots \vec{e}_b^k; \vec{e}_b^{k+1} \cdots \vec{e}_b^n$, then the two subbases are the partitions $\vec{e}_b^1 \cdots \vec{e}_b^k$ and $\vec{e}_b^{k+1} \cdots \vec{e}_b^n$, for which suitable notations will be developed. Usually we're only interested in one subbase, the remaining subbase is then its complementary subbase, complement for short. The base $b$ might well be the base $q$ in which we are presently working, and there may have been some reason for the partition. Or the base $b$ may have been produced during the course the analysis of some problem.

Whether a subbase is considered to consist of only the base vectors or of the larger full $n$-dimensional array containing also the zero vectors, doesn't really matter as far as the concept is concerned, but the algebraic properties are different due to the different dimensions, therefore, we need to distinguish by notation. We define partial bases of dimensions $k$ and $l$, respectively, by the augmented partitions $\vec{e}_b^1 \cdots \vec{e}_b^k; 0 = \vec{E}_b^k$, and $0; \vec{e}_b^{k+1} \cdots \vec{e}_b^n = \vec{E}_b^l$, such that

$$\vec{E}_b = \vec{E}_b^k + \vec{E}_b^l. \tag{19.1}$$

It is clear that the partial bases $\vec{E}_b^k$ and $\vec{E}_b^l$ are fully $n$-dimensional tensors, the dimensions $k$ and $l$ referring to the partial bases which are contained in them. A special case of a subbase is the 1-dimensional base $\vec{e}_b^1$, and a special case of a partial base is the base vector $\vec{e}_b^1$ augmented by zero vectors. The additional subscripts $k$ and $l$ are the equivalent of the subscript number 1 of the single dimensional subspace $e_b^1$. By the same token, we may regard the subbases as larger dimensional base vectors.
Any vector \( \vec{v} = \vec{E}_b \cdot \vec{v}^b \) consists of the \( n \) vector components \( \vec{e}_b^i \cdot v^b_i \), but now we define partial vectors as the components which are formed by

\[
\vec{E}_{k_b} \cdot \vec{v}^b \equiv \vec{v}^k, \quad \vec{E}_{l_b} \cdot \vec{v}^b \equiv \vec{v}^l,
\]

where only 2 partial bases are used to describe the full vector, instead of \( n \) base vectors \( \vec{e}_i \). From eqns.(19.1) and (19.2), it follows that

\[
\vec{v} = \vec{v}^k + \vec{v}^l = \vec{E}_{k_b} \cdot \vec{v}^b + \vec{E}_{l_b} \cdot \vec{v}^b.
\]

Each of the partial vectors \( \vec{v}^k \) and \( \vec{v}^l \) has the full \( n \) dimensions, similar to the vector components of \( \vec{v} \), see Section 15.1, and can be expressed by the full \( n \)-dimensional partial vector arrays of scalar components \( \vec{v}^k \) and \( \vec{v}^l \), respectively, in the relations

\[
\begin{align*}
\vec{v}^k & = \vec{E}_b \cdot \vec{v}^k, \\
\vec{v}^l & = \vec{E}_b \cdot \vec{v}^l,
\end{align*}
\]

where the partial vector arrays are related to the full vector by the partials

\[
\begin{bmatrix}
v^{b_1} \\
v^{b_k} \\
v^{b_k+1} \\
\vdots \\
v^{b_n}
\end{bmatrix} =
\begin{bmatrix}
v^{b_1} \\
v^{b_k} \\
0 \\
\vdots \\
v^{b_n}
\end{bmatrix} +
\begin{bmatrix}
0 \\
v^{b_k+1}
\vdots
\end{bmatrix}
\]

(19.5)

The only difference between the customary concept of a vector component of a vector and the partial vector is that a component is the part in one base vector direction, while the partial vector is the sum of all components in the one subspace. Being full \( n \)-dimensional arrays, partial bases and vectors can be expressed in any other full \( n \)-dimensional base, e.g.,

\[
\begin{align*}
\vec{E}_{k_s} & = \vec{E}_b \cdot \vec{E}_s, \\
\vec{E}_{l_s} & = \vec{E}_b \cdot \vec{E}_s, \\
\vec{v}^{k_s} & = \vec{E}_s \cdot \vec{v}^k, \\
\vec{v}^{l_s} & = \vec{E}_s \cdot \vec{v}^l.
\end{align*}
\]

The corresponding augmented partitions of the inverse base are the reciprocal partial bases

\[
\begin{bmatrix}
\vec{e}_1^{b_1} \\
\vdots \\
\vec{e}_1^{b_k} \\
\vdots \\
\vec{e}_1^{b_k+1} \\
\vdots \\
\vec{e}_1^{b_n}
\end{bmatrix} =
\begin{bmatrix}
\vec{e}_1^{b_1} \\
\vdots \\
\vec{e}_1^{b_k} \\
0 \\
\vdots \\
\vec{e}_1^{b_n}
\end{bmatrix} +
\begin{bmatrix}
0 \\
\vec{e}_1^{b_k+1}
\vdots
\end{bmatrix}
\]

(19.6)
It is important to note that the symbols $E_k$ and $E_l$ do not act like units in a multiplication any more, like $E$ did (Section 15.1). $E_k^b$ is not the inverse of $E_k^b$, they are like half-inverses of a rectangular matrix [5]. Their product is
\[
E_k^b \cdot E_k^b = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \overline{E}_k^b.
\tag{19.7}
\]

Similar $E_l^b$ is not the inverse of $E_l^b$,
\[
E_l^b \cdot E_l^b = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} = \overline{E}_l^b.
\tag{19.8}
\]

The half-inverse $E_k^b$ consists not only of other vectors as those of $E_k^b$, it also spans another subspace as $E_k^b$, and the half-inverse $E_l^b$ spans another subspace as $E_l^b$. This is similar to the different base vectors $e_{bl}$ and $e_{bl}$. We might well call $E_k^b$ and $E_l^b$ partial column bases and $E_k^b$ and $E_l^b$ partial row bases, the column and row referring to the space symbol. This nomenclature is used in matrix algebra, e.g., [48], who calls the collection of column vectors of a matrix column space and the the collection of the row vectors row space, but actually meaning bases that span the respective space. With the introduction of the half-inverses, there are now two additional subspaces defined, with the same indices $k$ and $l$. We call the subspaces to which $E_k^b$ and $E_l^b$ are tangent, column subspaces $k$ and $l$, and the subspaces to which $E_k^b$ and $E_l^b$ are tangent, row subspaces $k$ and $l$. While space is not dependent on any row or column representation, the subspace is, by the way it is constructed from a base. If it is created by other means, it will have to be decided whether it fits in with a column or a row subspace. In the special case that the partition is made in an orthogonal base, the two column subbases are orthogonal and the two row subbases can, at the most, be affine transformations of the column subbases, and therefore, the column and row subspaces are identical, and the name subspace $k$ or subspace $l$ is sufficient.

In the general case, the row subspaces are orthogonal to the column subspaces, but the two row subspaces and the two column subspaces, respectively, are skew to each other.

With the partial bases, the extraction of the partial vectors can be achieved formally by
\[
\overline{v}_k^b = E_k^b \cdot \overrightarrow{v}, \quad \overline{v}_l^b = E_l^b \cdot \overrightarrow{v}.
\tag{19.9}
\]

It is clear that the group of vectors that comprise the subbase, although from base $b$, cannot be denoted by $b$, which already denotes another group. We use another symbol to denote the subbases
\[
[\overrightarrow{e}_{b1} \cdots \overrightarrow{e}_{bb}] \equiv \overline{E}_k^b, \quad \text{and} \quad [\overrightarrow{e}_{bb+1} \cdots \overrightarrow{e}_{bn}] \equiv \overline{E}_l^b,
\tag{19.10}
\]
with which we distinguish between partial base and subbase. Similarly, for the reciprocal subbases:
\[
\begin{bmatrix} \overrightarrow{e}_{b1} \\ \vdots \\ \overrightarrow{e}_{bb} \end{bmatrix} = \overline{E}_k^b, \quad \begin{bmatrix} \overrightarrow{e}_{bb+1} \\ \vdots \\ \overrightarrow{e}_{bn} \end{bmatrix} = \overline{E}_l^b.
\tag{19.11}
\]

The subbases are smaller dimensional half-tensors and cannot be added like the partial bases, they are the partitions of the full dimensional base
\[
\overline{E}_b = \left[ \overline{E}_k^b ; \overline{E}_l^b \right],
\tag{19.12}
\]
and similarly, the reciprocal base is partitioned into subbases

\[
\overrightarrow{E}^{b}_{k} = \begin{bmatrix}
\overrightarrow{E}^{k}_{k} \\
\vdots \\
\overrightarrow{E}^{l}_{l}
\end{bmatrix}.
\] (19.13)

We call \(\overrightarrow{E}^{k}_{k}\) and \(\overrightarrow{E}^{l}_{l}\) sub column bases and \(\overrightarrow{E}^{b}_{k}\) and \(\overrightarrow{E}^{b}_{l}\) sub row bases. There may be an occasion when we want to indicate the full base explicitly in its partitioned form, then we use the symbols

\[
\overrightarrow{E}_{k,l} = \text{RHS of eqn.}(19.12), \quad \overrightarrow{E}^{k,l}_{_k} = \text{RHS of eqn.}(19.13).
\] (19.14)

The corresponding partitions of the vector array are called subvectors,

\[
\begin{bmatrix}
\overrightarrow{v}^{b_{1}} \\
\vdots \\
\overrightarrow{v}^{b_{k}}
\end{bmatrix} = \overrightarrow{v}^{k}, \quad \begin{bmatrix}
\overrightarrow{v}^{b_{k+1}} \\
\vdots \\
\overrightarrow{v}^{b_{n}}
\end{bmatrix} = \overrightarrow{v}^{l},
\] (19.15)

where a subvector in the extreme can become a scalar component. While the partial vectors can be transformed in full space to another base, the subvectors can’t. They can only be transformed to another subbase within the same subspace. With all the defined quantities the partitioning of a vector into subvectors can formally be done by the operation

\[
\overrightarrow{v}^{k} = \overrightarrow{E}^{k}_{k} \cdot \overrightarrow{v}, \quad \overrightarrow{v}^{l} = \overrightarrow{E}^{l}_{l} \cdot \overrightarrow{v},
\] (19.16)

and the partial vectors can be obtained from the partitions

\[
\overrightarrow{v}^{k} = \overrightarrow{E}^{k}_{k} \cdot \overrightarrow{v}^{k}, \quad \overrightarrow{v}^{l} = \overrightarrow{E}^{l}_{l} \cdot \overrightarrow{v}^{l}.
\] (19.17)

The total vector is again the sum of partial vectors

\[
\overrightarrow{v} = \overrightarrow{E}^{k}_{k} \cdot \overrightarrow{v}^{k} + \overrightarrow{E}^{l}_{l} \cdot \overrightarrow{v}^{l},
\] (19.18)

where the partial vectors are two vector components. The meaning of the term “partial” is taken from the definition of partial differentials. The partial bases and partial vectors correspond to partial differentials, whose sum make up a total, while subbases and subvectors correspond to scalar components as partitions of the total, which is obtained by stringing partitions together but not by their sum. The subbases have one full and one partial dimension. With this they form the connection between subvector arrays and full vectors in eqns.(19.16) and (19.18).

Another set of partial vectors can also be obtained from the inverse base,

\[
\overrightarrow{u} = \overrightarrow{v}^{k} = \overrightarrow{E}^{b}_{k} \cdot \overrightarrow{v}^{k} + \overrightarrow{E}^{b}_{l} \cdot \overrightarrow{v}^{l} \quad (19.19)
\]

\[
\overrightarrow{u} = \overrightarrow{v}^{k} = \overrightarrow{E}^{b}_{k} \cdot \overrightarrow{v}^{k} + \overrightarrow{E}^{b}_{l} \cdot \overrightarrow{v}^{l} \quad (19.20)
\]

\[
\overrightarrow{u} = \overrightarrow{u}^{b}_{k} = \overrightarrow{E}^{b}_{k} \cdot \overrightarrow{v}^{b}_{k} + \overrightarrow{E}^{b}_{l} \cdot \overrightarrow{v}^{b}_{l}
\] (19.18)

\[
\overrightarrow{u} = \overrightarrow{v} \cdot \overrightarrow{E}^{b}_{k} + \overrightarrow{v} \cdot \overrightarrow{E}^{b}_{l}.
\] (19.22)

corresponding to eqns.(19.3), (19.18), (19.2) and (19.4), respectively. The partial vectors are formally extracted from the full vector by

\[
\overrightarrow{v}^{b}_{k} = \overrightarrow{v}^{b}_{k} \cdot \overrightarrow{E}^{k}_{b}, \quad \overrightarrow{v}^{b}_{l} = \overrightarrow{v}^{b}_{l} \cdot \overrightarrow{E}^{l}_{b},
\] (19.23)
corresponding to eqn.(19.9), where

\[
\begin{bmatrix}
v_{b1} & \cdots & v_{bk} & v_{bk+1} & \cdots & v_{bn} \\
v_{b1} & \cdots & v_{bk} & 0 & \cdots & v_{bn}
\end{bmatrix}
\equiv
\begin{bmatrix}
v_k
\end{bmatrix} + \begin{bmatrix}
v_{b+k+1} & \cdots & v_{bn}
\end{bmatrix} + \begin{bmatrix}
v_l
\end{bmatrix}
\]  
(19.24)

The subvectors are extracted by

\[
v_k = v \cdot E_k, \quad v_l = v \cdot E_l,
\]  
(19.25)
corresponding to eqn.(19.16), where

\[
v_k = \begin{bmatrix} v_{b1} & \cdots & v_{bk} \end{bmatrix}, \quad v_l = \begin{bmatrix} v_{bk+1} & \cdots & v_{bn} \end{bmatrix}.
\]  
(19.26)

If the explicit indication of the partitioned form of the full vector is required, we use the symbol

\[
LHS \text{ of eqn.}(19.5), \quad \text{Top LHS of eqn.}(19.24). \quad (19.27)
\]

We are not able to define a subspace symbol, like the \( \rightarrow \) for the full dimensional vector, for the subvector. The subvector exists as an array only. This corresponds to the use of full 3-dimensional notation in 2-dimensional problems, e.g., a vector \( \mathbf{u} = i \mathbf{u}_x + j \mathbf{u}_y \), yet \( \mathbf{u} \) is still a full 3-dimensional vector, but the array \( \mathbf{u}_x, \mathbf{u}_y \) is not the vector array of a 3-dimensional vector. Similarly, we cannot express the subbases in space notation, but the partial bases we can. All full dimensional base symbols can be transformed to space notation, even though in higher than 3-dimensional abstract space the space dimension is abstract anyway.

In all of the equations above, the actual computation must be made in a particular base. The space notation \( \rightarrow \) in all of the equations above can be replaced by Cartesian base symbol \( \rightarrow s \) or by skew base symbol \( \rightarrow q \), using column and row bases.

Within the subspace, any rotation or deformation of the subbase is possible. The bases \( k \) or \( l \) can be transformed to other bases with the same dimensions, e.g., \( u \) and \( v \), producing a subbase with another name, e.g., \( \overrightarrow{E_k} u \) and \( \overrightarrow{E_l} v \). These bases are still in the same subspace, and still have \( \rightarrow \) or any other full dimensional base symbol. It can be shown that the subbase \( \overrightarrow{E_k} \) multiplied by any \( k \times k \) transformation matrix \( \overrightarrow{E^K} u \) will not lead out of the subspace. These considerations demonstrate the justification of using a subscript name as well as subscript base symbol for the subbase.

For the subspaces and subvectors only, the additional name \( k \) and \( l \) as part of the names is necessary. It is not possible to denote the subvectors as \( \overrightarrow{v^k} \) or \( \overrightarrow{v^l} \), because according to our base symbol rules they could then be transformed to full dimensional base by an equation like \( \overrightarrow{E_k} \cdot \overrightarrow{v} = \overrightarrow{v} \), so that \( \overrightarrow{v} \) would be a partial vector, while it was a full vector in the beginning.

As we have chosen our notation, the symbols \( \overrightarrow{k} \) and \( \overrightarrow{l} \) are not possible. The rules of sub and partial notation are different from the rules of base symbol notation.

The multiplication with the subbase symbol follows the rule that the \( E \) acts like a unit, but the subbase subscript may migrate over as in eqns.(19.2), (19.9) and (19.16), or it occurs two times in one product term but doesn't cancel, as in eqns.(19.18) and (19.20). There is no consistent rule of sub and partial symbols as there is for base symbols. This can be explained by the fact that the subbase symbols \( k \) and \( l \) are actually part of new names of bases or vectors, not like base symbol representations of the same vector in a new base.

Another question may be whether the sub and superscripts in eqns.(19.7) and (19.8) are justified. According to the transformation rules
\[
\mathbf{E}_b \cdot \mathbf{E}_b^b = \mathbf{E}_b^b,
\]
\[
\mathbf{E}_b^b \cdot \mathbf{E}_b = \mathbf{E}_b^b,
\]

but \(\mathbf{E}_b^b, \mathbf{E}_b^b\) and \(\mathbf{E}_b^b\) are all the same matrix of eqn.(19.7). Also the reverse multiplication of eqn.(19.7), \(\mathbf{E}_b^b \cdot \mathbf{E}_b\), transformed to base \(b\), produces the same matrix. The transformations to a Cartesian base \(s\) by

\[
\mathbf{E}_s^s \cdot \mathbf{E}_b = \mathbf{E}_s^s,
\]
\[
\mathbf{E}_b^b \cdot \mathbf{E}_s = \mathbf{E}_b^b,
\]
\[
\mathbf{E}_b \cdot \mathbf{E}_s^s = \mathbf{E}_b^s,
\]
\[
\mathbf{E}_b^s \cdot \mathbf{E}_b^b \cdot \mathbf{E}_s = \mathbf{E}_b^s,
\]
\[
\mathbf{E}_s^s \cdot \mathbf{E}_b^b \cdot \mathbf{E}_s = \mathbf{E}_b^s,
\]

produce the same tensor

\[
\mathbf{E}_s^s = \mathbf{E}_s^s = \mathbf{E}_s^s,
\]

and similarly,

\[
\mathbf{E}_s^s = \mathbf{E}_s^s = \mathbf{E}_s^s.
\]

This apparent overnotation is only due to the special tensor \(\mathbf{E}_s^s\). Continuing the transformation to a physical tensor

\[
\mathbf{E}_s^s \cdot \mathbf{E}_s^s = \mathbf{E}_s^s,
\]
\[
\mathbf{E}_s^s \cdot \mathbf{E}_s^s = \mathbf{E}_s^s,
\]
\[
\mathbf{E}_s^s \cdot \mathbf{E}_s^s = \mathbf{E}_s^s,
\]

shows that also the physical tensors are equal:

\[
\mathbf{E}_s^s = \mathbf{E}_s^s = \mathbf{E}_s^s.
\]

Now it may seem strange that the same physical tensor should represent both the column and row subspaces, which are different. But we must remember that a tensor never represents a base. That is, the same tensor \(\mathbf{E}_s^s\) may become base \(\mathbf{E}_s^s\) or base \(\mathbf{E}_s^s\) by different transformations.

More to the point in question here, the tensor \(\mathbf{E}_s^s\) transformed to \(\mathbf{E}_s^q\) and transformed to base \(\mathbf{E}_s^q\), produces different base vectors. In the case of \(\mathbf{E}_s^s\), the transformations to base \(\mathbf{E}_s^q\) and to \(\mathbf{E}_s^q\) produce different partial bases. We find that the column base \(\mathbf{E}_s^q\) transformed to any other full base in this form, \(\mathbf{E}_s^q\) or \(\mathbf{E}_s^q\), represents the same partial base, i.e., subspace. And the row base \(\mathbf{E}_s^q\), transformed to any other full base in this form, \(\mathbf{E}_s^q\) or \(\mathbf{E}_s^q\), represents the same other partial base and subspace. Indeed, we can show by consistent transformations that

\[
\mathbf{E}_s^q \cdot \mathbf{v}_s = \mathbf{v}_s^q,
\]
\[
\mathbf{v}_s \cdot \mathbf{E}_s^q = \mathbf{v}_s^q,
\]

shows that also the physical tensors are equal:

\[
\mathbf{E}_s^s = \mathbf{E}_s^s = \mathbf{E}_s^s.
\]
where \( \vec{v}_k^s \) is the same vector \( \vec{v}_k \) as in eqn.(19.2), and \( \vec{v}_k^a \) is the same vector \( \vec{v}_k \) as in eqn.(19.19), which are two physically different vectors. We note that \( \vec{E}_k^s \) is generally not a symmetric matrix, i.e., \( \vec{E}_k^a \) is not a symmetric tensor. We can also in a reverse manner produce the two partial bases

\[
\vec{E}_k^b \cdot \vec{E}_b = \vec{E}_k^b,
\]

\[
\vec{E}_b^b \cdot \vec{E}_k = \vec{E}_k^b.
\]

The need for the double subbase indices starts in the result of the multiplications

\[
\begin{align*}
\vec{E}_k^b \cdot \vec{I}_b &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \vec{E}_k^b, \\
\vec{I}_b^b \cdot \vec{E}_k &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \vec{E}_k^b,
\end{align*}
\]

(19.30)

(19.31)

although the two different matrix symbols on the right have the same value.

We extend now the sub and partial concepts to a (2\(^{\text{nd}}\) order) tensor \( \vec{K} \) by the transformations with the partial bases:

\[
\begin{align*}
\vec{E}_k^b \cdot \vec{E}_b &= \vec{K}_k^b = \begin{bmatrix} \vec{K}_k^b \\ 0 \end{bmatrix}, \\
\vec{I}_b^b \cdot \vec{E}_k &= \vec{K}_k^b = \begin{bmatrix} 0 \\ \vec{K}_k^b \end{bmatrix}, \\
\vec{E}_k^b \cdot \vec{K}_k^b &= \vec{I}_b^b = \begin{bmatrix} \vec{K}_k^b \\ \vec{K}_k^b \end{bmatrix}, \\
\vec{I}_b^b \cdot \vec{K}_k^b &= \vec{I}_b^b = \begin{bmatrix} 0 \\ \vec{K}_k^b \end{bmatrix}, \\
\vec{E}_k^b \cdot \vec{K}_k^b \cdot \vec{E}_b &= \vec{K}_k^b = \begin{bmatrix} \vec{K}_k^b \\ 0 \end{bmatrix}, \\
\vec{I}_b^b \cdot \vec{K}_k^b \cdot \vec{E}_b &= \vec{K}_k^b = \begin{bmatrix} 0 \\ \vec{K}_k^b \end{bmatrix}.
\end{align*}
\]

(19.32)

(19.33)

(19.34)

(19.35)

(19.36)

(19.37)

(19.38)

(19.39)
All the matrices that are produced in the above eqns. (19.32) to (19.39), i.e., the collection of all the partitions, represent full dimensional partial tensors, and can be transformed to any other skew base q or Cartesian base s. But generally the partitions will then not remain. The partial tensor matrices are full-dimensional matrices. They are all different, and both sub and superscripts as part of their names are required for complete distinction. We can also write them in space notation

\[
\begin{align*}
\bar{K} \cdot \bar{E}_k &= \bar{K}_k^b \\
\vdots \\
\bar{E}_l \cdot \bar{K} \cdot \bar{E}_l &= \bar{K}_l^l.
\end{align*}
\]

(19.40)

The partitioning of the matrix \( \bar{K}_b^b \) into submatrices can be done with the subbases k and l in base b only, by the transformations

\[
\begin{align*}
\bar{K}_b^b \cdot \bar{E}_k^b &= \bar{K}_k^b, \\
\bar{K}_b^b \cdot \bar{E}_l^b &= \bar{K}_l^b, \\
\bar{E}_k^b \cdot \bar{K}_b^b &= \bar{K}_k^k, \\
\bar{E}_l^b \cdot \bar{K}_b^b &= \bar{K}_l^l, \\
\end{align*}
\]

(19.41)

which are the separate column and row submatrices of eqns. (19.32) to (19.35). Further transformation produces the partitions

\[
\begin{align*}
\bar{E}_b^b \cdot \bar{K}_b^b \cdot \bar{E}_k^b &= \bar{K}_k^k, \\
\bar{E}_b^b \cdot \bar{K}_b^b \cdot \bar{E}_l^b &= \bar{K}_l^l, \\
\bar{E}_l^b \cdot \bar{K}_b^b \cdot \bar{E}_k^b &= \bar{K}_k^k, \\
\bar{E}_l^b \cdot \bar{K}_b^b \cdot \bar{E}_l^b &= \bar{K}_l^l, \\
\end{align*}
\]

(19.42)

which partitions the whole matrix as

\[
\bar{K}_b^b = \begin{bmatrix}
\bar{K}_{k_k}^k & \bar{K}_{k_l}^l \\
\bar{K}_{l_k}^k & \bar{K}_{l_l}^l
\end{bmatrix}
\]

(19.43)

In the equations above, the base b can be transformed to any other full base, while subbases k and l can only be transformed within the subspace, and the partitions remain partitions under these conditions.

The double subscripts on matrices are commonly used where partitions of matrices occur as typically in Finite Element applications [48]. We have extended them here by the base symbols to distinguish between partial and sub bases and vectors, and by the space notation to include the physical tensors, albeit abstract. The only dependence that we have not included in the subnotation is that k and l are partitions in base b. This is where we have made a compromise between short new names and long names that indicate the chain of dependence, as mentioned previously in Section 15.1.

The numerical form in which subbases are given depend on the base that is presently used. If this is a generally skew base \( \bar{E}_q \), then subbases may be given as \( \bar{E}_k^q \) and \( \bar{E}_l^q \) where k and l are the partitions of a combined base given as transformation matrix \( \bar{E}_b^q \). The corresponding full dimensional partial bases are \( \bar{E}_k^q_b \) and \( \bar{E}_l^q_b \). The partial bases formally satisfy the equations of partitioning a tensor,
\[
\begin{align*}
\overline{E}_b^q \cdot \overline{E}_b^b &= \overline{E}_b^q, \\
\overline{E}_b^q \cdot \overline{E}_b^l &= \overline{E}_b^l,
\end{align*}
\]  
\tag{19.44}

according to eqns.(19.32) and (19.33), and the subbases the equations for the partitions by

\[
\begin{align*}
\overline{E}_b^q \cdot \overline{E}_k^b &= \overline{E}_k^q, \\
\overline{E}_b^q \cdot \overline{E}_l^b &= \overline{E}_l^q,
\end{align*}
\]  
\tag{19.45}

according to eqns.(19.41). The partitioning is always defined in one particular base b.

The half-inverse subbases are found by inverting the transformation matrix \[\left[ \overline{E}_b^q \right]^{-1} = \overline{E}_q^b,\]
and then doing the same partitioning of rows, which correspond to the formal equations for the partial bases

\[
\begin{align*}
\overline{E}_b^k \cdot \overline{E}_q^b &= \overline{E}_q^k, \\
\overline{E}_b^l \cdot \overline{E}_q^b &= \overline{E}_q^l,
\end{align*}
\]  
\tag{19.46}

according to eqns.(19.34) and (19.35), and for the partitioning into subbases

\[
\begin{align*}
\overline{E}_b^k \cdot \overline{E}_q^b &= \overline{E}_q^k, \\
\overline{E}_b^l \cdot \overline{E}_q^b &= \overline{E}_q^l,
\end{align*}
\]  
\tag{19.47}

according to eqns.(19.41).

Note particularly that once the column base \(\overline{E}_b^q\) and its partition is given, all the remaining operations that lead to eqns.(19.46) and (19.47) are independent of the metric of base \(\overline{E}_q^b\). The corresponding operations can be carried out by the inverse base \(\overline{E}_q^b\) and its partitioning in base \(b\).

As a final remark, all the statements about subspace are interpreted geometrically, independent of time. Therefore, subspaces may change with time, or any other one or more scalar parameters that are identified as not being a coordinate of space, and all the equations derived are still valid.

### 19.2. Projection

The partial vector lying in a subspace is also called a projection. But in skew bases there are two kinds of projections, an orthogonal or normal projection, and a skew projection, depending on the way the subbase and its complement are given. The part of the vector lying in the subbase depends on the kind of projection done, just like the component of a vector in one base vector direction depends on the other base vectors.

A subspace on which a projection can be done, may be one partition \(\overline{E}_k^k\) of a present skew base \(\overline{E}_q\), so that the complementary subbase \(\overline{E}_l^l\) is the other partition of \(\overline{E}_q\). The result is that the subbase and its complement is known, and that they are skew. Analog to a geometrically skew projection in 3-dimensional Euclidean space, the skew projection of the physical vector \(v\) is defined as the vector component \(\overline{E}_k^k \cdot \overline{v}^k\). This definition is based on observation that complementary component is then \(\overline{E}_l^l \cdot \overline{v}^l\), according to eqn.(19.18), such that the vector is separated into the two skew components as shown diagramatically in Fig.13(a). The definition of the projection is that the projected vector is a physical vector (even if abstract), not a scalar vector array. Our definition of projection does not define the scalar components of the projection in any base, which are dependent on the magnitudes of the base vectors in which the final calculation is done.
The skew projection can only be done if the complement of the subspace is known. This concept is the generalization of the ordinary vector component $v_\alpha$, which is a skew projection on the 1-dimensional subspace (now the base vector) $e_\alpha$.

Unlike in Cartesian base, the skew projection consists of two steps, of which eqn.(19.17) is the second. The first step is to obtain the scalar components $v_k$ from $v$ by eqn.(19.16). This requires the row subbase $E_k k$, which can only be obtained from the given column subbase if the complement is also known, via the inversion that leads to eqn.(19.6) from which the half-inverse subspace is extracted.

Strangely enough, the eqn.(19.16) looks like a normal projection on the subbase $E_k k$ rather than a skew one on the subbase $E_k k$. But this is not the case for the following reason. The projection is not defined as the scalar vector array, and the half-inverse $E_k k$ of subbase $E_k k$, which appears in eqn.(19.16), does not span the same physical space. This is true even if the base $E_k k$ consisted of orthonormal base vectors, like a Cartesian subbase, because the half-inverse depends on the skewness of the full base $E_b$ according to eqn.(19.6). We find therefore that the projection of $v$ consists of the sequence of the two operations implied by eqns.(19.16) and (19.17), respectively, which result in

$$v^k = E_k k \cdot E_k k \cdot v^k,$$

where the product $E_k k \cdot E_k k$ can be interpreted as a partial space tensor of eqn.(19.29), operating on $v$ to produce the projection.

The actual computation must take place in a full base, which may be the skew base $E_b$. Then eqn.(19.48) is transformed to

$$v^b = E_k k \cdot E_k k \cdot v^b,$$

where the product $E_k k \cdot E_k k$ is the matrix $E_k k$ of eqn.(19.7). The conclusion is that the partial vector array $v^b$ represents the vector array, measured in base $b$, of the skew projection of $v$ on the partial base $E_k k$, where the skewness is due to base $E_b$. The projection is the vector $v^k$ lying in the column subspace $k$. The complementary vector component $v^{\perp}$ is the skew projection of $v$ on the partial base $E_{l \perp l}$, constructed by the equations

$$v^{\perp} = E_{l \perp l} \cdot E_{l \perp l} \cdot v,$$

Figure 13. Projections.
By the same steps, the partial vector array \( \mathbf{v}_k^b \) represents the vector array, measured in row base \( b \), of the skew projection of \( \mathbf{v} \) on the partial base \( \overline{E}_k^k \), the projection is the vector \( \mathbf{v}_k \) lying in the row subspace \( k \).

For the skew projections in eqns.(19.48) and (19.50), the computations can be done in any skew base \( q \) in the place of the indicated space notation. Since the metric of this base doesn’t enter the computations at all, the metric doesn’t even have to be known. Of course it will then also not be known if the subbases \( \overline{E}_k^k \) and \( \overline{E}_l^l \) are orthogonal, and the projections are normal projections.

If for some reason a subspace is produced during the course of an analysis, in which a skew base \( \overline{E}_q^q \) is used, a tangent subbase \( \overline{E}_k^q \) or partial base \( \overline{E}_b^q \) will exist or can be constructed for every point in the subspace, depending on the form in which the subspace is defined. The partial base \( \overline{E}_b^q \) is the matrix \( \overline{E}_k^q \) appended with zero columns, even though base \( b \) is not yet known. We may not call this partial base \( \overline{E}_k^q \), which can only be obtained by a further transformation.

If no complement is given, then a normal projection on the subspace is the only one that makes sense. However there is no way that a normal projection can be made without knowing the metric \( \overline{E}_q^q \) of base \( \overline{E}_q^q \). The vector to be projected can be given as row \( \mathbf{v}_q \), or it can be given as column \( \mathbf{v}_q^T \), from which it can be transformed into a row

\[
\mathbf{v}_q^T = \left[ \mathbf{v}^T \right]^T = \left[ \overline{E}_q^q \cdot \mathbf{v}_q \right]^T.
\]

The sequence of operations for the normal projection may be done as follows. First, a complementary column subbase which is normal to the given subbase \( \overline{E}_k^q \), must be found, so that eqn.(19.3), transformed to skew base \( q \) as

\[
\mathbf{v}_q^T = \mathbf{v}_q^T + \mathbf{v}_q^T = \overline{E}_k^q \cdot \mathbf{v}_q + \overline{E}_l^q \cdot \mathbf{v}_q^T
\]

contains two normal partial bases. Of course a normal rowbase \( \overline{E}_l^l \) could be constructed without the aid of the metric by solving

\[
\overline{E}_l^l \cdot \overline{E}_k^q = \overline{E}_k^l,
\]

which is the submatrix \( \overline{U} \) of eqn.(19.31), but by definition of the symbols, \( \overline{E}_l^l \) is the half-inverse, which is not generally a normal complement. We note here the similarity with the formula for the cross product in skew base, eqn.(15.69) of Section 15.8, that a vector normal to a given surface of two column vectors appears as rowvector.

The true normal complementary subbase can be found in transposed base \( q \), which is \( \tau q \), by solving the equation

\[
\left[ \overline{E}_k^q \right]^T \cdot \overline{E}_l^q \cdot \mathbf{v}_q = \mathbf{0}_l^k.
\]

The explicit name for the transposed matrix in the equation above would be \( \overline{E}_k^q \tau q \). The computed subbase is transformed to posed base \( q \) by

\[
\overline{E}_q^q \cdot \overline{E}_l^q \cdot \mathbf{v}_q = \overline{E}_l^q.
\]

Now continue as with the skew projection, assemble the subbases as in eqn.(19.12), but in base \( q \), to

\[
\overline{E}_b^q = \left[ \overline{E}_k^q \mid \overline{E}_l^q \right].
\]
The inverse $\overline{E}_q^b = \overline{E}_q^b$ is partitioned horizontally, and the upper partition extracted, which is formally be done by eqn.(19.41) in base $q$ by
\[ \overline{E}_q^k \cdot \overline{E}_q^b = \overline{E}_q^k. \]
The subvector is obtained by eqn.(19.16) in base $q$ by
\[ \overline{E}_q^k \cdot v^q = v^k, \]
and this is transformed to the partial vector in base $q$ by
\[ \overline{E}_q^k \cdot v^k = v^q, \]
which is the projection as required.

A slightly more satisfactory method is to call the given column subbase $\overline{E}_q^k u$, then to orthonormalize the column vectors, using the metric $\overline{E}_q^q$, and call the new subbase $\overline{E}_q^q_k$, which is now orthonormal within the subspace. Also give the complement row subbase from eqn.(19.54) the name $\overline{E}_q^q v$, then orthogonalize the rowvectors and call the result $\overline{E}_q^q_r$. The advantage is that the inverted matrix of eqn.(19.56) will not produce two other subspaces.

The operation of eqn.(19.49), which may be carried out in any other base, corresponds to a perfect filtering process of the total vector $v$, by the filter $\overline{E}_q^b_k$, such that all the components of the complementary partial base are filtered out. The choice of the collection of $k$ vectors corresponds to a “band” with bandwidth $k$ of a filter. Put the other way round, a perfect filtering process can be perceived as a projection on a partial base in abstract space. This gives a geometrical meaning to a filter. This filter concept is based on the corresponding process of frequency analysis, put into tensor form in Section 12.1 (Part I). A time base vector (sampled function of time) $\overline{f}^t$ is transformed to a frequency base vector $\overline{f}^\omega$ by eqn.(72). A partition can be extracted by the operation, using $F$ as name for function space base,
\[ \overline{F}_\omega^k \cdot \overline{f}^\omega = \overline{f}^k, \]
where $k$ is a partition of frequency base $\omega$. But a partition is a subvector that cannot be transformed in full space; therefore, the partial vector is obtained by the operation
\[ \overline{F}_\omega^k \cdot \overline{f}^\omega = \overline{f}^k, \]
which corresponds to the method of Fourier Transform of using the full dimensional frequency vector. The inverse transformation by eqn.(71) (Part I) produces the filtered time function
\[ \overline{f}^t = \overline{F}^t_{\omega} \cdot \overline{f}^\omega \]
\[ = \overline{F}^t_{\omega} \cdot \overline{F}_\omega^\omega \cdot \overline{F}_\omega^\omega \cdot \overline{f}^t. \]
The new tensor $\overline{F}_t^k = \overline{F}^t_{\omega} \cdot \overline{F}_\omega^\omega \cdot \overline{F}_\omega^\omega$ is the filter. Because the Fourier Transform uses an orthogonal matrix, both column subspace and the half-inverse row subspace are the same, and there is only one projection and one filtered time vector. For partitions in a skew base two types of filtering corresponding to the two possible skew projections would have to be considered.

The partitioning of a tensor equation like
\[ \overline{K} \cdot \overline{u} = \overline{f} \]
is performed formally by the corresponding premultiplication
\[ \overline{E}_q^k \cdot \overline{K} \cdot \overline{u} = \overline{E}_q^k \cdot \overline{f} = \overline{f}^k, \]
\[ \overline{E}_q^l \cdot \overline{K} \cdot \overline{u} = \overline{E}_q^l \cdot \overline{f} = \overline{f}^l. \]
which produces the subvectors $\vec{f}^k$ and $\vec{f}^l$, where the partitioning is made in some specified base $\vec{E}_b$. The vector $\vec{u}$ is the sum of the partial vectors according to eqn.(19.18)

$$\vec{u} = \vec{E}_{k_k} \cdot \vec{u}^k + \vec{E}_{l_l} \cdot \vec{u}^l,$$

which substituted in eqn.(19.58) becomes

$$\vec{E}_k \cdot K \cdot \vec{E}_{k_k} \cdot \vec{u}^k + \vec{E}_l \cdot \vec{E}_{l_l} \cdot \vec{u}^l = \vec{E}_k \cdot \vec{f} = \vec{f}^k,$$

$$\vec{E}_l \cdot \vec{E}_{k_k} \cdot \vec{u}^k + \vec{E}_l \cdot \vec{E}_{l_l} \cdot \vec{u}^l = \vec{E}_l \cdot \vec{f} = \vec{f}^l.$$  \hfill (19.59)

Using the projections of eqn.(19.16), the equations above become

$$\vec{E}_k \cdot \vec{E}_{k_k} \cdot \vec{u}^k + \vec{E}_l \cdot \vec{E}_{l_l} \cdot \vec{u}^l = \vec{E}_k \cdot \vec{f} = \vec{f}^k,$$

$$\vec{E}_l \cdot \vec{E}_{k_k} \cdot \vec{u}^k + \vec{E}_l \cdot \vec{E}_{l_l} \cdot \vec{u}^l = \vec{E}_l \cdot \vec{f} = \vec{f}^l,$$  \hfill (19.60)

which is the complete formal procedure to transform the eqn.(19.57) into an equation of partitions

$$\vec{K}_k \cdot \vec{u}^k + \vec{K}_l \cdot \vec{u}^l = \vec{f}^k,$$

$$\vec{K}_l \cdot \vec{u}^k + \vec{K}_l \cdot \vec{u}^l = \vec{f}^l.$$  \hfill (19.61)

where the partitions of the tensor are those of eqn.(19.42) and the partitions of the vectors those of eqn.(19.16).

Instead of subvectors and subtensors, the equations can be transformed by the same steps to partial vectors and partial tensors

$$\vec{K}_k \cdot \vec{u}^b + \vec{K}_k \cdot \vec{u}^b = \vec{f}^k,$$

$$\vec{K}_l \cdot \vec{u}^b + \vec{K}_l \cdot \vec{u}^b = \vec{f}^l,$$  \hfill (19.62)

where now the partials of the tensor are the complete matrices of eqns.(19.36) to (19.39), and the partials of the vectors those of eqn.(19.5). Eqn.(19.62) can be transformed to any other base

$$\vec{K}^k \cdot \vec{u}^k + \vec{K}^l \cdot \vec{u}^l = \vec{f}^k,$$

$$\vec{K}^l \cdot \vec{u}^k + \vec{K}^l \cdot \vec{u}^l = \vec{f}^l.$$  \hfill (19.63)

Similar to the ideal filtering or projection of a vector, the partial tensors are filtered or projected tensors, but if $\vec{E}_b$ is a skew base, the projections are on four different subspaces rather than two.

Projection and ideal filtering are equivalent operations corresponding to partition of equations in selected generally skew bases. Note, however, from the inequalities

$$\vec{K}^k \cdot \vec{u}^k \neq \vec{f}^k,$$

$$\vec{K}^l \cdot \vec{u}^l \neq \vec{f}^l,$$  \hfill (19.64)

that the projection of an equation is not equal to the equation of projections—unless the diagonal subtensors, $\vec{K}^k$ and $\vec{K}^l$, are zero.
20. PARTIAL TRANSPOSE

The limits that were imposed by vector algebra were removed by matrix algebra, but the limit is only pushed one order higher, while Tensor notation doesn’t have the order limits. Although higher than 2\textsuperscript{nd} order tensors may be less important, they do occur in simple mechanical situations, e.g., in the linear relation between stress and strain tensors. Every scalar product can be reduced to a matrix multiplication, and therefore it seems warranted to reduce higher order tensors to lower order arrays in order to maintain matrix algebra. This is, in fact, done in practice in the stress-strain relationship, where the two tensors are arranged as 6-dimensional vectors of their differing elements. Their relation is then a 6×6 matrix, called an elasticity matrix \cite{48}, which is, however, actually a 4\textsuperscript{th} order tensor and in tensor notation is denoted as such by $C_{ijkl}^{ij}$ (see \cite{2,26}).

Matrices that are stored in a 1-dimensional array, like in computer memory, typically use their indices as 2-digit counter to determine the logic sequence, e.g., the array $A_{ij}$, $i,j = 1$ to 3 is stored as the sequence $A_{11}, A_{12}, A_{13}, A_{21}, A_{22}, A_{23}, A_{31}, A_{32}, A_{33}$, practically using a trinary number. This operation corresponds to putting the rows of the matrix into one long row, called stretching (see \cite{23}). Flattening, used by \cite{50}, seems less appropriate for our purpose. Reference \cite{23} has developed an elaborate symbolic calculus using the symbols $V$ and $A$ to make all thinkable rearrangements of array elements possible.

To introduce this operation, consider a matrix as a row of columns, i.e., as a row vector that has column vectors as elements, a hyper-rowvector. Transposing only the hyper-rowvector, we obtain a hyper-columnvector with columnvectors as elements, which is one longer column. This operation can be written symbolically by conventional matrix symbols as the following sequence:

\[
\overline{A} = \begin{bmatrix} \bar{A} \end{bmatrix}, \text{ Transpose: } \begin{bmatrix} \bar{A} \end{bmatrix}^\top = \overline{A}^\top \equiv \bar{A}^\top
\]

We call this a \textit{partial transpose}, particularly the \textit{outer} partial transpose. The previous customary transpose of vectors and matrices is called \textit{total transpose} if a distinction is necessary or desirable.

The right columnvector we simply call a stretched matrix. It still represents the same matrix, only by a different structure; therefore, we don’t call it a vector. We will also identify the transposed part by attaching the transpose symbol $\tau$, which must go above the inner column symbol, by which the sequence of elements is strictly defined.

For our purpose, we only need the stretching of a matrix into a column and the inverse process, but for symmetry we define another partial transpose to stretch a matrix into a row, by applying the transpose to the hyper-column to obtain a hyper-row:

\[
\overline{A} = \begin{bmatrix} \bar{A} \end{bmatrix}, \text{ Transpose: } \begin{bmatrix} \bar{A} \end{bmatrix}^\top = \begin{bmatrix} \bar{A} \end{bmatrix}_{\tau} \equiv \bar{A}_{\tau}
\]

Again the transpose symbol $\tau$ must be attached to the transposed part which goes below the inner row symbol. How the partial transpose operation acts upon the base symbols is shown by tracing the base symbols in the following.
Outer Partial Transpose

Stretch matrix with base symbols into column:

\[ \overrightarrow{\mathcal{A}}^a_b = \overrightarrow{\mathcal{A}}^a_b \text{, Transpose: } \left[ \overrightarrow{\mathcal{A}}^a_b \right]^T = \overrightarrow{\mathcal{A}}^\tau_b \equiv \overrightarrow{\mathcal{A}}^\tau_b \]  

(20.1)

Stretch matrix with base symbols into row:

\[ \overrightarrow{\mathcal{A}}^a_b = \overrightarrow{\mathcal{A}}^a_b \text{, Transpose: } \left[ \overrightarrow{\mathcal{A}}^a_b \right]^T = \overrightarrow{\mathcal{A}}^\tau_b \equiv \overrightarrow{\mathcal{A}}^\tau_b \]  

(20.2)

When a stretched matrix is transposed, all the rules of the total transpose of a matrix apply, which means that the vector symbols must get a transposed symbol, and the matrix must also get a transposed symbol. We also specify that the sequence of vector bars from inside to outside remains the same, i.e., the transposed vector bars are a mirror image of the bars before transposing.

\[ \left[ \overrightarrow{\mathcal{A}}^a_b \right]^T = \overrightarrow{\mathcal{A}}^\tau_a, \quad \left[ \overrightarrow{\mathcal{A}}^a_b \right]^T = \overrightarrow{\mathcal{A}}^\tau_b \]  

(20.3)

On the other hand, a matrix can be transposed first by a total transpose and then by a partial transpose to obtain the same result. All these possible operations are shown diagramatically by eqns.(20.6), (20.7), (20.8), and (20.9) in the following Table 1. The 2-dimensional arrays we have used to show all details of element positions are sufficient for any dimension of array, and in the following, we use only 2-dimensional arrays in the examples.

For purpose of symmetry, or completeness, let us define the two corresponding operations where the inner elements of the hyper-row are transposed, which we call the Inner Partial Transpose.

Inner Partial Transpose

Stretch matrix with base symbols into row:

\[ \overrightarrow{\mathcal{A}}^a_b = \overrightarrow{\mathcal{A}}^a_b \text{, Transpose: } \left[ \overrightarrow{\mathcal{A}}^a_b \right]^T = \overrightarrow{\mathcal{A}}^\tau_a \equiv \overrightarrow{\mathcal{A}}^\tau_a \]  

(20.4)

Stretch matrix with base symbols into column:

\[ \overrightarrow{\mathcal{A}}^a_b = \overrightarrow{\mathcal{A}}^a_b \text{, Transpose: } \left[ \overrightarrow{\mathcal{A}}^a_b \right]^T = \overrightarrow{\mathcal{A}}^\tau_b \equiv \overrightarrow{\mathcal{A}}^\tau_b \]  

(20.5)

How the two operations of eqns.(20.4) and (20.5) work on the elements of the matrix is apparent from the first and last terms in eqns.(20.6) and (20.9), respectively; i.e., these operations do not produce a new form, they are just convenient. However, we notice that the transpose sign on the matrix must be added when the inner transpose is done. This is a consequence of the definition of the outer transpose being not to add the transpose sign, yet to be compatible. The result is that stretching to a column is not symmetric to stretching to a row, rather its transpose. But in any case, we obtain two forms of stretched rows and two forms of stretched columns, allowing all possible freedom.
Table 1.
In the end, we don’t have to indicate the formal operations of eqns. (20.1), (20.2), (20.4), and (20.5) any more, but simply use the stretched form.

Note also that in eqns. (20.6), (20.7), (20.8), and (20.9) the matrix elements are indicated as values of the initial posed matrix, not their names, because in each differently posed form the elements have different names. If the initially posed matrix has already a transpose sign, then this must be transposed as well according to the rules. As an example, let us stretch all 8 possible different forms of an unsymmetric tensor $\bar{K}$ in skew base $q$ into a column and into a row in the same sequence of outer transpose, inner transpose, outer transpose, inner transpose as shown in Table 2.

| Table 2. |
|-----------------|-----------------|-----------------|-----------------|
| $\bar{K}^\tau_a$ | $\Rightarrow$ | $\bar{K}^\tau_a$ | $\Rightarrow$ | $\bar{K}^\tau_a$ | $\Rightarrow$ | $\bar{K}^\tau_a$ | $\Rightarrow$ | $\bar{K}^\tau_a$ |
| $\bar{K}^\tau_a$ | $\Rightarrow$ | $\bar{K}^\tau_a$ | $\Rightarrow$ | $\bar{K}^\tau_a$ | $\Rightarrow$ | $\bar{K}^\tau_a$ | $\Rightarrow$ | $\bar{K}^\tau_a$ |
| $\bar{K}^\tau_a$ | $\Rightarrow$ | $\bar{K}^\tau_a$ | $\Rightarrow$ | $\bar{K}^\tau_a$ | $\Rightarrow$ | $\bar{K}^\tau_a$ | $\Rightarrow$ | $\bar{K}^\tau_a$ |
| $\bar{K}^\tau_a$ | $\Rightarrow$ | $\bar{K}^\tau_a$ | $\Rightarrow$ | $\bar{K}^\tau_a$ | $\Rightarrow$ | $\bar{K}^\tau_a$ | $\Rightarrow$ | $\bar{K}^\tau_a$ |
| $\bar{K}^\tau_a$ | $\Rightarrow$ | $\bar{K}^\tau_a$ | $\Rightarrow$ | $\bar{K}^\tau_a$ | $\Rightarrow$ | $\bar{K}^\tau_a$ | $\Rightarrow$ | $\bar{K}^\tau_a$ |
| $\bar{K}^\tau_a$ | $\Rightarrow$ | $\bar{K}^\tau_a$ | $\Rightarrow$ | $\bar{K}^\tau_a$ | $\Rightarrow$ | $\bar{K}^\tau_a$ | $\Rightarrow$ | $\bar{K}^\tau_a$ |
| $\bar{K}^\tau_a$ | $\Rightarrow$ | $\bar{K}^\tau_a$ | $\Rightarrow$ | $\bar{K}^\tau_a$ | $\Rightarrow$ | $\bar{K}^\tau_a$ | $\Rightarrow$ | $\bar{K}^\tau_a$ |

Choosing a particular stretched configuration we may write the invariant forms of eqn. (18.18) and (18.27) as

$$K_m \cdot \bar{u}^m + M_m \cdot \bar{u}^m = f_m, \quad (20.10)$$

$$K_e \cdot \bar{u}^e + M_e \cdot \bar{u}^e = f_e, \quad (20.11)$$

respectively, which have the same form as the corresponding tensor equations.

Finding the meanings of the stretched matrices from Table 2, it seems as if $\bar{K}_r$ can have two meanings, which are distinguished in Table 1 by $\bar{K}_a$ and $\bar{K}_b$, and this distinction is not visible any more if $a = b$. But tracing the operations by eqns. (20.2) and (20.4) shows that they are the same if $a = b$; therefore, the stretched matrix symbols are unique.

Not so are the multiplications. As soon as a matrix from an expression $\bar{K}_b \cdot \bar{u}$ is stretched, then multiplication must be incorporated in the stretching operations of eqns. (20.1) and (20.2). The stretched form from the outer partial transpose of $\bar{K}_b \cdot \bar{u}$ is from eqn. (20.2)

The outer partial transpose of $\bar{K}_b \cdot \bar{u}$ is from eqn. (20.2)

$$\bar{K}_b \cdot \bar{u} = \left[ \bar{K}_b \cdot \bar{u} \right]_a = \left[ K_{b,a1} \cdot \bar{u}, K_{b,a2} \cdot \bar{u} \right] = K_{b,a} \cdot \bar{u},$$

from which it is clear what $K_{b,a} \cdot \bar{u}$ means.
On the other hand the stretched form from the inner partial transpose of $\overline{K}_b^\tau \cdot \overline{u}^b$ from eqn.(20.4) is

inner partial transpose of $\overline{K}_b^\tau \cdot \overline{u}^b = \left[ K^\tau_{-a,b} \right] \overline{u}^b$

$$= \left[ K^\tau_{-a,b1} \cdot u^{b1} + K^\tau_{-a,b2} \cdot u^{b2} \right] = K^\tau_{-a,b} \cdot \overline{u}^b,$$

which clarifies the exact meaning of $K^\tau_{-a,b} \cdot \overline{u}^b$. But the problem is that the distinction between the two operations is only indicated by the different base symbols $a$ and $b$, where $a$ and $\tau a$ also count as different base symbols. When the expression is in only one base, $a = b$, as in $\overline{K}_r^\tau \cdot \overline{u}^r$, the multiplication is not clearly defined, unless the tensor matrix is symmetric.

For such distinction, we apply two rules which may be used alternatively according to the situation:

1. If the scalar product is with the inner row-bar, it is indicated by a $\cdot$, if it is with the outer row-bar, it is indicated by a $\circ$.

$$K_{rj} \cdot \overline{u}^r \equiv K_{r}^\circ \overline{u}^r, \quad (20.12)$$

$$K_{rj} \cdot \overline{u}^r \equiv K_{r}^\circ \overline{u}^r. \quad (20.13)$$

2. Append the affected base symbols with tensor indices.

$$\left[ K_{rj} \cdot \overline{u}^r \right] \equiv K_{rj}^\circ \overline{u}^r, \quad (20.14)$$

$$\left[ K_{rj} \cdot \overline{u}^r, \quad K_{rj} \cdot \overline{u}^r \right] \equiv K_{rj}^\circ \overline{u}^r. \quad (20.15)$$

where the tensor index overrides compatible base symbols so that certain equalities can be expressed. In all cases, the sequence as in matrix algebra must be kept. If the tensor matrix is symmetric, the multiplication of the elements is not the same but the result is, and then also no distinction is necessary.

By the same method, the product of a row by a matrix is stretched

outer partial transpose of $u^\tau a \cdot \overline{K}_b^\tau = \left[ K_{b,a1} \cdot u^{a1} + K_{b,a2} \cdot u^{a2} \right] = K^\tau_{b,a} \cdot \overline{u}^a$,

inner partial transpose of $u^\tau a \cdot \overline{K}_b^\tau = \left[ K^\tau_{-a,b1} \cdot \overline{u}^{b1}, \quad K^\tau_{-a,b2} \cdot \overline{u}^{b2} \right] = K^\tau_{-a,b} \cdot \overline{u}^a$.

The products when the matrix is stretched to a column, become

outer partial transpose of $\overline{K}_b^\tau \cdot \overline{u}^a = \overline{u}_b \cdot \overline{K}^\tau_a$,

inner partial transpose of $\overline{K}_b^\tau \cdot \overline{u}^a = \overline{u}_b \cdot \overline{K}^\tau_a$,

outer partial transpose of $u^a \cdot \overline{K}_b^\tau = u^a \cdot \overline{K}^\tau_a$,

inner partial transpose of $u^a \cdot \overline{K}_b^\tau = u^a \cdot \overline{K}^\tau_a$. 
The same rules apply to space notation, defining the expressions

\[ \frac{\vec{K}_a \cdot \vec{u}}{\vec{K}_a} \cdot \vec{u}, \frac{\vec{K}_a \cdot \vec{u}}{\vec{K}_a} \cdot \vec{u}, \frac{\vec{K}_a \cdot \vec{u}}{\vec{K}_a} \cdot \vec{u}, \frac{\vec{K}_a \cdot \vec{u}}{\vec{K}_a} \cdot \vec{u}, \frac{\vec{K}_a \cdot \vec{u}}{\vec{K}_a} \cdot \vec{u}, \text{etc.} \]

The stretched forms of the space tensor matrix, or transformation matrix, become

\[ \overrightarrow{E}_a^b \Rightarrow \overrightarrow{E}_{\tau a}^b, \overrightarrow{E}_{b a}^\tau, \overrightarrow{E}_{b a}^\tau \text{ or } \overrightarrow{E}_{b a}^\tau \Rightarrow \overrightarrow{E}_{b a}^\tau. \]

The stretched form of the metric assumes the form

\[ \overrightarrow{E}_a^\tau \Rightarrow \overrightarrow{E}_{a \tau}^a \text{ or } \overrightarrow{E}_{a \tau}^a \Rightarrow \overrightarrow{E}_{a \tau}^a, \]

which correspond to \( g_{ij} \) and \( g^{ij} \) of tensor notation, respectively. The transformation of column to row and vice versa becomes then

\[ \overrightarrow{E}_a^\tau \Rightarrow \overrightarrow{E}_a^\tau, \overrightarrow{E}_a^\tau \Rightarrow \overrightarrow{E}_a^\tau, \]

remembering that the metric matrix is symmetric, and therefore, the multiplication sequence doesn't have to be indicated. These forms are useful to show the connection between the tensor notation and the tensor matrix notation.

The stretched form of the identity transformation matrix becomes

\[ \overrightarrow{E}_a^a \Rightarrow \overrightarrow{E}_a^\tau, \overrightarrow{E}_a^\tau \Rightarrow \overrightarrow{I}. \]

To stretch a tensor \( \overrightarrow{K}_{ia} \) by outer partial transpose in the product

\[ \overrightarrow{A}^a_{\tau b} \cdot \overrightarrow{K}^b_{ic} = \overrightarrow{L}^a_{ic} \Rightarrow \overrightarrow{L}^\tau c \]

requires the outer expansion of \( \overrightarrow{A}^a_{\tau b} \) into a 4th order tensor

\[ \overrightarrow{A}^a_{\tau b} \cdot \overrightarrow{K}^b_{ic} = \left[ \overrightarrow{A}^a_{\tau b} \cdot \overrightarrow{K}^b_{ic} \right] \Rightarrow \left[ \overrightarrow{A}^a_{\tau b} \cdot \overrightarrow{K}^b_{ic} \right] \]

\[ = \left[ \overrightarrow{A}^a_{\tau b} \cdot \overrightarrow{K}^b_{ic} \right] \]

\[ = \left[ \overrightarrow{A}^a_{\tau b} \cdot \overrightarrow{K}^b_{ic} \right] \]

\[ \Rightarrow \overrightarrow{L}^\tau c = \overrightarrow{L}^\tau c, \]

where all the indices are determined by compatible multiplication and the form of the result. By eqn.(20.17), the expansion of \( \overrightarrow{A}^a_{\tau b} \) is defined. It is a hyperdiagonal \( \overrightarrow{I} \) multiplied by the hyperscalar \( \overrightarrow{A}^a_{\tau b} \). Of course it is easy enough to write out the linear equations implied by eqn.(20.16) and
to collect the coefficients of the tensor matrix elements of \( \overrightarrow{K} \); therefore, eqn.(20.16) can simply
be read as the consistent notation for just such a coefficient matrix.

The base symbols are compatible, but even without them, the sequence of multiplications are
defined by the sequence of inner and outer vector bars. If the expression is in one base only, then
the base symbols are not sufficient to indicate the required expansion of the tensor \( \overrightarrow{A} \). In that
case, the additional tensor indices are appended, i.e., in Cartesian base

\[
\overrightarrow{A}^s_j \cdot \overrightarrow{K}^s_j = \overrightarrow{L}^s \Rightarrow \overrightarrow{A}^s_j \cdot \overrightarrow{K}^{s_j} = \overrightarrow{L}^s.
\]

This is not the same as the tensor notation would be, because the repeated index \( j \) across the
\( \cdot \) sign means a contraction, while the third index \( j \) indicates only the matching symmetric part,
which is not allowed in tensor notation. We assume that the double letter indices \( sj \) and \( si \) are
not going to be confused with single letter tensor indices \( j \) and \( i \).

The difference with tensor notation is due to the fact that in tensor notation the expression
would be

\[
\overrightarrow{A}^k_i \cdot \overrightarrow{K}^i_j = \overrightarrow{L}^k.
\]

with the same result but the tensor matrix \( \overrightarrow{A} \) not expanded. Actually the tensor notation indicates
a much more effective multiplication, avoiding the unnecessary repetition of \( \overrightarrow{A} \) and the empty
spaces in the expanded matrix, corresponding to the same operation in matrix algebra using
hyperelements. But the use of hyperelements in matrix algebra requires brackets, destroying
the conventional meaning of matrix quantities. It is merely a matter to convert the required
multiplications into a matrix operation with scalar elements that the new notation accomplishes.
If the expanded matrix is also used in tensor notation, the tensor expression would become

\[
\overrightarrow{A}^t_k \cdot \overrightarrow{K}^t_l = \overrightarrow{L}^t,
\]

which does not indicate how the expansion of the matrix \( \overrightarrow{A} \) is constructed from the matrix.

Once the expansion is done, we go even one step further and write the larger matrices again
with single vector bars but indicating the larger dimensions, as

\[
\overrightarrow{A}^\tau_c \cdot \overrightarrow{K}^\tau_c = \overrightarrow{L}^\tau \equiv \overrightarrow{A}^{-\tau_c x_a} = \overrightarrow{L}^{-\tau_c x_b}.
\]

Applying the stretching of the matrix multiplication of eqn.(20.16) to a transformation equa-
tion, we obtain

\[
\overrightarrow{E}^a_b \cdot \overrightarrow{K}^b_c = \overrightarrow{K}^a_c \Rightarrow \overrightarrow{E}^\tau_c \cdot \overrightarrow{K}^\tau_c = \overrightarrow{K}^\tau_c \quad (20.18)
\]

\[
\overrightarrow{E}^{\tau_c x_a} \cdot \overrightarrow{K}^{\tau_c x b} = \overrightarrow{K}^{\tau_c x a} \quad (20.19)
\]

Not quite similar is the operation to stretch a tensor \( \overrightarrow{K} \) by inner partial transpose in the
product

\[
\overrightarrow{A}^a_b \cdot \overrightarrow{K}^b_c = \overrightarrow{L}^a_c \Rightarrow \overrightarrow{L}^\tau_a \quad (20.20)
\]

The inner expansion of \( \overrightarrow{A} \) into a 4\(^{th}\) order tensor is done by exposing the details of the multipli-
cation as follows.
where again all the indices are determined by compatible multiplication and the form of the result. By eqn.(20.18) the expansion of $\mathcal{E}_b$ is defined. It is a hypermatrix of diagonal matrices consisting of a single scalar times $\mathcal{I}$.

The matrix equation with single larger dimensional matrices become

$$\sim \mathcal{E}_b \cdot \mathcal{K}_c = \mathcal{K}_c =: \mathcal{E}_b \cdot \mathcal{K}_c$$

Applying the stretching of the matrix multiplication of eqn.(20.21) to a transformation equation, we obtain

$$\sim \mathcal{E}_b \cdot \mathcal{K}_c = \mathcal{K}_c =: \mathcal{E}_b \cdot \mathcal{K}_c$$

The previous operations can be changed easily to the partial transpose stretching the matrix to a row, by simply totally transposing those equations. The following expressions are obtained, after exchanging the base symbols to adapt to the given expression

$$\sim \mathcal{E}_b \cdot \mathcal{K}_c = \mathcal{K}_c =: \mathcal{E}_b \cdot \mathcal{K}_c$$

which can be shown to coincide with the result of a detailed analysis as done before, verifying that the symbols are consistent. The corresponding transformation equation is

$$\sim \mathcal{E}_b \cdot \mathcal{K}_c = \mathcal{K}_c =: \mathcal{E}_b \cdot \mathcal{K}_c$$

Finally, the inner partial transpose stretching the matrix to a row produces

$$\sim \mathcal{E}_b \cdot \mathcal{K}_c = \mathcal{K}_c =: \mathcal{E}_b \cdot \mathcal{K}_c$$
and the corresponding transformation equation

\[ \widehat{K}^a_b \cdot \widehat{E}^b_c = \widehat{K}^{\tau}_c \Rightarrow K^a_\tau \cdot \widehat{E}^b_\tau_c = K^{\tau}_c a \]

(20.30)

\[ \equiv K^{\tau}_b x a \cdot \widehat{E}^{b x a}_c x a = K^{\tau}_c x a . \]

(20.31)

Even with all these equations not all combinations that may occur are exhausted, as we demonstrate by the following problem, where the unknown tensor matrix \( \overline{X}_b \) must be determined from the matrix equation

\[ \overline{A}_a \cdot \overline{X}_b + \overline{X}_b \cdot \overline{B}_b = \overline{C}_b. \]

This equation cannot be solved by applying ordinary matrix algebra with the given matrices, but the equations can be written by expanding the unknown matrix into a column vector of unknowns. The notation of the expanded matrices is not going to be clear due to the repeated base symbols; therefore, tensor indices \( i \) and \( j \) are appended

\[ \overline{A}_{ai} \cdot \overline{X}^a_i + \overline{X}^a_i \cdot \overline{B}^{bj}_i = \overline{C}^a_i. \]

The first term is stretched with the outer partial transpose according to eqn.(20.17)

\[ \overline{A}_{ai} \cdot \overline{X}^a_i \Rightarrow \overline{A}^{\tau b}_{ai} \cdot \overline{X}^{\tau b}_i. \]

None of the previous formulas is available to stretch the matrix \( \overline{X}^a_b \) of the second term into the same column; therefore, the term is transposed first and then stretched with the inner partial transpose according to eqn.(20.21)

\[ \left[ \overline{X}^a_b \cdot \overline{B}^{bj}_i \right]^T = \overline{B}^{\tau bj}_i \cdot \overline{X}^{\tau bj}_a \Rightarrow \overline{B}^{\tau a}_b \cdot \overline{X}^{\tau a}_b, \]

which supplies an additional formula

\[ \overline{X}^a_b \cdot \overline{B}^{bj}_c = \overline{C}^a_c \Rightarrow \overline{B}^{\tau a}_b \cdot \overline{X}^{\tau a}_b = \overline{C}^{\tau a}. \]

(20.32)

The expanded matrix equation is now

\[ \left[ \overline{A}^{\tau b}_a + \overline{B}^{\tau a}_b \right] \cdot \overline{X}^{\tau bj}_a = \overline{C}^{\tau a}. \]

Had the bases all been the same, some more additional tensor symbols would have to be introduced.

The linear relation between stress and strain tensors contains a 4th order tensor, written in tensor notation as

\[ T^{ij} = C^{ijkl} \cdot S^{kl} \]

with contravariant stress and strain tensors \( T^{ij} \) and \( S^{kl} \), respectively, which correspond to stretched column tensor matrices in matrix tensor notation. Let the stress and strain tensors be given in Cartesian base as \( \overline{T}^s \) and \( \overline{S}^s \). The stretched tensor matrices are

\[ \overline{T}^s \Rightarrow \overline{T}^s, \quad \overline{S}^s \Rightarrow \overline{S}^s. \]
where the transpose has fallen away due to the Cartesian base. The stress strain relation is then in matrix form

\[
\bar{T} = \bar{C} \bar{\varepsilon}.
\]

(20.33)

If any transformation to another base \( a \) must be made, generally skew, then the stress and strain tensors are \( \bar{T}_a^a \) and \( \bar{\varepsilon}_a^a \), respectively, and the corresponding stretched forms are

\[
\bar{T}_a^a \Rightarrow \bar{\gamma}_a^a, \quad \bar{\varepsilon}_a^a \Rightarrow \bar{S}_a^a.
\]

The matrix form of stress strain relation in base \( a \) becomes

\[
\bar{T}_a^a = \bar{C}_a^a \cdot \bar{\varepsilon}_a^a.
\]

(20.34)

With the transformation matrices \( \bar{E}_a^s \) and \( \bar{E}_a^s \), the transformation of the stretched stress tensor is

\[
\bar{T}_a^a = \bar{E}_a^s \cdot \bar{T}_s^s \cdot \bar{E}_a^s
\]

\[
\Rightarrow \bar{T}_a^a = \bar{E}_a^s \cdot \bar{\gamma}_a^s \cdot \bar{\gamma}_a^s
\]

\[
= \left[ \begin{array}{c}
\bar{E}_s^s \\
\bar{E}_s^s
\end{array} \right] \cdot \left[ \begin{array}{c}
\bar{\gamma}_a^s \\
\bar{\gamma}_a^s
\end{array} \right]
\]

\[
= \bar{E}_s^s \cdot \bar{\gamma}_a^s \cdot \bar{\gamma}_a^s
\]

\[
= \bar{E}_s^s \cdot \bar{\gamma}_a^s
\]

\[
= \bar{E}_s^s \cdot \bar{\varepsilon}_a^s
\]

\[
= \bar{E}_s^s \cdot \bar{\varepsilon}_a^s
\]

where tensor indices have been necessary for distinction, but remembering that the symmetry of the tensor matrices is lost in skew base. Comparing the new transformation matrix \( \bar{E}_a^{\tau \alpha \times \alpha} \) with the tensor-like transformation equation

\[
\bar{T}_a^a = \bar{E}_a^s \cdot \bar{\gamma}_a^s \cdot \bar{\gamma}_a^s
\]

then it is clear that

\[
\bar{E}_a^s = \bar{E}_a^s \cdot \bar{\gamma}_a^s
\]

\[
= \left[ \begin{array}{c}
\bar{E}_s^s \\
\bar{E}_s^s
\end{array} \right] \cdot \left[ \begin{array}{c}
\bar{\gamma}_a^s \\
\bar{\gamma}_a^s
\end{array} \right]
\]

\[
= \bar{E}_s^s \cdot \bar{\varepsilon}_a^s
\]

\[
= \bar{E}_s^s \cdot \bar{\varepsilon}_a^s
\]

which can in fact be verified by simply multiplying out and comparing matrix elements.
With the larger transformation matrix of the stretched tensor matrix, the transformation of the 4th order tensor becomes

\[ \overrightarrow{C}^a_{\tau a} = \overrightarrow{E}^a_s \cdot \overrightarrow{C}^s_{\tau a} \cdot \overrightarrow{E}^s_{\tau a} \]  

(20.35)

To escape all further notation of stretched tensor matrices and their transformations with hypermatrices, we can at one stage simply define new vectors and tensors of higher dimension but lower order as long as we don’t want to have stretched tensor matrices in mixed bases. That is, we may define new vectors \( \overrightarrow{t t}, \overrightarrow{s s} \), and new tensor \( \overrightarrow{C C} \) such that

\[ \begin{align*}
\overrightarrow{t t}^s & \equiv \overrightarrow{t s}^s, \\
\overrightarrow{s s}^s & \equiv \overrightarrow{s s}^s, \\
\overrightarrow{C C}^s & \equiv \overrightarrow{C C}^s, \\
\overrightarrow{E E}^a & \equiv \overrightarrow{E E}^{a x a},
\end{align*} \]

and then carry out all the necessary operations with exactly the same Matrix Tensor Notation as before with up to second order tensors, remembering that the notation is good for any dimension. This applies as well to other 2nd order 6-dimensional tensors that may be defined, such as a combined force-moment vector and its translation matrix, which is partly defined by a translation theorem in [29], and the corresponding 6×6 equations of mechanics that will result if the force and moment equations are combined in this manner.

To end this section on partial transpose, we immediately recognize that it is much more complicated than tensor notation: first, because of the double dimensional tensors, and second, because of the transpose. Also the inclusion of higher order tensors made the addition of tensor indices necessary. The question whether the Matrix Tensor Notation is then still useful, depends perhaps not so much on what the complications are but what we want to accomplish. Matrix tensor notation doesn’t have any particular advantages in a single base; therefore, one complication is due to the recognition of different bases while still recognizing tensor names, which tensor notation cannot do. For this purpose, we might do the the addition of tensor symbols the other way round, i.e., add base symbols to tensor notation. But if we want to write the tensor equations as matrix equations, then the difficulty of distinguishing between covariant and contravariant requires all the complications we have presented. If the user decides that Matrix Tensor Notation is too complicated for higher order tensors, then there is still the large field of second order tensors where it is useful; the occasional occurrence of a higher order relation like the stress-strain relationship is then perhaps not too much a price to pay, considering the advantage of jumping back to 2nd order as proposed by the equivalence eqns.(20.36).
21. CURVED SKEW COORDINATES

For the purpose of demonstration, we use the 2-dimensional space with Cartesian x, y coordinates as shown in Fig.14. There are many reasons why curved coordinates are used which does not concern us here in the discussion of the notation. Let there be functions $\xi(x, y)$ and $\eta(x, y)$ which are geometrically interpreted as surfaces of constant values $\xi, \eta$. In 3 dimensions, the constant $\xi, \eta$ values describe curved surfaces; in n-dimensional space, they are subspaces of $n - 1$ dimensions called generalized surfaces, on which one variable has a constant value. The subspaces cross in lines, which are 1-dimensional subspaces in any n-dimensional space, and only one variable actually varies along such a line. In a higher-dimensional space, $n - 1$-dimensional subspaces are generalized surfaces, while lines are 1-dimensional subspaces. In 2-dimensional space, therefore, surfaces and lines happen to be the same. Having realized this, the appropriate generalizations from the 2-dimensional figures can be made.

The lines in Fig.14(a) are meant to be the coordinate lines, which we may regard as a projection of the lines in 3-dimensional or higher-dimensional space. The unitary bases derived from them are shown at points A and B. In Fig.14(b) the coordinate surfaces are shown, only in 2 dimensions they look like lines as in the figure, which we may regard as a crosssection of 3-dimensional or higher-dimensional space; the hatching indicating surfaces. The reciprocal unitary bases derived from them are shown at the same points A and B.

What follows is practically a generalization of Section 12 (Part I) to skew curved coordinates. The special cases of skew but rectilinear and uniform coordinates and of curvilinear orthogonal coordinates are not discussed separately. All simplifications that occur can readily be inferred and are well known, and do not require any extra symbols in our Matrix Tensor Notation beyond the ones introduced for the general case.

At any point $x, y$ the values of coordinates are collected in an array

$$\overline{s} = \begin{bmatrix} x \\ y \\ \vdots \\ s^n \end{bmatrix}, \quad \overline{q} = \begin{bmatrix} \xi \\ \eta \\ \vdots \\ q^n \end{bmatrix}. \tag{21.1}$$

These are not vector arrays, and although they may be called algebraic vectors according to Section 1 (Part I), we will call them coordinate columns, or simply coordinates, to prevent any confusion. The name of the skew coordinates has been chosen $\overline{q}$ as customary in Analytical Dynamics for abstract generalized n-dimensional coordinates. Consider the values at a point A
in Fig. 14(a), \( \overrightarrow{q}_A \), and at point \( B, \overrightarrow{q}_B \), then the difference \( \overrightarrow{q}_B - \overrightarrow{q}_A \) is meaningless as a measure of distance between the points. Therefore, \( \overrightarrow{q} \) cannot be a line vector; there is not a unique base in which \( \overrightarrow{q} \) could be any vector array in this coordinate system at both points \( A \) and \( B \). The column \( \overrightarrow{s} \), however, is like a very special coordinate column inasmuch as \( \overrightarrow{s}_B - \overrightarrow{s}_A \) does represent the linevector from \( A \) to \( B \) like components in Cartesian base. In our previous notation, the position vector was called \( \overrightarrow{s} \), and if the Cartesian base aligned with the \( x,y \) axes is \( \overrightarrow{E}_s \), then the position vector array is \( \overrightarrow{s} \), see Section 9 (Part I). The special relation between Cartesian coordinates and the position vector from eqn.(55) (Part I) is repeated here for convenience

\[
\frac{ds}{s} = \frac{\overrightarrow{s}}{\overrightarrow{s}}, \tag{21.2}
\]

where

\[
\frac{ds}{s} = \overrightarrow{E}_s \cdot \frac{\overrightarrow{s}}{s}. \tag{21.3}
\]

At any point \( \overrightarrow{s} \), or position \( \overrightarrow{s}_s \), let's say point \( B \) in Fig. 14(a), there may be any vector \( \overrightarrow{v} = \overrightarrow{E}_s \cdot \overrightarrow{v} \). Such a vector is perceived to be located at the single point \( B \) only, and not reaching to any other point, its dimension would not be length anyway. The "length" in graphical diagrams, at arbitrary scale, is not a length in space. This is where the position vector \( s \) is special, as it is associated with both points, from where and to where it is measured.

The relation between Cartesian base and Cartesian coordinates is shown in Fig. 15(a), where the column base

\[
\overrightarrow{E}_s = \left[ \begin{array}{c} \overrightarrow{e}_1 \\ \overrightarrow{e}_2 \\ \vdots \\ \overrightarrow{e}_n \end{array} \right] = \left[ \begin{array}{c} \frac{\partial \overrightarrow{s}}{\partial x} \\ \frac{\partial \overrightarrow{s}}{\partial y} \end{array} \right] = \frac{ds}{d\overrightarrow{s}}, \tag{21.4}
\]

and generally in \( n \)-dimensional space

\[
\overrightarrow{E}_s = \left[ \overrightarrow{e}_1, \overrightarrow{e}_2, \ldots, \overrightarrow{e}_n \right] = \left[ \frac{\partial \overrightarrow{s}}{\partial s^1}, \frac{\partial \overrightarrow{s}}{\partial s^2}, \ldots, \frac{\partial \overrightarrow{s}}{\partial s^n} \right] = \frac{ds}{d\overrightarrow{s}}. \tag{21.5}
\]

These equations are the same as eqn.(57) (Part I) in more detail. The use of partial and total differential symbols is explained in [51]. Using the relations in eqns.(21.2) and (21.3), we find all the special expressions

\[
\frac{ds}{d\overrightarrow{s}} = \frac{\overrightarrow{s}}{\overrightarrow{s}} \cdot \frac{d\overrightarrow{s}}{d\overrightarrow{s}} = \overrightarrow{E}_s = \overrightarrow{1}, \tag{21.6}
\]

\[
\frac{ds}{d\overrightarrow{s}} = \overrightarrow{E}_s, \tag{21.7}
\]

\[
\frac{d\overrightarrow{s}}{ds} = \overrightarrow{E}_s. \tag{21.8}
\]

### 21.1. Unitary Base

The well-known definition of unitary base is analogous to eqn.(21.5), as shown in Fig. 15(b). Choosing tangents to the coordinate lines, direction vectors \( \overrightarrow{e}_\xi \) and \( \overrightarrow{e}_\eta \) are defined and collected into a column base as

\[
\overrightarrow{E}_q = \left[ \overrightarrow{e}_\xi, \overrightarrow{e}_\eta \right] = \left[ \frac{\partial \overrightarrow{s}}{\partial \xi}, \frac{\partial \overrightarrow{s}}{\partial \eta} \right] = \frac{d\overrightarrow{s}}{d\overrightarrow{q}}, \tag{21.9}
\]

or generally,
This is the unitary base in our notation. It is a generalization of eqn.(59) (Part I).

A Unitary base $\vec{E}_q$ is shown at Point A in Fig.14(a). It is generally skew, has nonunit base vectors, and varies w.r.t. position; i.e., another unitary base $\vec{E}_q$ at point B in Fig.14(a) is different from the one at A.

The transformation to skew base is defined according to Section 15.4

$$\vec{E}_q = \vec{E}_s \cdot \vec{E}_q^s.$$  \hspace{1cm} (21.11)

Defining the transformation matrix $\vec{E}_q^s$, and using eqns.(21.4) and (21.10), we get

$$\vec{E}_q^s = \frac{d\vec{s}}{ds} \cdot \frac{d\vec{s}}{dqq} = \frac{d\vec{s}}{dqq},$$ \hspace{1cm} (21.12)

$$\vec{E}_q^q = \frac{d\vec{q}}{ds} \cdot \begin{bmatrix} \frac{d\vec{s}}{ds} \\ \frac{d\vec{s}}{dq} \end{bmatrix}^{-1} = \left[ \vec{E}_q^s \right]^{-1}. \hspace{1cm} (21.13)$$

The derivatives are Jacobian matrices according to Section 11 (Part I), which in this 2-dimensional example are

$$\frac{d\vec{q}}{ds} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix}, \quad \frac{d\vec{s}}{ds} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}. \hspace{1cm} (21.14)$$

From eqn.(21.9), the reciprocal base can be obtained by the inverse derivative

$$\vec{E}_q^o = \frac{d\vec{q}}{ds},$$ \hspace{1cm} (21.15)

$$= \vec{E}_s \cdot \vec{E}_q^s, \hspace{1cm} (21.16)$$

using eqn.(21.11). Note that the derivative w.r.t. to $\vec{s}$ can be expressed

$$\frac{d}{ds}() = \frac{d}{ds}() \cdot \frac{d\vec{s}}{ds} = \frac{d}{ds}() \cdot \vec{E}_q^s.$$ \hspace{1cm} (21.17)
The reciprocal bases are shown in Fig.14(b). They are also skew, have nonunit base vectors, and are different between any two points A and B. It is apparent in the figure that the directions of the base vectors of the reciprocal bases are normal to the surfaces of constant $\xi$ and $\eta$. This can be shown by the derivation of a base from the gradient. The gradient of any function $\xi(\vec{s})$, or $\eta(\vec{s})$, is defined in our notation as in Section 11 (Part I), $\nabla \xi = \frac{d\xi}{ds}$, and similar for the function $\eta$. Having two rowvectors, they can be collected to a base

$$\begin{bmatrix}
\nabla \xi \\
\nabla \eta
\end{bmatrix} = \begin{bmatrix}
e_\xi \\
e_\eta
\end{bmatrix} = \overrightarrow{E}_q^s,$$

(21.19)

where

$$\begin{bmatrix}
e_\xi \\
e_\eta
\end{bmatrix} = \begin{bmatrix}
\frac{d\xi}{ds} \\
\frac{d\eta}{ds}
\end{bmatrix} = \frac{d\vec{q}}{ds},$$

(21.20)

and from comparing the vector derivatives, it is clear that this is indeed the inverse of base $\overrightarrow{E}_q^s$.

Mathematically different bases are defined by eqns.(21.9) and (21.15), and either one can be used in any application; the explanations from Figs.14 and 15 are merely to give the mathematical definition a corresponding geometrical meaning. To know that coordinates $\vec{q}$ are curved, it must be known that $\vec{s}$ are Cartesian coordinates, or at least the metric of the base $\overrightarrow{E}_s(\vec{s})$ must be known. The unitary and reciprocal unitary bases can then be expressed in Cartesian components by replacing $\vec{s}$ in the definitions of eqns.(21.10) and (21.15) by $\vec{s} = \vec{s}$, which become then the transformation matrices in eqns.(21.11) and (21.16).

At any one point, the metric of unitary base according to definition in Section 15.3 is

$$\overrightarrow{E}_q^{\vec{q}} = \overrightarrow{E}_s^{\vec{s}} \cdot \overrightarrow{E}_q^s = \overrightarrow{E}_s^{\vec{s}} \cdot \overrightarrow{E}_q^s,$$

referred to space or a Cartesian base, respectively. If the base is orthogonal, it is customary to use normalized base vectors to express scalar components. We will introduce normalized bases even for a skew unitary base for the purpose of demonstrating the generality of formulas. In orthogonal coordinates, we have the choice of constructing the normalized base from the posed base or from the reciprocal base by two different transformations. The one base is the inverse and transpose of the other, therefore, geometrically the same. In skew coordinates, the two bases normalized by this method are skew and neither inverses nor transposes of each other, and the Matrix Tensor Notation will have to distinguish between them.

### 21.2. Unit Tangent Base

To construct a normalized base from a posed unitary base, we use the scale factor diagonal matrix $\overrightarrow{h}$ from eqn.(58) (Part I), and define the single scale factors $h_i$ by the ratio of $\|d\vec{s}\|/d\vec{q}_i$. This scale factor corresponds to the distances between coordinate lines in Fig.14(a). Let the name of the unit base be $h$, then the unitary base vectors are in terms of unit base vectors

$$\vec{e}_\xi = \overrightarrow{e}_{h_1} \cdot h_1, \quad \vec{e}_\eta = \overrightarrow{e}_{h_2} \cdot h_2, \quad \cdots \quad \vec{e}_q^i = \overrightarrow{e}_{h_i} \cdot h_i, \quad \cdots$$
which becomes a transformation equation like eqn.(58),

$$ E_q = E_h \cdot \bar{h}. $$

(21.21)

To construct the diagonal matrix $\bar{h}$, the scale factors are determined from the diagonal terms of the metric

$$ \bar{h} = \sqrt{\text{diag} \left( E_q^T \right)} . $$

(21.22)

The base $h$ has unit base vectors but is skew and corresponds to the skew base $E_q$ of Fig.8(a) of Section 15.1. It has the inverse $E_q^h$ and the transpose $E_q^{\tau h}$,

$$ E_q^h = \bar{h} \cdot E_q^q, \quad E_q^{\tau q} = \bar{h} \cdot E_q^{\tau h} . $$

(21.23)

Actually, the scale factor $\bar{h}$ is the tensor matrix of an affine deformation tensor $H$, corresponding to the deformation tensor $\bar{Q}$ of Section 17, here in the relation

$$ \bar{E}_q = \bar{H} \cdot \bar{E}_h $$

and a transformation matrix, where

$$ \bar{h} = \bar{H}_q = \bar{H}_h = \bar{E}_q^h . $$

21.3. Unit Normal Base

The normalized base from the reciprocal unitary base $E_q^p$, corresponding to $E_q^p$ of Fig.8(d) of Section 15.1, will have to have another name, for which we choose $g$, normal to $E_q$, with scale factors $g_j = |\text{grad } q^j|$ such that

$$ e_{\xi} = g_1 \cdot e_1^q, \quad e_\eta = g_2 \cdot e_2^q, \quad \cdots \quad e_{\gamma} = g_j \cdot e_j^q, \quad \cdots ; $$

therefore,

$$ E_q^g = g \cdot E_q^q, $$

(21.24)

with the diagonal matrix $g$ whose elements are other scale factors consisting of $|\text{grad } q^j|$. To construct the diagonal matrix $g$, the scale factors are determined from the diagonal terms of the inverse metric

$$ g = \sqrt{\text{diag} \left( E_q^{\tau q} \right)} , $$

(21.25)

which of course is only the inverse of $\bar{h}$ if the coordinates $q$ are orthogonal. The base $g$ is also skew and has the inverse $E_g$ and the transpose $E_{\tau g}$,

$$ E_g = E_q^q \cdot g, \quad E_{\tau g} = E_{\tau q} \cdot g. $$

(21.26)

The scale factor matrix $g$ can be interpreted to be the tensor matrix of another affine deformation tensor $G$ in the relation$^2$

$$ E_g = G \cdot E_q $$

$^2$Confusion of this tensor should not occur with the velocity gradient $\nabla \bar{q}$ used later.
and a transformation matrix, where
\[ \bar{g} = \bar{G}_q = \bar{G}_g = \bar{G}_g. \] (21.27)

Contrary to unitary base, there are no variables \( \bar{h} \) and \( \bar{g} \) from which the unit bases can be derived, generally no such variables can be constructed in curvilinear coordinates.

\( \bar{E}_h \) is the collection of unit base vectors denoted by \( e_1, e_2, e_3 \) in [27], \( \bar{e}_1, \bar{e}_2, \bar{e}_3 \) and called covariant unit vectors in [23], and \( h_i \) are the same as the customary scale factors, although rarely used in tensor analysis texts, but rather in treatises on curvilinear coordinates in vector analysis [27]. \( \bar{E}_g \) is the collection of unit base vectors denoted by \( E_1, E_2, E_3 \) in [27].

The unitary base vectors \( \bar{E}_q \) and their reciprocal \( \bar{E}_q^* \), are indicated by \( \alpha, \beta, \), respectively, [27], \( e_1, e_2, e_3, e^1, e^2, e^3 \) [12], \( a_1, a_2, a_3, a^1, a^2, a^3 \) [23], but mostly by \( g_i, g^j \) [2,26], which is consistent with the name \( g \) of the metric. In our notation, the consistency between the name of the metric and base vectors is preserved by having \( \bar{E}_h \) and \( \bar{E}_g \) as metric for bases \( h \) and \( g \), respectively, see also Section 15.3.

We call \( \bar{E}_h \) the tangent unit base and \( \bar{E}_g \) the normal unit base, which corresponds to covariant and contravariant unit bases; but the reciprocal bases \( \bar{E}_h^* \) and \( \bar{E}_g^* \) are not units.

Each of the unit bases can be transformed to Cartesian base as well, which for orthogonal coordinates defines a rotation matrix, by:

\[ \bar{E}_h^s = \bar{E}_h \cdot \bar{E}_h = \bar{E}_q \cdot \bar{h}^c = \frac{d \bar{s}}{d \bar{q}} \cdot \bar{h}^c, \quad \bar{E}_h^q = \bar{E}_q \cdot \bar{E}_h = \bar{g} \cdot \bar{h}^q = \bar{h} \cdot \frac{d \bar{q}}{d \bar{s}}, \]

\[ \bar{E}_g^s = \bar{E}_g \cdot \bar{E}_g = \bar{E}_q \cdot \bar{g} = \frac{d \bar{s}}{d \bar{q}} \cdot \bar{g}, \quad \bar{E}_g^q = \bar{E}_g \cdot \bar{E}_q = \bar{g} \cdot \bar{g} = \bar{g} \cdot \frac{d \bar{q}}{d \bar{s}}. \]

The scale factors in the diagonal matrices are usually employed to transform unitary scalar components to unit-based scalar components by

\[ \bar{v} = \bar{h} \cdot \bar{v}_q, \quad \bar{v}_q = \bar{v} \cdot \bar{g}. \] (21.28)

In skew coordinates, the vectors \( \bar{v}_h \) and \( \bar{v}_g \) are not unit-based, and in orthogonal coordinates components of \( \bar{v}_h, \bar{v}_h, \bar{v}_g, \bar{v}_g \) are all the same.

All the transformation laws for skew base of Sections 15.1 to 15.7 apply to the unitary base \( \bar{E}_q \) and the normalized bases of this section, the only difference being that, in previous sections, the skew base was not derived from coordinates,

\[ \bar{v} = \bar{E}_q \cdot \bar{v}_q, \quad \bar{v}_q = \bar{E}_q \cdot \bar{v}, \quad \bar{v} = \bar{E}_g \cdot \bar{v}_g, \]

\[ \bar{v}_g = \bar{v} \cdot \bar{g}, \quad \bar{v}_h = \bar{v} \cdot \bar{h}, \quad \bar{v}_g = \bar{v} \cdot \bar{g}. \] (21.30)
Similarly, all transformation laws of tensors in these bases are given by the equations of Section 15.7.

In the remainder of this section, we will use the base name $\alpha$ to include all three bases and their reciprocals, unitary $\overrightarrow{E}_q$, tangent $\overrightarrow{E}_h$, and normal unit $\overrightarrow{E}_n$ to express their common transformation properties.

But there is a question whether the transformation of the position vector
\[ \overrightarrow{s} = \overrightarrow{E}_\alpha \cdot \overrightarrow{s} = \overrightarrow{E}_\alpha \cdot \overrightarrow{s} \]  
(21.31)

can be allowed. The problem is that the base $\overrightarrow{E}_\alpha$ varies along $\overrightarrow{s}$. Consequently, a distance between $A$ and $B$ cannot be expressed in base $\alpha$ by $s_A - s_B$, because each vector has actually a different base, expressed explicitly by $s_A - s_B$. Eqn.(21.2) expresses the special property of Cartesian coordinates, that the coordinate column and the vector array are the same. Also the line vector from $A$ to $B$, $s_B - s_A$ is always associated with the point given by the coordinates $\xi, \eta$ only, not at the same time with the origin $(\xi = 0, \eta = 0)$. But compare this to the special case where a vector $v = f(s)$. The vector $v_B$ is located at point $B$ only, and by association, the position vector $s$ may be thought to be associated with the end point $B$ only. By the same argument, we restrict the meaning of $s$ to be a vector array associated with the point given by the coordinates $\xi, \eta$ only, not at the same time with the origin $(\xi = 0, \eta = 0)$.

At this stage, some clarification about the use of the position vector $\overrightarrow{s}$ may be in order. To indicate a position in curved coordinates, the vector $\overrightarrow{s}$ is replaced by the coordinate column $\overrightarrow{q}$. The finite distance vector is required only in rigid body dynamics, for which skew but rectilinear uniform coordinates may be used, where $\overrightarrow{s}$ is the position vector. Curved coordinates are applied in continuum mechanics where the distance vector is not required. In continuum mechanics we deal with forces but not with moments about a distant point. If there is really a need to calculate a finite distance vector between curved coordinates $\overrightarrow{q}_B$ and $\overrightarrow{q}_A$, then we must transform to linear coordinates $\overrightarrow{s}_B - \overrightarrow{s}_A = \overrightarrow{E}_s \cdot \left( \overrightarrow{q}_B - \overrightarrow{q}_A \right)$.

21.4. Differential Position Vector

Differentiating eqn.(21.3), or following the details of Fig.15(a), the differential vector $d\overrightarrow{s}$ can be expressed in terms of the differential Cartesian coordinates
\[ d\overrightarrow{s} = \overrightarrow{E}_s \cdot d\overrightarrow{s} = \overrightarrow{E}_s \cdot d\overrightarrow{s}. \]  
(21.32)

The differential $d\overrightarrow{q}$ is considered to be a quantity between the positions given by the coordinates $\overrightarrow{q}$ and $\overrightarrow{q} + d\overrightarrow{q}$, or by the position vectors $s$ and $s + ds$, and as $ds \to 0$ it seems from Fig.15(b) that $d\overrightarrow{q}$ indicates a straight line at one specified position in space. We want to use it as a linevector from point $s$ at coordinates $\overrightarrow{q}$, to point $s + ds$ at coordinates $\overrightarrow{q} + d\overrightarrow{q}$. Note that eqn.(21.9) may be interpreted as an equation for the differential vector
\[ ds = \overrightarrow{E}_q \cdot d\overrightarrow{q}, \]  
(21.33)

which satisfies our heuristic requirement. In words: $d\overrightarrow{q}$ are the components of $d\overrightarrow{s}$ measured in base $q$. The reason why this works in contrast to the finite coordinates is that $\overrightarrow{E}_q \to \text{Constant}$ as $ds \to 0$. Now we must not get carried away with base symbols because $d\overrightarrow{s} \neq \overrightarrow{E}_q \cdot d\overrightarrow{q}$, or generally, $ds \neq \overrightarrow{E}_\alpha \cdot d\overrightarrow{s}$. The problem is that base $\times$ differential column $\neq$ differential (base
The differential $ds^\alpha$ is not the differential vector $E^\alpha \cdot ds$, the differential sign cannot be carried over in a base transformation like a constant in curved coordinates. To comply with the meaning of the differential sign, we define

$$ ds^\alpha = \overline{\bar{s}}^\alpha \left( \overline{q} + \overline{dq} \right) - \overline{\bar{s}}^\alpha \left( \overline{q} \right) $$

but $\neq E^\alpha \cdot \left[ s^\alpha \left( \overline{q} + \overline{dq} \right) - s^\alpha \left( \overline{q} \right) \right]$

with the restricted meaning of $\overline{\bar{s}}^\alpha$. We find then that the quantity $ds^\alpha$ is defined only to be consistent in our notation, but it is pretty useless. This is due to the fact that $ds^\alpha$ may be zero when $dq$ is not. Therefore, it may happen that if $ds^\alpha$ is used in a derivative, a singular matrix will result. In fact, it will be shown that $ds^\alpha$ is not a vector, in the sense of a vector array. The same principle as eqn.(21.34) applies to any vector $v^\alpha$,

$$ dv^\alpha = v^\alpha \left( \overline{q} + \overline{dq} \right) - v^\alpha \left( \overline{q} \right) $$(21.35)

but $\neq E^\alpha \cdot \left[ v^\alpha \left( \overline{q} + \overline{dq} \right) - v^\alpha \left( \overline{q} \right) \right]$.

We conclude: The differential line vector in unitary base or normalized bases $\alpha$ cannot be expressed by the base symbol $-\alpha$, but instead can be expressed with the differential coordinate column $dq$, by a transformation of eqn.(21.33)

$$ ds = E_h \cdot h \cdot dq = E_g \cdot g^{-1} \cdot dq. $$

We express the differential vector $ds$ in base $\alpha$ by a notation with brackets, defined as

$$ ds = E^\alpha \cdot (ds)^\alpha, \quad (ds)^\alpha = E_s^\alpha \cdot ds, $$

and the differential vector $dv$ in base $\alpha$ as

$$ dv = E^\alpha \cdot (dv)^\alpha, \quad (dv)^\alpha = E_s^\alpha \cdot dv. $$

Eqn.(21.33), transformed to Cartesian base, shows that differentials of columns are related like the transformation of vectors

$$ ds = E^q_s \cdot dq, \quad dq = E^q_s \cdot ds; $$

therefore,

$$ (ds)^q = E^q_s \cdot ds = dq, $$

but

$$ (ds)^h = E^h_q \cdot (ds)^q = h \cdot dq, $$

$$ (ds)^g = E^g_q \cdot (ds)^q = g^{-1} \cdot dq. $$

Immediately the question comes up, if $\overline{q}$ is a function $\overline{q}(s)$, then

$$ \frac{d \overline{q}}{ds} = \frac{d \overline{q}}{d \overline{s}} \cdot \overline{s} = \frac{d \overline{q}}{d \overline{s}} \cdot d \overline{s}, $$

which we find coincides with the transformation of the differential vectors from eqn.(21.33). Therefore, $ds$ and $d \overline{q}$ are vector arrays of differential magnitude, and this is what we mean by differential vector array, in contrast to differential of a vector array $ds^\alpha$ or $dv^\alpha$. In space or in Cartesian base, the two are the same.
There is some discrepancy between the symbols of eqn. (21.33), because names and base symbols don’t agree. It is an equality, not a transformation. By this time the analogy with strain, Section 17.4, is apparent. Let us try to find a transformation by considering $\overrightarrow{q}$ as coordinates which have at some previous era coincided with the Cartesian coordinates $\overrightarrow{s}$. Then $\overrightarrow{q}(\overrightarrow{s})$ is a mapping of old Cartesian coordinates on new ones, but contrary to the mathematical concept of mapping from one space to another, we consider the mapping of one system on another within the identical orthonormal space, the two only being separated by a parameter, like time. We have then the situation as in Section 17.4, where the new position vector $\overrightarrow{r}$ indicated a deformation of the original unstrained vector $\overrightarrow{s}$. That vector, $\overrightarrow{s}'$ measured in base $s$, is the analog of the present proposed meaning of $\overrightarrow{q}$. Instead of a mapping, we prefer to interpret the change from previous orthonormal coordinates $\overrightarrow{q}$ to present curvilinear coordinates as a deformation, whether this is a real deformation, as in Section 17, or not, and all vectors are measured in the same fixed base $s$.

Going back into the past era, the position vector there may be called $\overrightarrow{q}'$, with the differential

$$d\overrightarrow{q}' = E_s \cdot d\overrightarrow{q}$$  (21.44)

like the vector $d\overrightarrow{s}'$ in Fig.12. But according to transformation symbols in Cartesian base,

$$d\overrightarrow{q} = E_s \cdot d\overrightarrow{q}'$$  (21.45)

From eqn.(21.44) and (21.45) follows

$$d\overrightarrow{q} \equiv d\overrightarrow{q}'$$  (21.46)

Our interpretation is therefore: $\overrightarrow{q}$ is a position vector in a past era, indicating another position in the same space as the curved coordinates $\overrightarrow{q}$.

The orthonormal coordinates $\overrightarrow{q}$ from a past era are deformed into curved strained coordinates $\overrightarrow{q}$ at position $\overrightarrow{s}$ in the present era in the same space. The differential vector $d\overrightarrow{q}$ is in a different orientation in the past era, and has now been deformed, in the general sense of Section 17, into its present orientation and deformed length into the differential vector $d\overrightarrow{s}$. The position of the differential vector $d\overrightarrow{s}'$ is at the present coordinates $\overrightarrow{q}$. In terms of mechanics, $\overrightarrow{q}$ serves as Lagrange coordinate for $\overrightarrow{s}$. In complete analogy with Section 17, $\frac{d\overrightarrow{s}}{d\overrightarrow{q}} = \overrightarrow{Q}$, the deformation tensor. And finally, the base symbol $\overrightarrow{q}$ in $E_q$ serves itself as a Lagrange coordinate column $\overrightarrow{q}$, or a pre-mapping coordinate column $\overrightarrow{q}$, a method of notation that proves extremely useful in mapping.

Although from eqn.(21.39) it can be inferred that a row differential exists as $d\overrightarrow{q} = d\overrightarrow{s} \cdot E_q^s$, we are not going to use it; it would define a new row variable $\overrightarrow{q}$ of coordinates normal to the curved coordinates $\overrightarrow{q}$ in the present Cartesian base. Instead, we are going to use the notation

$$\begin{align*}
d\overrightarrow{q} &= \left[\begin{array}{c}d\overrightarrow{q} \end{array}\right]^T, \\
\frac{d\overrightarrow{q}}{d\overrightarrow{s}} &= \left[\begin{array}{c}d\overrightarrow{s} \\
\frac{d\overrightarrow{q}}{d\overrightarrow{s}}\end{array}\right]^T, \\
\frac{d\overrightarrow{s}}{d\overrightarrow{q}} &= \left[\begin{array}{c}d\overrightarrow{q} \\
\frac{d\overrightarrow{s}}{d\overrightarrow{q}}\end{array}\right]^T.
\end{align*}$$

21.5. The Gradient

Derivatives in multiple variables occur in many applications that are not necessarily related to 3-dimensional physical space. Let $\overrightarrow{s}$ be a column of coordinates. The elements may be physical coordinates, generally curved, or they may be any independent variables which are taken as
coordinates in abstract higher-dimensional space. Considering a vector as a single quantity, the
column \( \mathbf{s} \) is regarded as a single quantity. Let \( \mathbf{q} \) be another column of coordinates in the same
space, related by a given function \( \mathbf{q}(\mathbf{s}) \) to coordinates \( \mathbf{s} \). The function, regarded as a single
function of a single argument, consists of the number of nonlinear functions that make up the
elements of the column \( \mathbf{q} \). In view of this perspective, the differentials \( d\mathbf{s} \) and \( d\mathbf{q} \) are total
differentials, and the Jacobian matrix \( \frac{d\mathbf{q}}{d\mathbf{s}} \) is a single total derivative, with the symbols for total
and partial differentials and derivatives as in [51]. If for one point \( \mathbf{s} \) is given and \( \mathbf{q} \) is required,
the nonlinear function \( \mathbf{q}(\mathbf{s}) \) is simply evaluated. If \( \mathbf{q} \) is given and \( \mathbf{s} \) is required, the solution
must be found by iterative methods, and the proper root of the multiple solution must be chosen.
Once this pair of values is known, all the derivatives can be evaluated explicitly.

The elements of the derivative \( \frac{d\mathbf{q}}{d\mathbf{s}} \) are obtained from the given function by partial derivatives
\( \frac{\partial \mathbf{q}_i}{\partial s_j} \). The inverse derivative is found by matrix inversion according to eqn.(21.13). Whether we
are dealing with real or abstract coordinates, the derivatives may be interpreted as transformation
matrices between different bases according to eqns.(21.12) and (21.13), even though neither the
metric of base \( \mathbf{s} \) or base \( \mathbf{q} \) have to be known. With the interpretation of the transformation
matrix as tensor in mixed bases, the derivatives are tensors. The columns \( \mathbf{s} \) and \( \mathbf{q} \) are vector
arrays of abstract position vectors \( \mathbf{s} \) and \( \mathbf{q} \) in abstract space, where one may be a deformation of
the other as in Section 17.4.

The gradient of any scalar function \( \phi(\mathbf{s}) \) in space with position vector \( \mathbf{s} \) is defined in Section 11
(Part I), repeated here for convenience, as the rowvector

\[
\begin{aligned}
\nabla \phi &= \mathbf{d} \phi \\
&= \frac{d\phi}{d\mathbf{s}}.
\end{aligned}
\]

Using the definition of reciprocal unitary base of eqn.(21.15),

\[
\begin{aligned}
\nabla \phi &= \mathbf{d} \phi \\
&= \frac{d\phi}{d\mathbf{q}} \cdot \frac{d\mathbf{q}}{d\mathbf{s}} = \frac{d\phi}{d\mathbf{q}} \cdot \mathbf{E}^q \phi,
\end{aligned}
\]

which compared to the transformation of a rowvector, eqn.(21.30), shows that

\[
\begin{aligned}
\nabla \phi &= \mathbf{d} \phi \\
&= \frac{d\phi}{d\mathbf{q}}.
\end{aligned}
\]

The gradient determined by this formula is always normal to the surface of \( \phi = \) constant in any
\( n \)-dimensional space, independent of the skewness of the unitary base or the curvature of the
coordinates \( \mathbf{q} \), without knowing the metric of the base. This is apparent from the equation

\[
\begin{aligned}
d\phi &= 0 \quad \text{on the surface } \phi = \text{constant} \\
&= \frac{d\phi}{d\mathbf{q}} \cdot \mathbf{d} \mathbf{q} = \frac{d\phi}{d\mathbf{q}} \cdot \mathbf{(d\mathbf{s})}^q \phi.
\end{aligned}
\]

The vector \( \mathbf{(d\mathbf{s})}^q \) is a column vector (contravariant), while \( \frac{d\phi}{d\mathbf{q}} \) is a rowvector (covariant), and
therefore, the scalar product in base \( q \) is a true scalar product. Then it is clear from the condition
of eqn.(21.50), that \( \frac{d\phi}{d\mathbf{q}} \) is normal to every infinitesimal vector \( \mathbf{(d\mathbf{s})}^q \) in the surface \( \phi(q) \).
A further part of the definition of \( \text{grad} \phi \) is that the derivative is w.r.t. space coordinates only. Therefore, if \( \phi \) is a function of space and time, \( \phi(\xi, \eta, \ldots, t) \), then the gradient is only the part

\[
\text{grad} \phi = \frac{\partial \phi}{\partial \xi}, \frac{\partial \phi}{\partial \eta}, \ldots
\]

excludes the partial derivative \( \frac{\partial \phi}{\partial t} \). The parameter \( t \) is not included in the differential \( ds \) of the definition of the gradient. In any abstract \( n \)-dimensional space, it must therefore be stated if anyone of the variables acts as a parameter like time in 3-dimensional space if the abstract concept of gradient is used. This is not a restriction, because by simply not defining any one variable of a multivariate problem as a special scalar parameter, the term gradient can be used without violating the definition of gradient in abstract space. This is, for instance, the case in the application of the Newton-Raphson iteration method for solution of a multivariate root of simultaneous nonlinear equations.

If any time parameter is involved, then the gradient should correctly be denoted as a partial derivative \( \frac{\partial \phi}{\partial s} \), the other partial derivative being \( \frac{\partial \phi}{\partial t} \), where the total differential is denoted by

\[
d\phi = \partial_s \phi + \partial_t \phi,
\]

the notation of [51] applied to the partial derivative w.r.t. a vector variable.

We will assume that, in the use of the gradient, the time differential \( dt = 0 \), and therefore, \( \partial_s \phi = d\phi \), and denote the gradient with the total derivative symbol \( \frac{d\phi}{ds} \). Only when time dependency must explicitly be excluded, will we use the symbol \( \frac{\partial \phi}{ds} \) for the gradient.

If the gradient vector array must be used as a column, the notation for the transposed is

\[
\text{grad} \phi^\tau_q = \left[ \frac{d\phi}{dq} \right]^T.
\]

If the gradient must be expressed as a column vector in posed base, then the metric of space is required to be able to transform according to

\[
\text{grad} \phi = E_q \cdot \text{grad} \phi^q,
\]

\[
\text{grad} \phi^q = E^q_q \cdot \text{grad} \phi^\tau_q = E^q_q \cdot \left[ \frac{d\phi}{dq} \right]^T.
\]

In the normalized bases

\[
\text{grad} \phi_h = \frac{d\phi}{dq} \cdot E^h_q = \frac{d\phi}{dq} \cdot \frac{\partial h}{\partial q},
\]

\[
\text{grad} \phi_g = \frac{d\phi}{dq} \cdot E^g_q = \frac{d\phi}{dq} \cdot \frac{\partial g}{\partial q},
\]

with the scale factors \( \frac{\partial h}{\partial q} \) and \( \frac{\partial g}{\partial q} \) of Sections 21.2 and 21.3. The gradient of a vector \( \vec{v} \) is a tensor according to Section 11 (Part I), repeated here for convenience,

\[
\vec{G} \equiv \frac{d\vec{v}}{ds} = \vec{E}_s \cdot \frac{d\vec{v}}{ds} \cdot \vec{E}_s
\]

\[
= \vec{E}_s \cdot \vec{G}_s \cdot \vec{E}_s,
\]

\[3\]Confusion of this tensor should not occur with the rate of deformation tensor \( \vec{G} \) used before.
where the tensor has the name $G^\alpha$, which can accordingly be transformed to any mixed bases

$$G^\alpha_b = E_s^\alpha \cdot G^s_b \cdot E_b^s. \quad (21.58)$$

To obtain the gradient of a vector in terms of its components in a variable base, $v^\alpha$, the differential $dv^\alpha$ must be analyzed first, see Section 21.7.

The Hamilton, or nabla, operator corresponds to the differentiation $\frac{d}{ds}$, so that the correctly posed relation is

$$\nabla = \frac{d}{ds}. \quad (21.59)$$

Therefore, $\nabla$ is a rowvector and transforms as

$$\nabla() = \frac{d}{dq}() \cdot \frac{d}{ds}() = \nabla_s() \cdot \overline{E}_q^s. \quad (21.60)$$

With this transformed operator, the gradient is obtained in operator form in unitary base by the same formula as in Cartesian coordinates

$$\text{grad} \phi = \nabla \phi_s, \quad (21.61)$$

$$\text{grad} \phi_q = \nabla_s \phi \cdot \overline{E}_q^s = \nabla_q \phi. \quad (21.62)$$

For the normalized bases $h$ and $g$, the transformations of eqns.(21.28) and (21.29) can be applied to the nabla operator by

$$\nabla_h() = \frac{d}{dq}() \cdot \overline{h}^{1} = \nabla_q() \cdot \overline{E}_h^q = \nabla_h, \quad (21.63)$$

$$\nabla_q() = \frac{d}{dq}() \cdot \overline{g} = \nabla_q() \cdot \overline{E}_g^q = \nabla_g. \quad (21.64)$$

Unfortunately, because of the sequence in matrix algebra and the sequence of differential operator and argument, the operand of the transformed nabla does not sit at the end of the actual terms. This complication seems to be preferable to two transposes to change the sequence of the differential and the matrix operation. Once this is recognized, the nabla operator can be transformed to nonunitary base.

With these definitions, the transformed eqn.(21.62) becomes

$$\text{grad} \phi_h = \nabla_h \phi, \quad (21.65)$$

$$\text{grad} \phi_q = \nabla_q \phi.$$

21.6. Second Order Derivatives

The expressions in this section up to now contain vectors and matrices which can be expressed efficiently by our Matrix Tensor Notation. Higher derivatives produce higher order tensors. We express these, and operations with them, essentially by tensor notation. The elements may be arranged in stretched matrices as explained in Section 20, but this is only one concrete storage method, it is not a requirement. The Matrix Tensor Notation is retained but the tensor indices are appended if the order of multiplication is not clearly defined by the arrangement of terms, or they can be used freely as auxiliary measure. We may interpret tensor notation as prescription for a computer program where the indices are the running counters in a loop. In matrix tensor notation, we have to add these indices anyway in a computer program. The Matrix Tensor Notation places more emphasis on the tensors as single quantities instead of the computation with elements.
The second derivative of a scalar function $\phi$ is
\[
\frac{d}{ds} \left[ \frac{d\phi}{ds} \right] = \frac{d^2\phi}{ds \, ds},
\]
and is stored in a row of rows according to Section 20. The storage of the second derivative becomes
\[
\frac{d^2\phi}{ds \, ds} = \left[ \begin{array}{ccc} \frac{\partial^2\phi}{d\xi \, ds} & \frac{\partial^2\phi}{d\eta \, ds} & \cdots \\ \end{array} \right],
\]
but the proper base symbols for such an expression are not obvious and will be worked out in the next section. Any post multiplication with a differential or finite vector $\frac{d^2\phi}{ds \, ds} \cdot \overrightarrow{a} = \overrightarrow{b}$ is unique because the stretched expression is symmetric w.r.t. $\overrightarrow{s}$. If the derivative is w.r.t. different columns, the expression becomes
\[
\frac{d}{dq} \left[ \frac{d\phi}{ds} \right] = \frac{d^2\phi}{ds \, dq} = \left[ \begin{array}{ccc} \frac{\partial^2\phi}{dx \, dq} & \frac{\partial^2\phi}{dy \, dq} & \cdots \\ \end{array} \right],
\]
while the derivative in the other sequence becomes
\[
\frac{d}{ds} \left[ \frac{d\phi}{dq} \right] = \frac{d^2\phi}{ds \, dq} = \left[ \begin{array}{ccc} \frac{\partial^2\phi}{dx \, dq} & \frac{\partial^2\phi}{dy \, dq} & \cdots \\ \end{array} \right].
\]
Any post multiplication with a vector must now either be clear from the coordinate symbols
\[
\frac{d^2\phi}{ds \, dq} \cdot \overrightarrow{a} = \overrightarrow{b},
\]
or it must be indicated with appended tensor indices
\[
\frac{d^2\phi}{ds \, dq} \cdot a^i = b^i \quad \text{and} \quad \frac{d^2\phi}{ds \, dq} \cdot a^j = b^j.
\]
To express the second order derivative of a scalar function as matrix, one of the derivatives must be transposed
\[
\frac{d^2\phi}{ds \, ds} = \left[ \begin{array}{ccc} \frac{\partial^2\phi}{d\xi \, ds} \\ \frac{\partial^2\phi}{d\eta \, ds} \\ \vdots \end{array} \right], \quad \frac{d^2\phi}{dq \, ds} = \left[ \begin{array}{ccc} \frac{\partial^2\phi}{dx \, dq} \\ \frac{\partial^2\phi}{dy \, dq} \\ \vdots \end{array} \right].
\]
The $\overrightarrow{s}$ column doesn't get a transpose sign if it is known that it is Cartesian.

Quite similarly, the second derivative of a column is obtained, whether this be a coordinate column or a vector array,
\[
\frac{d}{ds} \left[ \frac{d\overrightarrow{r}}{dq} \right] = \frac{d^2\overrightarrow{r}}{ds \, dq} = \left[ \begin{array}{ccc} \frac{\partial^2\overrightarrow{r}}{dx \, dq} & \frac{\partial^2\overrightarrow{r}}{dy \, dq} & \cdots \end{array} \right],
\]
in the generally unsymmetric case.

The second derivative is a convenient method to represent a stretched matrix in Matrix Tensor Notation.

For a continuous function $\phi(\overrightarrow{s})$ the sequence of partial derivatives may be changed $\frac{\partial^2\phi}{\partial x \partial y} = \frac{\partial^2\phi}{\partial y \partial x}$, but this is not true for total derivatives, and therefore,
and similarly, for the elements
\[
\frac{\partial^2 \phi}{\partial r_i \partial s_j} \neq \frac{\partial^2 \phi}{\partial s_j \partial r_i},
\]
because the variables \( r_i \) and \( s_j \) come from two different columns. The relation between second derivatives of different sequences can be obtained in the same manner as for total derivatives of scalar variables

\[
\frac{d^2 \phi}{ds \; dq} = \frac{d}{ds} \left[ \frac{d\phi}{ds} \; \frac{ds}{dq} \right] = \frac{d^2 \phi}{ds \; ds} \cdot \frac{ds}{dq} + \frac{d\phi}{ds} \cdot \frac{d^2 s}{ds \; dq} = \frac{d^2 \phi}{ds \; ds} \cdot \frac{ds}{dq} + \frac{d\phi}{ds} \frac{d^2 s}{ds \; dq} = \frac{d^2 \phi}{dq \; dq} \cdot \frac{dq}{ds} + \frac{d\phi}{dq} \frac{d^2 q}{dq \; ds} = \frac{d^2 \phi}{dq \; dq} \cdot \frac{dq}{ds} + \frac{d\phi}{dq} \frac{d^2 q}{dq \; ds},
\]

where the tensor index \( i \) has been used to explain the switch of order of multiplication with the symmetric second derivative. The sequence of scalar multiplication may not be changed. For comparison, this expression in tensor notation is

\[
\frac{\partial^2 \phi}{ds^k \; dq^l} = \frac{\partial^2 \phi}{dq^k \; ds^l} + \frac{\partial \phi}{dq^k} \frac{\partial^2 s^k}{ds^l \; dq^l} = \frac{\partial^2 \phi}{ds^k \; ds^l} + \frac{\partial^2 s^k}{ds^l \; dq^l} \frac{\partial \phi}{ds^k},
\]

where the sequence of multiplication may be changed. Eqn. (21.71) is equally valid for a coordinate column \( \mathbf{r}(q) \)

\[
\frac{d^2 r}{ds \; dq} = \frac{d^2 r}{dq \; ds} + \frac{d r}{dq} \cdot \frac{d^2 s}{ds \; dq}.
\]

The inverse second derivative is obtained from the posed second derivative as with single variable derivatives, by taking the derivative of the unit tensor expressed as a product

\[
\frac{ds}{dq} \cdot \frac{dq}{ds} = \mathbf{I},
\]

\[
\frac{d}{dq} \left[ \frac{ds}{dq} \; \frac{dq}{ds} \right] = \mathbf{0} = \frac{d^2 s}{dq \; dq} \cdot \frac{ds}{dq} + \frac{ds}{dq} \cdot \frac{d^2 q}{dq \; ds} = \frac{d^2 s}{dq \; dq} \cdot \frac{ds}{dq} + \frac{ds}{dq} \cdot \frac{d^2 q}{dq \; ds} \cdot \frac{ds}{dq},
\]

therefore,

\[
\frac{d^2 s}{dq \; dq} = - \frac{d s}{dq} \cdot \frac{d^2 q}{dq \; dq} \cdot \frac{d s}{dq} \cdot \frac{d s}{dq}.
\]
Eqn.(21.73) is the well-known inversion by chained derivatives which is used to invert dependent
and independent variables of a set of second order partial differential equations, put in com-
 pact Matrix Tensor Notation. Higher order inverse derivatives are obtained by differentiating
eqn.(21.73) again
\[
\frac{d}{d q} \left[ \text{LHS of eqn.}(21.73) \right] = \frac{d}{d s} \cdot \left[ \text{RHS of eqn.}(21.73) \right] \cdot \frac{d s}{d q},
\]
etc., to the order required.

An expression of the derivative of the inverse can be found
\[
\frac{d}{d s} \left[ \frac{d q}{d s} \right]^{-1} = \frac{d}{d s} \left[ \frac{d s}{d q} \right] = \frac{d}{d s} \left[ \frac{d s}{d q} \right] \cdot \frac{d q}{d s} = \frac{d^2 s}{d q d q} \cdot \frac{d q}{d s},
\]
and then using eqn.(21.73),
\[
\frac{d}{d s} \left[ \frac{d q}{d s} \right]^{-1} = \frac{d^2 s}{d s d q} = - \frac{d s}{d q} \cdot \frac{d^2 q}{d s d q} \cdot \frac{d s}{d q}.
\]
(21.74)

A change of derivative variable from \( \overline{q} \) to \( \overline{r} \) is obtained by
\[
\frac{d^2 s}{d q d q} \cdot \frac{d q}{d q} = \frac{d}{d r} \left[ \frac{d s}{d r} \right] \cdot \frac{d r}{d q} = \frac{d^2 s}{d r d r} \cdot \frac{d r}{d q} + \frac{d s}{d r} \cdot \frac{d^2 r}{d r d q} \cdot \frac{d r}{d q} = \frac{d^2 s}{d r d r} \cdot \frac{d r}{d q} \cdot \frac{d r}{d q} \cdot \frac{d r}{d q}.
\]
(21.75)

if \( \overline{s}(\overline{r}) \) and \( \overline{q}(\overline{r}) \) is known.

To obtain the derivative of variables connected by an implicit equation
\[
\overline{f}(\overline{q}, \overline{s}) = 0,
\]
consider that
\[
d \overline{f} = \frac{\partial \overline{f}}{\partial \overline{q}} \cdot d \overline{q} + \frac{\partial \overline{f}}{\partial \overline{s}} \cdot d \overline{s} = 0,
\]
from which follows that
\[
\frac{d \overline{q}}{d \overline{s}} = - \frac{\partial \overline{f}}{\partial \overline{q}} \bigg/ \frac{\partial \overline{f}}{\partial \overline{s}}.
\]
(21.76)

where the operation of left matrix division \( \frac{\partial \overline{f}}{\partial \overline{q}} \bigg/ \frac{\partial \overline{f}}{\partial \overline{s}} \) is equal to the operation of multiplication with
the inverse \( \left[ \frac{\partial \overline{f}}{\partial \overline{q}} \right]^{-1} \). Eqn.(21.76) is the tensor equivalent of the well-known scalar formula
\[
\frac{d z}{d s} = - \frac{\partial \phi}{\partial v} \bigg/ \frac{\partial \phi}{d z}
\]
for the implicit relation \( \phi(z, s) = 0 \).
While all these forms if written in tensor notation, look equally familiar for a person comfortable in tensor notation, they are in Matrix Tensor Notation in a form which makes them suitable to transformation to other bases. We also believe that the compact form with names, vector symbols and total differentials make the equations easier to read because they are exactly the same as for ordinary single variables with the same name without vector symbol. This is the purpose of our notation. In contrast, tensor notation represents the detail of computational procedure of the operations.

21.7. The Differential Vector

In Cartesian base \( s \), the differential of a vector \( \mathbf{v} \) is

\[
\mathbf{dv} = \frac{d}{ds} \left[ \mathbf{E}_s \cdot \mathbf{v}^s \right] = \mathbf{E}_s \cdot d\mathbf{v}^s. \tag{21.77}
\]

In this section, differentials and derivatives due to \( ds \) are considered only, not due to a change in time, \( dt \). In curved coordinates \( \mathbf{q} \), the variable bases \( \alpha \) change with position, and the differential must be expanded

\[
\mathbf{dv} = \frac{d}{ds} \left[ \mathbf{E}_\alpha \cdot \mathbf{v}^\alpha \right] = \mathbf{E}_\alpha \cdot \mathbf{v}^\alpha + \frac{d\mathbf{E}_\alpha}{ds} \cdot \mathbf{v}^\alpha \tag{21.78}
\]

which shows that \( \mathbf{dv}^\alpha \) is not a vector array because it differs from \( (\mathbf{dv})^\alpha \) by the amount \( \mathbf{dE}_\alpha \cdot \mathbf{v}^\alpha \). The term \( d\mathbf{E}_\alpha \) is a new differential quantity which is likewise not a base because it doesn’t behave like a base in eqn.(21.78), or in a base transformation, in fact \( d\mathbf{E}_q = 0 \), and the Cartesian \( d\mathbf{E}_s = 0 \). The result is that any differential of a tensor array is not a differential tensor array.

21.8. The Derivative of a Vector

Differentials of coordinate columns are differential vector arrays, and therefore, the pseudo-division of a differential vector by a differential column creates a gradient tensor

\[
\frac{\mathbf{dv}}{dq} = \frac{d\mathbf{v}^s}{ds} \cdot \frac{d\mathbf{s}}{dq} = \mathbf{G}_s \cdot \mathbf{E}_q^s = \mathbf{G}_q^s, \tag{21.79}
\]

but more generally,

\[
\mathbf{G}^\alpha = \mathbf{G}_s \cdot \mathbf{E}_q^s = \frac{d\mathbf{v}^\alpha}{ds} \cdot \mathbf{E}_\alpha^s. \tag{21.80}
\]

Applied to eqn.(21.78)

\[
\mathbf{G} = \frac{d\mathbf{v}^s}{ds} = \frac{d}{ds} \left[ \mathbf{E}_\alpha \cdot \mathbf{v}^\alpha \right] = \mathbf{E}_\alpha \cdot \mathbf{v}^\alpha + \frac{d\mathbf{E}_\alpha}{ds} \cdot \mathbf{v}^\alpha. \tag{21.81}
\]

Alternatively, using Cartesian base,

\[
\mathbf{G}_s^s = \frac{d\mathbf{v}^s}{ds} = \frac{d}{ds} \left[ \mathbf{E}_\alpha \cdot \mathbf{v}^\alpha \right] = \mathbf{E}_\alpha \cdot \mathbf{v}^\alpha + \frac{d\mathbf{E}_\alpha}{ds} \cdot \mathbf{v}^\alpha. \tag{21.82}
\]

Even more generally, using changing base \( \alpha \), the expression becomes slightly longer,

\[
\mathbf{G}^\alpha = \frac{d\mathbf{v}^\alpha}{ds} \cdot \mathbf{E}_\alpha^s \tag{21.83}
\]

\[
= \frac{d}{ds} \left[ \mathbf{E}_\alpha \cdot \mathbf{v}^\alpha \right] \cdot \mathbf{E}_\alpha^s = \frac{d}{ds} \mathbf{E}_\alpha \cdot \mathbf{v}^\alpha \cdot \mathbf{E}_\alpha^s + \frac{d\mathbf{E}_\alpha}{ds} \cdot \mathbf{E}_\alpha^s. \tag{21.84}
\]
and even transformed to any other mixed bases a and b,

\[
\overline{G}^a_s = \overline{E}^s_a \cdot \frac{d}{ds} \overline{E}^s_a \cdot \overline{\nu}^a_b + \overline{E}^a_s \cdot \frac{d}{ds} \cdot \overline{E}^s_b,
\]  

(21.85)

in which the quantities \( \frac{d}{ds} \overline{E}^a_s \) and \( \frac{d}{ds} \overline{\nu}^a_s \) are created, which are not tensors, but they can be transformed partly by

\[
\frac{d}{ds} \overline{E}^a_s \cdot \frac{d}{dq} = \frac{d}{dq} \overline{E}^a_s \quad \text{and} \quad \frac{d}{ds} \overline{\nu}^a_s \cdot \frac{d}{dq} = \frac{d}{dq} \overline{\nu}^a_s.
\]

The problem that causes nontensor quantities is the differential of a quantity expressed in changing base a in the numerator, not in the denominator. To distinguish the derivative

\[
\frac{d\overline{\nu}^a_s}{ds} \neq \overline{G}^a_s \quad \text{from} \quad \frac{\overline{(dv)}^a_s}{ds} = \overline{G}^a_s,
\]  

(21.86)

we use the symbol for the derivative of a vector array

\[
\frac{d\overline{\nu}^a_s}{ds} = \overline{G}^a_s, \quad \frac{d\overline{\nu}^a_s}{dq} = \overline{G}^a_s.
\]  

(21.87)

of which the lower base symbol can be transformed to

\[
\frac{d\overline{\nu}^a_s \cdot \frac{d}{dq}}{ds} = \frac{d\overline{\nu}^a_s}{dq} = \overline{G}^a_s, \quad \frac{d\overline{\nu}^a_s \cdot \overline{E}^s_s}{ds} = \overline{G}^a_s.
\]  

(21.88)

For the particular two bases h and g, the explicit expression becomes

\[
\overline{G}^h_n = \frac{d}{dq} \left[ \overline{h} \cdot \overline{\nu} \right], \quad \overline{G}^g_n = \frac{d}{dq} \left[ \overline{g} \cdot \overline{\nu} \right],
\]  

(21.89)

but obviously the advantage of the notation is to avoid the explicit forms and use the common general forms instead; any expansions are rather done separately, in a computational or mathematical subroutine.

The well-known fact from tensor analysis, that a derivative of a tensor array is not a tensor, does not point to the fact that the problem is the differential, not the derivative. In tensor notation, the distinction is indicated in different ways:

\[
\frac{d\overline{v}^q}{dq} = \frac{\partial v^i}{dx^j} \quad \text{but} \quad \frac{\overline{(dv)}^q}{dq} = \nabla_j v^i \quad [7,17]
\]

\[
\frac{d\overline{v}^q}{dq} = \frac{\partial v^i}{dx^j} \quad \text{but} \quad \frac{d\overline{(dv)}^q}{dq} = \overline{Dv}^i \quad [12]
\]

\[
\frac{d\overline{v}^q}{dq} = \frac{\partial v^i}{dx^j} \quad \text{but} \quad \frac{\overline{(dv)}^q}{dq} = v^i_j \quad [23,27]
\]

\[
\frac{d\overline{v}^q}{dq} = v^i_j \quad \text{but} \quad \frac{\overline{(dv)}^q}{dq} = v^i_j \quad [26]
\]

\[
\frac{d\overline{v}^q}{dq} = v^i_j \quad \text{but} \quad \frac{\overline{(dv)}^q}{dq} = v^i_j \quad [2]
\]

Our notation is not equivalent to any of these; our ' in eqn.(21.87) indicates the result of an operation on the produced quantity. In our notation, we allow differentiation w.r.t. different
coordinate columns \( \overline{q} \) and \( \overline{s} \), which is not possible by the comma "\( , \)" of tensor notation. In fact in the defining eqn. (21.87), the upper \( \alpha \) comes from \( \overline{v}^{\alpha} \), while the lower \( s \) comes from \( d\overline{s} \). These can be any other unitary bases

\[
\frac{d\overline{v}^{\alpha}}{dr} = \overline{G}_{\alpha r}, \quad \frac{d\overline{v}^{\alpha}}{dq} = \overline{G}_{\alpha q},
\]

(21.90)
due to the chained derivatives

\[
\frac{d}{ds}(\cdot) = \frac{d}{dq}(\cdot) \frac{d\overline{q}}{ds} = \frac{d}{dr}(\cdot) \frac{d\overline{r}}{ds}
\]

\[
= \frac{d}{dq}(\cdot) \overline{E}^{q}_{s} = \frac{d}{dr}(\cdot) \overline{E}^{r}_{s}.
\]

The \( \overline{\alpha} \) symbol denotes a column which is not transformable, i.e.,

\[
\overline{E}_{\alpha}^{\ell} \cdot \overline{G}_{\alpha q}^{\ell} \neq \overline{G}_{\alpha q},
\]

and a base \( \overline{E}_{\alpha}^{\ell} \) doesn't exist. The tensor notation that is nearest to our notation, is the \( | \), which separates the nontransformable index on the left from the transformable symbol on the right.
22. THE CHRISTOFFEL SYMBOL

Starting again with eqn.(21.78), the derivative w.r.t. position is
\[ \frac{d \vec{v}}{ds} = \frac{d}{ds} \frac{E_\alpha}{E_\alpha} \cdot \frac{v^\alpha}{E_\alpha} + \frac{\vec{E}_\alpha}{E_\alpha} \cdot \frac{d v^\alpha}{ds}. \] (22.1)

The well-known Christoffel symbol \( \Gamma \) is used to define the derivative of base \( \alpha \) and is presented here in the unfamiliar form
\[ \frac{d}{ds} \vec{E}_\alpha \equiv \frac{\Gamma}{\tau_\alpha}, \] (22.2)
where the first and second kind is not distinguished. The Christoffel symbol in tensor notation can be replaced by the derivative of the base. The tensor symbol \( g_{ij} \) is the same as our base symbol \( E_q \), and the metric symbols \( g^{ij} \) correspond to our \( E^{\tau q}_s \) and \( E^q_s \), respectively, but there is no single symbol in tensor notation of our base \( E^s_q \). If curved coordinates are given in the form \( \vec{q}(s) \), then it is much more convenient to express the gradient of a vector in terms of the derivative of base \( E^s_q \) than in terms of the metric.

We immediately recognize that the space symbols can be transformed to other bases, by expanding the term
\[ \frac{d}{ds} \vec{E}_\alpha = \frac{d}{ds} \left[ \vec{E}_s \cdot E^s_\alpha \right] \cdot \frac{d s}{d s} = \vec{E}_s \cdot \frac{d}{ds} E^s_\alpha \cdot E^s_b = \vec{E}_s \cdot \frac{d}{ds} \vec{E}^s_\alpha \cdot \vec{E}^s_b = \vec{E}_s \cdot \frac{\Gamma}{\tau_\alpha}, \]
where \( s \) is the Cartesian base. The customary definition of the Christoffel symbol corresponds to a transformation of the upper Cartesian base symbol, and therefore, the general definition of the Christoffel symbol in mixed bases is
\[ \vec{E}^s_s \cdot \frac{d}{d s} \vec{E}^s_\alpha \cdot \vec{E}^r_b = \frac{\Gamma}{\tau_\alpha}, \] (22.3)
where \( \alpha, a \) and \( b \) are all changing bases, and the \( -/i_\alpha \) symbol denotes the row which is not transformable. Note that the reference to other curved coordinates is included
\[ \vec{E}^s_s \cdot \frac{d}{d r} \vec{E}^r_\alpha \cdot \vec{E}^s_b = \frac{\Gamma}{\tau_\alpha}, \] (22.4)
because
\[ \frac{d}{d r} \vec{E}^s_s \cdot \vec{E}^r_b = \frac{d}{d s} \vec{E}^s_\alpha \cdot \frac{d s}{d r} \cdot \vec{E}^r_b = \frac{d}{d s} \vec{E}^s_\alpha \cdot \vec{E}^r_s \cdot \vec{E}^r_b = \frac{d}{d s} \vec{E}^s_\alpha \cdot \vec{E}^s_b. \]

Using the arbitrary transformation definition of the upper base, the Christoffel symbol can also be transformed to mixed bases in our notation by
\[ \vec{E}^{\tau s}_s \cdot \frac{d}{d s} \vec{E}^s_\alpha \cdot \vec{E}^{\tau b}_b = \frac{\Gamma}{\tau_\alpha}. \] (22.5)

Our definition of the Christoffel symbol is the complete generalization of the special kinds used in tensor analysis,
\[ \vec{E}^q_s \cdot \frac{d}{d s} \vec{E}^s_q \cdot \vec{E}^s_s = \frac{d q}{d s} \cdot \frac{d}{d q} \vec{E}^s_q \equiv \frac{\Gamma}{\tau_q}, \] (22.6)
which is the Christoffel symbol of the second kind in our notation, and

$$\Gamma^r_q = \frac{d}{dq} \left( \frac{d}{ds} \bar{E}_r^s \cdot \bar{E}_q^s \right) = \left[ \frac{d}{dq} \frac{d}{ds} \bar{E}_q^s \right]^T \cdot \frac{d}{dq} \bar{E}_q^s \equiv \Gamma^r_q \cdot \bar{E}_q^s, \tag{22.7}$$

which is the Christoffel symbol of the first kind in our notation. But particularly, the base transformation symbols allow us to define the Christoffel symbol in normalized bases

$$\Gamma^h_q = \frac{d}{dq} \cdot \frac{d}{ds} \left[ \frac{d}{dq} \cdot \bar{h}^{-1} \right] \cdot \bar{h}^{-1} = \frac{d}{dq} \cdot \frac{d}{ds} \left[ \frac{d}{dq} \cdot \bar{g}^{-1} \right] \cdot \bar{g}^{-1} \cdot \frac{d}{dq} \cdot \frac{d}{ds} \left[ \frac{d}{dq} \cdot \bar{g}^{-1} \right] \cdot \bar{g} = \Gamma^r_q \cdot \bar{g}. \tag{22.8}$$

Of course the actual determination of the symbol is quite lengthy, which is just the reason of having a new symbol. But in an application the final expression should be written as a chain of terms of which the first are the derivatives and second derivatives. These are determined numerically first, then the chain of terms is simply computed numerically. Avoiding thus very long analytic expressions of the final symbol in terms of the first, all equations become easier to read and to compute. Very seldom a complete analytic expansion is worthwhile. That is, to compute the symbol $\Gamma^h_q$ at one point $\bar{q}$ in the field, the eqn.(22.8) is only the last step.

The previous steps are in the following backward sequence, assuming either $\bar{q}(s)$ or $\bar{s}(q)$ is available analytically: determine $\frac{d\bar{q}}{ds}$ or its inverse analytically, then numerically; determine the base and the metric numerically; determine $\bar{h}$ and $\bar{h}^{-1}$ numerically from the diagonal of the metric; determine the second derivatives analytically, then numerically; expand

$$\frac{d}{dq} \left[ \bar{E}_q^s \cdot \bar{h}^{-1} \right] = \frac{d^2}{dq} \cdot \bar{h}^{-1} \cdot \bar{E}_q^s \cdot \frac{d}{dq} \bar{h}^{-1} ;$$

expand

$$\frac{d}{dq} \bar{h}^{-1} = -\bar{h}^{-1} \cdot \frac{d}{dq} \bar{h} \cdot \bar{h}^{-1} ;$$

obtain the derivatives of the diagonals $\frac{d}{dq} \bar{h}$ analytically as a combination of elements of first and second derivatives; then evaluate numerically. This is the beginning of the chain of numerical evaluation for the eqn.(22.8).

In all the preceding formulas, the sequence of multiplication after the stretched matrix is not indicated explicitly because it is unique by the base symbols as explained in Section 20; i.e.,

$$\frac{d}{ds} \bar{E}^r_s \cdot \bar{v}^a \cdot \bar{E}_a^s \equiv \frac{d}{ds} \bar{E}^r_s \cdot \bar{v}^a \circ \bar{E}_a^s,$$

$$\bar{E}^r_s \cdot \frac{d}{ds} \bar{E}_a^s \cdot \bar{E}_b^s \cdot \bar{E}_b^b \equiv \bar{E}^r_s \cdot \frac{d}{ds} \bar{E}_a^s \circ \bar{E}_b^s \cdot \bar{E}_b^b,$$

$$\bar{E}^r_s \cdot \Gamma^r_{qa} \cdot \bar{E}_b^b \equiv \bar{E}^r_s \cdot \Gamma^r_{qa} \circ \bar{E}_b^b,$$

The Christoffel symbol corresponds to the second derivative

$$\Gamma^r_q = \frac{d^2}{dq} \cdot \frac{d}{ds} \bar{E}_r^s,$$  \hspace{1cm} (22.9)

$$\Gamma^r_a = \frac{d}{ds} \cdot \frac{d^2}{dq} \bar{E}_r^s.$$  \hspace{1cm} (22.10)
\[
\Gamma^s_{rq} = \frac{d^2s}{dsdq}, \quad (22.11)
\]
\[
\frac{\Gamma^s}{q} = \frac{d^2s}{dqdq}, \quad (22.12)
\]

and has the transformation property
\[
\Gamma^a_{rq} = E^a_b \cdot \Gamma^b_{cq} \cdot E^c_r = E^a_b \cdot \Gamma^b_{cq} \cdot E^c_r, \quad (22.13)
\]
\[
\Gamma^{\tau a}_{rq} = E^{\tau a}_b \cdot \Gamma^b_{cq} \cdot E^c_r = E^{\tau a}_b \cdot \Gamma^b_{cq} \cdot E^c_r. \quad (22.14)
\]

The customary form of the Christoffel symbol of the second kind in tensor analysis corresponds to the expression in our notation in base \( q \)
\[
\Gamma^q_{rq} = \frac{d\bar{s}}{d\bar{q}} \cdot \frac{d^2\bar{s}}{d\bar{q}dq}, \quad (22.15)
\]
and of the first kind, to the expression in our notation
\[
\Gamma^{\tau q}_{rq} = \left[ \frac{d\bar{s}}{d\bar{q}} \right]^T \cdot \frac{d^2\bar{s}}{d\bar{q}dq}, \quad (22.16)
\]
noting that
\[
\frac{d\bar{s}}{d\bar{q}} = E^s_q, \quad E^{\tau q}_q = [E^s_q]^T. \quad (22.17)
\]

The transpose symbol on the upper base symbol corresponds to the lower index in tensor notation, which would become in our notation
\[
\Gamma^q_{rq} = \Gamma_{q \times q \times q} = \frac{ds}{d\bar{q}} \cdot \frac{d^2\bar{s}}{d\bar{q}dq}. \quad (22.17)
\]

Using the inverse of the 2nd derivative, the Christoffel symbols of the second and first kinds can also be expressed as
\[
\Gamma^q_{rq} = -\frac{d^2\bar{q}}{d\bar{s}d\bar{s}} \cdot \frac{d\bar{s}}{d\bar{q}} \cdot \frac{ds}{d\bar{q}}, \quad (22.18)
\]
\[
\Gamma^{\tau q}_{rq} = \frac{E^{\tau q}}{q} \cdot \Gamma^{\tau q}_{q}, \quad (22.19)
\]

The first and second kind are merely representations in the two different unitary bases. The transformation of the top base symbol had no logical reason that can be deduced from existing transformation laws. It defines a particular, convenient, grouping of quantities that does not violate existing transformation laws. We have done it here to comply with the customary definition of the Christoffel symbols, albeit generalized to arbitrary base. The consequence is that we are prevented from defining a similar symbol in the pre-era configuration of Cartesian coordinates \( \bar{q} \) of Section 21.4, i.e., \( \Gamma^q_{rs} \neq \frac{d^2\bar{q}}{d\bar{s}d\bar{s}} \); i.e., in the application of the Christoffel symbols, it must be strictly defined which base is Cartesian. This restriction doesn’t occur with the expressions as second derivatives.
In some texts, the definition of the Christoffel symbols corresponds to our the definitions given here by deriving them from the base \([2,26]\) as follows, compared to our notation:

\[
\begin{align*}
\text{Tensor Notation:} & & \text{Matrix Tensor Notation:} \\
\Gamma_{ij}^k & = \Gamma_{ij}^k g_n & \frac{d}{dq} \mathbf{E}_q^i = \mathbf{E}_q^i \cdot \frac{\Gamma^q_{ij}}{q} \\
g_{ij} \cdot g_k & = \Gamma_{ijk} & \frac{d}{dq} \mathbf{E}_q^i = \mathbf{E}_q^i \cdot \frac{\Gamma^q_{ij}}{q} 
\end{align*}
\]

Other notations for the Christoffel symbols of the first kind and second kind, respectively, are

\[
\begin{align*}
\{ i \} & \text{ and } \{ jk \} & \text{[23,27]} \\
\Gamma_{i,jk} & \text{ and } \Gamma_{i,j}^k & \text{[12]} \\
\Gamma_{kji} & \text{ and } \Gamma_{k,j}^i & \text{[2,26]} \\
\Gamma_{q,j}^i & \text{ and } \Gamma_{q,j}^i & \text{[51]}
\end{align*}
\]

where in the last row our symbols with matching indices appended are given.

The so-called transformation of the Christoffel symbols means that the symbol in base \(q\) must be transformed to the symbol in terms of base \(r\), using the explicit form \(\Gamma^r_q(q)\). From eqns.(22.9) to (22.12), it can be seen that this is merely a transformation of second derivatives, using the method and notation of Section 21.6,

\[
\frac{\Gamma^r_q}{\frac{d}{d\xi}} = \frac{d}{d\xi} \cdot \frac{d^2\xi}{d\xi^r d\xi^r} = \frac{d}{d\xi} \cdot \frac{d\xi^1}{d\xi^i} \cdot \frac{d\xi^j}{d\xi^i} + \frac{d}{d\xi} \cdot \frac{d^2\xi}{d\xi^r d\xi^r},
\]

which is the expression where all functions \(\xi^i(\xi)\) have been eliminated. In classical tensor notation only one of the second derivatives is a Christoffel symbol, so that eqn.(22.21) becomes

\[
\frac{\Gamma^r_q}{\frac{d}{d\xi}} = \frac{d}{d\xi} \cdot \frac{d\xi^1}{d\xi^i} \cdot \frac{d\xi^j}{d\xi^i} + \frac{d}{d\xi} \cdot \frac{d^2\xi}{d\xi^r d\xi^r}.
\]
Because of the arbitrary combination of terms in the definition of the Christoffel symbol, it cannot be used to express all the second derivatives occurring in its transformation equation.

With the definition of the Christoffel symbol of the first kind in eqn. (22.14), we obtain by the same method the so-called transformation equation

\[
\Gamma_{\gamma \alpha \beta} = \Gamma_{c,ab} \frac{\partial x^a}{\partial X^c} \frac{\partial x^b}{\partial X^d} + g_{ab} \frac{\partial^2 x^a}{\partial X^d \partial X^e},
\]

which in tensor notation is [12], changed to revised partial derivatives notation

\[
\Gamma_{\gamma \alpha \beta} = \Gamma_{c,ab} \frac{\partial x^a}{\partial X^c} \frac{\partial x^b}{\partial X^d} + g_{ab} \frac{\partial^2 x^a}{\partial X^d \partial X^e},
\]

where \(g_{ab}\) is the metric of base \(q\), corresponding to our \(E_q\).

Eqn. (21.81) provides the equation for the vector gradient i.t.o, the derivative of the vector array

\[
\overline{G}_q^s = \frac{d}{dq} \overline{E}_q^s \cdot \overline{v}^q + \overline{E}_q^s \cdot \overline{G}_q^q. \tag{22.24}
\]

which becomes with the Christoffel symbol the well-known formula for the tensor, called the covariant derivative (referring to the pseudo-division by a contravariant vector) of a contravariant vector,

\[
\overline{G}_q^s = \Gamma_{r,sq} \overline{v}^r + \frac{\partial \overline{v}^r}{\partial q}. \tag{22.25}
\]

Referring to our equivalence of column and row vectors with contra and covariant vectors, this is the row derivative of a column vector.

Again, the post multiplication of the Christoffel symbol is defined by the rules of Section 20, but even if another order is taken, the result is the same because the Christoffel symbol is symmetric in the derivatives, so that

\[
\Gamma_{\gamma \alpha \beta} = \Gamma_{\alpha,\gamma \beta} = \Gamma_{\beta,\alpha \gamma} \tag{22.26}
\]

From eqn. (21.81) and (22.2) follows our generalized transformation of the gradient of a vector in terms of the derivative in changing base, which is the whole purpose of the Christoffel symbol,

\[
\overline{G} = \Gamma_{r,sq} \overline{v}^r + \frac{\partial \overline{v}^r}{\partial q}. \tag{22.27}
\]

which can then be transformed to any mixed bases

\[
\overline{E}_a \cdot \overline{G}_b = \overline{E}_a \cdot \Gamma_{\alpha,\alpha} \overline{v}^\alpha + \overline{E}_a \cdot \overline{G}_a^a \overline{E}_b^b \tag{22.31}
\]

\[
\overline{G}_b^a = \Gamma_{\alpha,\alpha} \overline{v}^\alpha + \overline{E}_a \cdot \overline{G}_b^a. \tag{22.32}
\]
This includes the two special cases

\[
\overline{G}_h^h = \overline{r}_h^h \cdot \overline{v}_h^h + \overline{G}_h^h ,
\]

(22.33)

\[
\overline{G}_g^g = \overline{r}_g^g \cdot \overline{v}_g^g + \overline{G}_g^g ,
\]

(22.34)

where all necessary transformations are already included by the meaning of the symbols. The result is that the customary tensor equations can be applied directly to normalized bases.

A vector in reciprocal base, a covariant vector in tensor notation, is written in our transposed form as column \( \overline{v}^{\tau\alpha} \). The derivative of this vector and the vector array is defined with a \( \tau\alpha \) symbol the same way as for \( \overline{v}^{\alpha} \), considering \( \tau\alpha \) just another base as explained in Section 15.2 and 15.6. The gradient of a vector according to eqn.(16) becomes in any mixed bases

\[
\overline{G}_b^a = \overline{E}_s^a \cdot \frac{d}{ds} \overline{E}_{\tau\alpha}^s \cdot \overline{v}_{\tau\alpha} + \overline{E}_b^a \cdot \frac{d}{ds} \overline{G}_s^\tau \cdot \overline{E}_b^s .
\]

(22.35)

Therefore, comparing with eqn.(22.32), the vector array derivative is defined

\[
\frac{d \overline{v}^{\tau\alpha}}{ds} = \overline{G}_{s}^{\tau\alpha} ,
\]

(22.36)

and a new Christoffel symbol is defined

\[
\overline{E}_b^a \cdot \frac{d}{ds} \overline{E}_{\tau\alpha}^b \cdot \overline{E}_b^s + \overline{E}_b^a \cdot \frac{d}{ds} \overline{G}_s^\tau \cdot \overline{E}_b^s \equiv \overline{G}_b^{\tau\alpha} .
\]

(22.37)

which is not simply a base-transformation of the Christoffel symbol of eqn.(22.3), because the untransformable symbol is different. The gradient transformation with these symbols becomes

\[
\overline{G}_b^a = \overline{E}_b^a \cdot \frac{d}{ds} \overline{E}_{\tau\alpha}^b \cdot \overline{v}_{\tau\alpha} + \overline{E}_b^a \cdot \frac{d}{ds} \overline{G}_s^\tau \cdot \overline{E}_b^s ,
\]

(22.38)

or

\[
\overline{G}_b^{\tau\alpha} = \overline{E}_b^a \cdot \frac{d}{ds} \overline{E}_{\tau\alpha}^b \cdot \overline{v}_{\tau\alpha} + \overline{E}_b^a \cdot \frac{d}{ds} \overline{G}_s^\tau \cdot \overline{E}_b^s .
\]

(22.39)

These forms include the customary case when \( a = b = \alpha = q \), but also the important cases when \( a = b = \alpha = h \), and \( a = b = \alpha = g \).

Referred to Cartesian base,

\[
\overline{G}_s^s = \overline{G}_s^{(r\tau\alpha)} \cdot \overline{v}^{(r\tau\alpha)} + \overline{E}_s^s \cdot \overline{G}_s^{(r\tau\alpha)} .
\]

(22.40)

Instead, the derivative can be obtained from the row form, but then stretched matrices must be employed,

\[
\overline{G}_s^s = \frac{d \overline{v}_s^s}{ds} = \frac{d}{ds} \left[ \overline{v}_q \cdot \overline{E}_s^q \right] = \frac{d \overline{v}_s^s}{ds} \cdot \overline{E}_s^q + \overline{v}_q \cdot \frac{d}{ds} \overline{E}_s^q = \overline{G}_s^{(rq\tau\alpha)} \cdot \overline{E}_s^s + \overline{v}_q \cdot \frac{d}{ds} \overline{E}_s^q .
\]

(24.41)
In terms of space notation as in eqn. (22.2),

$$\Gamma^q_s = \frac{d}{ds} E^q_s = \frac{d^2 q}{ds^2} .$$

(22.42)

and consistent with the transformation of the Christoffel symbol, eqn. (22.3), both space symbols can be transformed to any base,

$$\Gamma^q_s = \frac{d^2 q}{ds}\cdot E^q_s .$$

(22.43)

Because of the arbitrary transformation definition of eqn. (22.3), this transformation is also not logically consistent with our other transformations.

We can get eqn. (22.41) back to matrix form of the derivative by a partial transpose of the new Christoffel symbol,

$$\mathcal{G}^s = E^s_{\tau q} \cdot \mathcal{G}^{\tau q} + \Gamma^{\tau q}_{s} \cdot \nu^{\tau q} ,$$

(22.45)

which shows that the new Christoffel symbol is but a partially transposed version of the previous one, a complication that doesn't occur in tensor notation. In Matrix Tensor Notation a mere rearrangement of terms requires a new name which is derived from the old name by the transpose rules of Section 20, applied now to the the third order array.

Because $\Gamma^q_s$ is defined as a second derivative, it can be transformed to the second derivative of the inverse by the rules of Section 21.6

$$\Gamma^q_s = -\frac{d}{ds} \frac{d^2 q}{ds} \cdot \frac{d}{ds} \cdot \frac{d}{ds} \cdot \frac{d}{ds} = -\mathcal{E}^q_s \cdot \mathcal{G}^{\tau q} \cdot \mathcal{E}^q_s \cdot \mathcal{G}^{\tau q}$$

$$= -\Gamma^{\tau q}_{s} \cdot \mathcal{E}^q_s ,$$

(22.46)

which contains the previous familiar Christoffel symbol. Again doing a partial transpose, we find that

$$\Gamma^{\tau q}_{s} = -\mathcal{E}^q_s \cdot \Gamma^{\tau q}_{s} ,$$

(22.47)

and the corresponding expression for the gradient becomes

$$\mathcal{G}^s = E^s_{\tau q} \cdot \mathcal{G}^{\tau q} - \mathcal{E}^s_{\tau q} \cdot \Gamma^{\tau q}_{s} \cdot \nu^{\tau q}$$

(22.48)

and

$$\mathcal{G}^{\tau q}_{s} = \mathcal{E}^{\tau q}_{s} \cdot \mathcal{G}^s_{s} \cdot \mathcal{E}^s_{q}$$

$$= \mathcal{G}^{\tau q}_{s} - \Gamma^{\tau q}_{s} \cdot \nu^{\tau q} ,$$

(22.49)

which corresponds to the customary tensor equation

$$v_{i j} = \frac{\partial v_i}{\partial x^j} - \Gamma^k_{i j} v_k$$
except that our Christoffel symbol is rearranged in such a way that it produces a matrix as derivative.

As before, by the proper transformations, it is possible to express in normalized bases

\[ G^{\tau h}_{\lambda} = \frac{\bar{E}^{\tau h}_{\lambda}}{s^\rho} \cdot G^\rho_{\lambda} \cdot \bar{E}^\rho_{\lambda} \]

\[ = \frac{G^{\tau h}_{\lambda}}{s^\rho} - \frac{\bar{T}^{\tau h}_{\lambda}}{s^\rho}, \quad (22.50) \]

\[ G^{\tau g}_{\lambda} = \frac{\bar{E}^{\tau g}_{\lambda}}{s^\rho} \cdot G^\rho_{\lambda} \cdot \bar{E}^\rho_{\lambda} \]

\[ = \frac{G^{\tau g}_{\lambda}}{s^\rho} - \frac{\bar{T}^{\tau g}_{\lambda}}{s^\rho}. \quad (22.51) \]

All the possible forms and transformations that may be required in any bases derived from curvilinear coordinates can thus be obtained by the generalized definition of the Christoffel symbol, where the base symbols define the proper transformation or arrangement.

In tensor analysis, a more familiar way to derive the Christoffel symbol and its properties is to refer to the metric of the base, while the foregoing expressions for the Christoffel symbol are based on the coordinate functions \( \bar{q}(s) \) or \( \bar{s}(q) \). There may be a theoretical reason that the coordinate relations to Cartesian base are not known, except for merely the metric \( \bar{E}^{\tau g}_{q} \). The metric is a variable of the coordinates

\[ \frac{d}{ds} \bar{E}^{\tau q}_{q} = \frac{d}{ds} \bar{E}^{\tau q}_{s} \cdot \bar{E}^{s}_{q} + \bar{E}^{\tau q}_{s} \cdot \frac{d}{ds} \bar{E}^{s}_{q}. \quad (22.52) \]

Because, in our notation, we cannot transpose elements of a 3rd order tensor, we cannot represent the familiar inverse result from eqn.(22.52) other than by appended tensor indices

\[ \Gamma^{\tau q i}_{q k} = \frac{1}{2} \left[ \frac{d}{dq} \bar{E}^{\tau q j}_{q i} + \frac{d}{dq} \bar{E}^{\tau q k}_{q i} - \frac{d}{dk} \bar{E}^{\tau q i}_{q j} \right]. \quad (22.53) \]

Our formulas are more suited to a particular Cartesian base to which the curved coordinates \( \bar{q}(s) \) are referred, while in tensor notation, the equation (20.53) gives the Christoffel symbol in terms of the metric without any reference to a particular coordinate system. Our definition of the Christoffel symbol in eqn.(22.15) apparently uses the Cartesian coordinates \( s, \) but in fact these can be transformed to any other Cartesian base \( \tau \) by

\[ \frac{d\bar{q}}{ds} \cdot \frac{d^2\bar{s}}{dq \, dq} = \frac{d\bar{q}}{d\tau} \cdot \frac{d^2\bar{r}}{d\bar{q} \, d\bar{q}} \quad (22.54) \]

because the transformation matrix

\[ \bar{E}^s_r = \frac{d\bar{s}}{d\bar{r}} \]

is independent of \( \bar{q} \).

While our Matrix Tensor Notation may be able to define a more general Christoffel symbol, the notation is not sufficient to describe unique manipulations with any third order, or higher, quantity. However the use of the second derivative instead provides a possible easier handling of third order quantities, even though auxiliary tensor indices may have to be appended. Alternatively perhaps, our base symbols may be appended to tensor notation.
22.1. Curl

In Cartesian base, curl of a vector can be obtained from the vector cross product

\[ \text{curl} \, \vec{v}^s \equiv \zeta^s = \nabla^s \times \vec{v}^s. \]

Let the 3-dimensional antisymmetric tensor of the elements corresponding to curl \( \vec{v}^s \) be \( \overline{C}_s \), then

\[ \overline{C}_s = \frac{d \vec{v}^s}{d s} - \left[ \frac{d \vec{v}^s}{d s} \right]^\top \]  

which transforms to unitary base as

\[ \overline{C}^q = \overline{E}^q_s \cdot \left[ \frac{d \vec{v}^s}{d q} - \left[ \frac{d \vec{v}^s}{d q} \right]^\top \right] \cdot \overline{E}_q^s, \]

where in view of the cross product only the one physical base \( q \) is used. Transforming

\[ \frac{d \vec{v}^q}{d q} = \frac{d}{d q} \left[ \overline{E}^q_s \cdot \vec{v}^s \right] \]

\[ = \frac{d}{d q} \overline{E}^q_s \cdot \vec{v}^s + \overline{E}^q_s \cdot \frac{d \vec{v}^s}{d q} \]

\[ = \frac{d^2 \vec{v}^s}{d q \, dq} \cdot \vec{v}^s + \overline{E}^q_s \cdot \frac{d \vec{v}^s}{d q} \cdot \overline{E}_q^s \]

\[ = \left[ \frac{d \vec{v}^q}{d q} \right]^\top \cdot \overline{E}^q_s \cdot \left[ \frac{d \vec{v}^s}{d q} \right]^\top \cdot \overline{E}_q^s. \]

Because of the symmetry of the matrix in the first term on the right, this matrix is eliminated in the difference

\[ \frac{d \vec{v}^q}{d q} - \left[ \frac{d \vec{v}^q}{d q} \right]^\top = \overline{E}^q_s \cdot \left[ \frac{d \vec{v}^s}{d q} - \left[ \frac{d \vec{v}^s}{d q} \right]^\top \right] \cdot \overline{E}_q^s \]

\[ = \overline{E}^q_s \cdot \overline{C}_s \cdot \overline{E}_q^s \]

\[ = \overline{C}^q. \]

In this form neither the metric nor the curvature enters the formula.

Using the nabla operator in unitary base as defined in Section 21.5, and the scale factor due to the cross product from Section 15.8, the equation for the transformed curl vector is

\[ \overline{\zeta}^q = \overline{E}^q_s \cdot \overline{\zeta}^s. \]

\[ (1/\sqrt{g}) \nabla_q \times \vec{u}_q = \overline{\zeta}^q. \]

Transformed to any other bases his formula does not have the same simple form, the curvature of the coordinates then enter the formula by way of the derivative of the base, or the Christoffel symbol.

22.2. Div

No simple formula for div \( \vec{v} \) exists. The well-known form from tensor analysis, [12,27], written in our notation
\[ \text{div } \bar{v} = \frac{1}{\sqrt{g}} \left( \frac{d}{dq} \sqrt{g} \right) \cdot \bar{v} + \text{trace} \left( \frac{d\bar{v}}{dq} \right) \]

is a compact form, but the actual differentiation of the square root of the determinant of the metric contains the curvature in terms of Christoffel symbols. We write, therefore, directly in our notation

\[ \text{div } \bar{v} = \text{trace} \left( \bar{G}^{\alpha} \right) \]
\[ = \text{trace} \left( \bar{E}_{\alpha}^{\prime} \cdot \bar{G}_{\alpha}^{\prime} \cdot \bar{E}_{\alpha}^{\prime} \right) \]
\[ = \text{trace} \left( \bar{G}_{\alpha}^{\prime} \right), \quad (22.60) \]

where the last line follows from the fact that the trace is invariant under a similarity transformation. Substituting the expression for the velocity gradient in terms of the derivative from eqn.(21.85), with \( a = b = a \),

\[ \text{div } \bar{v} = \text{trace} \left( \bar{E}_{\alpha}^{\prime} \right) \cdot \left[ \frac{d}{ds} \bar{E}_{\alpha}^{\prime} \right] \cdot \bar{v}_{\alpha} \cdot \bar{E}_{\alpha} + \text{trace} \left( \frac{d\bar{v}}{ds} \cdot \bar{E}_{\alpha}^{\prime} \right), \]

where the derivative of the base can be replaced by any of the forms for the Christoffel symbol.

22.3. Conclusions

The Christoffel symbol may not always be easy to read. Our notation provides the notation as derivative of a base, which is clear and explicit. But the matrix symbols become very complicated as soon as a quantity is more than second order, be it a tensor or a symbol. The transpose also is not defined for a higher than second order quantity. It seems, therefore, that our Matrix Tensor Notation is not suitable for higher than second order quantities, seeing that, in some cases, tensor indices are necessary anyway. But one way that Matrix Tensor Notation seems still to be useful in curved coordinates is the notation in terms of derivatives, avoiding the Christoffel symbol altogether. The distinction between physical tensors and components, and between different bases is shown clearly without too much complication of symbols.
23. CONCLUSION

Matrix Tensor Notation as introduced for skew bases provides a rational symbolic method to analyze problems where several skew bases occur, and distinguishing between base and space. Tensor notation in comparison has no rational provision for transformation between several bases. Applications may occur, like robotics or graphical projections, where coordinates are not involved at all, which is not the domain of tensor analysis.

The Matrix Tensor Notation of Part II, which includes the notation of Part I, has been demonstrated as a kind of notational algebra with the purpose of its application. No attempt has been made to actually formally prove this notational algebra. As far as we are aware, no such proof has been made for the consistent tensor notation either. But it might be possible to do so by Lie Algebra, such as has been made for rigid body rotations by Argyris and Poterasu [52], and with particular application to robotics extensively by Mladenova (see, e.g., [53,54]).

Matrix symbols become very complicated as soon as a quantity is more than second order, be it a tensor or a Christoffel symbol. The transpose also is not defined for a higher than second order quantity. It seems, therefore, that Matrix Tensor Notation is not suitable for higher than second order quantities, seeing that, in some cases, tensor indices are necessary anyway. Nevertheless, the complication of the transpose is usually not serious because, although discussed at length, it doesn’t occur much in the applications. Matrix Tensor Notation is still useful for any higher order quantities that can be expressed in terms of vectors and 2nd order tensors. The explicit notation of some higher order quantities in terms of second order derivatives based on curved coordinates that must somewhere be expressed explicitly by Cartesian coordinates anyway, is more explicit than the Christoffel symbol, which avoids reference to the Cartesian base.

Therefore, Matrix Tensor Notation seems to be a useful alternative, where expressions in terms of vectors and 2nd order tensors in several skew or Cartesian bases occur, and in the treatment of different curved coordinates which are related to Cartesian coordinates.

APPENDIX

MATRIX TENSOR NOTATION FOR PRINTING

In printed material, books and journals, physical tensors are often written in boldface, but relations are still written in matrix-vector mode, with the noncommutative sequence in multiplication, and distinguishing tensors of 2nd order in capital letters from physical vectors in small letters, and scalars in ordinary print, and using the transpose. The textbook [3] also uses this system. Texts on tensor analysis usually use the same small letter boldface notation for vectors while avoiding physical tensors altogether, e.g., [2,3]. Textbooks in mechanics use boldface small and capital letters for physical vectors, also avoiding physical tensors altogether, e.g., [8,21,27], while [16] uses boldface for physical vectors and tensors but doesn’t distinguish between them by notation. Many books on matrix algebra [5] and applications of matrix algebra in Finite Elements [48] use boldface for algebraic vector arrays and matrices. In applications of dynamics, boldface is sometimes used for the physical vector arrays as well as tensor arrays and transformation matrices [15,55], which makes it awkward to distinguish between tensors and bases.

One particular and important aim of Matrix Tensor Notation is to make it easily written by hand. But it can also be usefully applied to the printed boldface method to attain all the other aims as stated in Part I, with the following rules.

All vectors are still written in small letters and 2nd order tensors in Capital letters in boldface. The boldface symbol represents a physical part of the tensor, which is written with either one or two arrows in the Matrix Tensor Notation. How many arrows it represents, and on which side they are, is completely evident in the equation where they occur. When they are standing alone, it is either implied by the other symbols or it doesn’t matter. The base symbols are then added.

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4The author is indebted to C. D. Mladenova for drawing his attention to those publications.
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to the boldface symbols. A vector array or tensor matrix however is again in ordinary print as in this article. The following self-explanatory examples make this clear:

\[ \mathbf{v} \equiv \overrightarrow{v} \text{ or } \overrightarrow{v} \]
\[ \mathbf{E} \equiv \overrightarrow{E} \]
\[ \mathbf{E}_q \equiv \overrightarrow{E}_q \]
\[ \overrightarrow{\mathbf{E}}^q \equiv \overrightarrow{E}^q \]
\[ \mathbf{K} \equiv \overrightarrow{K} \]
\[ \mathbf{K}_q \equiv \overrightarrow{K}_q \]
\[ \overrightarrow{\mathbf{K}}^q \equiv \overrightarrow{K}^q \]

\[ \mathbf{E}_q = [e_{q1} \ e_{q2} \ e_{q3}] \equiv \overrightarrow{E}_q = [\overrightarrow{e_{q1}} \ \overrightarrow{e_{q2}} \ \overrightarrow{e_{q3}}] \]
\[ \overrightarrow{\mathbf{E}}^q = \left[ \begin{array}{c} e_{q1} \\ e_{q2} \\ e_{q1} \end{array} \right] \equiv \overrightarrow{E}^q = \left[ \begin{array}{c} \overrightarrow{e_{q1}} \\ \overrightarrow{e_{q2}} \\ \overrightarrow{e_{q1}} \end{array} \right] \]

\[ \mathbf{f} \cdot \mathbf{v} \equiv f \cdot \overrightarrow{v} \]
\[ \mathbf{K} \cdot \mathbf{u} \equiv \overrightarrow{K} \cdot \overrightarrow{u} \]
\[ \mathbf{u} \cdot \mathbf{K} \equiv \overrightarrow{u} \cdot \overrightarrow{K} \]
\[ \mathbf{u} \cdot \mathbf{K}^\top \equiv \overrightarrow{u} \cdot \overrightarrow{K}^\top \]

\[ \mathbf{v} = \mathbf{E}_q \cdot \overrightarrow{v}^q \equiv \overrightarrow{v} = \overrightarrow{E}_q \cdot \overrightarrow{v}^q \]
\[ \overrightarrow{v}^q = \overrightarrow{E}^q \cdot \mathbf{v} \equiv \overrightarrow{v} = \overrightarrow{E}^q \cdot \overrightarrow{v} \]
\[ \mathbf{v} = \overrightarrow{v}_q \cdot \mathbf{E}_q^q \equiv \overrightarrow{v} = \overrightarrow{v}_q \cdot \overrightarrow{E}_q \]
\[ \overrightarrow{v}_q = \mathbf{v} \cdot \mathbf{E}_q \equiv \overrightarrow{v}_q = \overrightarrow{v} \cdot \overrightarrow{E}_q \]
\[ \mathbf{E}_q = \overrightarrow{E}_q \cdot \overrightarrow{E}_q \equiv \overrightarrow{E}_q = \overrightarrow{E}_q \cdot \overrightarrow{E}_q \]
\[ \overrightarrow{\mathbf{K}}_q = \overrightarrow{\mathbf{E}}^q \cdot \mathbf{K} \cdot \mathbf{E}_q \equiv \overrightarrow{\mathbf{K}}_q = \overrightarrow{\mathbf{E}}^q \cdot \overrightarrow{\mathbf{K}} \cdot \overrightarrow{E}_q \]
\[ \overrightarrow{\mathbf{K}}^q = \overrightarrow{\mathbf{E}}^q \cdot \mathbf{K} \cdot \mathbf{E}_q \equiv \overrightarrow{\mathbf{K}}^q = \overrightarrow{\mathbf{E}}^q \cdot \overrightarrow{\mathbf{K}} \cdot \overrightarrow{E}_q \]
\[ \overrightarrow{\mathbf{K}}_s = \overrightarrow{\mathbf{E}}_s \cdot \mathbf{K} \cdot \mathbf{E}_q \equiv \overrightarrow{\mathbf{K}}_s = \overrightarrow{\mathbf{E}}_s \cdot \overrightarrow{\mathbf{K}} \cdot \overrightarrow{E}_q \]
\[ \mathbf{K} = \overrightarrow{\mathbf{E}}_s \cdot \overrightarrow{\mathbf{K}}_q \cdot \overrightarrow{\mathbf{E}}_q \equiv \overrightarrow{\mathbf{K}} = \overrightarrow{\mathbf{E}}_s \cdot \overrightarrow{\mathbf{K}}_q \cdot \overrightarrow{\mathbf{E}}_q \]

\[ \frac{d\mathbf{s}}{dq} = \frac{d\overrightarrow{s}}{dq} \]
\[ \frac{d\mathbf{v}}{ds} = \frac{d\overrightarrow{v}}{ds} \]
\[ \frac{d}{ds} \mathbf{E}_q = \frac{d}{ds} \overrightarrow{E}_q \]
\[
\frac{d^2 s}{ds \, dq} \equiv \frac{d^2 s}{ds \, q}
\]

\[
\frac{d}{ds} Q \equiv \frac{d}{ds} \vec{Q}.
\]

REFERENCES