Two-weight imbedding inequalities for solutions to the $A$-harmonic equation

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Abstract

We obtain the two-weight imbedding inequalities for solutions to the $A$-harmonic equation and explore their applications to the $K$-quasiregular mappings in $\mathbb{R}^n$. These results can be used to study the properties of the homotopy operator $T : L^p(D, \Lambda^l) \rightarrow W^{1,p}(D, \Lambda^{l-1}), l = 1, 2, \ldots, n$, and to establish the weighted $L^p$-estimate for solutions to the $A$-harmonic equation.

Keywords: Two-weights; Inequalities; $A$-harmonic equation; Differential forms

1. Introduction and notation

The purpose of this paper is to prove the two-weight versions of imbedding inequalities for differential forms satisfying the $A$-harmonic equation and to establish two-weight norm estimates for the homotopy operator $T$. The $A$-harmonic equation is an interesting and important extension of $P$-harmonic equation, $p > 1$. Many interesting results about differential forms and their applications in fields including the potential theory and partial differential equations have been found, see [1–5,7,10,14]. Differential forms also play an important role in the general relativity and the theory of elasticity, see [9] and [12] for details.
In this paper, we keep using traditional notations developed in the field of differential forms. Specifically, we assume that \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) and \( \{e_1, e_2, \ldots, e_n\} \) is the standard unit basis of \( \mathbb{R}^n \), \( n \geq 2 \). We write \( \mathbb{R}^1 = \mathbb{R}^1 \). The linear space of \( l \)-vectors, spanned by the exterior products \( e_1 = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l} \), corresponds to all ordered \( l \)-tuples \( I = (i_1, i_2, \ldots, i_l) \), \( 1 \leq i_1 < i_2 < \cdots < i_l \leq n \), is denoted by \( \Lambda^l = \Lambda^l(\mathbb{R}^n) \), \( l = 0, 1, \ldots, n \). The Grassman algebra \( \Lambda = \bigoplus \Lambda^l \) is a graded algebra with respect to the exterior products. Suppose \( \alpha = \sum \alpha^l e_1 \wedge \cdots \wedge e_l \in \Lambda^l \), \( \beta = \sum \beta^l e_1 \wedge \cdots \wedge e_l \in \Lambda^l \), the inner product in \( \Lambda \) is defined by \( \langle \alpha, \beta \rangle = \sum \alpha^l \beta^l \) with summation over all \( l \)-tuples \( I = (i_1, i_2, \ldots, i_l) \) and all integers \( l = 0, 1, \ldots, n \). The Hodge star operator \( \star : \Lambda \rightarrow \Lambda \) is defined by the rule \( \star y = e_1 \wedge e_2 \wedge \cdots \wedge e_n \) and \( \alpha \wedge \star \beta = \beta \wedge \star \alpha \) for all \( \alpha, \beta \in \Lambda \). The ball is denoted by \( B \) and \( kB \) is the ball with the same center as \( B \) and with diameter \( kB \). The \( n \)-dimensional Lebesgue measure of a set \( E \subset \mathbb{R}^n \) is denoted by \( |E| \). Also, we do not distinguish balls from cubes.

As usual, the norm of \( \alpha \in \Lambda \) is given by the formula \( |\alpha|^2 = \langle \alpha, \alpha \rangle = \star (\alpha \wedge \star \alpha) \in \Lambda^0 = \mathbb{R} \). The Hodge star is an isometric isomorphism on \( \Lambda \) with \( \star : \Lambda^l \rightarrow \Lambda^{n-l} \) and \( \star(\star) = (-1)^{l(n-l)} \Lambda^l \rightarrow \Lambda^l \). We call \( \omega \) a weight if \( \omega \in L^1_{\text{loc}}(\mathbb{R}^n) \) and \( \omega > 0 \) a.e. For \( 0 < p < \infty \) and a weight \( w(x) \), we denote the weighted \( L^p \)-norm of a measurable function \( f \) over \( \Omega \) by \( \|f\|_{p, \mathbb{R}, w^\alpha} = \left( \int_\Omega |f(x)|^p w^{\alpha x} \, dx \right)^{1/p} \), where \( \alpha \) is a real number.

It is well known that a differential \( l \)-form \( \omega \) on \( \Omega \) is a de Rham current, see [11] or [13], on \( \Omega \) with values in \( \Lambda^l(\mathbb{R}^n) \). We denote the space of differential \( l \)-forms by \( D'(\Omega, \Lambda^l) \). We write \( L^p(\Omega, \Lambda^l) \) for the \( l \)-forms \( \omega(x) = \sum_i \omega_{ij_1,\ldots,j_l}(x) dx_{j_1} \wedge \cdots \wedge dx_{j_l} \in L^p(\Omega, \mathbb{R}^1) \) for all \( i \). Thus \( L^p(\Omega, \Lambda^l) \) is a Banach space with norm \( \|\omega\|_{p, \Omega} = (\int_\Omega |\omega(x)|^p dx)^{1/p} = (\int_\Omega (\sum_i |\omega_{ij_1,\ldots,j_l}(x)|^2)^{p/2} dx)^{1/p} \). If \( \omega \in D'(\Omega, \Lambda^l) \), the vector-valued differential form \( \nabla \omega \) is defined by \( \nabla \omega = \left( \frac{\partial \omega}{\partial x_1}, \ldots, \frac{\partial \omega}{\partial x_n} \right) \), where the partial differentiation is applied to the coefficients of \( \omega \).

Also, \( W^{1,p}(\Omega, \Lambda^l) \) is used to denote the space of all differential \( l \)-forms on \( \Omega \) whose coefficients are in \( W^{1,p}(\Omega, \mathbb{R}^1) \). The notations \( W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^1) \) and \( W^{1,p}_{\text{loc}}(\Omega, \Lambda^l) \) are self-explanatory. For \( 0 < p < \infty \) and a weight \( w(x) \), the weighted norm of \( \omega \in W^{1,p}(\Omega, \Lambda^l) \) over \( \Omega \) is denoted by

\[
\|\omega\|_{W^{1,p}(\Omega, \Sigma), w^\alpha} = |\omega|_{W^{1,p}(\Omega, \Sigma), w^\alpha} = \text{diam}(\Omega)^{-1} \|\omega\|_{p, \Omega, w^\alpha} + \|\nabla \omega\|_{p, \Omega, w^\alpha},
\]

where \( \alpha \) is a real number.

We denote the exterior derivative by \( d : D'(\Omega, \Lambda^l) \rightarrow D'(\Omega, \Lambda^{l+1}) \) for \( l = 0, 1, \ldots, n \), and the codifferential operator \( d^* : D'(\Omega, \Lambda^{l+1}) \rightarrow D'(\Omega, \Lambda^l) \) is given by \( d^* = (-1)^{l+1} \star d \star \) on \( D'(\Omega, \Lambda^{l+1}) \), \( l = 0, 1, \ldots, n \).

In 1993, T. Iwaniec and A. Lutoborski proved the following result in [8]. Let \( D \subset \mathbb{R}^n \) be a bounded, convex domain. For each \( y \in D \) there is a linear operator \( K_y : C^\infty(D, \Lambda^l) \rightarrow C^\infty(D, \Lambda^{l-1}) \) defined by \( (K_y \omega)(x; \xi_1, \ldots, \xi_{l-1}) = \int_0^1 t^{l-1} \partial^\omega (tx + y - ty; x - y, \xi_1, \ldots, \xi_{l-1}) \, dt \) and the decomposition: \( \omega = d(K_y \omega) + K_y (d \omega) \). T. Iwaniec and A. Lutoborski introduce a homotopy operator \( T : C^\infty(D, \Lambda^l) \rightarrow C^\infty(D, \Lambda^{l-1}) \) by averaging \( K_y \) over all points \( y \) in \( D \),

\[
T \omega = \int_D \varphi(y) K_y \omega \, dy,
\]
where \( \varphi \in C_0^\infty (D) \) is normalized by \( \int_D \varphi(y) \, dy = 1 \). By substituting \( z = tx + y - ty \), (1.2) reduces to

\[
T \omega(x, \xi) = \int_D \omega(z, \xi(z, x-z), \xi) \, dz, \tag{1.3}
\]

where the vector function \( \zeta : D \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is given by \( \zeta(z, h) = h \int_0^\infty s^{\lambda-1} (1 + s)^{\alpha-1} \varphi(z - sh) \, ds \).

Proposition 4.1 in [8] says that the integral (1.3) defines a bounded operator \( T : L^1(D, \wedge^l) \rightarrow W^{1,s}(D, \wedge^{l-1}) \), \( l = 1, 2, \ldots, n \) with norm estimated by \( \|Tu\|_{W^{1,s}(D)} \leq C |D| \|u\|_{s,D} \).

The imbedding inequalities have been playing important roles in developing the \( L^p \) theory of differential forms. Some new results about harmonic forms have been established in the study of the \( A \)-harmonic equation

\[
d^* A(x, d \omega) = 0 \tag{1.4}
\]

for differential forms, where \( A : \Omega \times \wedge^l(\mathbb{R}^n) \rightarrow \wedge^l(\mathbb{R}^n) \) satisfies the following conditions:

\[
|A(x, \xi)| \leq a |\xi|^{p-1} \quad \text{and} \quad \langle A(x, \xi), \xi \rangle \geq |\xi|^p \quad (1.5)
\]

for almost every \( x \in \Omega \) and all \( \xi \in \wedge^l(\mathbb{R}^n) \). Here \( a > 0 \) is a constant and \( 1 < p < \infty \) is a fixed exponent associated with (1.4). A solution to (1.4) is an element of the Sobolev space \( W^{1,p}_{\text{loc}}(\Omega, \wedge^{l-1}) \) such that \( \int_\Omega \langle A(x, d \omega), d \varphi \rangle = 0 \) for all \( \varphi \in W^{1,p}_{\text{loc}}(\Omega, \wedge^{l-1}) \) with compact support.

Definition 1.1. We call \( u \) an \( A \)-harmonic tensor in \( \Omega \) if \( u \) satisfies the \( A \)-harmonic equation (1.4) in \( \Omega \).

2. Statements of the main results

Definition 2.1. We say a pair of weights \((w_1(x), w_2(x))\) satisfies \( A_r(\lambda, \Omega) \)-conditions for some \( r > 1 \) and \( \lambda > 0 \), write \((w_1, w_2) \in A_r(\lambda, \Omega)\), if \( w_1(x), w_2(x) > 0 \) a.e., and

\[
\sup_B \left( \frac{1}{|B|} \int_B w_1^{\alpha}(x) \, dx \right) \left( \frac{1}{|B|} \int_B \left( \frac{1}{w_2(x)} \right)^{1/(r-1)} \, dx \right)^{r-1} < \infty \tag{2.1}
\]

for all balls \( B \subset \Omega \).

Note that if we choose \( w_1 = w_2 \), the \( A_r(\lambda, \Omega) \)-weight reduces to the weight introduced by S. Ding and P. Shi in [4]. Also, if we set \( w_1 = w_2 \) and \( \lambda = 1 \), it becomes the usual \( A_r(\Omega) \)-weight. We will need the following generalized Hölder inequality appearing in [6].

Lemma 2.2. Let \( 0 < \alpha < \infty, 0 < \beta < \infty \) and \( s^{-1} = \alpha^{-1} + \beta^{-1} \). If \( f \) and \( g \) are measurable functions on \( \mathbb{R}^n \), then

\[
\|fg\|_{s,\Omega} \leq \|f\|_{\alpha,\Omega} \cdot \|g\|_{\beta,\Omega} \tag{2.2}
\]

for any \( \Omega \subset \mathbb{R}^n \).
From results appearing in [8], we have the following lemma.

**Lemma 2.3.** Let \( u \in L^s_{\text{loc}}(B, \wedge^l), l = 1, 2, \ldots, n, 1 < s < \infty \), be a differential form in a ball \( B \subset \mathbb{R}^n \). Then

\[
\| \nabla (T u) \|_{s, B} \leq C |B| \| u \|_{s, B},
\]

(2.3)

\[
\| T u \|_{s, B} \leq C |B| \text{diam}(B) \| u \|_{s, B}.
\]

(2.4)

We also need the following weak reverse Hölder inequality that appears in [10].

**Lemma 2.4.** Let \( u \) be an \( A \)-harmonic tensor in \( \Omega \), \( \rho > 1 \) and \( 0 < s, t < \infty \). Then there exists a constant \( C \), independent of \( u \), such that

\[
\| u \|_{s, B} \leq C |B|^{(t-s)/st} \| u \|_{t, \rho B},
\]

(2.5)

for all balls or cubes \( B \) with \( \rho B \subset \Omega \).

We need the following properties of the Whitney covers appearing in [10].

**Lemma 2.5.** Each \( \Omega \) has a modified Whitney cover of cubes \( V = \{ Q_i \} \) such that \( \bigcup Q_i = \Omega \), \( \sum_{Q \in V} \chi_{\sqrt{2} Q} \leq N \chi_{\Omega} \) for all \( x \in \mathbb{R}^n \) and some \( N > 1 \).

**Theorem 2.6.** Let \( u \in L^s_{\text{loc}}(\Omega, \wedge^l), l = 1, 2, \ldots, n, 1 < s < \infty \), be a differential form satisfying (1.4) in a bounded domain \( \Omega \subset \mathbb{R}^n \) and \( T \) be the homotopy operator defined in (1.2). Assume that \( \rho > 1 \) and \((w_1(x), w_2(x)) \in A_r(\lambda, \Omega)\) for some \( r > 1 \) and \( \lambda > 1 \). Then, there exists a constant \( C \), independent of \( u \), such that

\[
\left( \int_B |\nabla (T u)|^s w_1^\alpha (x) \, dx \right)^{1/s} \leq C |B| \left( \int_{\rho B} |u|^t w_2^{\alpha / \lambda} (x) \, dx \right)^{1/t}
\]

(2.6)

for any real number \( \alpha \) with \( 0 < \alpha < 1 \).

**Theorem 2.7.** Let \( u \in L^s_{\text{loc}}(\Omega, \wedge^l), l = 1, 2, \ldots, n, 1 < s < \infty \), be a differential form satisfying (1.4) in a bounded domain \( \Omega \subset \mathbb{R}^n \) such that \( du \in L^r_{\text{loc}}(\Omega, \wedge^{l+1}) \) and \( T \) be the homotopy operator defined in (1.2). Assume that \( \rho > 1 \) and \((w_1(x), w_2(x)) \in A_r(\lambda, \Omega)\) for some \( r > 1 \) and \( \lambda > 1 \). Then, there exists a constant \( C \), independent of \( u \), such that

\[
\| T u \|_{W^{1, s}(B), \wedge^l} \leq C |B| \| u \|_{s, \rho B, \wedge^{l+1}}
\]

(2.7)

for all balls \( B \) with \( \rho B \subset \Omega \) and any real number \( \alpha \) with \( 0 < \alpha < 1 \).

**Theorem 2.8.** Let \( u \in L^s(D, \wedge^l), l = 1, 2, \ldots, n, 1 < s < \infty \), be a differential form satisfying (1.4) in a bounded, convex domain \( D \subset \mathbb{R}^n \) and \( T \) be the homotopy operator defined by (1.3). Assume that \((w_1(x), w_2(x)) \in A_r(\lambda, D)\) for some \( r > 1 \) and \( \lambda > 1 \). Then, there exists a constant \( C \), independent of \( u \), such that
\[
\begin{align*}
\|\nabla (Tu)\|_{s,D,w_1^\alpha(x)} & \leq C \|u\|_{s,D,w_2^\alpha(x)}, \\
\|Tu\|_{W^{1,s}(D),w_1^\alpha(x)} & \leq C \|u\|_{s,D,w_2^\alpha(x)}
\end{align*}
\]
for any real number \(\alpha\) with \(0 < \alpha < 1\).

### 3. The proofs of the main results

In this section, we prove our theorems presented in the previous section.

**Proof of Theorem 2.6.** Let \(t = s\lambda/(\lambda - \alpha)\), using Lemma 2.2 yields

\[
\left( \int_B |\nabla (Tu)|^t w_1^\alpha \, dx \right)^{1/t} = \left( \int_B \left( |\nabla (Tu)| w_1^{\alpha/(t-1)} \right)^{t/(t-1)} \, dx \right)^{1/t} \leq \|\nabla (Tu)\|_{l,B} \left( \int_B w_1^{\alpha/(t-1)} \, dx \right)^{1/(t-1)} = \|\nabla (Tu)\|_{l,B} \left( \int_B w_1^{\alpha} \, dx \right)^{\alpha/s\lambda}.
\]

(3.1)

From Lemma 2.3, we obtain

\[
\|\nabla (Tu)\|_{l,B} \leq C_1 |B| \|u\|_{l,B}.
\]

(3.2)

Choose \(m = s\lambda/(\lambda + \alpha(r-1))\), then \(1/m = 1/s + \alpha(r-1)/s\lambda\), and \(m < s\). Substituting (3.2) into (3.1) and using Lemma 2.4, we find that

\[
\left( \int_B |\nabla (Tu)|^t w_1^\alpha \, dx \right)^{1/t} \leq C_1 |B| \|u\|_{l,B} \left( \int_B w_1^{\alpha} \, dx \right)^{\alpha/s\lambda} \leq C_2 |B| |B|^{(m-1)/mt} \|u\|_{m,\rho_B} \left( \int_B w_1^{\alpha} \, dx \right)^{\alpha/s\lambda}.
\]

Using Lemma 2.2 with \(1/m = 1/s + (s - m)/sm\), we have

\[
\|u\|_{m,\rho_B} = \left( \int_{\rho_B} |u|^m \, dx \right)^{1/m} = \left( \int_{\rho_B} \left( |u|^\alpha w_2^{\alpha/s\lambda} - |u|^{\alpha/s\lambda} \right)^m \, dx \right)^{1/m} \leq \left( \int_{\rho_B} |u|^\alpha w_2^{\alpha/s\lambda} \, dx \right)^{1/s} \left( \int_{\rho_B} \frac{1}{w_2^\alpha} \, dx \right)^{\alpha/s\lambda} \leq \left( \int_{\rho_B} |u|^\alpha w_2^{\alpha/s\lambda} \, dx \right)^{1/s} \left( \int_{\rho_B} \frac{1}{w_2^\alpha} \, dx \right)^{1/(r-1)} \leq C_3 |B| |B|^{(m-1)/mt} \|u\|_{m,\rho_B} \left( \int_B w_1^{\alpha} \, dx \right)^{\alpha/s\lambda}.
\]

(3.4)
for all balls $B$ with $\rho B \subset \Omega$. Substituting (3.4) into (3.3), we obtain

$$
\left( \int_B |\nabla (Tu)|^s w_1^{\alpha} \, dx \right)^{1/s} \leq C_2 |B|^{(m-t)/mt} \left( \int_{\rho B} |u|^t w_2^{a/\lambda} \, dx \right)^{1/s}
\times \left( \int_B w_1^{\alpha/\lambda} \, dx \right)^{\alpha/\lambda} \left( \int_{\rho B} \left( \frac{1}{w_2} \right)^{1/(r-1)} \, dx \right)^{a/(r-1)/s\lambda}.
$$

(3.5)

Since $(w_1, w_2) \in A_r(\lambda, \Omega)$, we find that

$$
\|u_1\|^{\alpha/s\lambda}_{1, B} \cdot \|1/w_2\|^{\alpha/s\lambda}_{1/(r-1), \rho B} \leq \left( \int_{\rho B} w_1^{\alpha/\lambda} \, dx \right)^{\alpha/\lambda} \left( \int_{\rho B} \left( \frac{1}{w_2} \right)^{1/(r-1)} \, dx \right)^{a/(r-1)/s\lambda}
\leq \left( \int_{\rho B} \left( \frac{1}{|\rho B|} \int_{\rho B} w_1^{\alpha/\lambda} \, dx \right) \left( \frac{1}{|\rho B|} \int_{\rho B} \left( \frac{1}{w_2} \right)^{1/(r-1)} \, dx \right)^{(r-1)/\alpha/s\lambda} \right)^{\alpha/s\lambda}
\leq C_3 |B|^{ar/s\lambda}.
$$

(3.6)

Combining (3.6) and (3.5), we have

$$
\left( \int_B |\nabla (Tu)|^s w_1^{\alpha} \, dx \right)^{1/s} \leq C_2 |B|^{(m-t)/mt} \cdot C_3 |B|^{ar/s\lambda} \|u\|_{s, \rho B, w_2^{a/\lambda}}
\leq C_4 |B| \left( \int_{\rho B} |u|^t w_2^{a/\lambda} \, dx \right)^{1/s}.
$$

(3.7)

for all balls $B$ with $\rho B \subset \Omega$. We have proved that (2.6) is true if $0 < \alpha < 1$. □

Using the method similar to the proof of Theorem 2.6, we obtain

$$
\|Tu\|_{s, B, w_2^{\alpha}} \leq C |B| \text{diam}(B) \|u\|_{s, \rho B, w_2^{a/\lambda}},
$$

(3.8)

where $\alpha$ is any real number with $0 < \alpha < 1$ and $\rho > 1$.  

Proof of Theorem 2.7. From (1.1), (2.6) and (3.8), we have

\[ \| Tu \|_{W^{1,s}(B),w_1} = \text{diam}(B)^{-1} \| Tu \|_{s,B,w_1} + \| \nabla(Tu) \|_{s,B,w_1} \]
\[ \leq \text{diam}(B)^{-1} \cdot C_1|B| \text{diam}(B) \| u \|_{s,\rho B,w_2^{1/\lambda}} + C_2|B|\| u \|_{s,\rho B,w_2^{1/\lambda}} \]
\[ \leq C_1|B|\| u \|_{s,\rho B,w_2^{1/\lambda}} + C_2|B|\| u \|_{s,\rho B,w_2^{1/\lambda}} \leq C_3|B|\| u \|_{s,\rho B,w_2^{1/\lambda}}, \]

which is equivalent to (2.7). The proof of Theorem 2.7 is completed. □

Note that the parameter \( \alpha \) in both of Theorems 2.6 and 2.7 is any real number with \( 0 < \alpha < 1 \). Therefore, we can choose \( \alpha \) to be different values to obtain the required versions of the two-weight imbedding inequality. For example, set \( \alpha = \frac{1}{r} \) in Theorem 2.6. Then, inequality (2.6) becomes

\[ \left( \int_B |\nabla(Tu)|^s w_1^{1/r} \, dx \right)^{1/s} \leq C|B| \left( \int_{\rho B} |u|^s w_2^{1/r} \, dx \right)^{1/s}. \]  \hspace{1cm} (3.9)

If we choose \( \alpha = 1/s \) in Theorem 2.6, then (2.6) reduces to

\[ \left( \int_B |\nabla(Tu)|^s w_1^{1/s} \, dx \right)^{1/s} \leq C|B| \left( \int_{\rho B} |u|^s w_2^{1/s} \, dx \right)^{1/s}. \]  \hspace{1cm} (3.10)

If we choose \( \alpha = \frac{\lambda}{s+\lambda} \) in Theorem 2.6, then \( 0 < \alpha < 1 \). Thus, (2.6) reduces to the following version:

\[ \left( \int_B |\nabla(Tu)|^s w_1^{\lambda/(s+\lambda)} \, dx \right)^{1/s} \leq C|B| \left( \int_{\rho B} |u|^s w_2^{1/\lambda} \, dx \right)^{1/s}. \]  \hspace{1cm} (3.11)

If we choose \( \alpha = \frac{\lambda}{1+\lambda} \) in Theorem 2.6, then (2.6) becomes

\[ \left( \int_B |\nabla(Tu)|^s w_1^{\lambda/(1+\lambda)} \, dx \right)^{1/s} \leq C|B| \left( \int_{\rho B} |u|^s w_2^{1/(1+\lambda)} \, dx \right)^{1/s}. \]  \hspace{1cm} (3.12)

Remark. Choosing \( \alpha \) to be some special values in Theorem 2.7, we shall have some similar results. Considering the length of the paper, we do not list these similar results here.

Finally, we use Lemma 2.5 to prove the following global \( A_r(\lambda,D) \)-weighted imbedding inequality in a bounded domain \( D \) for differential forms.

Proof of Theorem 2.8. Using (2.6) and Lemma 2.5, we find that

\[ \| \nabla(Tu) \|_{s,D,w_1} = \left( \int_D |\nabla(Tu)|^s w_1 \, dx \right)^{1/s} \leq \sum_{Q \in V} C_1|Q| \left( \int_{\rho Q} |u|^s w_2^{1/\lambda} \, dx \right)^{1/s}. \]
\[ \leq C_1 |D| \sum_{Q \in V} \left( \int_{\rho Q} |u|^s w_2^{\alpha/\lambda} \, dx \right)^{1/s} \leq C_1 |D| \sum_{Q \in V} \left( \int_{D} |u|^s w_2^{\alpha/\lambda} \, dx \right)^{1/s} \]
\[ \leq C_2 \left( \int_{D} |u|^s w_2^{\alpha/\lambda} \, dx \right)^{1/s} = C_2 \|u\|_{s,D,w_2^{\alpha/\lambda}} \]  
(3.13)
since \( D \) is bounded. We have proved that (2.8) holds. Similarly, using Lemma 2.5 and (3.8), we obtain
\[ \|Tu\|_{s,D,w_1^{\alpha}} \leq C_3 \text{diam}(D) \|u\|_{s,D,w_2^{\alpha/\lambda}}. \]  
(3.14)
Combining (1.1), (3.13) and (3.14), we have
\[ \|Tu\|_{W^{1,s}(D),w_1^{\alpha}} = \text{diam}(D)^{-1} \|Tu\|_{s,D,w_1^{\alpha}} + \|\nabla(Tu)\|_{s,D,w_1^{\alpha}} \]
\[ \leq C_3 \|u\|_{s,D,w_2^{\alpha/\lambda}} + C_2 \|u\|_{s,D,w_2^{\alpha/\lambda}} \leq C_4 \|u\|_{s,D,w_2^{\alpha/\lambda}}, \]
which says that (2.9) holds. The proof of Theorem 2.8 has been completed. \( \square \)

**Remark.** Choosing \( \alpha \) to be some special values in (2.8) and (2.9), we will have some global results similar to the local case.

### 4. Applications

The study of the \( A \)-harmonic equation is closely related to the theory of quasiconformal and quasiregular mappings. Many interesting results of solutions to the \( A \)-harmonic equation and their applications in fields such as quasiconformal mappings and the theory of elasticity have been found, see [2–5,10,14]. In fact, some properties of the solutions to the \( A \)-harmonic equation are extensions of quasiconformal and quasiregular mappings in \( \mathbb{R}^n \).

It is well known that if \( f(x) = (f^1, f^2, \ldots, f^n) \) is \( K \)-quasiregular in \( \mathbb{R}^n \), then
\[ u = f^1 df^1 \wedge df^2 \wedge \cdots \wedge df^{l-1} \]  
(4.1)
is a solution of the \( A \)-harmonic equation (1.4) with a fixed exponent \( p = n/l, l = 1, 2, \ldots, n-1 \), where \( A \) is some operator satisfying (1.5). By Theorem 2.8, we conclude that there exists a constant \( C \), independent of \( f \), such that
\[ \|\nabla(T(f^1 df^1 \wedge df^2 \wedge \cdots \wedge df^{l-1}))\|_{s,D,w_1^{\alpha}(x)} \]
\[ \leq C \|f^1 df^1 \wedge df^2 \wedge \cdots \wedge df^{l-1}\|_{s,D,w_2^{\alpha/\lambda}(x)}, \]  
(4.2)
\[ \|T(f^1 df^1 \wedge df^2 \wedge \cdots \wedge df^{l-1})\|_{W^{1,r}(D),w_1^{\alpha}(x)} \]
\[ \leq C \|f^1 df^1 \wedge df^2 \wedge \cdots \wedge df^{l-1}\|_{s,D,w_2^{\alpha/\lambda}(x)} \]  
(4.3)
for any real number \( \alpha \) with \( 0 < \alpha < 1 \) and the weights \( (w_1(x), w_2(x)) \in A_r(\lambda, D) \) with \( r > 1, \lambda > 1 \).
Next, we explore some applications to the $p$-harmonic tensors. Let $A : Ω × Λ^l(ℝ^n) → Λ^l(ℝ^n)$ be an operator defined by

$$A(x, ξ) = ξ |ξ|^{p-2}.$$ 

Then, $A$ satisfies two conditions in (1.5) and the $A$-harmonic equation (1.4) reduces to the following $p$-harmonic equation (or $p$-Laplace equation):

$$d^*(du|du|^{p-2}) = 0.$$  (4.4) 

A differential form $u ∈ D'(D, Λ^l), l = 0, 1, \ldots, n$, is called a $p$-harmonic tensor if $u$ satisfies (4.4). Hence, if $u$ is a $p$-harmonic tensor, inequalities (2.6), (2.7), (2.8) and (2.9) hold, respectively.

Finally, we study applications to the harmonic functions. If $u ∈ D'(D, Λ^0)$, that is, $u$ is a differentiable function in $D$, then the $A$-harmonic equation (1.4) reduces to

$$\text{div} A(x, \nabla u) = 0$$  (4.5)

and the $p$-harmonic equation (4.4) becomes

$$\text{div}(\nabla u|\nabla u|^{p-2}) = 0.$$  (4.6)

The solutions of (4.5) and (4.6) are called $A$-harmonic functions and $p$-harmonic functions, respectively [7]. Also, if we choose $p = 2$ in (4.6), Eq. (4.6) becomes the traditional harmonic equation

$$Δu = 0.$$ 

From above discussion, we know that our main results, inequalities (2.6), (2.7), (2.8) and (2.9), still hold for the $A$-harmonic functions, the $p$-harmonic functions and the usual harmonic functions, respectively.

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References