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2-factors and hamiltonicity

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Abstract

We prove the following generalization of a result of Faudree and van den Heuvel. Let G be a 2-connected graph with a 2-factor. If $d(u)+d(v)\geqslant n-2$ for all pairs of non-adjacent vertices u,v contained in an induced $K_{1,3}$, in an induced $K_{1,3}+e$ or as end-vertices in an induced P_4, then G is Hamiltonian. © 1998 Published by Elsevier Science B.V. All rights reserved

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1. Terminology and notation

We use [2] for terminology and notation not defined here and consider finite simple graphs only.

Let G be a graph on n vertices. We say that G is *hamiltonian* if G has a *Hamilton cycle*, i.e. a cycle containing all vertices of G. If X is a graph, we say that G is *X-free* if G does not contain an induced subgraph isomorphic to X. In this paper we use $K_{1,3}$, $Z_1 \simeq K_{1,3}+e$ and P_4 to denote the graphs of Fig. 1. According to the labeling of the vertices we will write $\langle a,b,c,d\rangle \simeq K_{1,3}$, $\langle a,b,c,d\rangle \simeq Z_1$ and $\langle a,b,c,d\rangle \simeq P_4$, respectively.

We will use $\omega(G)$ to denote the number of components of G. A graph G is said to be *t-tough* (cf. [3]) if $t\cdot\omega(G-S)\leqslant |S|$ for every subset S of $V(G)$ with $\omega(G-S)>1$. If $v\in V(G)$, then $N(v)$ denotes the set of vertices adjacent to v (the *neighborhood* of v) and $d(v)=|N(v)|$ denotes the degree of v. If we restrict $N(v)$ and $d(v)$ to a subgraph $F\subset G$, then we will use $N_F(v)$ and $d_F(v)$, respectively. We say that a subgraph $H\subset G$ is a *2-factor* of G if H is a spanning subgraph of G and $d_H(v)=2$ for every $v\in V(G)$.

Let C be a cycle of G. If an orientation of C is fixed and $u,v\in V(C)$, then by $u\vec{C}v$ we denote the consecutive vertices on C from u to v in the orientation specified by

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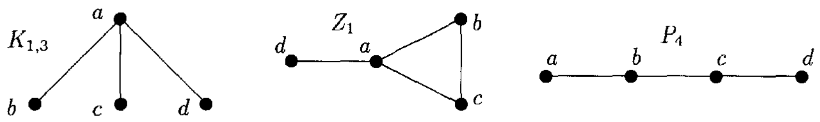

Fig. 1.

the orientation of C. The same vertices, in reverse order, are given by $v \overleftarrow{C} u$. If $C \subset G$ is a cycle with a fixed orientation and $v \in V(C)$, then v^+ and v^- denotes the successor and predecessor of v on C, respectively.

2. Main result

Our research was motivated by the following famous conjecture by Chvátal.

Conjecture (*Chvátal* [3]). Every 2-tough graph is hamiltonian.

For the class of 2-tough graphs Enomoto, Jackson, Katerinis and Saito proved the following result.

Theorem 1 (Enomoto et al. [5]). *Every 2-tough graph has a 2-factor.*

Obviously, having a 2-factor is a necessary condition for a graph to be hamiltonian. Moreover, it can be decided in polynomial time whether a given graph G has a 2-factor (see [1]).

The first result for hamiltonicity of graphs having a 2-factor is due to Hoede.

Theorem 2 (Hoede [7]). *Let G be a connected graph with a 2-factor and let $G_1,\dots,G_{11}$ be the graphs shown in Fig. 2. If G is $G_1,\dots,G_{11}$-free, then G is hamiltonian.*

We now turn our attention to degree conditions. The following result by Faudree and van den Heuvel shows that Ore's [8] and Dirac's [4] degree conditions for hamiltonicity can be relaxed under the additional assumption that G has a 2-factor.

Theorem 3 (Faudree and van den Heuvel [6]). *Let G be a 2-connected graph with a 2-factor. If $d(u)+d(v) \geqslant n-2$ for all pairs of non-adjacent vertices $u,v \in V(G)$, then G is hamiltonian.*

Motivated by Theorem 2, we got the impression that it might be sufficient to require the condition $d(u)+d(v) \geqslant n-2$ for all pairs of non-adjacent vertices u,v which are contained in an induced P_4 or Z_1 (cf. G_1 and G_2 in Fig. 2). However, examples can be given showing that this is not the case even with the requirement $d(u)+d(v) \geqslant n-1$.

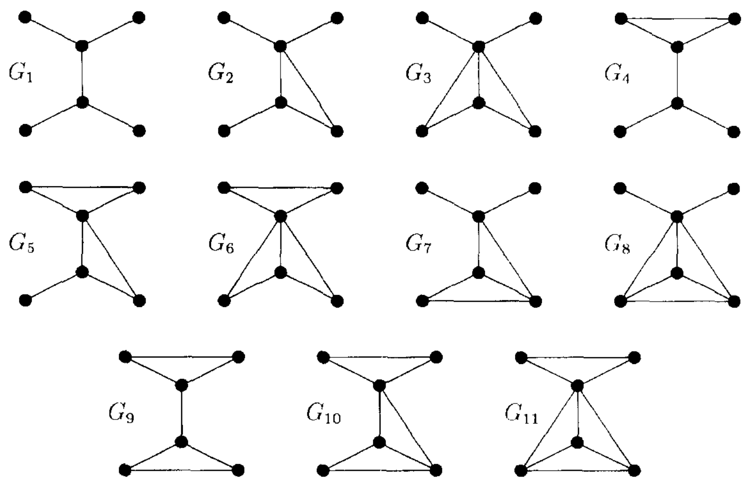

Fig. 2.

A class of such graphs can be obtained by joining two additional vertices u, v to two prescribed vertices of an arbitrary clique on at least 5 vertices (notice that u and v are contained in an induced $K_{1,3}$ and have $d(u) + d(v) = 4 \leqslant n - 3$). Thus, the degree condition required for the induced claw is necessary.

Next, consider the class of graphs $G_{p,q,r}$ which consist of three complete graphs K_p, K_q, K_r for $p \geqslant q \geqslant r \geqslant 3$ and the additional edges u_iv_i, u_iw_i, v_iw_i for $i = 1, 2$ and vertices $u_1, u_2 \in V(K_p)$, $v_1, v_2 \in V(K_q)$ and $w_1, w_2 \in V(K_r)$. These graphs are 2-connected, claw-free with a 2-factor, but the degree condition is not satisfied for all induced P_4 and induced Z_1.

Finally, the complete bipartite graph $K_{p,q}$ with $p = \lfloor (n-1)/2 \rfloor$ and $q = \lceil (n+1)/2 \rceil$ for $n \geqslant 5$ is 2-connected, satisfies $d(u) + d(v) \geqslant n - 2$ for every pair of nonadjacent vertices u, v, but it has no 2-factor.

These examples show that all the assumptions of the following theorem are, in some sense, best possible.

Theorem 4. *Let G be a 2-connected graph with a 2-factor. If $d(u) + d(v) \geqslant n - 2$ for all pairs of non-adjacent vertices u, v contained in a $K_{1,3}$, in a Z_1 or as endvertices in a P_4, then G is hamiltonian.*

Example. Let i_0, i_1, i_2, i_3, i_4 be integers such that $i_0, i_4 \geqslant 1$, $i_2 \geqslant 2$, $i_1 \geqslant i_2 + i_4 - 1$, $i_3 \geqslant i_0 + i_2 - 1$. Let G be the graph obtained by taking vertex-disjoint graphs H_0, H_1, H_2, H_3, H_4, where $H_j \simeq K_{i_j}$ for $j = 0, 1, 3, 4$ and $H_2 \simeq \overline{K_{i_2}}$, and by adding all edges xy for $x \in V(H_i)$, $y \in V(H_{i+1})$, $i = 0, 1, 2, 3$. Then the graph G satisfies the assumptions of Theorem 4, but not of Theorem 3. Note that G has diameter $\operatorname{diam}(G) = 4$ while the assumptions of Theorem 3 imply $\operatorname{diam}(G) \leqslant 3$.

3. Proofs

We first prove some lemmas which will be useful for the proof of Theorem 4.

Lemma 1. *Let C_p, C_q and C be three vertex-disjoint cycles with $V(C_p)=\{u_1,\dots,u_p\}$ and $V(C_q)=\{v_1,\dots,v_q\}$. If $u_pv_q\in E(G)$ and $d_C(u_1)+d_C(v_1)\geqslant|V(C)|+1$, then there is a cycle C' such that $V(C')=V(C_p)\cup V(C_q)\cup V(C)$.*

Proof. Since $d_C(u_1)+d_C(v_1)\geqslant|V(C)|+1$, there exists a pair of consecutive vertices $w_1,w_2\in V(C)$ such that $u_1w_1, v_1w_2\in E(G)$ or $u_1w_2, v_1w_1\in E(G)$ and we can easily construct the desired cycle C'. □

Lemma 2. *Let C_p and C_q be vertex-disjoint cycles with vertices labeled $u_1,\dots,u_p$ and $v_1,\dots,v_q$. Suppose $u_pv_q\in E(G)$; $u_pv_1, u_1v_q, u_1v_1\notin E(G)$. If $d_{C_p\cup C_q}(u_1)+d_{C_p\cup C_q}(v_1)\geqslant p+q-1$, then there is a cycle C such that $V(C)=V(C_p)\cup V(C_q)$.*

Proof. Suppose there is no such cycle. Then $v_1u_{p-1}, v_{q-1}u_1\notin E(G)$. Let

$$S=\{i\mid v_1u_i\in E(G),\ 2\leqslant i\leqslant p-2\},\qquad T=\{i\mid u_1u_{i+1}\in E(G),\ 1\leqslant i\leqslant p-2\}.$$

If there is some $i\in T\cap S$, then $C=v_1\,u_i\,\overleftarrow{C_p}\,u_1\,u_{i+1}\,\overrightarrow{C_p}\,u_p\,v_q\,\overleftarrow{C_q}\,v_1$ would be the desired cycle. Hence we can assume that $S\cap T=\emptyset$. Now $d_{C_p}(v_1)=|S|$ and $d_{C_p}(u_1)=|T|+1$, from which $d_{C_p}(u_1)+d_{C_p}(v_1)=|S|+|T|+1=|S\cup T|+1\leqslant p-1$. By the same argument we obtain $d_{C_q}(u_1)+d_{C_q}(v_1)\leqslant q-1$ and thus $d_{C_p\cup C_q}(u_1)+d_{C_p\cup C_q}(v_1)\leqslant p+q-2$, a contradiction. □

Let C^1, C^2 be two vertex-disjoint cycles. We say that a vertex $v\in V(C^1)$ is C^2-*universal*, if v is adjacent to all vertices of C^2.

Assume now that there are two vertex-disjoint cycles C^1, C^2 and a C^2-universal vertex $v\in V(C^1)$. If v^- or v^+ has a neighbor on C^2, then we can again easily construct a cycle C such that $V(C)=V(C^1)\cup V(C^2)$.

Lemma 3. *Let G be a non-hamiltonian graph with a 2-factor consisting of $k\geqslant2$ cycles $C^1,C^2,\dots,C^k$, where k is minimal. Then for every pair of cycles C^i,C^j, $1\leqslant i<j\leqslant k$, and every C^j-universal vertex $v\in V(C^i)$, neither v^- nor v^+ has a neighbor on C^j.*

Corollary 4. *Let G be a non-hamiltonian graph with a 2-factor consisting of $k\geqslant2$ cycles $C^1,C^2,\dots,C^k$, where k is minimal. Then for every pair of cycles C^i,C^j, $1\leqslant i<j\leqslant k$, all C^j-universal vertices of $V(C^i)$ are pairwise non-consecutive.*

Corollary 5. *Let G be a non-hamiltonian graph with a 2-factor consisting of $k\geqslant2$ cycles $C^1,C^2,\dots,C^k$, where k is minimal. Then there is no pair of cycles C^i,C^j, $1\leqslant i$*

$<j\leqslant k$, such that there is both a C^j-universal vertex $v_i\in V(C^i)$ and a C^i-universal vertex $v_j\in V(C^j)$.

We will also use the following simple lemma.

Lemma 6. *Let C be a cycle in a graph G and let $x,y\in V(C)$ be such that there is no x,y-path P with $V(P)=V(C)$. Then $x^+y^+\notin E(G)$ and $d_C(x^+)+d_C(y^+)\leqslant|V(C)|$.*

Proof. If $x^+y^+\in E(G)$, then $P=x\overleftarrow{C}y^+\,x^+\overrightarrow{C}y$ is a x,y-path with $V(P)=V(C)$. Hence $x^+y^+\notin E(G)$. Put $M=\{z\in V(C)\,|\,zx^+\in E(G)\}$ and $N=\{z\in x^{++}\overrightarrow{C}y^+\,|\,z^-y^+\in E(G)\}\cup\{z\in y^{++}\overrightarrow{C}x\,|\,z^+y^+\in E(G)\}$. Then $|M|=d_C(x^+)$, $|N|=d_C(y^+)-1$ and $x^+\notin M\cup N$. Thus, if $d_C(x^+)+d_C(y^+)\geqslant|V(C)|+1$, there is a vertex $z\in M\cap N$, but then the path $x\overleftarrow{C}y^+z^-\overleftarrow{C}x^+z\overrightarrow{C}y$ (if $z\in x^+\overrightarrow{C}y^+$) or $x\overleftarrow{C}z^+\,y^+\overrightarrow{C}zx^+\overrightarrow{C}y$ (if $z\in y^+\overrightarrow{C}x^+$) yields a contradiction. Hence $d_C(x^+)+d_C(y^+)\leqslant|V(C)|$. □

Proof of Theorem 4. Assume G is not hamiltonian and choose a 2-factor of G with $k\geqslant 2$ cycles $C^1,C^2,\ldots,C^k$ such that k is minimal. We distinguish the following cases.

Case 1: There are two cycles C^{t_1},C^{t_2}, $1\leqslant t_1<t_2\leqslant k$, which are connected by two vertex-disjoint edges.

Subcase A: There is an edge xy such that $x\in V(C^{t_1})$, $y\in V(C^{t_2})$ and neither x is C^{t_2}-universal nor y is C^{t_1}-universal.

Subcase B: Every vertex $x\in V(C^{t_1})$ with $N(x)\cap V(C^{t_2})\neq\emptyset$ is C^{t_2}-universal.

Case 2: No pair of cycles C^i,C^j, $1\leqslant i<j\leqslant k$, is connected by two vertex-disjoint edges.

By Corollary 5, no other possibilities can occur.

Throughout the proof, we denote $n_i=|V(C^i)|$, $1\leqslant i\leqslant k$. For convenience we set $p=n_1$ and $q=n_2$.

Case 1: We can, without loss of generality, suppose that $C^{t_1}=C^1\simeq C_p$ with vertices labeled $u_1,\ldots,u_p$, $C^{t_2}=C^2\simeq C_q$ with vertices labeled $v_1,\ldots,v_q$, $u_pv_q\in E(G)$ and $u_iv_j\in E(G)$ for some i,j with $1\leqslant i\leqslant p-1$, $1\leqslant j\leqslant q-1$.

Subcase A: Suppose (without loss of generality) that $u_pv_1,u_1v_q,u_1v_1\notin E(G)$. Thus $\langle u_1,u_p,v_q,v_1\rangle\simeq P_4$, from which $d(u_1)+d(v_1)\geqslant n-2$. Since k is minimal, by Lemma 1 and Lemma 2 we have $d_{C^1}(u_1)+d_{C^1}(v_1)=p-1$, $d_{C^2}(u_1)+d_{C^2}(v_1)=q-1$. If u_1u_{i+1}, $v_1v_{j+1}\in E(G)$, then the cycle $u_1u_{i+1}\overrightarrow{C^1}u_p\,v_q\overleftarrow{C^2}v_{j+1}\,v_1\overrightarrow{C^2}v_j\,u_i\overleftarrow{C^1}u_1$ contradicts the minimality of k. Hence, we can, without loss of generality, assume that $u_1u_{i+1}\notin E(G)$. Since, equality holds in Lemma 2, this implies $v_1u_i\in E(G)$ and thus $2\leqslant i\leqslant p-2$. Moreover, since $v_1u_{p-1}\notin E(G)$, there exists $r>i$ such that $u_{r-1}v_1,u_{r+1}u_1\in E(G)$ and $u_1u_r,v_1u_r\notin E(G)$. Since there is no cycle C such that $V(C)=V(C^1)\cup V(C^2)$, we have $u_rv_2,u_rv_q\notin E(G)$. By symmetry and since $u_rv_1\notin E(G)$, we conclude $u_rv_{q-1}\notin E(G)$. Now, $C=v_1\,u_{r-1}\overleftarrow{C^1}u_1\,u_{r+1}\overrightarrow{C^1}u_p\,v_q\overleftarrow{C^2}v_1$ is a cycle such that $V(C)=V(C^1)\cup V(C^2)\setminus\{u_r\}$. If $u_ru_i,u_ru_{i+1}\in E(G)$ for some i with $2\leqslant i\leqslant r-2$ or $r+1\leqslant i\leqslant p-1$, then u_r can be *inserted* into the cycle C by replacing the edge u_iu_{i+1} by the path $u_iu_ru_{i+1}$. Hence, we conclude that $u_ru_{r-2}, u_ru_{r+2} \notin E(G)$ and $d_{C^1}(u_r) \leqslant p/2$. Likewise u_r can be inserted if $u_rv_i, u_rv_{i+1} \in E(G)$ for some i with $2 \leqslant i \leqslant q-3$. Hence $d_{C^2}(u_r) \leqslant (q-4+1)/2 = (q-3)/2$. For any other cycle C^j, $3 \leqslant j \leqslant k$, if $u_rw_1, u_rw_2 \in E(G)$ for two consecutive vertices w_1, w_2 on C^j, then u_r can be inserted into C^j, contradicting the minimality of k. Hence $d_{C^j}(u_r) \leqslant n_j/2$ and thus $d(u_r) \leqslant p/2 + (q-3)/2 + \sum_{j=3}^{k} n_j/2 = (n-3)/2$. Now $\langle u_{r-1}, u_{r-2}, u_r, v_1 \rangle$ and $\langle u_{r+1}, u_r, u_{r+2}, u_1 \rangle$ are isomorphic to $K_{1,3}$ or Z_1 implying $d(v_1) \geqslant (n-1)/2$ and $d(u_1) \geqslant (n-1)/2$. Altogether we obtain $n-1 \leqslant d(u_1) + d(v_1) \leqslant p + q - 2 + \sum_{j=3}^{k} n_j = n-2$, a contradiction.

Subcase B: Let $M = \{x \in V(C^1) \mid N_{C^2}(x) \neq \emptyset\}$. Then, by the assumptions of Case 1, $|M| \geqslant 2$, $u_p \in M$ and (recall Corollary 5 and Corollary 4), no two vertices in M are consecutive on C^1. Suppose first that there are $x, y \in M$, $x \neq y$, such that both $x^-x^+ \notin E(G)$ and $y^-y^+ \notin E(G)$. Then, since (by Lemma 3) both $\langle x, x^-, x^+, v_q \rangle \simeq K_{1,3}$ and $\langle y, y^-, y^+, v_q \rangle \simeq K_{1,3}$, we have $d(x^-) + d(x^+) + d(y^-) + d(y^+) \geqslant 2(n-2) \geqslant 2(p+q-2+n-p-q) \geqslant 2(p+1) + 2(n-p-q)$. On the other hand, by the minimality of k, there is no hamiltonian x,y-path in $G[V(C^1)]$ and hence, by Lemma 6, $d_{C^1}(x^+) + d_{C^1}(y^+) + d_{C^1}(x^-) + d_{C^1}(y^-) \leqslant 2p$. Together we obtain $2(p+1) + 2(n-p-q) \leqslant d(x^+) + d(y^+) + d(x^-) + d(y^-) \leqslant 2p + 2(n-p-q)$, which is a contradiction.

Hence we can suppose that $x^-x^+ \in E(G)$ for every $x \in M$, $x \neq u_p$. But then, for any $x \in M$, $x \neq u_p$, we have $u_1x \notin E(G)$ and $u_1x^{++} \notin E(G)$ (otherwise the cycles $u_1xv_1 \overrightarrow{C^2} v_q u_p \overleftarrow{C^1} x^+x^- \overleftarrow{C^1} u_1$ and $u_1x^{++} \overrightarrow{C^1} u_p v_q \overleftarrow{C^2} v_1xx^+x^- \overleftarrow{C^1} u_1$ contradict the minimality of k). Now, $x^{++} \notin M$, since $x^{++} \overrightarrow{C^1} x^-x^+x$ is a hamiltonian path in $G[V(C^1)]$. Since also (by Lemma 6) $u_1x^+ \notin E(G)$ and, by Lemma 3, $d_{C^2}(u_1) = 0$, we have $d_{C^1 \cup C^2}(u_1) \leqslant p - 1 - 3(|M|-1)$. Since every vertex in M is C^2-universal, we have $d_{C^1 \cup C^2}(v_q) \leqslant q - 1 + |M|$. If there is a cycle C^i, $3 \leqslant i \leqslant k$, such that u_1 and v_q have consecutive neighbors on C^i, then we easily construct a cycle C' with $V(C') = V(C^1) \cup V(C^2) \cup V(C^i)$, contradicting the minimality of k; hence, $d_{C^3 \cup \cdots \cup C^k}(u_1) + d_{C^3 \cup \cdots \cup C^k}(v_q) \leqslant |V(C^3) \cup \cdots \cup V(C^k)| = n - p - q$. Since $\langle u_p, v_q, v_1, u_1 \rangle \simeq Z_1$, we have $d(u_1) + d(v_q) \geqslant n-2$. Altogether we obtain $n - 2 \leqslant d(u_1) + d(v_q) \leqslant p - 1 - 3(|M|-1) + q - 1 + |M| + n - p - q$, from which $|M| \leqslant 3/2$, a contradiction.

Case 2: Since G is 2-connected, there are m cycles, $3 \leqslant m \leqslant k$, say, $C^1, C^2, \ldots, C^m$, with vertices labeled $v_1^i, \ldots, v_{n_i}^i$, and pairs of vertices $v_{r_i}^i, v_{s_i}^i \in V(C^i)$ such that $v_{s_i}^i v_{r_{i+1}}^{i+1} \in E(G)$ (modulo m). If $s_i = r_i \pm 1$ for all $1 \leqslant i \leqslant m$, then there is a cycle C such that $V(C) = \bigcup_{i=1}^{m} V(C^i)$, e.g. $C = v_{s_1}^1 v_{r_2}^2 \overrightarrow{C^2} v_{s_2}^2 v_{r_3}^3 \ldots v_{s_m}^m v_{r_1}^1 \overrightarrow{C^1} v_{s_1}^1$, if $s_i = r_i + 1$ for $1 \leqslant i \leqslant m$, which contradicts the minimality of k.

Now suppose, without loss of generality, that $s_1 \neq r_1 \pm 1$. Thus, $n_1 \geqslant 4$. If $v_{r_1+1}^1 v_{s_1+1}^1 \in E(G)$ or $d_{C^1}(v_{r_1+1}^1) + d_{C^1}(v_{s_1+1}^1) \geqslant n_1 + 1$, then, by Lemma 6, there is a hamiltonian path in $G[V(C^1)]$ with endvertices $v_{r_1}^1$, $v_{s_1}^1$.

Suppose such a path does not exist. With a repeat of previous arguments we will show that $v_{s_1}^1, v_{r_1}^1$ are both universal vertices and that $n_1 = 4$. Suppose first that $v_{s_1}^1$ is not universal. Then there is a vertex $x \in V(C^2)$ such that $v_{s_1}^1 x \in E(G)$, but $v_{s_1}^1 x^+ \notin E(G)$.

As in Subcase A we obtain this time $d(v^1_{s_1+1})+d(x^+)\leqslant(n_1-2)+(n_2-1)+\sum_{j=3}^k n_j < n-2$, a contradiction. The same argument holds for $v^1_{r_1}$. Thus, both $v^1_{s_1}$ and $v^1_{r_1}$ are universal vertices. Suppose next that $n_1\geqslant 5$. By Lemma 6 we have $d_{C^1}(v^1_{s_1+1}) + d_{C^1}(v^1_{r_1+1})\leqslant n_1$. Hence we may assume that $d_{C^1}(v^1_{s_1+1})\leqslant n_1/2$. But then $\langle v^1_{s_1}, x, x^+, v^1_{s_1+1}\rangle \simeq Z_1$ for any pair of consecutive vertices $x, x^+\in V(C^2)$ and $d_{C^1\cup C^2}(v^1_{s_1+1})+d_{C^1\cup C^2}(x^+) \leqslant n_1/2+(n_2-1+1)+\sum_{j=3}^k n_j<(n_1-2)+n_2+\sum_{j=3}^k n_j\leqslant n-2$, a contradiction. Hence $n_1=4$.

Let $\{s_1, r_1\}=\{2,4\}$. Then $d_{C^1}(v^1_1)=d_{C^1}(v^1_3)=2$ and both v^1_2 and v^1_4 are contained in an induced Z_1, say, $\langle v^1_2, v^1_1, v^m_{n_m}, v^m_1\rangle$ and $\langle v^1_4, v^1_3, v^2_{n_2}, v^2_1\rangle$. Since $N_{C^m}(v^1_3)=\emptyset$, $N_{C^m}(v^1_1)=\emptyset$, $N_{C^2}(v^1_3)=\emptyset$, $N_{C^2}(v^1_1)=\emptyset$, we have $d_{C^1\cup C^2\cup C^3}(v^1_1) + d_{C^1\cup C^2\cup C^3}(v^1_3)=4$, where $n_1+n_2+n_3\geqslant 4+3+3=10$. Since $d(v^1_1)+d(v^1_3)\geqslant n-2$, we have $k\geqslant 4$ and $\sum_{j=4}^k d_{C^j}(v^1_1)+d_{C^j}(v^1_3)\geqslant \sum_{j=4}^k n_j+4$. Hence there exists a cycle C^j and two consecutive vertices w_1, w_2 on C^j such that (without loss of generality) $v^1_1 w_1, v^1_3 w_2\in E(G)$. Then $C^a=v^1_4 v^2_1 \overrightarrow{C^2} v^2_{n_2} v^1_4$ and $C^b=v^1_1 v^1_2 v^1_3 w_2 \overleftarrow{C^j} w_1 v^1_1$ are two cycles such that $V(C^a)\cup V(C^b)=V(C^1)\cup V(C^2)\cup V(C^j)$, which contradicts the minimality of k.

This shows that, for each cycle C^i, the vertices $v^i_{r_i}$ and $v^i_{s_i}$ are connected by a hamiltonian path in $G[V(C^i)]$, $1\leqslant i\leqslant m$. But then there is a cycle C such that $V(C)=\bigcup_{j=1}^m V(C^j)$, contradicting again the minimality of k. This contradiction completes the proof of Theorem 4. $\square$

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