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On a characterization of the essential spectra of the sum and the product of two operators

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ABSTRACT

In this paper, we consider the sum and the product of two operators acting on a Banach space and we present some new and quite general conditions to investigate their Wolf, Schechter and Browder essential spectra.

1. Introduction

Let X and Y be two Banach spaces. We denote by $\mathcal{L}(X,Y)$ (resp. $\mathcal{C}(X,Y)$) the set of all bounded (resp. closed, densely defined) linear operators from X into Y and we denote by $\mathcal{K}(X,Y)$ the subspace of compact operators from X into Y. For $A \in \mathcal{C}(X,Y)$, we write $\mathcal{D}(A) \subset X$ for the domain, $N(A) \subset X$ for the null space and $R(A) \subset Y$ for the range of A. The nullity, $\alpha(A)$, of A is defined as the dimension of $N(A)$ and the deficiency, $\beta(A)$, of A is defined as the codimension of $R(A)$ in Y. The set of Fredholm operators from X into Y is defined by

$$\Phi(X,Y) := \{A \in \mathcal{C}(X,Y) \text{ such that } \alpha(A) < \infty,\ \beta(A) < \infty \text{ and } R(A) \text{ is closed in } Y\}.$$

For $A \in \Phi(X,Y)$, the number $i(A) := \alpha(A) - \beta(A)$. If $X = Y$ then $\mathcal{L}(X,Y)$, $\mathcal{C}(X,Y)$, $\mathcal{K}(X,Y)$, and $\Phi(X,Y)$ are replaced by $\mathcal{L}(X)$, $\mathcal{C}(X)$, $\mathcal{K}(X)$, and $\Phi(X)$ respectively. A complex number λ is in Φ_A if $\lambda - A$ is in $\Phi(X)$.

We denote by $\mathcal{R}(X)$ the class of all Riesz operators which is characterized in [2] by

$$\mathcal{R}(X) := \{A \in \mathcal{L}(X) \text{ such that } \lambda - A \in \Phi(X) \text{ for each } \lambda \neq 0\}.$$

Let $\sigma(A)$ (resp. $\rho(A)$) denote the spectrum (resp. the resolvent set) of A.

In this paper, we are concerned with the following essential spectra:

- $\sigma_e(A) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi(X)\} = \mathbb{C} \setminus \Phi_A$,
- $\sigma_w(A) := \mathbb{C} \setminus \rho_w(A)$,
- $\sigma_b(A) := \mathbb{C} \setminus \rho_b(A)$,

where $\rho_w(A) := \{\lambda \in \Phi_A \text{ such that } i(\lambda - A) = 0\}$ and $\rho_b(A)$ denotes the set of those $\lambda \in \rho_w(A)$ such that all scalars near λ are in $\rho(A)$.

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The subset $\sigma_e(.)$ is the Wolf essential spectrum [8,28], $\sigma_w(.)$ is the Schechter essential spectrum [9–12,17,22,23] and $\sigma_b(.)$ denote the Browder essential spectrum [8,14,15,19].

Note that, if A is a self-adjoint operator on a Hilbert space, then

$$\sigma_e(A) = \sigma_w(A) = \sigma_b(A).$$

Also, note that all these sets are closed and in general they satisfy the following inclusions

$$\sigma_e(A) \subseteq \sigma_w(A) \subseteq \sigma_b(A).$$

When dealing with essential spectra of linear operators on Banach space, one of the main problems consists in studying the essential spectra of the sum and the product of two operators. It is well known that if A and B are commuting bounded operators on a Banach space then

$$\sigma(A+B) \subseteq \sigma(A) + \sigma(B), \qquad \sigma(AB) \subseteq \sigma(A)\sigma(B);$$

the spectrum of their sum and product are subsets of the sum and product of their spectra (see [18]). But, in general, if A and B don't commute there's no reason to expect a simple relationship between their spectra and the spectra of their sum or their product.

Among the works in this direction we quote [25]. In fact, the authors are concerned with the case where A and B are respectively closed and bounded operators and which commute modulo the compact operators. Under some conditions, they obtain

$$\sigma_e(A+B) \subseteq \sigma_e(A) + \sigma_e(B), \qquad \sigma_e(AB) \subseteq \sigma_e(A)\sigma_e(B).$$

We mentioned that these results don't include the usual spectrum's case and remain valid for bounded operators A and B.

Let $A \in \mathcal{C}(X)$. It follows from the closeness of A that $\mathcal{D}(A)$ (the domain of A) endowed with the graph norm $\|.\|_A$ ($\|x\|_A = \|x\| + \|Ax\|$) is a Banach space denoted by X_A. Clearly, for $x \in \mathcal{D}(A)$ we have $\|Ax\| \leqslant \|x\|_A$, so $A \in \mathcal{L}(X_A, X)$. Furthermore, we have the obvious relations

$$\alpha(\hat{A}) = \alpha(A), \qquad \beta(\hat{A}) = \beta(A), \qquad R(\hat{A}) = R(A).$$

Furthermore, if A is a closed operator such that Φ_A is not empty, then by [21, Theorem 2.9], Φ_A is open, thus it is the union of a disjoint collection of connected open sets. Each such set, $\Phi_i(A)$, will be called a component of Φ_A. In each $\Phi_i(A)$, a fixed point, λ_i, is chosen in a prescribed manner. Since $\alpha(\lambda_i - A) < \infty$, $R(\lambda_i - A)$ is closed and $\beta(\lambda_i - A) < \infty$, then there exist a closed subspace X_i and a subspace Y_i such that $\dim Y_i = \beta(\lambda_i - A)$ satisfying

$$X = N(\lambda_i - A) \oplus X_i \quad \text{and} \quad X = Y_i \oplus R(\lambda_i - A).$$

Now, let P_{1i} be the projection of X onto $N(\lambda_i - A)$ along X_i and let P_{2i} be the projection of X onto Y_i along $R(\lambda_i - A)$. P_{1i} and P_{2i} are bounded finite rank operators. It is shown in [21] that $(\lambda_i - A)|_{\mathcal{D}(A) \cap X_i}$ has a bounded inverse, A_i,

$$A_i : R(\lambda_i - A) \longrightarrow \mathcal{D}(A) \cap X_i.$$

Let T_i the bounded operator defined by: $T_i x := A_i(I - P_{2i})x$ satisfying:

$$T_i(\lambda_i - A) = I - P_{1i} \quad \text{on } \mathcal{D}(A), \tag{1.1}$$

$$(\lambda_i - A)T_i = I - P_{2i} \quad \text{on } X. \tag{1.2}$$

Hence, T_i is a quasi-inverse of $(\lambda_i - A)$. Moreover, when $\lambda \in \Phi_i(A)$ and $\frac{-1}{\lambda - \lambda_i} \in \rho(T_i)$, the operator $R'_\lambda(A) := T_i[(\lambda - \lambda_i)T_i + I]^{-1}$ is shown in [24] to be a quasi-inverse of $(\lambda - A)$ (see Definition 2.1). In fact $R'_\lambda(A)$ is defined and analytic for all $\lambda \in \Phi_A$ except for at most an isolated set, $\Phi^0(A)$, having no accumulation point in Φ_A.

This work is devoted to extend the results started in [25] to various essential spectra of closed operators. In fact, we are in the position to characterize the sum and the product of Wolf, Schechter and Browder essential spectra of two operators, $A \in \mathcal{C}(X)$ and $B \in \mathcal{L}(X)$, which commute modulo the Fredholm perturbations.

Hence, we show that the description of the essential spectra of the sum and the product of two operators can be further improved to generalize the results obtained by [25].

We organize our paper in the following way: In Section 2 we gather some results and notations from Fredholm theory connected with the third section. The main results are presented in Section 3.

2. Preliminary results

Definition 2.1. A bounded operator B is called a quasi-inverse of the closed operator A if

(i) $R(B) \subset \mathcal{D}(A)$ and $AB = I + K_1$, $K_1 \in \mathcal{K}(X)$.
(ii) $BA = I + K_2$, $K_2 \in \mathcal{K}(X)$. $\diamond$

Definition 2.2. Let X and Y be two Banach spaces and let $F \in \mathcal{L}(X,Y)$, F is said to be a Fredholm perturbation if $T+F \in \Phi(X,Y)$ whenever $T \in \Phi(X,Y)$. $\diamond$

The set of Fredholm perturbations is denoted by $\mathcal{F}(X,Y)$. In Definition 2.2, if we replace $\Phi(X,Y)$ by $\Phi^b(X,Y) := \Phi(X,Y) \cap \mathcal{L}(X,Y)$ we obtain the set $\mathcal{F}^b(X,Y)$. These classes of operators are introduced and investigated by I.C. Gohberg, A.S. Markus and I.A. Feldman in [6]. Recently, it is shown in [3] that $\mathcal{F}^b(X,Y)$ is a closed subset of $\mathcal{L}(X,Y)$ and if $X=Y$, then $\mathcal{F}^b(X) := \mathcal{F}^b(X,X)$ is a closed two-sided ideal of $\mathcal{L}(X)$.

The following identity was established in [3, Theorem 2.4].

Lemma 2.1. *Let X and Y two Banach spaces. Then*

$$\mathcal{F}^b(X,Y) = \mathcal{F}(X,Y). \qquad \diamond$$

Definition 2.3. Let X and Y be Banach spaces.

(i) An operator $A \in \mathcal{L}(X,Y)$ is said to be weakly compact if $A(U)$ is relatively weakly compact in X for every bounded subset $U \subset X$. The family of weakly compact operators from X to Y is denoted by $\mathcal{W}(X,Y)$.
(ii) An operator $S \in \mathcal{L}(X,Y)$ is called strictly singular if, for every infinite-dimensional subspace M of X, the restriction of S to M is not a homeomorphism. The family of strictly singular operators from X to Y is denoted by $\mathcal{S}(X,Y)$.
(iii) An operator $S \in \mathcal{L}(X,Y)$ is said to be strictly cosingular if there exists no closed subspace N of Y with $\operatorname{codim}(N) = \infty$ such that $\pi_N S : X \longrightarrow Y/N$ is surjective. The family of strictly cosingular operators from X to Y is denoted by $\mathcal{CS}(X,Y)$. $\diamond$

The concept of strictly singular operators was introduced in the pioneering paper by T. Kato [16] as a generalization of the notion of compact operators. The class of strictly cosingular operators was introduced by A. Pelczynski [20]. If $X=Y$, the family of weakly compact, strictly singular and strictly cosingular operators on X are denote by $\mathcal{W}(X) := \mathcal{W}(X,X)$, $S(X) := S(X,X)$ and $CS(X) := CS(X,X)$ respectively. The three families are closed two-sided ideals of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$ (cf. [5–7,27]).

Remark 2.1. (i) If $X=Y$, we conclude from Lemma 2.1 is that $\mathcal{F}(X)$ is a closed two-sided ideals of $\mathcal{L}(X)$.

(ii) It follows from [23, Theorem 2.1, p. 167] that we have

$$\mathcal{K}(X,Y) \subseteq \mathcal{F}(X,Y).$$

(iii) If $X=Y$, $\mathcal{R}(X)$ is not an ideal (see [4]). In [22], it is proved that $\mathcal{F}(X)$ is a largest closed two-sided ideal contained in $\mathcal{R}(X)$. Most of the results on ideal structure deal with the well-known closed ideals which have arisen from applied work with operators. We can quote, for example, compact operators, weakly compact operators, strictly singular operators (see [7,12,16]), strictly cosingular operators (see [20,27]). In general, we have

$$\mathcal{K}(X) \subset \mathcal{S}(X) \subset \mathcal{F}(X) \subset \mathcal{J}(X) \quad \text{and} \quad \mathcal{K}(X) \subset \mathcal{CS}(X) \subset \mathcal{F}(X) \subset \mathcal{J}(X),$$

where $\mathcal{J}(X)$ denotes the set

$$\mathcal{J}(X) = \bigl\{F \in \mathcal{L}(X) \text{ such that } I-F \in \Phi(X) \text{ and } i(I-F)=0\bigr\}. \qquad \diamond$$

Lemma 2.2. *Let $A \in \mathcal{C}(X,Y)$. Suppose that there are operators $A_1, A_2 \in \mathcal{L}(Y,X)$, $F_1 \in \mathcal{J}(X)$ and $F_2 \in \mathcal{J}(Y)$ such that*

$$A_1 A = I - F_1 \quad \text{on } \mathcal{D}(A), \tag{2.1}$$

$$A A_2 = I - F_2 \quad \text{on } Y. \tag{2.2}$$

Then $A \in \Phi(X,Y)$. $\diamond$

Proof. The proof is similar to that in [13, Theorem 2.1]. $\square$

Lemma 2.3. *(See [25, Lemma 2.1].) Let $A \in \mathcal{C}(X)$ such that Φ_A is not empty and n be a positive integer. Then for each $\lambda \in \Phi_A \setminus \Phi^0(A)$, there exists a subspace V_λ dense in X and depending on λ such that for all $x \in V_\lambda$, $R'_\lambda(A)x \in \mathcal{D}(A^n)$.* $\diamond$

3. Main results

Lemma 3.1. *Let $A \in \Phi(X)$, $B \in \mathcal{L}(X)$ and $F \in \mathcal{F}(X)$. Suppose that $AB_{|V} = F_{|V}$ where V is a dense subspace of X. Then $B \in \mathcal{F}(X)$.* $\diamond$

Proof. Since $A \in \Phi(X)$, then there exists $A_0 \in \mathcal{L}(X)$ such that $A_0 A = I - K_1$ where $K_1 \in \mathcal{K}(X)$. Hence,

$$A_0 A B_{|V} = A_0 F_{|V}, \qquad (I - K_1) B_{|V} = F_{1|V},$$

where $F_1 \in \mathcal{F}(X)$, so we obtain,

$$B_{|V} = (K_1 B + F_1)_{|V}.$$

Now using the fact that the operators B and $K_1 B + F_1$ are bounded and the subspace V is dense, we have by continuity that $B = K_1 B + F_1$. Hence, it is clear that B is a Fredholm perturbation. □

Lemma 3.2. *Let $A \in \mathcal{C}(X)$, $B \in \mathcal{L}(X)$, $\lambda \in \Phi_A \setminus \Phi^0(A)$ and $\mu \in \Phi_B \setminus \Phi^0(B)$. If there exist a positive integer n and a Fredholm perturbation F_1, such that $B : \mathcal{D}(A^n) \longrightarrow \mathcal{D}(A)$ and $ABx = BAx + F_1 x$, for all $x \in \mathcal{D}(A^n)$. Then there exists a Fredholm perturbation F depending analytically on λ and μ such that*

$$R'_\lambda(A) R'_\mu(B) = R'_\mu(B) R'_\lambda(A) + F. \qquad \diamond$$

Proof. Using Lemma 2.3, we infer that there exists a subspace V_λ dense in X such that for all $x \in V_\lambda$, we have $R'_\lambda(A)x \in \mathcal{D}(A^n)$.

Now, let $x \in V_\lambda$, then we have

$$(\lambda - A) B R'_\lambda(A) x = \big[B(\lambda - A) - F_1\big] R'_\lambda(A) x = \big[B(I - K_1) - F_1 R'_\lambda(A)\big] x,$$

where $K_1 \in \mathcal{K}(X)$. Set $F_2 = -F_1 R'_\lambda(A) \in \mathcal{F}(X)$ and $F_3 = -BK_1 + F_2 \in \mathcal{F}(X)$.

Hence, we get

$$(\lambda - A) B R'_\lambda(A) x = Bx + F_3 x.$$

Moreover, $(\lambda - A) R'_\lambda(A) Bx = (I - K_1) Bx = Bx - K_2 x$, where $K_2 = K_1 B \in \mathcal{K}(X)$.

This make us conclude that

$$\begin{aligned}(\lambda - A)\big[B R'_\lambda(A) - R'_\lambda(A) B\big] x &= (F_3 + K_2) x \\ &= F_4 x, \end{aligned} \tag{3.1}$$

where $F_4 \in \mathcal{F}(X)$. In fact Eq. (3.1) holds for all $x \in V_\lambda$, then the use of Lemma 3.1 make us conclude that

$$B R'_\lambda(A) - R'_\lambda(A) B = F_5,$$

where $F_5 \in \mathcal{F}(X)$. On the other hand,

$$\begin{aligned}(\mu - B)\big[R'_\mu(B) R'_\lambda(A) - R'_\lambda(A) R'_\mu(B)\big] &= (I - K_3) R'_\lambda(A) - (\mu - B) R'_\lambda(A) R'_\mu(B) \\ &= R'_\lambda(A) - K_4 - \big[R'_\lambda(A)(\mu - B) + F_5\big] R'_\mu(B) \\ &= R'_\lambda(A) - K_4 - R'_\lambda(A)(I - K_5) + F_6 \\ &= -K_4 + K_6 + F_6 \\ &= F_7 \end{aligned}$$

where $K_i \in \mathcal{K}(X)$ for $i = 3, 4, 5, 6$ and $F_i \in \mathcal{F}(X)$ for $i = 6, 7$. Hence,

$$R'_\mu(B) R'_\lambda(A) - R'_\lambda(A) R'_\mu(B) = F, \quad \text{where } F \in \mathcal{F}(X).$$

Therefore

$$R'_\mu(B) R'_\lambda(A) = R'_\lambda(A) R'_\mu(B) + F.$$

Furthermore, the analyticity of F in λ and μ follows from the analyticity of $R'_\mu(B)$ and $R'_\lambda(A)$. □

Theorem 3.1. *Let $A \in \mathcal{C}(X)$ and $B \in \mathcal{L}(X)$. Suppose that there exist a positive integer n and $F \in \mathcal{F}(X)$ such that $B : \mathcal{D}(A^n) \longrightarrow \mathcal{D}(A)$ and $ABx = BAx + Fx$, for all $x \in \mathcal{D}(A^n)$. Then*

(i) $\quad \sigma_e(A + B) \subseteq \sigma_e(A) + \sigma_e(B).$

If $\sigma_e(A)$ is empty, then $\sigma_e(A) + \sigma_e(B)$ is also empty set.

(ii) *If in the addition $\mathbb{C} \backslash \sigma_e(A)$, $\mathbb{C} \backslash \sigma_e(B)$ and $\mathbb{C} \backslash \sigma_e(A + B)$ are connected, $\rho(A)$ and $\rho(A + B)$ are nonempty sets, then*

$$\sigma_w(A + B) \subseteq \sigma_w(A) + \sigma_w(B).$$

(iii) *Moreover, if* $\mathbb{C}\backslash\sigma_w(A)$, $\mathbb{C}\backslash\sigma_w(B)$ *and* $\mathbb{C}\backslash\sigma_w(A+B)$ *are connected,* $\rho(A)$ *and* $\rho(A+B)$ *are nonempty sets, then*

$$\sigma_b(A+B) \subseteq \sigma_b(A) + \sigma_b(B).$$

Proof. (i) First, it is clear that the theorem is trivially true if we suppose that $\sigma_e(A)+\sigma_e(B)$ is the entire complex plane. Hence, we assume in the next that $\sigma_e(A)+\sigma_e(B)$ is not the entire plane.

Second, we fix a point γ such that $\gamma \notin \sigma_e(A)+\sigma_e(B)$ and we define the operator A_1 as $A_1 := \gamma - A$. Hence, it is easy to verify that if $\lambda \in \sigma_e(B)$, the element $\gamma - \lambda$ will be in Φ_A which is equivalent to say that $\lambda \in \Phi_{A_1}$. In the following, we will find a Cauchy domain $\mathcal{D}$ such that $R'_\lambda(A_1)$ and $R'_\lambda(B)$ are analytic on $B(\mathcal{D})$, the boundary of $\mathcal{D}$.

In fact, $\sigma_e(A)$ is closed and $\sigma_e(B)$ is compact, then there exists an open set $U \supset \sigma_e(B)$ such that $B(U)$, the boundary of U is bounded and when $\lambda \in U$, $(\gamma - \lambda) \in \Phi_A$.

Therefore $\sigma_e(B) \subset U \subset \Phi_{A_1}$. Using [26, Theorem 3.3], we infer that there exists a bounded Cauchy domain $\mathcal{D}$ such that $\sigma_e(B) \subset D \subset U$.

Note that $\Phi^0(A_1)$ (resp. $\Phi^0(B)$) does not accumulate in Φ_{A_1} (resp. Φ_B), so we can chose $\mathcal{D}$ such that $R'_\lambda(A_1)$ and $R'_\lambda(B)$ are analytic on $B(\mathcal{D})$. We claim also that $R'_\lambda(A_1)$ is of the form $TC(\lambda)$ where $C(\lambda)$ is bounded operator valued analytic function of λ and T is a fixed bounded operator such that $T : X \longrightarrow \mathcal{D}(A_1) = \mathcal{D}(A)$.

Now, let us define the operators M_1 and M_2 as follows

$$M_1 = -\frac{1}{2\pi i} \int\limits_{+B(\mathcal{D})} R'_\lambda(A_1) R'_\lambda(B)\, d\lambda$$

and

$$M_2 = -\frac{1}{2\pi i} \int\limits_{+B(\mathcal{D})} R'_\lambda(B) R'_\lambda(A_1)\, d\lambda.$$

In order to prove this assertion, we shall show that $\gamma \in \Phi_{A+B}$, hence, it suffices to find two Fredholm perturbations F_1 and F_2 such that

$$(\gamma - B - A)M_1 = I + F_1$$

and

$$M_2(\gamma - B - A) = I + F_2 \quad \text{on } \mathcal{D}(A).$$

Now, writing the operator $\gamma - B - A$ as follows:

$$(\gamma - B - A) = (\gamma - \lambda - A) + (\lambda - B) = -(\lambda - A_1) + (\lambda - B),$$

we get

$$(\gamma - B - A)M_1 = -\frac{1}{2\pi i} \int\limits_{+B(\mathcal{D})} -(\lambda - A_1) R'_\lambda(A_1) R'_\lambda(B)\, d\lambda - \frac{1}{2\pi i} \int\limits_{+B(\mathcal{D})} -(\lambda - B) R'_\lambda(A_1) R'_\lambda(B)\, d\lambda. \tag{3.2}$$

Obviously, $(\lambda - A_1)R'_\lambda(A_1) = I + \mathfrak{F}$ where $\mathfrak{F}$ is a bounded finite rank operator depending analytically on λ. Then the first integral of the above equality is of the form

$$-\frac{1}{2\pi i} \int\limits_{+B(\mathcal{D})} -(I + \mathfrak{F}) R'_\lambda(B)\, d\lambda.$$

Using [24, Theorem 13], we deduce that

$$\frac{1}{2\pi i} \int\limits_{+B(\mathcal{D})} R'_\lambda(B)\, d\lambda = I + K_1,$$

where $K_1 \in \mathcal{K}(X)$. Moreover, we mention also that $\int_{+B(\mathcal{D})} -(I+\mathfrak{F})R'_\lambda(B)\, d\lambda$ is a compact operator. So, we infer that the first integral of (3.2) is of the form $I + K_2$, $K_2 \in \mathcal{K}(X)$.

Applying Lemma 3.2, we get

$$R'_\lambda(A_1) R'_\lambda(B) = R'_\lambda(B) R'_\lambda(A_1) + F,$$

where F is a Fredholm perturbation, then the second integral is equal to

$$-\frac{1}{2\pi i} \int\limits_{+B(\mathcal{D})} -(\lambda - B) R'_\lambda(B) R'_\lambda(A_1)\, d\lambda - \frac{1}{2\pi i} \int\limits_{+B(\mathcal{D})} (\lambda - B) F\, d\lambda.$$

Since $\int_{+B(\mathcal{D})} R'_\lambda(A_1)\,d\lambda$ is compact (see [24, Theorem 13]), then a same reasoning as the first part allows us to write

$$-\frac{1}{2\pi i}\int_{+B(\mathcal{D})} -(\lambda - B)R'_\lambda(B)R'_\lambda(A_1)\,d\lambda = I + K_3, \quad K_3 \in \mathcal{K}(X).$$

Using the fact that $\frac{1}{2\pi i}\int_{+B(\mathcal{D})}(\lambda - B)F\,d\lambda$ is also a Fredholm perturbation, we have

$$(\gamma - B - A)M_1 = I + F_1, \quad F_1 \in \mathcal{F}(X).$$

By a similar argument we obtain

$$M_2(\gamma - B - A) = I + F_2, \quad F_2 \in \mathcal{F}(X).$$

Therefore $(\gamma - B - A) \in \Phi(X)$, and we deduce that

$$\sigma_e(A + B) \subseteq \sigma_e(A) + \sigma_e(B).$$

(ii) This assertion follows immediately from [1, Theorem 2.1].

(iii) The proof of this assertion holds from [14, Lemma 3.1]. □

Theorem 3.2. *Let $A \in \mathcal{C}(X)$ and $B \in \mathcal{L}(X) \cap \Phi(X)$. Let $B : \mathcal{D}(A) \longrightarrow \mathcal{D}(A)$ and suppose that there exist $F \in \mathcal{F}(X)$ such that $ABx = BAx + Fx$, for all $x \in \mathcal{D}(A)$. Then BA is closable and*

(i)
$$\sigma_e(\overline{BA}) \subseteq \sigma_e(A)\sigma_e(B),$$
$$\sigma_e(AB) \subseteq \sigma_e(A)\sigma_e(B).$$

(ii) *If in the addition $\mathbb{C}\backslash\sigma_e(\overline{BA})$, $\mathbb{C}\backslash\sigma_e(AB)$, $\mathbb{C}\backslash\sigma_e(A)$, $\mathbb{C}\backslash\sigma_e(B)$ are connected, $\rho(A)$, $\rho(\overline{BA})$ and $\rho(AB)$ are nonempty sets, then*

$$\sigma_w(\overline{BA}) \subseteq \sigma_w(A)\sigma_w(B),$$
$$\sigma_w(AB) \subseteq \sigma_w(A)\sigma_w(B).$$

(iii) *Moreover, if $\mathbb{C}\backslash\sigma_w(\overline{BA})$, $\mathbb{C}\backslash\sigma_e(AB)$, $\mathbb{C}\backslash\sigma_w(A)$, $\mathbb{C}\backslash\sigma_w(B)$ are connected, $\rho(A)$, $\rho(\overline{BA})$ and $\rho(AB)$ are nonempty sets, then*

$$\sigma_b(\overline{BA}) \subseteq \sigma_b(A)\sigma_b(B),$$
$$\sigma_b(AB) \subseteq \sigma_b(A)\sigma_b(B). \quad \diamond$$

Proof. (i) Since the operator F is bounded and the restriction of the operator AB on $\mathcal{D}(A)$ is closable, then BA is closable.

Furthermore it is clear that $0 \notin \sigma_e(B)$ and $\sigma_e(B)$ is not empty, so the theorem is trivially true if $\sigma_e(A) = \mathbb{C}$ and we will assume in the next that $\sigma_e(A) \neq \mathbb{C}$.

Now, let γ a fixed point not in $\sigma_e(B)\sigma_e(A)$. In what follows, we will show that $\gamma \in \Phi(\overline{BA})$. Observing that $\sigma_e(A)$ is closed, $\sigma_e(B)$ is compact and $0 \notin \sigma_e(B)$, we infer that there exists an open set U, with bounded boundary $B(U)$, containing $\sigma_e(B)$ and satisfying that $0 \notin U$ and $(\gamma - \mu A) \in \Phi(X)$, $\forall \mu \in U$. Let $\mathcal{D}$ be a bounded Cauchy domain such that $\sigma_e(B) \subset \mathcal{D} \subseteq U$.

Writing $(\gamma - \mu A)$ as follows

$$(\gamma - \mu A) = \mu\gamma\left(\frac{1}{\mu} - \frac{1}{\gamma}A\right) = \frac{\gamma}{\lambda}\left(\lambda - \frac{1}{\gamma}A\right), \quad \lambda = \frac{1}{\mu}$$

and taking $\mathcal{D}'$ the image of $\mathcal{D}$ under the map $\lambda = \frac{1}{\mu}$, we can assume that $R'_\lambda(A_1)$ is analytic in λ on $B(\mathcal{D}')$ where $A_1 := \frac{1}{\gamma}A$. This assumption holds true thanks to the fact that $\forall \mu \in \overline{\mathcal{D}}$, $\frac{1}{\mu} \in \Phi_{A_1}$ and the operator $R'_\lambda(A_1)$ is analytic in λ throughout Φ_{A_1} except for at most an isolated set having no accumulation in Φ_{A_1}.

Let us define the operators M_1 and M_2 as follows

$$M_1 = -\frac{1}{2\pi i}\int_{+B(\mathcal{D}')} \frac{1}{\gamma\lambda} R'_\lambda(A_1) R'_{\frac{1}{\lambda}}(B)\,d\lambda$$

and

$$M_2 = -\frac{1}{2\pi i}\int_{+B(\mathcal{D}')} \frac{1}{\gamma\lambda} R'_{\frac{1}{\lambda}}(B) R'_\lambda(A_1)\,d\lambda.$$

Since $R(M_1) \subset \mathcal{D}(A)$, the operator $(\gamma - \overline{BA})M_1$ is well defined and we have

$$(\gamma - \overline{BA})M_1 = (\gamma - BA)M_1,$$

moreover

$$(\gamma - BA) = (\gamma - B\gamma A_1) = \gamma B(\lambda - A_1) - \gamma\lambda B + \gamma I = \gamma B(\lambda - A_1) + \gamma(I - \lambda B).$$

Then

$$(\gamma - \overline{BA})M_1 = -\frac{1}{2\pi i}\int_{+B(\mathcal{D}')}\left(\frac{1}{\lambda}B(\lambda - A_1)R'_\lambda(A_1)R'_{\frac{1}{\lambda}}(B) + \left(\frac{1}{\lambda} - B\right)R'_\lambda(A_1)R'_{\frac{1}{\lambda}}(B)\right)d\lambda.$$

On the one hand, the first part of the integrand can be written as follows:

$$\begin{aligned}\int_{+B(\mathcal{D}')}\frac{1}{\lambda}B(\lambda - A_1)R'_\lambda(A_1)R'_{\frac{1}{\lambda}}(B)\,d\lambda &= \int_{+B(\mathcal{D}')}\frac{1}{\lambda}B\big(I + K_1(\lambda)\big)R'_{\frac{1}{\lambda}}(B)\,d\lambda\\ &= \int_{+B(\mathcal{D}')}\frac{1}{\lambda}BR'_{\frac{1}{\lambda}}(B)\,d\lambda + K_2\\ &= \int_{+B(\mathcal{D})}\frac{1}{\mu}BR'_\mu(B)\,d\mu + K_2.\end{aligned}$$

On the other hand, since $0 \notin D$, hence using [24, Theorems 14.9 and 13] we get

$$\frac{1}{2\pi i}\int_{+B(\mathcal{D})}\frac{1}{\mu}BR'_\mu(B)\,d\mu = I + K_3,$$

where $K_i \in \mathcal{K}(X)$, $i = 1, 2, 3$.

Note that the second part of the integrand can be also written as:

$$\begin{aligned}\int_{+B(\mathcal{D}')}\left(\frac{1}{\lambda} - B\right)R'_\lambda(A_1)R'_{\frac{1}{\lambda}}(B)\,d\lambda &= \int_{+B(\mathcal{D}')}\left(\frac{1}{\lambda} - B\right)\left[R'_{\frac{1}{\lambda}}(B)R'_\lambda(A_1) + F_1\right]d\lambda\\ &= \int_{+B(\mathcal{D}')}\big[I + K_4(\lambda)\big]R'_\lambda(A_1)\,d\lambda + F_2\\ &= \int_{+B(\mathcal{D}')}R'_\lambda(A_1)\,d\lambda + F_3,\end{aligned}$$

where $K_4 \in \mathcal{K}(X)$ and $F_i \in \mathcal{F}(X)$, $i = 1, 2, 3$.

We claim that $R'_\lambda(A_1)$ is analytic in $\mathcal{D}'$ except for at most a finite number of points, then we deduce by Theorem 7.4 in [24] that

$$\frac{1}{2\pi i}\int_{+B(\mathcal{D}')}R'_\lambda(A_1)\,d\lambda = K_5 \in \mathcal{K}(X).$$

Therefore, $(\gamma - \overline{BA})M_1 = I + F_4$, where $F_4 \in \mathcal{F}(X)$.

Now, we can easily check that $\mathcal{D}(\overline{BA}) \subseteq \mathcal{D}(AB)$ and $\overline{BA}x = ABx + Fx\ \forall x \in \mathcal{D}(\overline{BA})$, so

$$(\gamma - \overline{BA}) = \gamma\overline{B(\lambda - A_1)} + \gamma(I - \lambda B) = \gamma(\lambda - A_1)B + \gamma(I - \lambda B) + F_5,$$

where $F_5 \in \mathcal{F}(X)$. Hence,

$$\begin{aligned}M_2(\gamma - \overline{BA}) &= \frac{1}{2\pi i}\int_{+B(\mathcal{D}')}\frac{1}{\gamma\lambda}R'_{\frac{1}{\lambda}}(B)R'_\lambda(A_1)\big[\gamma(\lambda - A_1)B + \gamma(I - \lambda B) + F_5\big]\,d\lambda\\ &= -\frac{1}{2\pi i}\int_{+B(\mathcal{D}')}\frac{1}{\lambda}R'_{\frac{1}{\lambda}}(B)(I + K_6)B\,d\lambda - \frac{1}{2\pi i}\int_{+B(\mathcal{D}')}\frac{1}{\gamma\lambda}\big(R'_\lambda(A_1)R'_{\frac{1}{\lambda}}(B) - F_1\big)\big(\gamma(I - \lambda B) + F_5\big)\,d\lambda\end{aligned}$$

$$= \left[\frac{1}{2\pi i}\int_{+B(\mathcal{D})}\frac{1}{\mu}R'_{\mu}(B)\,d\mu\right]B + K_7 - \frac{1}{2\pi i}\int_{+B(\mathcal{D}')}R'_{\lambda}(A_1)R'_{\frac{1}{\lambda}}(B)\left(\frac{1}{\lambda}-B\right)d\lambda + F_6$$

$$= I + F_7 - \frac{1}{2\pi i}\int_{+B(\mathcal{D}')}R'_{\lambda}(A_1)(I+K_8)\,d\lambda$$

$$= I + F_8$$

where $K_i \in \mathcal{K}(X)$, $i = 6, 7$ and $F_i \in \mathcal{F}(X)$, $i = 6, 7, 8$.

Therefore, we conclude that $(\gamma - \overline{BA}) \in \Phi(X)$, and the proof of the first inclusion is completed.

Now, to show that $\sigma_e(AB) \subseteq \sigma_e(B)\sigma_e(A)$, we will prove that $(\gamma - AB) \in \Phi(X)$.

Since $R(M_1) \subset \mathcal{D}(A)$ and $ABx = BAx - Fx$, $\forall x \in \mathcal{D}(A)$, we obtain

$$(\gamma - AB)M_1 = (\gamma - BA + F)M_1 = (\gamma - BA)M_1 + FM_1 = I + F_4 + F_9 = I + F_{10},$$

where $F_i \in \mathcal{F}(X)$, $i = 9, 10$. Furthermore, we have

$$M_2(\gamma - \overline{AB}) = M_2\big[\gamma(\lambda - A_1)B + \gamma(I - \lambda B)\big] = I + F_{11},$$

where $F_{11} \in \mathcal{F}(X)$. Hence, $(\gamma - \overline{AB}) \in \Phi(X)$ and we deduce that $\sigma_e(AB) \subseteq \sigma_e(A)\sigma_e(B)$.

(ii) The proof of this assertion holds from [1, Theorem 2.1].

(iii) This assertion follows immediately from [14, Lemma 3.1]. □

In the sequel, we give an example to illustrate our obtained results.

Example 3.1. We can give an immediate example to Theorem 3.1 and Theorem 3.2 by introducing the class of Riesz operators. In fact, this class of operators is not generally a closed ideal of $\mathcal{L}(X)$, but it is shown in [2] that the sum and the product of Riesz operators are still Riesz operators if we assume the commutativity modulo $\mathcal{K}(X)$. More precisely, if $A, B \in \mathcal{R}(X)$ and $AB - BA \in \mathcal{K}(X)$ then $A + B \in \mathcal{R}(X)$. Furthermore, if $A \in \mathcal{R}(X)$, $B \in \mathcal{L}(X)$ and $AB - BA \in \mathcal{K}(X)$ then $AB, BA \in \mathcal{R}(X)$. Since the essential spectrum of a Riesz operator is reduced to zero, then our results are well satisfied. ◇

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