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Entanglement-assisted codeword-stabilized quantum codes with imperfect ebits☆

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Abstract

In quantum communication systems, quantum error-correcting codes (QECCs) are known to exhibit improved performance with the use of error-free entanglement bits (ebits). In practical situations, ebits inevitably suffer from errors, and as a result, the error-correcting capability of the code is diminished. Previous studies have proposed two different schemes as a solution. One study uses only one QECC to correct errors on the receiver side (i.e., Bob) and sender side (i.e., Alice). The other uses different QECCs on each side. In this paper, we present a method to correct errors on both sides by using single nonadditive entanglement-assisted codeword stabilized quantum error-correcting code (EACWS QECC). We use the property that the number of effective error patterns decreases as much as the number of ebits. This property results in a greater number of logical codewords using the same number of physical qubits.

Keywords: Entanglement-assisted codeword-stabilized code; Imperfect ebits; Quantum error-correcting code

1. Introduction

Over the past two decades, there has been increased research on quantum computing and communications systems. Quantum error-correcting codes (QECCs) are required to implement practical quantum computing and communication systems because it is not feasible to maintain a quantum state, compute with qubits, or experiment with quantum phenomena without QECCs. Over the past two decades, there has also been rapid developments in QECC research.

The stabilizer formalism [1,2] provides a general framework to construct a QECC as well as a unified view of quantum and classical-error correcting codes. A classical linear block code with the dual-containing property [3] can be converted into a QECC by using stabilizer formalism.

Furthermore, codeword stabilized (CWS) quantum codes [4] have also been introduced. A CWS quantum code offers the first unified framework that includes both additive and non-additive code. It is defined by both a graph [5,6] and classical binary code. Word stabilizers for the CWS code are generated according to the graph, and they change any Pauli errors consisting of X, $Y(=XZ)$, and Z operators into effective errors consisting of only the Z operator. Using this feature, any Pauli error can be transformed into a binary error, with bit 1 for the Z operator and bit 0 for the I operator.

Entanglement-assisted quantum error-correcting codes (EAQECCs) [7–9] are an extended version of standard QECC. EAQECC uses maximally entangled qubits (ebits) that are shared by the transmitter and receiver. By using these ebits, the EAQECC is not subject to the dual-containing constraint, and has a larger minimum distance.

Entanglement-assisted codeword stabilized (EACWS) quantum codes [10] have been recently established. EACWS quantum codes can be constructed as nonadditive codes of a higher dimension than that of EAQECC with the same number of physical qubits.

Most studies on entanglement-assisted quantum codes have assumed that errors do not occur on the shared ebits from the receiver's side because ebits on the receiver's side do not pass through the transmit channel. However, in practice,

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receiver-side ebits also suffer from errors, and this reduces the error-correcting capability of the code. The following works have considered the imperfect ebits.

Shaw et al. [11] presented an EAQECC that corrects errors on both the sender's qubits and the receiver's shared ebits. They showed, for the first time, that a Steane code is equivalent to a $[\![6,1,3;1]\!]$ EAQEC code for correcting a single error on the receiver's (i.e., Bob's) ebits. Wilde et al. [12] simulated entanglement-assisted quantum turbo codes when the ebits on Bob's side are imperfect. Their aim was to analyze the effect of ebit noise on entanglement-assisted quantum turbo-code performance. Lai and Brun studied a practical case where errors on the receiver's side can be corrected. They presented two different schemes [13] to correct errors on the receiver's side, and showed an equivalent relationship between $[\![n,k,d;c]\!]$ EAQECC and $[\![n+c,k,d]\!]$ standard stabilizer code. Based on this equivalence, EAQECCs can correct errors on the ebits of the receiver's side. However, when this equivalence does not exist, the transmitter uses separate EAQECCs to protect the information qubits, while the receiver uses a standard stabilizer code to protect the ebits.

In this paper, we consider EACWS codes that correct errors on both sides simultaneously. We use the property that the total number of error patterns decreases through a transition from Pauli errors to binary errors. The transition relation between them is based on a simple ring graph. Using this property, we can generate nonadditive quantum code that has more logical codewords than additive quantum code with the same number of physical qubits. In addition, we show that ((6, 4, 3; 1)) EACWS QECC can correct both side's errors, even though the $[\![6,2,3;1]\!]$ EAQECC does not have equivalent $[\![7,2,3]\!]$ code.

The remainder of this paper is organized as follows. In Section 2, we introduce the basics of entanglement-assisted codeword-stabilized quantum codes. In Section 3, we provide an overview of entanglement-assisted quantum error-correcting codes with imperfect ebits. In Section 4, we describe the proposed scheme for EACWS code with imperfect ebits. We then provide some numerical examples. Finally, we summarize the paper in Section 5.

2. Entanglement-assisted codeword-stabilized (EACWS) quantum code

EACWS code is a class of quantum error-correcting code that covers both additive and nonadditive code. The purpose of this code is to increase the capacity of QECCs by using c ebits as CWS quantum codes. An $((n,K,d;c))$ EACWS quantum code encodes K-dimensional code space into n physical qubits with minimum distance d. In an EACWS code, it is assumed that the receiver's ebits are error free because the ebits on the receiver's side do not pass through the channel. We can think of the encoding process for EACWS codes in the following way.

The initial base state of the EACWS code with $n-c$ ancilla qubits and c ebits can be represented by

$$\left|S'\right\rangle = |0\rangle^{\otimes n-c}\,|\Phi_+\rangle^{\otimes c}\,, \tag{1}$$

where $|\Phi_+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$.

The c maximally entangled pairs $|\Phi_+\rangle$ are shared between Alice and Bob.

The set of word stabilizers S' for the initial base state that corresponds to the ancilla qubits of $|0\rangle^{\otimes n-c}$ is generated by

$$\begin{matrix} Z_1 I I \ldots I | I^{\otimes c} \\ I Z_2 I \ldots I | I^{\otimes c} \\ \vdots \\ I\, I \ldots Z_{n-c}\, I \ldots I | I^{\otimes c} \end{matrix} \tag{2}$$

where the operators to the right and the left of "|" respectively act on Alice's and Bob's qubits.

The set of word stabilizers S'_{EA} for the initial base state that acts on the ebits is generated by

$$\begin{matrix} I\, I \ldots I Z_{n-c+1}\, I \ldots I | Z_1 I \ldots I \\ I\, I \ldots I I Z_{n-c+2}\, I \ldots I | I Z_2 I \ldots I \\ \vdots \\ I I \ldots I I Z_n | I I Z_c \end{matrix} \tag{3}$$

$$\begin{matrix} I\, I \ldots I X_{n-c+1}\, I \ldots I | X_1 I \ldots I \\ I\, I \ldots I I X_{n-c+2}\, I \ldots I | I X_2 I \ldots I \\ \vdots \\ I I \ldots I I X_n | I \ldots I X_c. \end{matrix} \tag{4}$$

For CWS code in a standard form, the initial basis vectors span the code space, and are formed by applying the word operators w'_l to the initial base state. Hence, the number of word operators is equal to the dimension of the code space. The initial word operator $\left\{w'_l\right\}$ of an EACWS code can be represented by

$$w'_l = X^{X_l} \otimes Z^{V_l} X^{U_l} | I^{\otimes c}, \quad \text{for } l = 1, \ldots, K, \tag{5}$$

where X_l is a binary vector of length $n-c$, and V_l and U_l are binary vectors of length c. The X^{X_l} operators are applied to $n-c$ ancilla qubits, and the $Z^{V_l} X^{U_l}$ operators are applied to the c ebits on Alice's side. The identity operator $I^{\otimes c}$ on the right side means that the word operators are not applied to Bob's ebits.

The initial basis vectors (i.e., the base states) are given by

$$w'_l\left|S'\right\rangle \equiv \left|w'_l\right\rangle = X^{X_l} \otimes Z^{V_l} X^{U_l}\,|0\rangle^{\otimes n-c}\,|\Phi_+\rangle^{\otimes c}\,. \tag{6}$$

The base state does not involve any information qubits. Therefore, we need to encode an information state $|\phi\rangle$ into state $\left|\varphi'\right\rangle$. In this case, the code space is spanned by a linear combination of the states $\left|w'_l\right\rangle$. We swap the state $|\phi\rangle$ into the codeword by defining a unitary transformation $U_{w'}$ [9] as follows:

$$U_{w'}\left(|\phi\rangle \otimes \left|S'\right\rangle\right) = |0\rangle \otimes \sum_{l=0}^{K-1} \alpha_l \left|w'_l\right\rangle \equiv |0\rangle \otimes \left|\varphi'\right\rangle. \tag{7}$$

One additional step is needed to enable the codewords to correct errors. A unitary encoding operator U_E is drawn from the Clifford group, and maps the stabilizer generators for the base state to those of the CWS code in the standard form. By applying the operator U_E, each stabilizer generator has an X operator on one qubit in a different position, and Z operators on qubits that have relationships in the associated graph. In this paper, we consider a simple ring graph.

After the unitary encoding process, the word stabilizer is represented as

$$\begin{aligned} &X_1 Z_2 I \dots Z_n | I^{\otimes c} \\ &Z_1 X_2 Z_3 I \dots I | I^{\otimes c} \\ &\vdots \\ &I \dots I Z_{n-c-1} X_{n-c} Z_{n-c+1} I \dots I | I^{\otimes c}. \end{aligned} \tag{8}$$

In Eq. (8), the word stabilizers are generated by encoding them for the initial base state corresponding to the ancilla qubits.

$$\begin{aligned} &I \dots I Z_{n-c} X_{n-c+1} Z_{n-c+2} I \dots I | Z_1 I I \dots I \\ &\vdots \\ &Z_1 I \dots I Z_{n-1} X_n | I \dots I Z_c . \end{aligned} \tag{9}$$

$$\begin{aligned} &I \dots I Z_{n-c+1} I \dots I | X_1 I I \dots I \\ &\vdots \\ &I \dots I I Z_n | I \dots I X_c . \end{aligned} \tag{10}$$

In Eqs. (9) and (10), the word stabilizers are generated by encoding the word stabilizer of the initial base state that corresponds to the ebits.

After applying the unitary encoding operator U_E, the base state $\left|S'\right\rangle$ is converted into a state $|S\rangle$:

$$U_E \left|S'\right\rangle = |S\rangle . \tag{11}$$

Likewise, the word operators are generated by

$$w_l = U_E w_l' U_E^{\dagger}. \tag{12}$$

3. Entanglement-assisted quantum error-correcting codes with imperfect ebits

In practical settings, receiver-side ebits also suffer from errors, and this reduces their error-correcting capability. In this section, we review previous work [13] that considered two schemes for error correction on the receiver's imperfect ebits.

3.1. EAQECCs that are equivalent to standard stabilizer codes

Bowen's $[\![3,1,3;2]\!]$ EAQECC [14] is equivalent to $[\![5,1,3]\!]$ stabilizer code, and it can correct an arbitrary single error on both sides. The stabilizer generators of the $[\![5,1,3]\!]$ stabilizer code are

XZZXI, IXZZX

XIXZZ, ZXIXZ. (13)

The check matrix for the $[\![5,1,3]\!]$ stabilizer code can be expressed as follows:

$$\left(\begin{array}{ccccc|ccccc} 1&0&0&1&0&0&1&1&0&0\\ 0&1&0&0&1&0&0&1&1&0\\ 1&0&1&0&0&0&0&0&1&1\\ 0&1&0&1&0&1&0&0&0&1 \end{array}\right). \tag{14}$$

After row exchange and Gaussian elimination, the check matrix changes into

$$\left(\begin{array}{ccccc|ccccc} 1&0&0&1&0&0&1&1&0&0\\ 0&0&1&0&1&1&1&0&0&0\\ 0&1&1&0&0&1&1&1&1&0\\ 1&1&0&0&0&1&1&1&0&1 \end{array}\right). \tag{15}$$

The stabilizer generators that correspond to the changed check matrix are as follows:

XZZ|XI, ZZX|IX

ZYY|ZI, YYZ|IZ. (16)

Based on this result, Theorem 2 in Ref. [13] showed that the $[\![n-c,k,d;c]\!]$ EAQECC is equivalent to the $[\![n,k,d]\!]$ standard stabilizer code, and can correct qubit errors up to $\left\lfloor \frac{d-1}{2} \right\rfloor$ from both sides.

The process for finding the proof is as follows. Assume that the $[\![n,k,d]\!]$ standard stabilizer code has the set of stabilizer generators $\{g_1, g_2, \dots, g_{n-k}\}$. Then, suppose the check matrix of the stabilizer generators can be expressed by $[\mathrm{H}_X|\mathrm{H}_Z]$. After Gaussian elimination, the check matrix turns into the following form:

$$\left(\begin{array}{cc|cc} A & I_{S\times S} & D & 0\\ C & 0 & B & I_{S\times S}\\ E & 0 & F & 0 \end{array}\right) \tag{17}$$

for $0 \le S \le n-k$. Stabilizer generators can be represented as $g_1' \otimes Z_1, \dots, g_c' \otimes Z_c, h_1' \otimes X_1, \dots, h_c' \otimes X_c, g_{c+1}' \otimes I, \dots, g_{n-k-c}' \otimes I$ with simplified generators $g_j' = U Z_j U^{\dagger}$, $h_j' = U X_j U^{\dagger}$ $(j = 1, \dots, c)$ Therefore, the set of simplified generators is $\left\{g_1', \dots, g_{n-k-c}', h_1', \dots, h_c'\right\}$, which indicates $[\![n-c,k,d;c]\!]$ EAQECC.

In addition, they found some optimal EAQECCs that satisfy the linear programming bounds and the equivalent relation between the $[\![n,k,d]\!]$ standard stabilizer code and $[\![n-c,k,d;c]\!]$ EAQECC as follows:

$[\![15,10,4,5]\!]$, $[\![14,11,3,3]\!]$, $[\![13,9,4,4]\!]$, $[\![13,10,3,3]\!]$, $[\![12,9,3,3]\!]$, $[\![11,8,3,3]\!]$, $[\![10,6,4,4]\!]$, $[\![10,7,3,3]\!]$, $[\![9,6,3,3]\!]$, $[\![7,4,3,3]\!]$, $[\![8,4,4,4]\!]$, $[\![6,2,4,4]\!]$, $[\![7,3,3,1]\!]$, $[\![6,3,3,2]\!]$, $[\![6,1,5,5]\!]$, $[\![4,1,3,1]\!]$, $[\![4,1,3,3]\!]$, $[\![3,1,3,2]\!]$.

3.2. EAQECCs using another quantum code to protect Bob's ebits

The equivalent relationship is not always satisfied for the optimal $[\![n-c;k;d;c]\!]$ EAQECCs and $[\![n;k;d]\!]$ standard stabilizer code. When there is no equivalence, it was proposed that a separate QECC be used in order to protect the ebits.

Lai and Brun referred to this scheme as a combination code, where the sender uses an $[\![n,k,d_A;c]\!]$ EAQECC with encoding operator U_A to protect the information qubits, while the receiver uses a separate $[\![m,c,d_B]\!]$ standard stabilizer code with encoding operator U_B to protect the ebits. Thus, the entire encoding operator is represented by $U_A \otimes U_B$, and the notation of the combination code is $[[n, k, d_A; c]] + [[m, c, d_B]]$.

They also found EAQECCs that are not satisfied by the equivalent relationship between the $[[n, k, d; c]]$ EAQECCs and $[[n + c, k, d]]$ standard stabilizer code [15,16], and these are as follows:

$[[n, 1, n, n-1]]$ for n odd, $[[n, 1, n-1, n-1]]$ for n even,
$[[5, 1, 5, 4]]$, $[[5, 1, 4, 3]]$, $[[5, 1, 4, 2]]$, $[[5, 2, 3, 2]]$,
$[[6, 1, 5, 4]]$, $[[6, 1, 4, 3]]$, $[[6, 2, 4, 3]]$, $[[6, 2, 3, 1]]$,
$[[7, 1, 5, 2]]$, $[[7, 1, 5, 3]]$, $[[7, 1, 7, 6]]$, $[[7, 2, 5, 5]]$,
$[[7, 3, 4, 4]]$, $[[7, 3, 4, 3]]$, $[[7, 4, 3, 2]]$,
$[[8, 1, 6, 6]]$, $[[8, 2, 6, 6]]$, $[[8, 1, 6, 5]]$, $[[8, 3, 5, 5]]$,
$[[8, 2, 5, 4]]$, $[[8, 1, 4, 1]]$, $[[8, 3, 4, 3]]$, $[[8, 5, 3, 2]]$,
$[[9, 1, 7, 4]]$, $[[9, 1, 7, 5]]$, $[[9, 1, 7, 6]]$, $[[9, 1, 7, 7]]$,
$[[9, 1, 9, 8]]$, $[[9, 1, 7, 6]]$, $[[9, 1, 7, 7]]$, $[[9, 2, 6, 6]]$,
$[[9, 1, 6, 5]]$, $[[9, 1, 6, 6]]$, $[[9, 2, 5, 4]]$, $[[9, 5, 3, 1]]$,
$[[10, 1, 8, 8]]$, $[[10, 1, 7, 6]]$, $[[10, 1, 6, 5]]$, $[[10, 1, 6, 4]]$,
$[[10, 2, 7, 7]]$, $[[10, 2, 6, 5]]$, $[[10, 2, 5, 3]]$, $[[10, 2, 5, 2]]$,
$[[10, 3, 6, 7]]$, $[[10, 3, 6, 6]]$, $[[10, 4, 5, 5]]$, $[[10, 4, 5, 4]]$,
$[[13, 3, 9, 10]]$, $[[13, 1, 11, 10]]$, $[[13, 1, 11, 11]]$,
$[[13, 1, 9, 8]]$, $[[13, 1, 9, 9]]$, $[[15, 7, 6, 8]]$, $[[15, 8, 6, 7]]$,
$[[15, 9, 5, 6]]$.

$A[[n + m, k, d]]$ standard stabilizer code can correct $\left\lfloor \frac{d-1}{2} \right\rfloor$ arbitrary errors. Compared with the $[[n + m, k, d]]$ standard stabilizer code, the $[[n, k, d_A; c]] + [[m, c, d_B]]$ quantum code uses a smaller number of qubits going through the noisy channel in order to correct the same number of errors on the transmit channel.

4. Entanglement-assisted codeword-stabilized quantum codes with imperfect ebits

In this section, we show an EACWS code that simultaneously corrects qubit errors on the transmitter's side and ebit errors on the receiver's side. Our scheme corrects $\left\lfloor \frac{d-1}{2} \right\rfloor$ arbitrary errors on the receiver's side as well as on the sender's side. According to the properties of the EACWS code, any Pauli error can be turned into a binary error, and we identified binary codewords to correct the binary errors based on an exhaustive search. The advantage of this scheme is that it uses only one QECC to correct errors on both sides, regardless of whether the equivalent relation is satisfied.

4.1. EACWS quantum code with imperfect ebits using the property of stabilizer generators

Our scheme corrects errors on Bob's side as well as on Alice's side by using only one QECC. To this end, we use the property of the EACWS code in such a way that each stabilizer generator g_i (for $i = 1, \ldots, n$) has a single X operator and multiple Z operators on the qubits corresponding to the neighboring vertices of the graph. To correct the ebit errors, we need additional word stabilizers $(h_1, h_2, \ldots, h_c)$ as well as standard word stabilizers $(g_1, g_2, \ldots, g_n)$. The stabilizer generators for the standard form EACWS code that consists of the following:

$$\begin{aligned} g_1 &= X_1 Z_2 I \ldots I Z_n | I^{\otimes c} \\ g_2 &= Z_1 X_2 Z_3 \ldots I I | I^{\otimes c} \\ g_3 &= I Z_2 X_3 Z_4 \ldots I I | I^{\otimes c} \\ &\vdots \\ g_{n-c} &= I \ldots I Z_{n-c-1} X_{n-c} Z_{n-c+1} I \ldots I I | I^{\otimes c} \end{aligned} \tag{18}$$

$$\begin{aligned} g_{n-c+1} &= I \ldots I I Z_{n-c} X_{n-c+1} Z_{n-c+2} I \ldots I I | Z_1 I \ldots I \\ &\vdots \\ g_n &= Z_1 I \ldots I Z_{n-1} X_n | I \ldots I Z_c \\ h_1 &= I \ldots I Z_{n-c+1} I \ldots I | X_1 I \ldots I \\ &\vdots \\ h_{c-1} &= I \ldots I Z_{n-1} I | I \ldots I I X_{c-1} I \\ h_c &= I \ldots I I Z_n | I \ldots I I I X_c \end{aligned} \tag{19}$$

where Eq. (18) is derived from Eqs. (8) and (9). These stabilizer generators correspond to a simple ring graph. Eq. (19) is identical to Eq. (10).

The stabilizer generator can transform any single Pauli error on both sides into one or more Z errors, and these Z only errors are referred to as effective errors [4]. The effective errors are represented as binary errors because of the property that turns Z and I operators into 1 and 0. Thus, binary codewords can be found to correct these binary errors. These binary codewords are converted into word operators that formed the basis of the code space. Because the encoding process needs only to be applied to Alice's side, the word operators cannot have Z operators on the qubits in Bob's side; thus, the stabilizer generators are repeatedly applied to the word operators until all of the Z operators on Bob's side are removed.

As mentioned above, finding EA-CWS code with imperfect ebits is very similar to the code with perfect ebits [10]. However, some pairs of Pauli errors on the receiver and transmitter side have the same effective error. In the case of our scheme with a minimum distance of three, the number of these pairs is the same as the number of ebits, as shown in Table 1. Therefore, the total number of effective errors is smaller than the total number of correctable Pauli errors, and it results in a higher number of codewords. This is the difference from EA-CWS code with perfect ebits.

Table 1
The case of $((n, K, 3; c))$ EACWS quantum code, pairs of errors that have the same effective error.

Nu..	Single X error on Bob's side	Stabilizer generator applies to two equivalent errors	Equivalent single error on Alice's side
1	$I \ldots I_n \| X_1 I \ldots I_c$	h_1	$Z_1 I \ldots I_n \| I^{\otimes c}$
2	$I \ldots I_n \| I X_2 \ldots I_c$	h_2	$I Z_2 \ldots I_n \| I^{\otimes c}$
...	...	...	...
$c-1$	$I \ldots I_n \| I \ldots X_{c-1} I_c$	h_{c-1}	$I \ldots Z_{c-1} \ldots I_n \| I^{\otimes c}$
c	$I \ldots I_n \| I \ldots I I_c$	h_c	$I \ldots Z_c \ldots I_n \| I^{\otimes c}$

For instance, consider the code with $n = 7$, $d = 3$, and $c = 2$. Assume that the error that occurs on Bob's side is IIIIIII|XI. We can get an equivalent Pauli error IIIIIZI|II on Alice's side using the stabilizer generator h_1 = IIIIIZI|XI. Therefore, two equivalent errors, IIIIIII|XI and IIIIIZI|II, correspond to the same effective error IIIIIZI|II. For this reason, the total number of effective errors is smaller by the number of ebits than the total number of correctable Pauli errors, resulting in a larger number of codewords. In ((7, 9, 3; 2)) EACWS code, we consider 27 single Pauli errors that consist of 21 errors on the transmitter's side and 6 errors on the receiver's side. Then, all Pauli errors are converted into effective errors, including Z and I operators. In this process, two errors with a single X operator on the receiver's side have the same effective error as a single Pauli error on the sender's side. Because of the presence of two equivalent error patterns, the total number of effective error patterns is 25.

In the following section, we consider examples of our scheme with a minimum distance of three.

4.2. Examples of EACWS quantum code with imperfect ebits

In this section, we provide some examples of the EACWS codes based on our construction. All of the example codes use a base state on a simple ring graph that is identical to a CWS code in standard form. We consider a classical binary-error set, and then find classical codes that can correct it through a numerical search. Then, we construct the word operators from the set of binary codewords.

4.2.1. ((7, 9, 3; 2)) EACWS code

We can construct a ((7, 9, 3; 2)) code from a simple ring graph having seven vertices using two ebits with a minimum distance of three. This nonadditive code has one more dimension of code space than does the additive [[9, 3, 3]] code.

The initial base state is

$$\left|S'\right\rangle = |00000\rangle \, |\Phi_+ \Phi_+\rangle \,. \tag{20}$$

The stabilizer generators are generated based on the ring graph as follows:

g_1 = XZIIIIZ|II,
g_2 = ZXZIIII|II,
g_3 = IZXZIII|II,
g_4 = IIZXZII|II,
g_5 = IIIZXZI|II,
g_6 = IIIIZXZ|ZI,
g_7 = ZIIIIZX|IZ,
h_1 = IIIIIZI|XI,
h_2 = IIIIIIZ|IX.

All single errors can be corrected on both sides. Based on the effective errors, nine codewords can be found as follows:

0000000|00, 1110101|01, 1111000|01, 0001001|11,
0010010|11, 0011111|10, 0101100|10, 0111110|01,
1100010|00.

We can discover the word operators from these binary codewords. The word operators w_l' for the base state $\left|S'\right\rangle$ (before applying U_E) are

IIIIIII|II, XXXIXIY|II, XXXXIIZ|II, IIIXIZY|II,
IIXIIYZ|II, IIXXXYX|II, IXIXXZI|II, IXXXXXZ|II,
XXIIIXI|II.

and the word operators w_l for this code (after applying U_E) are

IIIIIII|II, IZZIZZY|II, IZZZIZX|II, ZIIZZYX|II,
ZIZIZXY|II, IIZZIYI|II, IZIZIXZ|II, ZZZZZIX|II,
ZZIIIZI|II.

4.2.2. ((9, 20, 3; 1)) EACWS code

The ((9, 20, 3; 1)) code can also be constructed from a simple ring graph with nine vertices. This code has two more dimensions of code space than ((10, 18, 3)) CWS quantum code with a simple ring graph and the same number of physical qubits.

The initial base state for this code is

$$\left|S'\right\rangle = |00000000\rangle \, |\Phi_+\rangle \,. \tag{21}$$

After the encoding operation U_E, the stabilizer generators for this code are as follows:

g_1 = XZIIIIIIZ|I,
g_2 = ZXZIIIIII|I,
g_3 = IZXZIIIII|I,
g_4 = IIZXZIIII|I,
g_5 = IIIZXZIII|I,
g_6 = IIIIZXZII|I,
g_7 = IIIIIZXZI|I,
g_8 = IIIIIIZXZ|I,
g_9 = ZIIIIIIZX|Z,
h_1 = IIIIIIIIZ|X.

Thirty Pauli error patterns can be corrected with this code. In this case, one error pair has the same effective error, and thus, 30 single Pauli errors can be changed into 29 effective errors (or binary errors). Then, the classical code correcting these effective errors is

110000100|1, 110001000|0, 110010111|0, 110011011|1,
111000010|1, 111011101|1, 111100001|0, 111111110|0,
000011111|0, 000100011|1, 000111100|1, 001100101|1,
001101001|0, 001110110|0, 001111010|1, 010101100|0,
010110011|0, 101001101|0, 101010010|0, 000000000|0.

The word operators w_l' for the base state $\left|S'\right\rangle$ (before applying U_E) are

XXIIIIXIZ|I, XXIIIXIII|I, XXIIXIXXX|I, XXIIXXIXY|I,
XXXIIIIXZ|I, XXXIXXXIY|I, XXXXIIIIX|I,
XXXXXXXXI|I, IIIIXXXXX|I, IIIXIIIXY|I,
IIIXXXXIZ|I, IIXXIIXIY|I, IIXXIXIIX|I,
IIXXXIXXI|I, IIXXXXIXZ|I, IXIXIXXII|I,
IXIXXIIXX|I, XIXIIXXIX|I, XIXIXIIXI|I, IIIIIIIII|I

and the word operators w_l for this code (after applying U_E) are

IZIIIIZZX|I, ZZIIIZIII|I, ZZIIZIZZZ|I, IZIIZZIIY|I, IZZIIIIIX|I, IZZIZZZZY|I, ZZZZIIIIZ|I, ZZZZZZZZI|I, IIIIZZZZZ|I, ZIIZIIIIY|I, ZIIZZZZZX|I, ZIZZIIZZY|I, IIZZIZIIZ|I, IIZZZIZZI|I, ZIZZZZIIX|I, IZIZIZZII|I, IZIZZIIZZ|I, ZIZIIZZIZ|I, ZIZIZIIZI|I, IIIIIIIII|I.

4.2.3. ((6, 4, 3; 1)) *EACWS code*

According to Ref. [13], a [[6, 2, 3; 1]] EAQECC is not equivalent to standard [[7, 2, 3]] code. Therefore, when the sender uses a [[6, 2, 3; 1]] code to protect the information qubits, the receiver has to use a separate standard stabilizer code to protect the ebits. On the other hand, our ((6, 4, 3; 1)) EACWS code can simultaneously protect qubits and ebits on both sides. Based on the simple ring graph, a ((6, 4, 3; 1)) EACWS code can be generated with six vertices using one ebit.

The initial base state of this code is

$$\left|S'\right\rangle = |00000\rangle\,|\Phi_+\rangle\,. \tag{22}$$

After the encoding operation U_E, the stabilizer generators of this code are as follows:

$g_1 =$ XZIIIZ|I,
$g_2 =$ ZXZIII|I,
$g_3 =$ IZXZII|I,
$g_4 =$ IIZXZI|I,
$g_5 =$ IIIZXZ|I,
$g_6 =$ ZIIIZX|Z,
$h_1 =$ IIIIIZ|X.

The total number of single qubit Pauli errors for Alice's and Bob's qubits is 21. In this case, the total number of binary errors is 20 because two Pauli errors, IIIIIZ|I and IIIIII|X, have the same binary error 000001|0.

The codewords are

000000|0, 001100|1, 110111|0, 111011|1.

The word operators before encoding, which is constructed from classical code, are

IIIIII|I, IIXXIZ|I, XXIXXX|I, XXXIXY|I,

and the word operators of this code (after applying) are

IIIIII|I, ZIZZZX|I, ZZIZZZ|I, IZZIIY|I.

5. Conclusion

In this paper, we presented EACWS codes with imperfect ebits. Based on the simple ring graph, the proposed scheme uses only one QECC to correct errors on both sides. Because of the property whereby two different Pauli errors correspond to the same effective error, we can construct two example codes, a ((7, 9, 3; 2)) and a ((9, 20, 3; 1)) code, which have larger codewords than their additive counterparts with the same number of physical qubits. We also presented a ((6, 4, 3; 1)) EACWS code to protect qubits and ebits on both sides. In the future, we will attempt to find a new code that has a better parameter K by applying a different form of graph. We will also find another nonadditive EACWS quantum code that has a higher minimum distance.

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