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Remark on the limit case of positive mass theorem for manifolds with inner boundary

Eui Chul Kim

Department of Mathematics, College of Education, Andong National University, Andong 760-749, South Korea

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Abstract

In (Comm. Math. Phys. 188 (1997) 121–133) Herzlich proved a new positive mass theorem for Riemannian 3-manifolds (N, g) whose mean curvature of the boundary allows some positivity. In this paper we study what happens to the limit case of the theorem when, at a point of the boundary, the smallest positive eigenvalue of the Dirac operator of the boundary is strictly larger than one-half of the mean curvature (in this case the mass $m(g)$ must be strictly positive). We prove that the mass is bounded from below by a positive constant $c(g)$, $m(g) \geq c(g)$, and the equality $m(g) = c(g)$ holds only if, outside a compact set, (N, g) is conformally flat and the scalar curvature vanishes. The constant $c(g)$ is uniquely determined by the metric g via a Dirac-harmonic spinor.

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1. Introduction

Let (N, g) be a complete Riemannian 3-manifold with boundary which is diffeomorphic to the Euclidean space \mathbb{R}^3 minus an open 3-ball centered at the origin. Let $r(y) = \sqrt{\sum_{i=1}^3 y_i^2}$, $y = (y_1, y_2, y_3) \in \mathbb{R}^3$, be the standard distance function to the origin of \mathbb{R}^3 . Then (N, g) is called *asymptotically flat* of order $\tau > 1/2$, if there is a diffeomorphism $\Phi : N \rightarrow \mathbb{R}^3 \setminus \{\text{an open 3-ball}\}$ such that the coefficients of the metric g in the induced rectangular coordinates satisfy

$$g_{ij} = \delta_{ij} + O(r^{-\tau}), \quad g_{ij,k} = O(r^{-\tau-1}), \quad g_{ij,k,l} = O(r^{-\tau-2})$$

E-mail address: eckim@andong.ac.kr (E.C. Kim).

as $r = r(\Phi) \rightarrow \infty$. Let $S(r) \subset N$ denote the Φ -inverse image of a round 2-sphere in \mathbb{R}^3 , centered at the origin and of sufficiently large radius $r > 0$. Throughout the paper we identify

$$N = \bigcup_{r \geq r_0} S(r) \quad \text{for some fixed constant } r_0 > 0.$$

The mass of (N, g) is usually defined by [1]

$$m(g) = \frac{1}{16\pi} \lim_{r \rightarrow \infty} \sum_{i,j=1}^3 \int_{S(r)} (g_{ij,j} - g_{jj,i}) \nu^i dS, \quad (1.1)$$

where ν is the outward unit normal to spheres $S(r) \subset N$ and dS is the area form of spheres $S(r)$. We remark here that one can express this definition in a coordinate-independent way, by considering a flat metric on N as a reference metric. Let g_{eu} be a metric on N which is the pullback of the Euclidean metric on $\mathbb{R}^3 \setminus \{\text{an open 3-ball}\}$ via the diffeomorphism $\Phi : N \rightarrow \mathbb{R}^3 \setminus \{\text{an open 3-ball}\}$. Then Eq. (1.1) is in fact equal to

$$m(g) = \frac{1}{16\pi} \lim_{r \rightarrow \infty} \int_{S(r)} g_{\text{eu}}(\text{div}_{g_{\text{eu}}}(g) - \text{grad}_{g_{\text{eu}}}(\text{Tr}_{g_{\text{eu}}}(g)), V_{\text{eu}}) \mu_{S(r)}(g_{\text{eu}}) \quad (1.2)$$

$$= \frac{1}{16\pi} \lim_{r \rightarrow \infty} \int_{S(r)} g(\text{div}_{g_{\text{eu}}}(g) - \text{grad}_{g_{\text{eu}}}(\text{Tr}_{g_{\text{eu}}}(g)), V_g) \mu_{S(r)}(g), \quad (1.3)$$

where V_{eu} (respectively V_g) is the outward unit normal to spheres $(S(r), g_{\text{eu}})$ (respectively $(S(r), g)$) and $\mu_{S(r)}(g_{\text{eu}})$ (respectively $\mu_{S(r)}(g)$) is the area form of spheres $(S(r), g_{\text{eu}})$ (respectively $(S(r), g)$). When one applies the Witten-type spinor method to prove positivity of the mass, one should use the latter Eq. (1.3) [2,5,6,9,11]. Note that Eqs. (1.2)–(1.3) are independent of deformation of the foliation $N = \bigcup_{r \geq r_0} S(r)$ via a diffeomorphism $F : N \rightarrow N$, since Stokes' theorem implies that

$$\begin{aligned} m(g) &= \frac{1}{16\pi} \lim_{r \rightarrow \infty} \int_{S(r)} g(\text{div}_{g_{\text{eu}}}(g) - \text{grad}_{g_{\text{eu}}}(\text{Tr}_{g_{\text{eu}}}(g)), V_g) \mu_{S(r)}(g) \\ &= \frac{1}{16\pi} \int_{\partial N} g(\text{div}_{g_{\text{eu}}}(g) - \text{grad}_{g_{\text{eu}}}(\text{Tr}_{g_{\text{eu}}}(g)), V_g) \mu_{\partial N}(g) \\ &\quad + \frac{1}{16\pi} \int_N \text{div}_g \{ \text{div}_{g_{\text{eu}}}(g) - \text{grad}_{g_{\text{eu}}}(\text{Tr}_{g_{\text{eu}}}(g)) \} \mu_N(g) \end{aligned}$$

whose right-hand side is independent of a choice of foliation on N by 2-spheres.

The mass is a geometric invariant of Riemannian asymptotically flat manifolds and of importance in Riemannian geometry as well as in general relativity. In [3,7] one finds an excellent exposition of the positive mass conjecture as well as the Penrose conjecture and a full list of related papers. A fundamental problem about the mass is to investigate the relation between the scalar curvature S_g of the manifold (N, g) , the mean curvature $\text{Tr}_g(\Theta)$ of the inner boundary $(\partial N, g|_{\partial N})$ and the mass $m(g)$ (here Θ indicates the second fundamental form of the boundary). The Riemannian positive mass theorem, proved by Schoen and Yau [10], states that, if (N, g) is an asymptotically flat 3-manifold of non-negative scalar

curvature $S_g \geq 0$ with minimal boundary $\text{Tr}_g(\Theta) \equiv 0$, then the mass is non-negative $m(g) \geq 0$. In fact, the limit case of zero mass can not be attained and so the mass must be strictly positive. The Penrose conjecture, recently proved by Huisken and Ilmanen [7], improves the positive mass theorem and states that, if the boundary is not only minimal but also outermost (i.e., N contains no other compact minimal hypersurfaces), then

$$m(g) \geq 4\sqrt{\frac{\text{Area}(\partial N, g)}{\pi}}$$

with equality if and only if (N, g) is isometric to the spatial Schwarzschild manifold.

In [5] Herzlich proved a new positive mass theorem for manifolds with inner boundary (see Theorem 2.1), making use of Dirac-harmonic spinors with well-chosen spectral boundary condition (see the PDE system (2.7) below). A remarkable feature of the theorem is that the mass $m(g)$ is non-negative even if there is some positivity of the mean curvature of the boundary. The limit case of zero mass (the flat space) occurs only if the smallest positive eigenvalue λ of the Dirac operator of the boundary is equal to one-half of the mean curvature $\text{Tr}_g(\Theta)$, i.e.,

$$\lambda = 2\sqrt{\frac{\pi}{\text{Area}(\partial N, g)}} = \frac{1}{2}\text{Tr}_g(\Theta).$$

The object of this paper is to study what happens to the limit case of the theorem when

$$2\sqrt{\frac{\pi}{\text{Area}(\partial N, g)}} \geq \frac{1}{2}\sup_{\partial N}\{\text{Tr}_g(\Theta)\} \quad \text{and} \quad 2\sqrt{\frac{\pi}{\text{Area}(\partial N, g)}} \neq \frac{1}{2}\text{Tr}_g(\Theta),$$

in which case the zero mass $m(g) = 0$ cannot be attained. We will prove (see Theorem 3.1) that there exists a positive constant $c(g) > 0$, uniquely determined by the metric g via a Dirac-harmonic spinor, such that $m(g) \geq c(g)$ and the equality $m(g) = c(g)$ occurs only if, outside a compact set, (N, g) is conformally flat and the scalar curvature $S_g \equiv 0$ vanishes. It will also be shown that the equality $m(g) = c(g)$ is indeed attained if (N, g) is conformally flat, the conformal factor being constant on the inner boundary ∂N , and the scalar curvature is everywhere zero. The idea to prove the rigidity statement is that, near infinity, one can conformally deform the considered metric as well as the connection, using the length of a harmonic spinor without zeros as the conformal factor.

2. The Witten–Herzlich method

In this section we recall some basic facts concerning the Witten-type spinor method used by Herzlich to prove a positive mass theorem for manifolds with inner boundary [2,5,6,9,11]. Let $(\partial_\theta, \partial_\phi, \partial_r)$ be a frame field on (N, g) determined by spherical coordinates (θ, ϕ, r) . Applying the Gram–Schmidt orthogonalization process to $(\partial_\theta, \partial_\phi, \partial_r)$, we obtain a g -orthonormal frame $(E_1, E_2, -E_3)$, defined on an open dense subset of N , such that $V := -E_3$ is the outward unit normal to hypersurfaces $(S(r), g)$, $r \geq r_0$, and each E_j , $j = 1, 2$, is tangent to $S(r)$, where $(S(r), g)$ denotes hypersurface $S(r)$ equipped with the metric induced by g . Let ∇ and ∇^∂ be the Levi-Civita connection of (N, g) and $(\partial N, g)$, respectively. Let D be the Dirac operator of (N, g) and D^∂ the induced Dirac operator of $(\partial N, g)$, respectively. Let $\Theta := \nabla V$ be the second fundamental form of $(\partial N, g)$. Then we have

$$\nabla_X \psi = \nabla_X^\partial \psi + \frac{1}{2}\Theta(X) \cdot E_3 \cdot \psi$$

for all vectors X on ∂N and so

$$D\psi - E_3 \cdot \nabla_{E_3} \psi = \sum_{i=1}^2 E_i \cdot \nabla_{E_i}^\partial \psi - \frac{1}{2}(\text{Tr}_g \Theta) E_3 \cdot \psi. \quad (2.1)$$

Let $\Sigma(N)$ and $\Sigma(\partial N)$ be the spinor bundle of (N, g) and $(\partial N, g)$, respectively. Recall that the Clifford bundle $\text{Cl}(\partial N)$ may be thought of as a subbundle of $\text{Cl}(N)$, the Clifford multiplication $\text{Cl}(\partial N) \times \Sigma(\partial N) \rightarrow \Sigma(\partial N)$ being naturally related to the one $\text{Cl}(N) \times \Sigma(N) \rightarrow \Sigma(N)$ via either

$$\pi_*(E_i \cdot E_3 \cdot \psi) = E_i \cdot (\pi_* \psi), \quad i = 1, 2, \quad (2.2)$$

or

$$-\pi_*(E_i \cdot E_3 \cdot \psi) = E_i \cdot (\pi_* \psi), \quad (2.3)$$

where $\pi_* : \Sigma(N) \rightarrow \Sigma(\partial N)$ is the restriction map. Eq. (2.1) is then projected to ∂N as

$$\pi_*(E_3 \cdot D\psi + \nabla_{E_3} \psi) = \mp \sum_{i=1}^2 D^\partial(\pi_* \psi) + \frac{1}{2}(\text{Tr}_g \Theta)(\pi_* \psi). \quad (2.4)$$

Regarding $\nabla^\partial \psi$, $\psi \in \Gamma(\Sigma(\partial N))$, as spinor fields on N , not projected to the boundary ∂N , one verifies easily that the formula

$$\nabla_X^\partial(E_3 \cdot \psi) = E_3 \cdot \nabla_X^\partial \psi$$

makes sense. Therefore D^∂ anticommutes with the action of the unit normal E_3 , and hence the discrete eigenvalue spectrum of D^∂ is symmetric with respect to zero. Moreover, we note that, since the smallest absolute value of eigenvalues of D^∂ must satisfy

$$\lambda \geq 2 \sqrt{\frac{\pi}{\text{Area}(\partial N, g)}}, \quad (2.5)$$

there is no non-trivial solutions to the equation $D^\partial \varphi = 0$.

Let $\langle \cdot, \cdot \rangle_g = \text{Re} \langle \cdot, \cdot \rangle_g$ be the real part of the standard Hermitian product $\langle \cdot, \cdot \rangle_g$ on the spinor bundle $\Sigma(N)$ over (N, g) . Then, using the scalar product $(\cdot, \cdot) = \langle \cdot, \cdot \rangle_g$, one can describe the asymptotic behaviour of spinor fields as

$$|\psi| = \sqrt{(\psi, \psi)} = O(r^{-\kappa}), \quad |\nabla \psi| = O(r^{-1-\kappa}), \quad \text{etc.}, \quad \kappa > 0. \quad (2.6)$$

Remark. Using the formulas in Propositions 2.1 and 2.3 of [8], one verifies that (2.6) is in fact equivalent to the decay condition

$$|\psi|_{g_{\text{eu}}} = \sqrt{(\psi, \psi)_{g_{\text{eu}}}} = O(r^{-\kappa}), \quad |\nabla^{g_{\text{eu}}} \psi|_{g_{\text{eu}}} = O(r^{-1-\kappa}), \quad \text{etc.},$$

described in terms of the flat metric g_{eu} .

Let P_\pm be the L^2 -orthogonal projection onto the subspace of positive (respectively negative) eigenspinors of the induced Dirac operator D^∂ . Let $W_{-\tau}^{1,2}$ be the weighted Sobolev space defined in [2]. In the rest of the paper, we fix a constant spinor ψ_o with $|\psi_o| = 1$ (i.e., ψ_o is a parallel spinor with respect to the flat metric g_{eu}), all the components of which are constant with respect to a spinor frame field induced

by rectangular coordinates, and we use the rule (2.2) for the Clifford multiplication. Now we consider the PDE system:

$$D\psi = 0, \quad \text{with boundary condition } \lim_{|x| \rightarrow \infty} \psi(x) = \psi_o, \quad P_- \psi = 0, \tag{2.7}$$

where ψ is a section of $\Sigma(N)$ with $\psi - \psi_o \in W_{-\tau}^{1,2}$, $\tau > 1/2$. (If one uses the rule (2.3) for the Clifford multiplication, then the spectral boundary condition $P_- \psi = 0$ must be replaced by $P_+ \psi = 0$ to guarantee positivity of the boundary term in Eq. (2.8) below for the mass).

Proposition 2.1 (see [5]). *Let (N, g) be a Riemannian asymptotically flat 3-manifold of order $\tau > 1/2$. Let the scalar curvature S_g of (N, g) be non-negative and the mean curvature $\text{Tr}_g(\Theta)$ of the boundary $(\partial N, g)$ satisfy*

$$\lambda \geq \frac{1}{2} \sup_{\partial N} \{ \text{Tr}_g(\Theta) \},$$

where λ is the smallest absolute value of eigenvalues of the induced Dirac operator D^∂ . Then there exists a unique solution to the PDE system (2.7).

Let ψ be a solution to the system (2.7). Let $\mu_{S(r)}(g)$, $\mu_{\partial N}(g)$, $\mu_N(g)$ denote the area form of $(S(r), g)$, $(\partial N, g)$, (N, g) , respectively. Then, applying Stokes' theorem, the Schrödinger–Lichnerowicz formula and the spectral boundary condition, we have

$$\begin{aligned} m(g) &= \frac{1}{8\pi} \lim_{r \rightarrow \infty} \int_{S(r)} g(\text{grad}_g(\psi, \psi), V) \mu_{S(r)}(g) \\ &= \frac{1}{4\pi} \int_{\partial N} \left(D^\partial(\pi_*\psi) - \frac{1}{2} \text{Tr}_g(\Theta)(\pi_*\psi), \pi_*\psi \right) \mu_{\partial N}(g) \\ &\quad + \frac{1}{4\pi} \int_N \left\{ (\nabla\psi, \nabla\psi) + \frac{1}{4} S_g(\psi, \psi) \right\} \mu_N(g) \\ &\geq \frac{1}{4\pi} \int_{\partial N} \left\{ \lambda - \frac{1}{2} \text{Tr}_g(\Theta) \right\} (\pi_*\psi, \pi_*\psi) \mu_{\partial N}(g), \end{aligned} \tag{2.8}$$

which proves the following positive mass theorem.

Theorem 2.1 (see [5]). *If (N, g) is asymptotically flat of order $\tau > 1/2$ with $S_g \geq 0$ and the mean curvature $\text{Tr}_g(\Theta)$ satisfies*

$$2\sqrt{\frac{\pi}{\text{Area}(\partial N, g)}} \geq \frac{1}{2} \sup_{\partial N} \{ \text{Tr}_g(\Theta) \},$$

then $m(g) \geq 0$, with equality if and only if (N, g) is flat.

Note that, if

$$2\sqrt{\frac{\pi}{\text{Area}(\partial N, g)}} \geq \frac{1}{2} \sup_{\partial N} \{ \text{Tr}_g(\Theta) \} \quad \text{and} \quad 2\sqrt{\frac{\pi}{\text{Area}(\partial N, g)}} \neq \frac{1}{2} \text{Tr}_g(\Theta) \tag{2.9}$$

on the boundary ∂N , then the equality $m(g) = 0$ of Theorem 2.1 cannot be attained, and hence one may find a reasonable positive constant $c(g) > 0$ depending on the metric g with $m(g) \geq c(g)$. In the next section, we investigate situation (2.9) and improve the rigidity statement of Theorem 2.1.

3. Conformal change of metric using length of a spinor without zeros as the conformal factor

We consider a conformal metric $\bar{g} = e^f g$ on N with $f \in W_{-\tau}^{1,2}$, $\tau > 1/2$. The scalar curvatures $S_{\bar{g}}$ and S_g are related by

$$\Delta_g(e^{kf}) = -(\operatorname{div}_g \circ \operatorname{grad}_g)(e^{kf}) = \frac{k}{2} e^{(k+1)f} S_{\bar{g}} - \frac{k}{2} e^{kf} S_g + \frac{k(1-4k)}{4} e^{kf} |df|_g^2, \quad (3.1)$$

where $k \in \mathbb{R}$ is an arbitrary real number, and the mean curvatures $\operatorname{Tr}_{\bar{g}}(\Theta_{\bar{g}})$ and $\operatorname{Tr}_g(\Theta_g)$ on the boundary ∂N are related by

$$\operatorname{Tr}_{\bar{g}}(\Theta_{\bar{g}}) = e^{-f/2} \operatorname{Tr}_g(\Theta_g) - e^{-f/2} df(E_3), \quad (3.2)$$

where E_3 is the inward unit normal to $(\partial N, g)$. Moreover, applying (3.1) to (1.3), one verifies that the masses $m(\bar{g})$ and $m(g)$ are related as follows:

$$\begin{aligned} m(\bar{g}) - m(g) &= \frac{1}{k} \cdot \frac{1}{8\pi} \int_{\partial N} g(\operatorname{grad}_g(e^{kf}), E_3) \mu_{\partial N}(g) + \frac{1}{k} \cdot \frac{1}{8\pi} \int_N \Delta_g(e^{kf}) \mu_N(g) \\ &= \frac{1}{8\pi} \int_{\partial N} e^{kf} df(E_3) \mu_{\partial N}(g) + \frac{1}{16\pi} \int_N e^{kf} \left(e^f S_{\bar{g}} - S_g + \frac{1-4k}{2} |df|_g^2 \right) \mu_N(g). \end{aligned} \quad (3.3)$$

Now let $\Sigma(N)_g$ and $\Sigma(N)_{\bar{g}}$ denote the spinor bundle of (N, g) and (N, \bar{g}) , respectively. Then there are natural isomorphisms $j : T(N) \rightarrow T(N)$ and $j : \Sigma(N)_g \rightarrow \Sigma(N)_{\bar{g}}$ preserving the inner products of vectors and spinors as well as the Clifford multiplication

$$\begin{aligned} \bar{g}(jX, jY) &= g(X, Y), \quad \langle j\psi_1, j\psi_2 \rangle_{\bar{g}} = \langle \psi_1, \psi_2 \rangle_g, \\ (jX) \cdot (j\psi) &= j(X \cdot \psi), \quad X, Y \in \Gamma(T(N)), \quad \psi, \psi_1, \psi_2 \in \Gamma(\Sigma(N)_g). \end{aligned}$$

We fix the notation $\bar{X} := j(X)$ and $\bar{\psi} := j(\psi)$ to denote the corresponding vector fields and spinor fields on (N, \bar{g}) , respectively. For shortness we also introduce the notation $\psi_p := e^{pf} \psi$, $p \in \mathbb{R}$. Then, one verifies that the connections $\bar{\nabla}, \nabla$ and the Dirac operators \bar{D}, D are related as follows.

Proposition 3.1.

- (i) $\overline{\operatorname{grad}(e^f)} = e^{-f/2} \overline{\operatorname{grad}(e^f)}$,
- (ii) $\bar{\nabla}_X \bar{\psi}_p = e^{pf} \{ \bar{\nabla}_X \bar{\psi} + \frac{4p-1}{4} e^{-f} \bar{g}(\overline{\operatorname{grad}(e^f)}, X) \bar{\psi} - \frac{1}{4} e^{-f} X \cdot \overline{\operatorname{grad}(e^f)} \cdot \bar{\psi} \}$,
- (iii) $\bar{D} \bar{\psi}_p = e^{pf} \{ e^{-f/2} \bar{D} \bar{\psi} + \frac{2p+1}{2} e^{-f} \overline{\operatorname{grad}(e^f)} \cdot \bar{\psi} \}$.

Let $\varphi = \varphi_0 + \varphi_1$ be a spinor field on (N, g) with $|\varphi_0| = 1$ and $\varphi_1 \in W_{-\tau}^{1,2}$, $\tau > 1/2$. Since $|\varphi| \rightarrow 1$ as $r \rightarrow \infty$, there exists a positive constant $r_* \geq r_0$ such that φ has no zeros in $N(r_*) := \bigcup_{r \geq r_*} S(r)$. Define

a conformal metric \bar{g} on $N(r_*)$ by

$$\bar{g} = (\varphi, \varphi)^q g, \quad q \in \mathbb{R}.$$

Then the connections $\bar{\nabla}, \nabla$ and the Dirac operators \bar{D}, D are related by

$$\begin{aligned} \bar{\nabla}_X \bar{\varphi}_p &= (\varphi, \varphi)^{pq} \left\{ \overline{\nabla}_X \varphi + \frac{q(4p-1)}{4} (\varphi, \varphi)^{-1} \bar{g}(\overline{\text{grad}}(\varphi, \varphi), X) \bar{\varphi} \right. \\ &\quad \left. - \frac{q}{4} (\varphi, \varphi)^{-1} X \cdot \overline{\text{grad}}(\varphi, \varphi) \cdot \bar{\varphi} \right\}, \end{aligned} \tag{3.4}$$

$$\bar{D} \bar{\varphi}_p = (\varphi, \varphi)^{pq} \left\{ (\varphi, \varphi)^{-q/2} \overline{D} \varphi + \frac{q(2p+1)}{2} (\varphi, \varphi)^{-1} \overline{\text{grad}}(\varphi, \varphi) \cdot \bar{\varphi} \right\}, \tag{3.5}$$

where $\varphi_p = (\varphi, \varphi)^{pq} \varphi$. On the other hand, we know (see [4]) that, if φ is an eigenspinor of D on $(N(r_*), g)$, then

$$\begin{aligned} \nabla_X \varphi &= -\frac{1}{2} (\varphi, \varphi)^{-1} T_\varphi(X) \cdot \varphi + \frac{3}{4} (\varphi, \varphi)^{-1} g(\text{grad}(\varphi, \varphi), X) \varphi \\ &\quad + \frac{1}{4} (\varphi, \varphi)^{-1} X \cdot \text{grad}(\varphi, \varphi) \cdot \varphi, \end{aligned} \tag{3.6}$$

where T_φ is the energy-momentum tensor defined by

$$T_\varphi(X, Y) = (X \cdot \nabla_Y \varphi + Y \cdot \nabla_X \varphi, \varphi).$$

Making use of Eqs. (3.4)–(3.6), we obtain the following proposition immediately.

Proposition 3.2. *In the notations above, we have:*

- (i) *If $p = -1/2$ and $D\varphi = 0$, then $\bar{D} \bar{\varphi}_p = 0$.*
- (ii) *If $\bar{\nabla}_X \bar{\varphi}_p = 0$ and $D\varphi = 0$, then $p = -1/2$ and $q = 1$.*
- (iii) *If $\bar{\nabla}_X \bar{\varphi}_p = 0$ with $p = -1/2$ and $q = 1$, then $D\varphi = 0$.*

We now find that, in order to improve the rigidity statement of Theorem 2.1, the optimal parameters p, q , are

$$p = -\frac{1}{2}, \quad q = 1. \tag{3.7}$$

For this choice of parameters, Eq. (3.4) gives

$$(\varphi, \varphi)^2 (\bar{\nabla}_{\bar{\varphi}_p}, \bar{\nabla}_{\bar{\varphi}_p}) = (\nabla \varphi, \nabla \varphi) + \frac{1}{2} (\varphi, \varphi)^{-1} (D\varphi, \text{grad}(\varphi, \varphi) \cdot \varphi) - \frac{3}{8} (\varphi, \varphi)^{-1} |\text{grad}(\varphi, \varphi)|^2.$$

Applying the Schrödinger–Lichnerowicz formula

$$\Delta(\varphi, \varphi) = -2(\nabla \varphi, \nabla \varphi) + 2(D^2 \varphi, \varphi) - \frac{1}{2} S_g(\varphi, \varphi),$$

where $\Delta = -\text{div} \circ \text{grad}$, one proves the following lemma.

Lemma 3.1. For the choice (3.7) of parameters, we have

$$\begin{aligned} \frac{1}{2} \operatorname{div} \{(\varphi, \varphi)^r \operatorname{grad}(\varphi, \varphi)\} &= (\varphi, \varphi)^r \left\{ (\varphi, \varphi)^2 (\overline{\nabla} \overline{\varphi}_p, \overline{\nabla} \overline{\varphi}_p) + \frac{1}{4} S_g(\varphi, \varphi) - (D^2 \varphi, \varphi) \right. \\ &\quad \left. - \frac{1}{2} (\varphi, \varphi)^{-1} (D\varphi, \operatorname{grad}(\varphi, \varphi) \cdot \varphi) + \frac{3}{8} (\varphi, \varphi)^{-1} |\operatorname{grad}(\varphi, \varphi)|^2 \right\} \\ &\quad + \frac{r}{2} (\varphi, \varphi)^{r-1} |\operatorname{grad}(\varphi, \varphi)|^2, \end{aligned}$$

where $r \in \mathbb{R}$ is an arbitrary real number.

Now we can prove the main result of the paper.

Theorem 3.1. Let (N, g) be a Riemannian asymptotically flat 3-manifold of order $\tau > 1/2$. If the scalar curvature S_g of (N, g) is non-negative and the mean curvature $\operatorname{Tr}_g(\Theta)$ of $(\partial N, g)$ satisfies

$$2 \sqrt{\frac{\pi}{\operatorname{Area}(\partial N, g)}} \geq \frac{1}{2} \sup_{\partial N} \{\operatorname{Tr}_g(\Theta)\}, \quad 2 \sqrt{\frac{\pi}{\operatorname{Area}(\partial N, g)}} \neq \frac{1}{2} \operatorname{Tr}_g(\Theta), \quad (3.8)$$

then there exists a positive constant $c(g) > 0$ uniquely determined by the metric g (as well as a beforehand fixed constant spinor ψ_o) such that

- (i) $m(g) \geq c(g)$, and
- (ii) the equality $m(g) = c(g)$ occurs only if, outside a compact set, g is conformally flat and the scalar curvature $S_g \equiv 0$ vanishes.

In case that $(N, g = e^{-f} g_{\text{eu}})$ is conformally flat, $f \in W_{-\tau}^{1,2}$, $\tau > 1/2$, and the conformal factor e^{-f} is constant on the boundary ∂N , then the equality $m(g) = c(g)$ holds.

Proof. Let ψ be a unique solution to the PDE system (2.7). We choose the parameter $r = -3/4$ in the formula of Lemma 3.1 so as to remove the terms involving $|\operatorname{grad}(\psi, \psi)|^2$. Then we have

$$\begin{aligned} m(g) &= \frac{1}{8\pi} \lim_{r \rightarrow \infty} \int_{S(r)} (\psi, \psi)^{-3/4} g(\operatorname{grad}(\psi, \psi), V) \mu_{S(r)}(g) \\ &= \frac{1}{4\pi} \int_{S(r_*)} (\pi_* \psi, \pi_* \psi)^{-3/4} \left(D^\partial(\pi_* \psi) - \frac{1}{2} \operatorname{Tr}_g(\Theta)(\pi_* \psi), \pi_* \psi \right) \mu_{S(r_*)}(g) \\ &\quad + \frac{1}{4\pi} \int_{N(r_*)} (\psi, \psi)^{-3/4} \left\{ (\psi, \psi)^2 (\overline{\nabla} \overline{\psi}_p, \overline{\nabla} \overline{\psi}_p) + \frac{1}{4} S_g(\psi, \psi) \right\} \mu_{N(r_*)}(g) \end{aligned}$$

for all sufficiently large constants $r_* \geq r_o$. On the other hand, we know that

$$m(g) = \frac{1}{8\pi} \lim_{r \rightarrow \infty} \int_{S(r)} g(\operatorname{grad}(\psi, \psi), V) \mu_{S(r)}(g)$$

$$\begin{aligned}
 &= \frac{1}{4\pi} \int_{\partial N} \left(D^\partial(\pi_*\psi) - \frac{1}{2} \text{Tr}_g(\Theta)(\pi_*\psi), \pi_*\psi \right) \mu_{\partial N}(g) \\
 &\quad + \frac{1}{4\pi} \int_N \left\{ (\nabla\psi, \nabla\psi) + \frac{1}{4} S_g(\psi, \psi) \right\} \mu_N(g) \\
 &> \frac{1}{4\pi} \int_{\partial N} \left\{ 2 \sqrt{\frac{\pi}{\text{Area}(\partial N, g)}} - \frac{1}{2} \text{Tr}_g(\Theta) \right\} (\pi_*\psi, \pi_*\psi) \mu_{\partial N}(g) > 0,
 \end{aligned}$$

since $\int_N (\nabla\psi, \nabla\psi) > 0$ is strictly positive. Therefore, there exists a positive constant $r_\infty \geq r_o$ satisfying the following two conditions: ψ has no zeros in $N(r_\infty) = \bigcup_{r \geq r_\infty} S(r)$ and

$$\begin{aligned}
 &\frac{1}{4\pi} \int_{S(r_\infty)} (\pi_*\psi, \pi_*\psi)^{-3/4} \left(D^\partial(\pi_*\psi) - \frac{1}{2} \text{Tr}_g(\Theta)(\pi_*\psi), \pi_*\psi \right) \mu_{S(r_\infty)}(g) \\
 &> \frac{1}{4\pi} \int_{\partial N} \left\{ 2 \sqrt{\frac{\pi}{\text{Area}(\partial N, g)}} - \frac{1}{2} \text{Tr}_g(\Theta) \right\} (\pi_*\psi, \pi_*\psi) \mu_{\partial N}(g) > 0.
 \end{aligned}$$

Let r_{glb} be the greatest lower bound of the set of all the constants r_∞ satisfying these two conditions and define

$$c(g) = \frac{1}{4\pi} \int_{S(r_{\text{glb}})} (\pi_*\psi, \pi_*\psi)^{-3/4} \left(D^\partial(\pi_*\psi) - \frac{1}{2} \text{Tr}_g(\Theta)(\pi_*\psi), \pi_*\psi \right) \mu_{S(r_{\text{glb}})}(g).$$

Then it is clear that the statements (i) and (ii) of the theorem are true. Now it remains to prove the last statement of the theorem. Let $\varphi = e^{f/2}\psi_o$. Then Proposition 3.1(iii) implies $D\varphi = 0$. Furthermore,

$$0 = \bar{\nabla}_{\bar{E}_i} \bar{\psi}_o = \bar{\nabla}_{\bar{E}_i}^\partial \bar{\psi}_o + \frac{1}{2} \Theta_{g_{\text{eu}}}(\bar{E}_i) \cdot \bar{E}_3 \cdot \bar{\psi}_o = \bar{\nabla}_{\bar{E}_i}^\partial \bar{\psi}_o + \frac{1}{2r_o} \bar{E}_i \cdot \bar{E}_3 \cdot \bar{\psi}_o, \quad i = 1, 2,$$

gives

$$\begin{aligned}
 \nabla_{E_i}^\partial(\pi_*\varphi) &= -\frac{1}{2r_o} e^{f/2} E_i \cdot (\pi_*\varphi) + \frac{3}{4} df(E_i)(\pi_*\varphi) + \frac{1}{4} E_i \cdot \left(\sum_{j=1}^2 df(E_j) E_j \right) \cdot (\pi_*\varphi) \\
 &= -\frac{1}{2r_o} e^{f/2} E_i \cdot (\pi_*\varphi),
 \end{aligned}$$

since the function f is constant on ∂N . Consequently, $\varphi = e^{f/2}\psi_o$ is the unique solution to the system (2.7) and the equality $m(g) = c(g)$ holds indeed. \square

Remark. Let $(N, g = e^{-f} g_{\text{eu}})$ be conformally flat, $f \in W_{-\tau}^{1,2}$, $\tau > 1/2$, and let the function f be constant on the boundary ∂N . Assume that $S_g \geq 0$ and the boundary condition (3.8) is satisfied. Then the scalar curvature S_g is given by (see (3.1))

$$\Delta_g(e^{f/4}) = -\frac{1}{8} e^{f/4} S_g$$

and so the mass by (see (3.3))

$$m(g) = -\frac{1}{8\pi} \int_{\partial N} e^{f/4} df(E_3) \mu_{\partial N}(g) + \frac{1}{16\pi} \int_N e^{f/4} S_g \mu_N(g).$$

Substituting Eq. (3.2) into (3.8), one verifies easily that $-df(E_3) \geq 0$, $df(E_3) \neq 0$, and the constant $c(g)$ in Theorem 3.1 is in fact equal to

$$\begin{aligned} c(g) &= -\frac{1}{8\pi} \int_{\partial N} e^{f/4} df(E_3) \mu_{\partial N}(g) \\ &= \frac{1}{4\pi} \int_{\partial N} (\pi_* \psi, \pi_* \psi)^{-3/4} \left(D^\partial(\pi_* \psi) - \frac{1}{2} \text{Tr}_g(\Theta)(\pi_* \psi), \pi_* \psi \right) \mu_{\partial N}(g), \end{aligned}$$

where $\psi = e^{f/2} \psi_o$ is a unique solution to system (2.7). In particular, if g is the spacelike Schwarzschild metric with

$$e^{-f} = \left(1 + \frac{m}{2r} \right)^4, \quad m > 0,$$

then a direct computation, on the minimal boundary $\partial N = S(r = m/2)$, shows that $c(g) = m$.

Remark. It might be possible to compare the constant $c(g)$ in Theorem 3.1 with the lower bound $4\sqrt{\text{Area}(\partial N, g)/\pi}$ of the Penrose inequality [3,7], in case that the boundary $(\partial N, g)$ is minimal. It seems that

$$4\sqrt{\frac{\text{Area}(\partial N, g)}{\pi}} \geq c(g),$$

since the boundary condition (outermost minimal surface) for the constant $4\sqrt{\text{Area}(\partial N, g)/\pi}$ is stronger than that (minimal surface) for $c(g)$.

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