Existence results for fractional integrodifferential equations with nonlocal condition via resolvent operators

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ABSTRACT

In this paper, we prove the existence of solutions of fractional integrodifferential equations by using the resolvent operators and fixed point theorem. An example is given to illustrate the abstract results.

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1. Introduction

In recent years, considerable interest in fractional calculus has been stimulated by the applications it finds in numerical analysis and different areas of applied sciences like physics and engineering [1–4]. Fractal phenomena often can be centered in the field of linear viscoelasticity. One of the major advantages of fractional calculus is that it can be considered as a super set of integer-order calculus. Thus, fractional calculus has the potential to accomplish what integer-order calculus cannot. The interested reader in this topic can consult the excellent books [5,6] and the most prominent encyclopedic treatise by Samko et al. [7]. Mathematical modeling of time fractional reaction–diffusion systems was established by Gafiychuk et al. [8] and Nakagawa et al. [9] published an overview of mathematical analysis for fractional diffusion equations with new mathematical aspects.

In fact a fractional differential equation is considered as an alternative model to a nonlinear differential equation [10]; Metzler and Klafter [11,12] have discussed in detail about the recent developments in the description of anomalous transport and the random walk’s guide to anomalous diffusion using the fractional dynamics approach. The nonlocal Cauchy problem for an abstract fractional evolution equation was discussed in [13] where as in [14,15] the authors have studied the existence of solutions of fractional impulsive evolution equations and integrodifferential equations in Banach spaces by using fixed point techniques. Hernández et al. [16] investigated the recent developments in the theory of abstract fractional differential equations in which the resolvent operator plays a key role in proving their existence results. Numerical experiments for fractional models on population dynamics are examined in [17] and some of the applications of nonlinear fractional differential equations with their approximations have been found in [18]. In this paper we study the existence of solutions of abstract fractional integrodifferential equations with a nonlocal condition by using a variant type of Krasnoselskii fixed point theorem.

2. Preliminaries

In this section we include some definitions and properties needed to establish our results. Let \( C(J; X) \) denote the space of all continuous functions from \( J := [0, b] \) into a Banach space \( X \) with the supnorm denoted by \( \| \cdot \|_{C(J; X)} \). The notation \( X_A \) denotes the domain of \( A \) endowed with the graph norm \( \| x \|_A = \| x \| + \| Ax \| \). In addition, \( B_r(x, X) \) represents the closed ball with center at \( x \) and radius \( r \) in \( X \).
**Definition 2.1.** The Riemann–Liouville fractional integral operator of order \( \alpha > 0 \), of function \( f \in L_1(\mathbb{R}_+) \) is defined as

\[
I_{0+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,
\]

where \( \Gamma(\cdot) \) is the Euler gamma function.

**Definition 2.2.** The Riemann–Liouville fractional derivative of order \( \alpha > 0, n - 1 < \alpha < n, n \in \mathbb{N} \), is defined as

\[
D_{0+}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} f(s) ds,
\]

where the function \( f(t) \) have absolutely continuous derivatives up to order \( (n-1) \).

The Riemann–Liouville fractional derivatives have singularity at zero. Physical interpretations of the Riemann–Liouville fractional derivatives are more complicated than Caputo fractional derivatives. But the Riemann–Liouville fractional derivatives naturally appear for real physical systems in electrodynamics. To overcome this difficulty, Caputo [19] introduced the fractional derivative in the following way.

**Definition 2.3.** The Caputo fractional derivative of order \( \alpha > 0, n - 1 < \alpha < n \), is defined as

\[
^C D_{0+}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^n(s) ds,
\]

where the function \( f(t) \) have absolutely continuous derivatives up to order \( (n-1) \). If \( 0 < \alpha < 1 \), then

\[
^C D_{0+}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t f'(s) (t-s)^{1-\alpha} ds.
\]

We observe from the above that both the Riemann–Liouville and the Caputo fractional operators do not possess neither semigroup nor commutative properties, which are inherent to the derivatives on integer order. For simplicity of notations, we shall take \( ^C D_{0+}^\alpha x(t) \), \( l^\alpha_{0+} x(t) \) as \( D^\alpha x(t) \), \( l^\alpha_{0+} x(t) \) operators.

**Properties 2.4.** For \( \alpha, \beta > 0 \) and \( f \) as a suitable function (for instance [2,7]) we have

1. \( l^\alpha_{0+} I_{0+}^\beta f(t) = I_{0+}^{\alpha+\beta} f(t) \)
2. \( l^\alpha_{0+} f(t) = I_{0+}^\alpha l^\beta_{0+} f(t) \)
3. \( l^\alpha_{0+} (f(t) + g(t)) = l^\alpha_{0+} f(t) + l^\alpha_{0+} g(t) \)
4. \( f(t) + ^C D_{0+}^\alpha f(t) = f(t) - f(0), \quad 0 < \alpha < 1 \)
5. \( ^C D_{0+}^\alpha l^\alpha_{0+} f(t) = f(t) \)
6. \( ^C D_{0+}^\alpha f(t) = I_{0+}^{1-\alpha} D^\alpha f(t) = I_{0+}^{1-\alpha} f'(t), \quad 0 < \alpha < 1 \)
7. \( ^C D_{0+}^\alpha l^\alpha_{0+} f(t) \neq ^C D_{0+}^\alpha f(t) \)
8. \( ^C D_{0+}^\alpha l^\alpha_{0+} f(t) \neq ^C D_{0+}^\beta l^\alpha_{0+} f(t) \)

Consider the fractional differential equation

\[
D^\alpha u(t) = Au(t) + f(t), \quad t \in J,
\]

where \( A \) is a closed linear unbounded operator in \( X \) and \( f \in C(J, X) \). Eq. (2.1) is equivalent to the following integral equation

\[
u(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t A u(s) (t-s)^{1-q} ds + \frac{1}{\Gamma(q)} \int_0^t f(s) (t-s)^{1-q} ds, \quad t \in J.
\]

This equation can be written in the following form of integral equation

\[
u(t) = h(t) + \frac{1}{\Gamma(q)} \int_0^t A u(s) (t-s)^{1-q} ds, \quad t \geq 0,
\]

where \( h(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t f(s) (t-s)^{1-q} ds \). Let us assume that the integral equation (2.3) has an associated resolvent operator \( S(t)_{t \geq 0} \) on \( X \).

Next we give examples where the exact solution of (2.1) and the integral representation (2.2) are the same.
Example 2.1. Consider the fractional differential equation (the derivative is in the Caputo sense) related to (2.3) and assume

\[ D^\alpha x(t) = x(t), \quad t \in [0, b], \]

\[ x(0) = 1, \quad 0 < \alpha < 1. \]

The corresponding integral solution of (2.4) is

\[ x(t) = 1 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds. \]  

The exact solution of (2.4) is (see [2]) given by

\[ x(t) = E_{\alpha,1}(t^\alpha) = \sum_{k=0}^{\infty} \frac{(t^\alpha)^k}{\Gamma(\alpha k + 1)} \]

where \( E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \). On expanding

\[ x(t) = \frac{(t^\alpha)^0}{\Gamma(\alpha + 1)} + \frac{(t^\alpha)^1}{\Gamma(2\alpha + 1)} + \frac{(t^\alpha)^2}{\Gamma(3\alpha + 1)} + \cdots = 1 + \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \cdots. \]

We want to show that both the solutions are same. For that, using (2.6) in (2.5), we get

\[
1 + \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \cdots = 1 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(1 + \frac{s^\alpha}{\Gamma(\alpha + 1)} + \frac{s^{2\alpha}}{\Gamma(2\alpha + 1)} + \cdots\right) ds
\]

\[
= 1 + \frac{1}{\Gamma(\alpha)} \left[\frac{-(t-s)^{\alpha-1}}{\alpha}\right]_0^t + \frac{1}{\Gamma(\alpha) \Gamma(\alpha + 1)} \int_0^t (t-s)^{\alpha-1} s^\alpha ds + \cdots
\]

\[
= 1 + \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha) \Gamma(\alpha + 1)} \int_0^t (t-s)^{\alpha-1} s^\alpha ds + \cdots. \]  

Next we prove that

\[
\frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} = \frac{1}{\Gamma(\alpha) \Gamma(\alpha + 1)} \int_0^t (t-s)^{\alpha-1} s^\alpha ds. \]  

The value of the integral

\[
\int_0^t (t-s)^{\alpha-1} s^\alpha ds = t^{\alpha-1} \int_0^t \left(1 - \frac{s}{t}\right)^{\alpha-1} s^\alpha ds
\]

\[
= t^{\alpha-1} \int_0^1 (1-x)^{\alpha-1} x^\alpha dx
\]

\[
= t^{\alpha-1} \int_0^1 (1-x)^{\alpha-1} t^{\alpha+1} x^\alpha dx
\]

\[
= t^{2\alpha} \int_0^1 x^{\alpha+1-1} t^{\alpha+1} x^\alpha dx
\]

\[
= t^{2\alpha} \frac{\Gamma(\alpha+1) \Gamma(\alpha)}{\Gamma(2\alpha+1)}. \]

From the above we get,

\[
\frac{1}{\Gamma(\alpha + 1) \Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^\alpha ds = \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)}. \]

Thus, we have proved (2.8). Similarly, we can evaluate the other terms on the right hand side of the series in (2.7) and it is equal to

\[
1 + \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \cdots. \]

Hence the solution representations (2.5) and (2.6) for (2.4) are the same. Note that the study on the solvability of (2.4) is frequently realized by associating it to the integral equation (2.5) [20].
Example 2.2. Consider the fractional differential equation (2.3) with \( X = R, f(t) = t, \ x(0) = 1, \ A \equiv 1, \) then
\[
D^\alpha x(t) = x(t) + t, \quad t \in [0, b].
\]
where \( 0 < \alpha < 1. \) The corresponding integral solution of (2.9) is
\[
x(t) = 1 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{x(s)}{(t-s)^{1-\alpha}} ds + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{s}{(t-s)^{1-\alpha}} ds.
\]
The exact solution of (2.9) is
\[
x(t) = E_{\alpha,1}(t^\alpha) + \int_0^t (t-s)^{\alpha-1} [E_{\alpha,\alpha}(t-s)^\alpha] ds.
\]
On expanding
\[
x(t) = 1 + \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \cdots + \int_0^t (t-s)^{\alpha-1} \left( \frac{1}{\Gamma(\alpha)} + \frac{(t-s)^\alpha}{\Gamma(2\alpha)} + \frac{(t-s)^{2\alpha}}{\Gamma(3\alpha)} + \cdots \right) s ds.
\]
Using the similar procedure as in Example 2.1 with judicious integral evaluation, we can show that both the solutions (2.11) and (2.12) for (2.9) are the same.

Next we define the resolvent operator of the integral equation (2.3).

Definition 2.5 ([21], Definition 1.1.3). A one parameter family of bounded linear operators \( S(t)_{t \geq 0} \) on \( X \) is called a resolvent operator for (2.3) if the following conditions hold:

(i) \( S(\cdot)x \in C([0, \infty); X) \) and \( S(0)x = x \) for all \( x \in X, \)

(ii) \( S(t)D(A) \subset D(A) \) and \( AS(t)x = S(t)Ax \) for all \( x \in D(A) \) and every \( t \geq 0, \)

(iii) for every \( x \in D(A) \) and \( t \geq 0, \)
\[
S(t)x = x + \frac{1}{\Gamma(q)} \int_0^t \frac{AS(s)x}{(t-s)^{1-q}} ds.
\]

In this paper we assume that the resolvent operator \( S(t)_{t \geq 0} \) is analytic ([21], Chapter 2) and there exists a function \( \varphi_A \) in \( L^1_{\text{loc}}([0, \infty); \mathbb{R}^+) \) such that
\[
\|S'(t)x\| \leq \varphi_A(t)\|x\|_{\mathcal{X}_A}, \quad \text{for all } t > 0.
\]

We have the following concept of solution using Definition 1.1.1 in [21].

Definition 2.6. A function \( u \in C(J; X) \) is called a mild solution of the integral equation (2.3) on \( J \) if \( \int_0^t (t-s)^{\alpha-1}u(s)ds \in D(A) \) for all \( t \in J, \ h(t) \in C(J, X) \) and
\[
u(t) = \frac{A}{\Gamma(q)} \int_0^t \frac{u(s)}{(t-s)^{1-q}} ds + h(t), \quad \forall t \in J.
\]
The next result follows from ([21], Proposition I.1.2, Corollary II.2.6 and Proposition I.1.3] which plays a key role in the subsequent sections of this work.

Lemma 2.1. Under the above conditions the following properties are valid:

(i) If \( u(\cdot) \) is a mild solution of (2.3), on \( J, \) then the function \( t \rightarrow \int_0^t S(t-s)h(s)ds \) is continuously differentiable on \( J, \) and
\[
u(t) = \frac{d}{dt} \int_0^t S(t-s)h(s)ds, \quad \forall t \in J.
\]

(ii) If \( h \in C^\beta(J; X) \) for some \( \beta \in (0, 1), \) then the function defined by
\[
u(t) = S(t)(h(t) - h(0)) + \int_0^t S'(t-s)[h(s) - h(t)]ds + S(t)h(0), \quad t \in J,
\]
is a mild solution of (2.3), on \( J. \)
If $h \in C(J; X_{\lambda})$, then the function $u : J \rightarrow X$ defined by
\[ u(t) = \int_0^t S(t - s)h(s)ds + h(t), \quad t \in J, \tag{2.16} \]
is a mild solution of (2.3), on $J$.

In this paper, we study the existence of “mild” solutions for a class of abstract fractional integrodifferential equation of the form
\[ D^q u(t) + e(t, u(t))) = Au(t) + f(t, u(t), \int_0^t k(t, s, u(s))ds), \quad t \in J \tag{2.17} \]
u(0) + g(u) = u_0 \tag{2.18}
where $D^q$ is the Caputo fractional derivative of order $0 < q < 1$. $A$ is a closed linear unbounded operator in a Banach space $X$ with dense domain $D(A)$, $u_0 \in X$ and $f : J \times X \rightarrow X$, $e : J \times X \rightarrow X$, $k : J \times J \times X \rightarrow X$, $g : C(J; X) \rightarrow X$ are continuous. Here $J = \{(t, s) : 0 \leq s \leq t \leq b\}$. For brevity let us take $Ku(t) = \int_0^t k(t, s, u(s))ds$.

### 3. Existence of solutions

Now we introduce the concept of mild solution for Eqs. (2.17)–(2.18). This equation is equivalent to the following integral equation
\[ u(t) = u_0 - g(u) + e(0, u_0) - e(t, u(t)) + \frac{1}{\Gamma(q)} \int_0^t \frac{Au(s)}{(t - s)^{1-q}}ds + \frac{1}{\Gamma(q)} \int_0^t \frac{f(s, u(s), Ku(s))}{(t - s)^{1-q}}ds, \quad \text{for all } t \in J. \tag{3.1} \]
Motivated by the Lemma 2.1 and the above representation (3.1), we introduce the concept of mild solution.

**Definition 3.1.** A function $u \in C(J; X)$ is said to be a mild solution of (2.17)–(2.18), on $J$ if $\int_0^t \frac{u(i)}{(t-s)^{1-q}}ds \in D(A)$ for all $t \in J$ and satisfies the integral equation (3.1).

Suppose there exists a resolvent operator $(S(t))_{t \geq 0}$ which is differentiable and the functions $f$, $g$, $k$ and $e$ are continuous in $X_{\lambda}$ then
\[ u(t) = u_0 - g(u) + e(0, u_0) - e(t, u(t)) + \frac{1}{\Gamma(q)} \int_0^t \frac{f(s, u(s), Ku(s))}{(t - s)^{1-q}}ds \\
+ \int_0^t S(t - s) \left( u_0 - g(u) + e(0, u_0) - e(s, u(s)) + \frac{1}{\Gamma(q)} \int_0^s \frac{f(\tau, u(\tau), Ku(\tau))}{(s - \tau)^{1-q}}d\tau \right)ds. \]
Assume the following conditions:

(H1) The function $f : J \times X^2 \rightarrow X_{\lambda}$ is completely continuous, there exists a constant $L_1 > 0$, such that
\[ \|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leq L_1(\|x_1 - x_2\| + \|y_1 - y_2\|), \quad \forall (t, x_1, y_1) \in J \times X^2, \ i = 1, 2. \]

(H2) The function $k : J \times X \rightarrow X_{\lambda}$ is continuous and there exists a constant $L_2 > 0$, such that
\[ \left\| \int_0^t [k(t, x_1) - k(t, x_2)]ds \right\| \leq L_2 \|x_1 - x_2\|, \quad \forall (t, x_1, x_2) \in J \times X, \ i = 1, 2. \]

(H3) There exists a constant $L_3 > 0$, of the function $e : J \times X \rightarrow X_{\lambda}$, such that
\[ \|e(t, x_1) - e(t, x_2)\| \leq L_3 \|x_1 - x_2\|, \quad \forall (t, x_1) \in J \times X, \ i = 1, 2. \]

(H4) There exists a constant $G > 0$, of the function $g : C(J; X) \rightarrow X_{\lambda}$, such that
\[ \|g(x_1) - g(x_2)\| \leq G \|x_1 - x_2\|, \quad \forall x_1, x_2 \in X, \ i = 1, 2. \]

(H5) $2(1 + \|\varphi_\lambda\|_{L_1^1})(1 + L_2)\gamma L_1 + G + L_3 \leq 1$.

For our convenience, let $\gamma = \frac{\|\varphi_\lambda\|_{L_1^1}}{\|\varphi_\lambda\|_{L_1^1}}$, $N = \max_{t \in J} f(t, 0, 0)$, $N^* = \max_{t \in J} e(t, 0)$, $N^{**} = \max_{t \in J} [\int_0^t k(t, s, 0)ds]$.

**Theorem 3.2.** Assume $u_0 \in D(A)$, $f$, $g$, $e$, $k$ satisfies the (H1)–(H5). Then there exists a mild solution of (2.17)–(2.18) on $J$.

**Proof.** First we transform the existence of solutions of (2.17)–(2.18) into a fixed point problem. For that, by considering Lemma 2.1(iii), we introduce the map $\Phi : C(J, X) \rightarrow C(J, X)$ by...
\[ \Phi u(t) = u_0 - g(u) + e(0, u_0) - e(t, u(t)) + \frac{1}{\Gamma(q)} \int_0^t \frac{f(s, u(s), Ku(s))}{(t-s)^{1-q}} ds + \int_0^t S'(t-s) \left( u_0 - g(u) + e(0, u_0) - e(s, u(s)) \right) ds \]

Now we decompose \( \Phi \) as \( \Phi_1 + \Phi_2 \) on \( B_r(0; C(J, X)) \) where

\[ \Phi_1 u(t) = u_0 - g(u) + e(0, u_0) - e(t, u(t)) + \frac{1}{\Gamma(q)} \int_0^t f(s, u(s), Ku(s)) (t-s)^{1-q} ds + \int_0^t S'(t-s) \left( u_0 - g(u) + e(0, u_0) - e(s, u(s)) \right) ds, \]
\[ \Phi_2 u(t) = \frac{1}{\Gamma(q)} \int_0^t f(s, u(s), Ku(s)) (t-s)^{1-q} ds + \int_0^t S'(t-s) \frac{1}{\Gamma(q)} \int_0^s f(\tau, u(\tau), Ku(\tau)) (s-\tau)^{1-q} d\tau ds. \]

Obviously \( h(t) = u_0 - g(u) + e(0, u_0) - e(t, u(t)) + \frac{1}{\Gamma(q)} \int_0^t f(s, u(s), Ku(s)) (t-s)^{1-q} ds \) is in \( C(J, X) \). Let \( Z = C(J, X) \) and \( B_r(0, Z) = \{ z \in Z : \|z\| \leq r \} \). Choose \( r \geq 2(1 + \|\varphi_h\|_1)(\|u_0\| + \|g(0)\| + \|e(0, u_0)\| + N^* + \gamma L_1(N^{**} + N)). \)

For any \( u, v \in Z \), we have

\[ \|\Phi_1 u(t) + \Phi_2 v(t)\| \leq \|u_0\| + \|g(u) - g(0)\| + \|g(0)\| + \|e(0, u_0)\| + \|e(t, u(t)) - e(t, 0)\| \]
\[ + \|e(0, t)\| + \frac{1}{\Gamma(q)} \int_0^t \|f(s, u(s), Ku(s)) - f(s, 0, 0)\| (t-s)^{1-q} ds + \int_0^t \|S'(t-s)\| \left( \|u_0\| + \|g(u) - g(0)\| + \|g(0)\| + \|e(0, u_0)\| + \|e(s, u(s)) - e(s, 0)\| \right) \]
\[ + \|e(s, 0)\| + \frac{1}{\Gamma(q)} \int_0^t \|f(\tau, u(\tau), Ku(\tau)) - f(\tau, 0, 0)\| (s-\tau)^{1-q} d\tau ds \]
\[ \leq \|u_0\| + Gr + \|g(0)\| + \|e(0, u_0)\| + L_3\|u(t)\| + N^* \]
\[ + \frac{L_1 b^q}{q \Gamma(q)} \left( \|u(s)\| + \|\int_0^t k(t, s, u(s)) ds\| + N \right) + \int_0^t \|S'(t-s)\| \left[ \|u_0\| + Gr + \|g(0)\| \right. \]
\[ + \|e(0, u_0)\| + L_3\|u(s)\| + N^* + \frac{L_1 b^q}{q \Gamma(q)} \left( \|u(s)\| + \|\int_0^t k(s, \tau, u(\tau)) d\tau\| \right) + N \] \]
\[ \leq \|u_0\| + Gr + \|g(0)\| + \|e(0, u_0)\| + L_3\|u(t)\| + N^* \]
\[ + \gamma L_1 \left( \|u(s)\| + \|\int_0^t [k(t, s, u(s)) - k(t, s, 0)] ds\| + \|\int_0^t k(t, s, 0) ds\| + N \right) \]
\[ + \int_0^t \|S'(t-s)\| \left[ \|u_0\| + Gr + \|g(0)\| + \|e(0, u_0)\| + L_3\|u(t)\| + N^* \right. \]
\[ + \gamma L_1 \left( \|u(s)\| + \|\int_0^t [k(s, \tau, u(\tau)) - k(s, \tau, 0)] d\tau\| + \|\int_0^t k(s, \tau, 0) d\tau\| \right) + N \] \]
\[ \leq \|u_0\| + Gr + \|g(0)\| + \|e(0, u_0)\| + L_3\|u(t)\| + N^* \]
\[ + \|\varphi_h\|_1 \left( \|u_0\| + Gr + \|g(0)\| + \|e(0, u_0)\| + L_3\|u(t)\| + N^* \right. \]
\[ + \gamma L_1 \left( \|u(s)\| + \|\int_0^t [k(s, \tau, u(\tau)) - k(s, \tau, 0)] d\tau\| + \|\int_0^t k(s, \tau, 0) d\tau\| \right) + N \] \]
\[ \leq (1 + \|\varphi_h\|_1) \left( \|u_0\| + Gr + \|g(0)\| + \|e(0, u_0)\| + L_3\|u(t)\| + N^* + \gamma L_1 (1 + L_2) \right. \]
\[ + \gamma L_1 (N^{**} + N) \] \]
\[ \leq r. \]

Thus \( \Phi \) maps \( B_r(0, Z) \) into itself and so \( \Phi_1 u + \Phi_2 v \in B_r \).

From the assumptions (H3) and (H4) we see that, for any \( u \in Z \),

\[ \int_0^t S'(t-s)(u_0 + g(u) + e(0, u_0) + e(s, u(s))) ds \leq \|\varphi_h\|_1 (\|u_0\| + Gr + g(0)\| + \|e(0, u_0)\| + L_3 r + N^*) \]

which implies that the function \( s \to S'(t-s)(u_0 + g(u) + e(0, u_0) + e(s, u(s))) \) is integrable on \( J \), for all \( t \in J \) and \( \Phi_1 u \in Z \).

Moreover, for \( u, v \in Z \) and \( t \in J \) we get

\[ \|\Phi_1 u(t) - \Phi_1 v(t)\| \leq \|g(u) - g(v)\| + \|e(t, u(t)) - e(t, v(t))\| \]
\[ + \int_0^t \|S'(t-s)\| \left( \|g(u) - g(v)\| + \|e(t, u(t)) - e(t, v(t))\| \right) ds \]
\[ \leq G\|u - v\| + L_3\|u - v\| + \|\varphi_h\|_1 (G\|u - v\| + L_3\|u - v\|) \]
\[ \leq (1 + \|\varphi_h\|_1)(G + L_3)(\|u - v\|). \]

By (H5), \( \Phi_1 \) is a contraction on \( B_r(0; Z) \).
Now we show that the operator $\Phi_2$ is completely continuous. Note that the function $s \rightarrow \int_0^t S'(t - s)\int_0^s \frac{\|f(s, u(s), Ku(s))\|}{(s-t)^{1-q}} \, dr \, ds$ is integrable from the assumption of $f(\cdot)$ and $k(\cdot)$ as shown above. First we show that $\Phi_2$ is uniformly bounded. For $t \in J$,

$$
\|\Phi_2 u(t)\| \leq \frac{1}{\Gamma(q)} \int_0^t \frac{\|f(s, u(s), Ku(s))\|}{(t-s)^{1-q}} \, ds + \int_0^t \|S'(t-s)\| \frac{1}{\Gamma(q)} \int_0^s \frac{\|f(\tau, u(\tau), Ku(\tau))\|}{(s-\tau)^{1-q}} \, d\tau \, ds
$$

$$
\leq \|\psi_k\|_1 (\gamma L_1 (1+L_2) + \gamma L_1 (N^{**} + N)).
$$

This shows that $\Phi_2$ is uniformly bounded.

Let $(u_n)$ be a sequence in $B_r(0; Z)$, such that $u_n \rightarrow u$ in $B_r(0; Z)$. Since the functions $f$ and $k$ are continuous,

$$
\|f(s, u_n(s), Ku_n(s)) - f(s, u(s), Ku(s))\| \rightarrow 0,
$$

as $n \rightarrow \infty$.

Now for each $t \in J$, we have

$$
\|\Phi_2 u_n(t) - \Phi_2 u(t)\| \leq \frac{1}{\Gamma(q)} \int_0^t \frac{\|f(s, u_n(s), Ku_n(s)) - f(s, u(s), Ku(s))\|}{(t-s)^{1-q}} \, ds
$$

$$
+ \int_0^t \|S'(t-s)\| \frac{1}{\Gamma(q)} \int_0^s \frac{\|f(\tau, u_n(\tau), Ku_n(\tau)) - f(\tau, u(\tau), Ku(\tau))\|}{(s-\tau)^{1-q}} \, d\tau \, ds
$$

$$
\rightarrow 0
$$

as $n \rightarrow \infty$.

From the above it is clear that $\Phi_2$ is continuous.

We need to prove that the set $\{\Phi_2 u(t): u \in B_r(0; Z)\}$ is relatively compact in $X$ for all $t \in J$. Obviously, $\{\Phi_2 u(0): u \in B_r(0; Z)\}$ is compact. Fix $t \in (0, b]$ and $u \in B_r(0; Z)$, define the operator $\Phi_2^e$ by

$$
\Phi_2^e u(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s, u(s), Ku(s))}{(t-s)^{1-q}} \, ds + \int_0^t S'(t-s) \frac{1}{\Gamma(q)} \int_0^s \frac{f(\tau, u(\tau), Ku(\tau))}{(s-\tau)^{1-q}} \, d\tau \, ds.
$$

Since by (H1), $f(\cdot)$ is completely continuous, the set $X_e = \{\Phi_2^e u(t): u \in B_r(0; Z)\}$ is precompact in $X$, for every $e > 0$, $0 < e < t$. Moreover, for every $u(\cdot)$ in $B_r(0; Z)$, we have

$$
\|\Phi_2 u(t) - \Phi_2^e u(t)\| \leq \frac{1}{\Gamma(q)} \int_t^{t+e} \frac{\|f(s, u(s), Ku(s))\|}{(t-s)^{1-q}} \, ds + \int_t^{t+e} \|S'(t-s)\| \frac{1}{\Gamma(q)} \int_0^s \frac{\|f(\tau, u(\tau), Ku(\tau))\|}{(s-\tau)^{1-q}} \, d\tau \, ds.
$$

This shows that precompact sets $X_e$ are arbitrarily close to the set $\{\Phi_2 u(t): u \in B_r(0; Z)\}$. Hence the set $\{\Phi_2 u(t): u \in B_r(0; Z)\}$ is precompact in $X$.

Next, let us prove that $\Phi_2(B_r(0; Z))$ is equicontinuous. The functions $\Phi_2 u$, $u \in B_r(0; Z)$ are equicontinuous at $t = 0$. For $t < t + h \leq b$, $h > 0$ we have

$$
\|\Phi_2 u(t + h) - \Phi_2 u(t)\| \leq \frac{1}{\Gamma(q)} \int_0^{t+h} \frac{\|f(s, u(s), Ku(s))\|}{(t+h-s)^{1-q}} \, ds - \int_0^t \frac{\|f(s, u(s), Ku(s))\|}{(t-s)^{1-q}} \, ds
$$

$$
+ \int_0^t \frac{\|f(\tau, u(\tau), Ku(\tau))\|}{(t-h+\tau-s)^{1-q}} \, d\tau
$$

$$
- \int_0^t \frac{\|f(\tau, u(\tau), Ku(\tau))\|}{(t-h+\tau-s)^{1-q}} \, d\tau
$$

$$
\leq \frac{1}{\Gamma(q)} \int_0^{t+h} \frac{\|f(s, u(s), Ku(s))\|}{(t+h-s)^{1-q}} \, ds
$$

$$
+ \int_0^t \frac{\|f(\tau, u(\tau), Ku(\tau))\|}{(t-h+\tau-s)^{1-q}} \, d\tau
$$

$$
+ \int_0^h \frac{\|S'(t+h-s)\|}{\Gamma(q)} \int_0^s \frac{\|f(\tau, u(\tau), Ku(\tau))\|}{(s-\tau)^{1-q}} \, d\tau
$$

$$
- \int_0^t \frac{\|f(\tau, u(\tau), Ku(\tau))\|}{(t-h+\tau-s)^{1-q}} \, d\tau
$$

which tends to zero as $h \rightarrow 0$, since by (H1) $f(\cdot)$ is completely continuous, the set $\{\Phi_2 u: u \in B_r(0; Z)\}$ is equicontinuous. Thus we have proved $\Phi_2(B_r(0; Z))$ is relatively compact for $t \in J$. By Arzela-Ascoli theorem $\Phi_2$ is compact. Hence by the Krasnoselskii fixed point theorem [22] there exists a fixed point $u \in Z$ such that $\Phi u = u$ which is a mild solution to the problem (2.17) with nonlocal condition (2.18). \qed
4. Application

Consider the following partial integro-differential equation with fractional temporal derivative of the form

$$\frac{\partial^q}{\partial^q t}(u(t, x) + a_1(t)u(t, x)) = \frac{\partial^2}{\partial x^2}u(t, x) + \int_0^t a_2(t - s)e^{-u(s, x)}ds + a_3(t) \sin u(t, x), \quad t > 0$$  \hspace{1cm} (4.1)

$$u(t, 0) = u(t, \pi) = 0, \quad (t, x) \in [0, b] \times [0, \pi],$$  \hspace{1cm} (4.2)

$$u(0, x) + \sum_{i=1}^n \int_0^{\tau_i} b_i(\tau) u(\tau, x) d\tau = z(\tau),$$  \hspace{1cm} (4.3)

where \( q \in (0, 1), z \in L^2[0, \pi] \) and \( a_i \in L^2(f), \quad i = 1, 2, 3 \) and \( b_j \in L^2(j, \mathbb{R}), \quad j = 1, 2 \). Take \( X = L^2[0, \pi] \) and let \( A \) be the operator given by \( Aw = w''' \) with domain

$$D(A) := \{ w \in X : w'' \in X, \quad w(0) = w(\pi) = 0 \}.$$

It is well known that \( A \) is the infinitesimal generator of an analytic semigroup \( (T(t))_{t \geq 0} \) on \( X \). Furthermore, \( A \) has a discrete spectrum with eigenvalues of the form \(-n^2, \quad n \in \mathbb{N}\), and the corresponding normalized eigenfunctions are given by

$$w_n(x) := \left( \frac{2}{\pi} \right)^{\frac{1}{2}} \sin(nx).$$

In addition, \( \{ w_n : n \in \mathbb{N} \} \) is an orthogonal basis for \( X \),

$$T(t)w = \sum_{n=1}^\infty e^{-n^2t}(w, w_n)w_n, \quad \text{for all} \quad w \in X \quad \text{and} \quad \text{every} \quad t > 0.$$

From these expressions it follows that \( (T(t))_{t \geq 0} \) is uniformly bounded compact semigroup, so that \( R(\lambda, A) = (\lambda - A)^{-1} \) is compact operator for all \( \lambda \in \rho(A) \).

From [21], we know that the integral equation

$$u(t) = f(t) + \frac{1}{\Gamma(q)} \int_0^t \frac{Au(s)}{(t - s)^{1-q}} ds, \quad s \geq 0,$$

has an associated analytic resolvent operator \( (S(t))_{t \geq 0} \) on \( X \) given by

$$S(t) = \begin{cases} \frac{1}{2\pi i} \int_{\chi \cup \gamma} e^{\lambda q}(\lambda^q - A)^{-1}d\lambda, & t > 0, \\ I, & t = 0, \end{cases}$$  \hspace{1cm} (4.4)

where \( \Gamma_{\chi, \gamma} \) denotes a contour consisting of the rays \( \{ re^{i\theta} : r \geq 0 \} \) and \( \{ re^{-i\theta} : r \geq 0 \} \) for some \( \theta \in (\pi, \frac{\pi}{2}) \). It is easy to see that \( S(t) \) is differentiable (Proposition 2.15 in [23], Theorem 2.2 in [21]) and there exists a constant \( N > 0 \) such that \( ||S'(t)|| \leq N||x|| \), for \( x \in D(A), \quad t > 0 \).

To represent the differential system (4.1)-(4.3) in the abstract form (2.17)-(2.18) we introduce the functions \( e : J \times X \rightarrow X, \quad f : J \times X^2 \rightarrow X, \quad g : Z \rightarrow X \) and \( k : \Delta \times X \) defined by

$$e(t, w)(x) = a_1(t)w(x),$$

$$f(t, w, Kw)(x) = w(x) + a_3(t) \sin w(x),$$

$$g(u(x)) = \sum_{i=1}^n \int_0^{\tau_i} b_i(\tau)u(\tau, x) d\tau \quad \text{and} \quad k(t, s, w(x)) = a_2(t - s)e^{-w(x)}.$$  \hspace{1cm} (4.5)

Note that \( ||g(u(x)) - g(v(x))|| \leq \sum_{i=1}^n t_i ||b_i|| ||u - v|| \) and \( L_3 = \sup_{\tau \in J} \|a_3(t)\| \). Here \( \|\varphi_k\|_{L^1} = N, L_1 = (1 + \sup_{\tau \in J} ||a_3(t)||), \quad L_2 = \sup_{\tau \in J} ||a_2(t)||, \quad G = \sum_{i=1}^n t_i ||b_i|| \) and choose \( t_i \) such that

$$r \geq 2(1 + ||\varphi_3||_{L^1})(N + \gamma L_1(N_\beta + N)) \quad \text{and} \quad 2(1 + ||\varphi_4||_{L^1})(1 + L_2) g L_1 + G + L_3 < 1.$$  \hspace{1cm} (4.6)

Thus the conditions (H1)-(H5) of Theorem 3.2 are satisfied. Hence there is a function \( u \in C(J, L^2[0, \pi]) \) which is a mild solution of (4.1)-(4.3) on \( J \).

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