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Isomorphisms from the Eulerian Umbral Algebra onto Formal Newton Series*

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1. INTRODUCTION

Let $\mathscr{P}$ denote the commutative algebra of all polynomials in the single variable x, with coefficients in a field K of characteristic zero. We usually think of K as either the real or complex numbers. Let $\mathscr{P}^*$ denote the set of all linear functionals mapping $\mathscr{P}$ into K. The action of the linear functional L on the polynomial $p(x)$ will be denoted by $\langle L \mid p(x)\rangle$. Thus $\langle L \mid p(x)\rangle$ is an element of K. The set $\mathscr{P}^*$ is made into a vector space over K in the usual manner by defining scalar multiplication $\cdot : K \times \mathscr{P}^* \to \mathscr{P}^*$ by

$$\langle \alpha \cdot L \mid p(x)\rangle = \alpha\langle L \mid p(x)\rangle, \tag{1.1}$$

where juxtaposition on the right-hand side of this equation stands for multiplication in the field K; and vector addition $+ : \mathscr{P}^* \times \mathscr{P}^* \to \mathscr{P}^*$ by

$$\langle L + M \mid p(x)\rangle = \langle L \mid p(x)\rangle + \langle M \mid p(x)\rangle, \tag{1.2}$$

where $+$ on the right-hand side of this equation is the addition defined in K.

Roman and Rota [10] make $\mathscr{P}^*$ into an algebra by defining the product $\oplus : \mathscr{P} \times \mathscr{P}^* \to \mathscr{P}^*$ by

$$\langle L \oplus M \mid x^n\rangle = \sum_{k=0}^{n} \binom{n}{k} \langle L \mid x^k\rangle\langle M \mid x^{n-k}\rangle. \tag{1.3}$$

Roman and Rota [10] and later Roman [8, 9] have made a detailed study of this algebra in order to obtain a deeper insight into a class of polynomial sets known as binomial polynomial sequences. The set $\{p_n(x)\}_{n=0}^{\infty}$ is called a

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polynomial sequence if for each nonnegative integer n, the degree of $p_n(x)$ is n. A polynomial sequence is known as a *binomial polynomial sequence* if for all nonnegative integers n

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x)\, p_{n-k}(y). \tag{1.4}$$

Allaway [1] calls the product $\oplus$ as defined on $\mathscr{P}^*$ by Eq. (1.3) the *binomial umbral product* and the algebra $\mathscr{P}^*$ defined over K by the operations $\cdot$, $\oplus$, and $+$ the *binomial umbral algebra*. This binomial umbral algebra will be denoted by $(\mathscr{P}^*, \cdot, +, \oplus, K)$.

Andrews [3], and later Allaway and Yuen [2] studied *Eulerian polynomial sequences* $\{p_n(x)\}_{n=0}^{\infty}$ defined by

$$p_n(xy) = \sum_{k=0}^{n} \binom{n}{k} p_k(x)\, y^k p_{n-k}(y), \tag{1.5}$$

for n a nonnegative integer. The technique they used for studying Eulerian polynomial sequences was analogous to what Rota *et al.* [11] used to study polynomials of binomial type. In order to investigate the properties of Eulerian polynomial sequences, Ihrig and Ismail [4] define a new product $\otimes : \mathscr{P}^* \times \mathscr{P}^* \to \mathscr{P}^*$ by

$$\langle L \otimes M \mid x^n \rangle = \langle L \mid x^n \rangle \langle M \mid x^n \rangle. \tag{1.6}$$

Allaway [1] uses this product in order to start an investigation of Eulerian polynomial sequences that is analogous to what Roman and Rota [10]; Roman [8, 9]; Garcia and Joni [6]; and Joni [5] did for binomial polynomial sequences. As was done in [1], we will call $\otimes$, defined by Eq. (1.6), the *Eulerian umbral product* and the algebra $\mathscr{P}^*$ over K defined by the operations $\cdot$, $+$, and $\otimes$ the *Eulerian umbral algebra*. This Eulerian umbral algebra will be denoted by $(\mathscr{P}^*, \cdot, +, \otimes, K)$.

Roman and Rota [10], after defining a *binomial delta functional* L as any functional having the property that

$$\langle L \mid 1 \rangle = 0 \tag{1.7}$$

and

$$\langle L \mid x \rangle \neq 0 \tag{1.8}$$

proved the following isomorphism theorem for the binomial umbral algebra $(\mathscr{P}^*, \cdot, +, \oplus, K)$.

THEOREM 1.1 (Roman and Rota [10, Theorem 3]). *Let L be any* *binomial delta functional. Then the mapping ϕ which associates to every linear functional*

$$M = \sum_{k=0}^{\infty} a_k L^k, \qquad a_k \in K,$$

the formal power series

$$f(t) = \sum_{k=0}^{\infty} a_k t^k$$

is a continuous isomorphism of the binomial umbral algebra onto the algebra of formal power series.

The main purpose of this paper is to give a number of isomorphisms for the Eulerian umbral algebra that are analogous to the isomorphism ϕ for the binomial umbral algebra as given by Theorem 1.1. The main difference between the isomorphism Roman and Rota [10] obtained for the binomial umbral algebra and the isomorphism we obtain for the Eulerian umbral algebra is that the range space for the Eulerian umbral algebra case is the set of all formal Newton series of the form

$$\sum_{k=0}^{\infty} a_k \frac{s^{(k)}}{k!},$$

where a_k belongs to K and for nonnegative integers k,

$$\begin{aligned} s^{(k)} &= s(s-1)(s-2) \cdots (s-k+1), \qquad && \text{if} \quad k \geqslant 1, \\ &= 1, && \text{if} \quad k = 0. \end{aligned} \tag{1.9}$$

Just as Theorem 1.1 in the binomial umbral algebra theory is suggested by the isomorphism theorem for shift invariant operators (see [11, Theorem 3]), the isomorphisms we obtain in this paper are suggested by the isomorphisms Allaway and Yuen [2] obtained for Eulerian shift invariant operators. In the last section of the paper, we show the relationship between the Eulerian umbral algebra and the algebra of Eulerian shift invariant operators.

2. The Eulerian Umbral Algebra

In order to make this paper self contained, we will give in this section a brief survey of some of the parts of [1] that we will use in the development in this paper. The notation, definitions, and theorems are as given in [1].

In [1], we developed the theory of the Eulerian umbral algebra

$(\mathscr{P}^*, \cdot, +, \otimes, K)$, as defined in Section 1, in a manner analogous to what Roman and Rota did for the binomial umbral algebra $(\mathscr{P}^*, \cdot, +, \oplus, K)$ in the first part of their paper [10]. Functional L is an *Eulerian delta functional* if

$$\langle L \mid 1 \rangle = 0 \tag{2.1}$$

and for all nonnegative integers n

$$\langle L \mid x^n \rangle \neq 0. \tag{2.2}$$

In the Eulerian umbral algebra theory the notation $L(k)$, defined by

$$\begin{aligned} \langle L(k) \mid x^n \rangle &= \prod_{i=0}^{k-1} \langle L \mid x^{n-i} \rangle, \qquad \text{if} \quad n \geqslant k, \\ &= 0, \qquad \qquad \qquad \qquad \text{if} \quad n < k, \end{aligned} \tag{2.3}$$

is the analogue of L^k in the binomial umbral algebra theory. The polynomial sequence $\{p_n(x)\}_{n=0}^{\infty}$ is the *associated polynomial sequence* for the Eulerian delta functional L if the bi-orthogonality relation

$$\langle L(k) \mid p_n(x) \rangle = n! \, \delta_{n,k} \tag{2.4}$$

holds for all nonnegative integers n and k.

THEOREM 2.1 (Allaway [1, Theorem 9.1]). *Every Eulerian delta functional has a unique associated polynomial sequence.*

Any linear functional in $\mathscr{P}^*$ can be expanded in terms of a given Eulerian delta functional and its associated polynomial sequence.

THEOREM 2.2 (Allaway [1, Theorem 10.1]). *Let M be any linear functional in $\mathscr{P}^*$ and let L be any Eulerian delta functional with associated polynomial sequence $\{p_n(x)\}_{n=0}^{\infty}$. Then*

$$M = \sum_{k=0}^{\infty} \langle M \mid p_k(x) \rangle \frac{L(k)}{k!}. \tag{2.5}$$

The right-hand side of Equation (2.5) is the limit of the partial sums $\sum_{k=0}^{n} \langle M \mid p_k(x) \rangle \, L(k)/k!$ in the topology on $\mathscr{P}^*$ defined by

DEFINITION 2.1. A sequence of linear functional $\{L_n\}_{n=0}^{\infty}$ converges to L in the *pointwise discrete* topology on $\mathscr{P}^*$ if for all $p(x)$ belonging to $\mathscr{P}$, there exist $n_0(p(x))$ depending on $p(x)$ such that for all $n \geqslant n_0$

$$\langle L_n \mid p(x) \rangle = \langle L \mid p(x) \rangle. \tag{2.6}$$

This topology was used by Roman and Rota [10] in their study of the binomial umbral algebra. It is important to note that n_0 depends on $p(x)$. For this reason and because of Eq. (2.6) we call this topology the pointwise discrete topology.

For the pointwise discrete topology, the Eulerian umbral product $\otimes$ is separately continuous. That is, if there exists a sequence of linear functionals $\{M_n\}_{n=0}^{\infty}$ converging in the pointwise discrete topology to M, then for all linear functionals R, both $M_n \otimes R$ and $R \otimes M_n$ converge in the pointwise discrete topology to $R \otimes M$. To show that this is true, let λ be any nonnegative integer. Because M_n converges to M, there exists an $n_0(\lambda)$ such that for all $n \geqslant n_0(\lambda)$,

$$\langle M_n \mid x^\lambda \rangle = \langle M \mid x^\lambda \rangle.$$

Thus for all $n \geqslant n_0(\lambda)$

$$\langle M_n \otimes R \mid x^\lambda \rangle = \langle M_n \mid x^\lambda \rangle \langle R \mid x^\lambda \rangle = \langle M \mid x^\lambda \rangle \langle R \mid x^\lambda \rangle = \langle M \otimes R \mid x^\lambda \rangle.$$

Thus, by the usual spanning argument (see [1, Proposition 2.1]), $M_n \otimes R$ converges to $M \otimes R$ in the pointwise discrete topology. Because $\otimes$ is commutative, it follows that $R \otimes M_n$ converges to $M \otimes R$. A similar result holds for the binomial product $\oplus$. Thus we have

PROPOSITION 2.1. *Both the binomial umbral product $\oplus$ and the Eulerian umbral product $\otimes$ are separately continuous for the pointwise discrete topology.*

THEOREM 2.3 (Allaway [1, Corollary 1 of Theorem 10.1]). *Let M be any linear functional and let L be an Eulerian delta functional. If for $a_k \in K$*

$$M = \sum_{k=0}^{\infty} a_k L(k),$$

then $a_k = \langle M \mid p_k(x) \rangle / k!$, where $\{p_k(x)\}_{k=0}^{\infty}$ is the associated sequence for L.

Just as in the binomial theory, $\{p_n(x)\}_{n=0}^{\infty}$ is an associated polynomial sequence for a binomial delta functional if and only if $\{p_n(x)\}_{n=0}^{\infty}$ is a binomial polynomial sequence (see Roman and Rota [10, Theorem 2]); we have the following analogous result for the Eulerian theory:

THEOREM 2.4 (Allaway [1, Theorem 10.3]). (a) *If L is an Eulerian delta funtional and $\{p_n(x)\}_{n=0}^{\infty}$ is L's associated polynomial sequence, then $\{p_n(x)\}_{n=0}^{\infty}$ is an Eulerian polynomial sequence.*

(b) *Every Eulerian polynomial sequence is the associated polynomial sequence for one and only one Eulerian delta functional.*

The class of Eulerian polynomial sequences is characterized by Eq. (1.5). The following theorem gives another important characterization of the class of Eulerian polynomial sequence in terms of the Eulerian umbral product $\otimes$:

THEOREM 2.5 (Allaway [1, Theorem 1.3]). *The sequence $\{p_n(x)\}_{n=0}^{\infty}$ is an Eulerian polynomial sequence if and only if for all linear functionals L and M in $\mathscr{P}^*$*

$$\langle L \otimes M \mid p_n(x)\rangle = \sum_{k=0}^{n} \binom{n}{k} \langle L \mid p_k(x)\rangle \langle M \mid x^k p_{n-k}(x)\rangle,$$

where n is any nonnegative integer.

The analogy between the binomial umbral algebra theory and the Eulerian umbral theory is very close up until the isomorphism theorem. The two theories seem to diverge when one begins to consider isomorphism results.

3. Formal Newton Series

In Section 4 we will show the existence of a continuous isomorphism from the Eulerian umbral algebra $(\mathscr{P}^*, \cdot, +, \otimes, K)$ onto the algebra of formal Newton series. In this section let us define the algebra of formal Newton series using some of the ideas of formal Newton series developed in [2].

We will denote the set of formal Newton series by $\mathscr{N}$. That is,

$$\mathscr{N} = \left\{ \sum_{k=0}^{\infty} a_k \frac{s^{(k)}}{k!} \,\middle|\, a_k \in K \right\}, \tag{3.1}$$

where K is any field of characteristic zero and $s^{(k)}$ is defined by Eq. (1.9). Equality is defined on $\mathscr{N}$ by

$$\sum_{k=0}^{\infty} a_k \frac{s^{(k)}}{k!} = \sum_{k=0}^{\infty} b_k \frac{s^{(k)}}{k!}$$

if and only if for all nonnegative integers n,

$$a_n = b_n.$$

The set $\mathscr{N}$ is made into a vector space by defining scalar multiplication $\cdot : K \times \mathscr{N} \to \mathscr{N}$ by

$$a \cdot \left(\sum_{k=0}^{n} a_k \frac{s^{(k)}}{k!} \right) = \sum_{k=0}^{\infty} a a_k \frac{s^{(k)}}{k!}, \tag{3.2}$$

and vector addition $+ : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ by

$$\sum_{k=0}^{\infty} a_k \frac{s^{(k)}}{k!} + \sum_{k=0}^{\infty} b_k \frac{s^{(k)}}{k!} = \sum_{k=0}^{\infty} (a_k + b_k) \frac{s^{(k)}}{k!}. \tag{3.3}$$

The set $\mathcal{N}$ is made into an algebra by noting that vector multiplication $\hat{\times}$: $\mathcal{N} \times \mathcal{N} \to \mathcal{N}$ is completely defined by using the well-known form of Saalschutz's theorem (see [7, p. 15]),

$$\left(\frac{s^{(m)}}{m!}\right)\left(\frac{s^{(n)}}{n!}\right) = \sum_{k=0}^{n+m} \frac{k!}{(k-m)!(k-n)!(m+n-k)!} \frac{s^{(k)}}{k!}, \tag{3.4}$$

where the juxtaposition on the right-hand side of Eq. (3.4) is polynomial multiplication in the ring $K[x]$ and $1/l! = 0$ if $l = -1, -2, -3, \ldots$. Also we note that for all nonnegative integers m, n, and k

$$\frac{k!}{(k-m)!(k-n)!(m+n-k)!} = \binom{k}{m, n}$$

is the trinomial coefficient, which is a nonnegative integer. If we let e be the multiplicative identity in K and interpret

$$\frac{k!}{(k-m)!(k-n)!(m+n-k)!}$$

to be

$$\binom{k}{m, n} e,$$

then the coefficient of $s^{(k)}/k!$ in the expression on the right-hand side of Eq. (3.4) is an element of $\mathcal{N}$. From this we define the vector product $\hat{\times} : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ by

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} a_n \frac{s^{(n)}}{n!}\right) \hat{\times} \left(\sum_{k=0}^{\infty} b_k \frac{s^{(k)}}{k!}\right) \\
&\quad = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_n b_k \left(\frac{s^{(n)}}{n!} \hat{\times} \frac{s^{(k)}}{k!}\right) \\
&\quad = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_n b_k \sum_{i=0}^{n+k} \frac{i!}{(i-n)!(i-k)!(n+k-i)!} \frac{s^{(i)}}{i!} \\
&\quad = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{i=0}^{n} \frac{a_{n-k} b_k i!}{(i-n+k)!(i-k)!(n-i)!} \frac{s^{(i)}}{i!}
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \sum_{i=0}^{n} \sum_{k=0}^{n} \frac{a_{n-k} b_k i!\, s^{(i)}}{(i-n+k)!(i-k)!(n-i)!\, i!}$$
$$= \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \sum_{k=0}^{n+i} \frac{a_{n+i-k} b_k i!\, s^{(i)}}{(k-n)!(i-k)!\, n!\, i!}$$
$$= \sum_{i=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{n+i} \frac{a_{n+i-k} b_k i!\, s^{(i)}}{(k-n)!(i-k)!\, n!\, i!}$$
$$= \sum_{i=0}^{\infty} \sum_{n=0}^{i} \sum_{k=0}^{i} \frac{a_{n+i-k} b_k i!\, s^{(i)}}{(k-n)!(i-k)!\, n!\, i!}$$
$$= \sum_{i=0}^{\infty} \sum_{n=0}^{i} \sum_{k=0}^{i} \binom{i}{n, k} a_{n+i-k} b_k \frac{s^{(i)}}{i!}.$$

Thus,

$$\left(\sum_{n=0}^{\infty} a_n \frac{s^{(n)}}{n!}\right) \hat{\times} \left(\sum_{k=0}^{\infty} b_k \frac{s^{(k)}}{k!}\right) = \sum_{i=0}^{\infty} c_i \frac{s^{(i)}}{i!}, \tag{3.5}$$

where

$$c_i = \sum_{n=0}^{i} \sum_{k=0}^{i} \binom{i}{n, k} a_{n+i-k} b_k.$$

We note that, if for all nonnegative integers k, a_k and b_k belong to the field K, then c_k also belongs to the field K. In fact, if we define the linear functional A and B belonging to $\mathscr{P}^*$ by

$$\langle A \mid (x-1)^n \rangle = a_n$$

and

$$\langle B \mid (x-1)^n \rangle = b_n,$$

then we obtain from Theorem 2.5 and the fact that $\{(x-1)^n\}_{n=0}^{\infty}$ is an Eulerian polynomial sequence that

$$\langle A \otimes B \mid (x-1)^i \rangle$$
$$= \sum_{k=0}^{i} \binom{i}{k} \langle A \mid (x-1)^k \rangle \langle B \mid x^k (x-1)^{i-k} \rangle$$
$$= \sum_{k=0}^{i} \binom{i}{k} \langle A \mid (x-1)^k \rangle \left\langle B \,\middle|\, \sum_{n=0}^{k} \binom{k}{n} (x-1)^{n+i-k} \right\rangle$$

$$= \sum_{k=0}^{i} \sum_{n=0}^{k} \binom{i}{k}\binom{k}{n} \langle A \mid (x-1)^k \rangle \langle B \mid (x-1)^{n+i-k} \rangle$$

$$= \sum_{k=0}^{i} \sum_{n=0}^{i} \binom{i}{n, k} a_{n+i-k} b_k.$$

That is, $c_i = \langle A \otimes B \mid (x-1)^i \rangle$. This is discussed more fully in [1, Sect. 4].

It is easy to see that $\mathscr{N}$ endowed with the scalar multiplication, vector addition, and vector multiplication as defined by Eqs. (3.2), (3.3), and (3.5), respectively, is an algebra over the field K. We will denote this algebra by $(\mathscr{N}, \cdot, +, \hat{\times}, K)$. It is made into a topological algebra in much the same way as Roman and Rota [10] make the algebra of formal power series into a topological algebra.

DEFINITION 3.1. A sequence of Newton series

$$f_n(s) = \sum_{k=0}^{\infty} a_{n,k} \frac{s^{(k)}}{k!}$$

converges in the *discrete coefficient topology* to the Newton series

$$f(s) = \sum_{k=0}^{\infty} a_k \frac{s^{(k)}}{k!}, \; n = 0, 1, 2, \ldots,$$

if and only if for each k there exists an $n_0(k)$ such that if $n \geqslant n_0(k)$, then $a_{n,k} = a_k$. In other words, for all nonnegative integers k, $\{a_{n,k}\}_{n=0}^{\infty}$ converges to a_k in the discrete topology on K.

4. The Isomorphism Theorem

Let n be any nonnegative integer and define the linear functional D by

$$\langle D \mid x^n \rangle = n. \tag{4.1}$$

Recalling that $D(k)$ is defined by Eq. (2.3), we wish to show that the mapping ϕ, which associates with every linear functional

$$M = \sum_{k=0}^{\infty} a_k \frac{D(k)}{k!}, \qquad a_k \in K$$

the formal Newton series

$$f(s) = \sum_{k=0}^{\infty} a_k \frac{s^{(k)}}{k!}$$

is a continuous isomorphism of the Eulerian umbral algebra $(\mathscr{P}^*, \cdot, +, \otimes, K)$ onto the algebra of formal Newton series $(\mathscr{N}, \cdot, +, \hat{\times}, K)$.

From the definitions of the Eulerian umbral algebra and the algebra of formal Newton series, we see by using Theorems 2.1–2.3 that ϕ is one-to-one and onto. To show that ϕ is an algebra homomorphism, we must show that for all linear functionals M and N belonging to $\mathscr{P}^*$

$$\phi(M \otimes N) = \phi(M) \,\hat{\times}\, \phi(N).$$

We note that the linear functional D defined by Eq. (4.1) is an Eulerian delta functional with associated polynomial sequence $\{(x-1)^n\}_{n=0}^{\infty}$. Therefore by Theorem 2.2

$$\frac{D(m)}{m!} \otimes \frac{D(n)}{n!} = \sum_{k=0}^{\infty} \left\langle \frac{D(m)}{m!} \otimes \frac{D(n)}{n!} \,\middle|\, (x-1)^k \right\rangle \frac{D(k)}{k!}. \tag{4.2}$$

Also from Theorem 2.4 we have that $\{(x-1)^n\}_{n=0}^{\infty}$ is an Eulerian polynomial sequence and thus by Theorem 2.5 we have that

$$\left\langle \frac{D(m)}{m!} \otimes \frac{D(n)}{n!} \,\middle|\, (x-1)^k \right\rangle = \sum_{r=0}^{k} \binom{k}{r} \left\langle \frac{D(m)}{m!} \,\middle|\, (x-1)^r \right\rangle \left\langle \frac{D(n)}{n!} \,\middle|\, x^r (x-1)^{k-r} \right\rangle \tag{4.3}$$

From the bi-orthogonality relation Eq. (2.4), the right-hand side of Eq. (4.3) has only one nonzero term, namely,

$$\binom{k}{m} \left\langle \frac{D(m)}{m!} \,\middle|\, (x-1)^m \right\rangle \left\langle \frac{D(n)}{n!} \,\middle|\, x^m (x-1)^{k-m} \right\rangle.$$

Thus,

$$\begin{aligned} \left\langle \frac{D(m)}{m!} \otimes \frac{D(n)}{n!} \,\middle|\, (x-1)^k \right\rangle &= \binom{k}{m} \sum_{i=0}^{k} \binom{m}{i} \left\langle \frac{D(n)}{n!} \,\middle|\, (x-1)^{i+k-m} \right\rangle \\ &= \binom{k}{m} \binom{m}{n+m-k} \\ &= \frac{k!}{(k-m)!(k-n)!(m+n-k)!}. \end{aligned}$$

Therefore, Eq. (4.2) becomes

$$\frac{D(m)}{m!} \otimes \frac{D(n)}{n!} = \sum_{k=0}^{m+n} \frac{k!}{(k-m)!(k-n)!(m+n-k)!} \frac{D(k)}{k!}.$$

By comparing this with Eq. (3.4), we obtain

$$\begin{aligned}
\phi\left(\frac{D(m)}{m!} \otimes \frac{D(n)}{n!}\right) &= \phi\left(\sum_{k=0}^{n+m} \frac{k!}{(k-m)!(k-n)!(m+n-k)!} \frac{D(k)}{k!}\right) \\
&= \sum_{k=0}^{n+m} \frac{k!}{(k-m)!(k-n)!(m+n-k)!} \phi\left(\frac{D(k)}{k!}\right) \\
&= \sum_{k=0}^{n+m} \frac{k!}{(k-m)!(k-n)!(m+n-k)!} \frac{s^{(k)}}{k!} \\
&= \frac{s^{(m)}}{m!} \hat{\times} \frac{s^{(n)}}{n!}.
\end{aligned}$$

That is,

$$\phi\left(\frac{D(m)}{m!} \otimes \frac{D(n)}{n!}\right) = \phi\left(\frac{D(m)}{m!}\right) \hat{\times} \phi\left(\frac{D(n)}{n!}\right).$$

In order to finish showing that ϕ is an algebra homomorphism, let M and N be any two linear functionals in $\mathscr{P}^*$ and consider $\phi(M \otimes N)$. By expansion Theorem 2.2 and the fact that D is an Eulerian delta functional, we have

$$M = \sum_{k=0}^{\infty} a_k \frac{D(k)}{k!}, \qquad a_k \in K,$$

and

$$N = \sum_{k=0}^{\infty} b_k \frac{D(k)}{k!}, \qquad b_k \in K.$$

Because $\otimes$ is bi-continuous in the pointwise discrete topology, we have by Proposition 2.1 that

$$\begin{aligned}
\phi(M \otimes N) &= \phi\left(M \otimes \sum_{k=0}^{\infty} b_k \frac{D(k)}{k!}\right) \\
&= \phi\left(\sum_{k=0}^{\infty} b_k \left(M \otimes \frac{D(k)}{k!}\right)\right)
\end{aligned}$$

$$= \phi\left(\sum_{k=0}^{\infty} b_k \sum_{n=0}^{\infty} a_n \frac{D(n)}{n!} \otimes \frac{D(k)}{k!}\right)$$

$$= \phi\left(\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} b_k a_n \sum_{r=0}^{n+k} \frac{r!}{(r-k)!(r-n)!(k+m-n)!} \frac{D(r)}{r!}\right).$$

By using the same interchange of summation technique as was used in obtaining Eq. (3.5) we obtain

$$\begin{aligned}
\phi(M \otimes N) &= \phi\left(\sum_{i=0}^{\infty} \sum_{n=0}^{i} \sum_{k=0}^{i} \binom{i}{n, k} a_{n+i-k} b_k \frac{D(i)}{i!}\right) \\
&= \sum_{i=0}^{\infty} \sum_{n=0}^{i} \sum_{k=0}^{i} \binom{i}{n, k} a_{n+i-k} b_k \frac{s^{(i)}}{i!} \\
&= \left(\sum_{n=0}^{\infty} a_n \frac{s^{(n)}}{n!}\right) \hat{\times} \left(\sum_{k=0}^{\infty} b_k \frac{s^{(k)}}{k!}\right) \\
&= \phi(M) \hat{\times} \phi(N).
\end{aligned}$$

Thus ϕ is an algebra homomorphism.

To show that ϕ is continuous for the topologies defined by Definitions 2.1 and 3.1, we use the same method as Roman and Rota used (see [10, proof of Theorem 3]). Let $\{M_n\}_{n=0}^{\infty}$ be a sequence of linear functional converging to M in the pointwise discrete topology defined by Definition 2.1. To show that ϕ is continuous we must show that the sequence $\{\phi(M_n)\}_{n=0}^{\infty}$ converges to $\phi(M)$ in the discrete coefficient topology defined by Definition (3.1). We note that $\{(x-1)^n\}_{n=0}^{\infty}$ is the associated polynomial sequence for the Eulerian delta functional D and thus by Theorem 2.2

$$M_n = \sum_{k=0}^{\infty} \langle M_n \mid (x-1)^k \rangle \frac{D(k)}{k!}$$

and

$$M = \sum_{k=0}^{\infty} \langle M \mid (x-1)^k \rangle \frac{D(k)}{k!} .$$

By Definition 2.1 for the pointwise discrete topology on $\mathscr{P}^*$, we have for a fixed positive integer j the existence of an $n_0((x-1)^j)$, such that for all $n \geqslant n_0$

$$\langle M_n \mid (x-1)^j \rangle = M \mid (x-1)^j \rangle.$$

Thus $\phi(M_n)$ converges to $\phi(M)$ in the discrete coefficient topology defined on $(\mathcal{N}, \cdot, +, \hat{\times}, K)$ by Definition 2.1.

We have proven

THEOREM 4.1. *Let D be the Eulerian delta functional defined by $\langle D \mid x^n \rangle = n$, where n is any nonnegative integer. Then the mapping ϕ which associates to every linear functional*

$$M = \sum_{k=0}^{\infty} a_k \frac{D(k)}{k!}, \qquad a_k \in K,$$

the formal Newton series

$$f(s) = \sum_{k=0}^{\infty} a_k \frac{s^{(k)}}{k!}, \qquad a_k \in K,$$

is continuous isomorphism of the Eulerian umbral algebra $(\mathscr{P}^*, \cdot, +, \otimes, K)$ onto the algebra of formal Newton series $(\mathscr{N}, \cdot, +, \hat{\times}, K)$.

When one compares Theorem 4.1 to the analogous binomial result, Theorem 1.1, the important difference is that in the binomial case any binomial delta functional can be used in the representation of the elements in $(\mathscr{P}^*, \cdot, +, \oplus, K)$, whereas in the Eulerian case the representation of the elements in $(\mathscr{P}^*, \cdot, +, \otimes, K)$ has to be given in terms of the specific delta functional D.

We now wish to give an example that shows that Theorem 4.1 cannot be extended to all Eulerian delta functionals. An example of an Eulerian delta functional L such that the mapping ψ, which associates to every linear functional

$$M = \sum_{k=0}^{\infty} a_k \frac{L(k)}{k!}$$

the formal Newton series

$$f(s) = \sum_{k=0}^{\infty} a_k \frac{s^{(k)}}{k!}$$

is *not* a continuous isomorphism of the Eulerian umbral algebra $(\mathscr{P}^*, \cdot, +, \otimes, K)$ onto the formal Newton series $(\mathscr{N}, \cdot, +, \hat{\times}, K)$ is

$$\begin{aligned} \langle L \mid x^n \rangle &= 0, \qquad \text{if} \quad n = 0, \\ &= 1, \qquad \text{if} \quad n > 0. \end{aligned}$$

To show this, we note that for all nonnegative integers n

$$\left\langle L(1) \otimes \frac{L(2)}{2!} \,\middle|\, x^n \right\rangle = \langle L \mid x^n \rangle \left\langle L(2) \,\middle|\, \frac{x^n}{2!} \right\rangle = \left\langle L(2) \,\middle|\, \frac{x^n}{2!} \right\rangle.$$

Thus

$$\psi\left(L(1)\otimes\frac{L(2)}{2!}\right)=\psi\left(\frac{L(2)}{2!}\right)=\frac{s^{(2)}}{2!}.$$

But,

$$\psi(L(1))\,\hat{\times}\,\psi\left(\frac{L(2)}{2!}\right)=\frac{s^2(s-1)}{2!}$$

$$\neq\frac{s^{(2)}}{2!}=\psi\left(L(1)\otimes\frac{L(2)}{2!}\right).$$

Thus ψ is not an algebra isomorphism from the Eulerian umbral algebra $(\mathscr{P}^*, \cdot, +, \otimes, K)$ onto the formal Newton series $(\mathscr{N}, \cdot, +, \hat{\times}, K)$.

Our last result in this section gives the continuous algebra isomorphism between the Eulerian umbral algebra $(\mathscr{P}^*, \cdot, +, \otimes, K)$ and the topological algebra of formal Newton series $(\mathscr{N}, \cdot, +, \hat{\times}, K)$, when the elements of $\mathscr{P}^*$ are written in terms of $\{L(k)\}_{k=0}^{\infty}$ and L is an arbitrary Eulerian delta functional in $\mathscr{P}^*$.

COROLLARY I. *Let L be an arbitrary Eulerian delta functional belonging to the Eulerian umbral algebra* $(\mathscr{P}^*, \cdot, +, \otimes, K)$. *The mapping* Ω *which associates to every linear functional*

$$\sum_{n=0}^{\infty} a_n L(n)$$

the formal Newton series

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k\langle L(k)\mid(x-1)^n\rangle\frac{s^{(n)}}{n!}$$

is a continuous algebra isomorphism from the Eulerian umbral algebra $(\mathscr{P}^*, \cdot, +, \otimes, K)$ *onto the topological algebra of formal Newton series* $(\mathscr{N}, \cdot, +, \hat{\times}, K)$.

Proof. Because the linear functional D defined by $\langle D\mid x^n\rangle=n$, $n=0, 1, 2, \ldots$, is an Eulerian delta functional with associated polynomial sequence $\{(x-1)^n\}_{n=0}^{\infty}$, we have from Theorem 2.2 that

$$\sum_{n=0}^{\infty} a_n L(n)=\sum_{n=0}^{\infty}\left\langle\sum_{k=0}^{\infty} a_k L(k)\;\middle|\;(x-1)^n\right\rangle\frac{D(n)}{n!},$$

which, due to Eq. (2.3) and linearity, becomes

$$\sum_{n=0}^{\infty} a_n L(n) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k \langle L(k) \mid (x-1)^n \rangle \frac{D(n)}{n!}.$$

The results now follow by Theorem 3.1. Q.E.D.

5. Eulerian Shift Invariant Operators

Let b be any element of the field K of characteristic zero. The *Eulerian shift operator* $\eta^b\colon \mathscr{P} \to \mathscr{P}$ is a linear operator defined by

$$\eta^b p(x) = p(bx). \tag{5.1}$$

A linear operator $\sigma\colon \mathscr{P} \to \mathscr{P}$ is called an *Eulerian shift invariant operator* if for all b belonging to K

$$\sigma \circ \eta^b = \eta^b \circ \sigma.$$

A detailed study of Eulerian shift invariant operators is given in [2, 3]. As an application of the theory developed in this paper and in [1], we give in this section another proof of the algebra isomorphism theorem for the space of Eulerian shift invariant operators. We will also show in this section the close relationship between the space of Eulerian shift invariant operators and the Eulerian umbral algebra.

Let $\mathscr{E}$ be the space of all Eulerian shift invariant operators. The space $\mathscr{E}$ is made into a vector space in the usual manner by defining scalar multiplication and vector addition linearly. It becomes an algebra by taking composition $\circ$ to be vector multiplication. The space $\mathscr{E}$ is made into a topological algebra by defining convergence as follows:

Definition 5.1. A sequence of operators $\{\tau_n\}_{n=0}^{\infty}$ all belonging to $\mathscr{E}$ converge to the operator τ if for all polynomials $p(x)$ there exists an n_0 depending on $p(x)$ such that for all $n \geqslant n_0$, $\tau_n(p(x)) = \tau(p(x))$.

We will denote the topological algebra of Eulerian shift invariant operators by $(\mathscr{E}, \cdot, +, \circ, K)$.

We can show the relationship between the topological algebra of shift invariant operators $(\mathscr{E}, \cdot, +, \circ, K)$ and the Eulerian umbral algebra $(\mathscr{P}^*, \cdot, +, \otimes, K)$. Let M be any linear functional belonging to $(\mathscr{P}^*, \cdot, +, \otimes, K)$ and define the linear operator $\nu(M)\colon \mathscr{P} \to \mathscr{P}$ by

$$\nu(M)\, x^n = \langle M \mid x^n \rangle\, x^n, \tag{5.2}$$

where n is any nonnegative integer. The operator $\nu(M)$ is extended linearly to all polynomials. The adjoint of $\nu(M)$, which we will denote by $\nu(M)^*$, is a linear operator mapping $\mathscr{P}^*$ into $\mathscr{P}^*$ and is uniquely defined by

$$\langle \nu(M)^*N \mid p(x)\rangle = \langle N \mid \nu(M)\, p(x)\rangle, \tag{5.3}$$

where N is any linear functional in $\mathscr{P}^*$ and $p(x)$ is any polynomial in $\mathscr{P}$. The next proposition shows that $\nu(M)^*$ is intimately related to the Eulerian umbral product $\otimes$.

PROPOSITION 5.1. *Let M be any linear functional and $\nu(M)^*$ be the adjoint of $\nu(M)$ defined by Eqs.* (5.2) *and* (5.3). *For all linear functionals N*

$$\nu(M)^*(N) = M \otimes N,$$

where $\otimes$ is the Eulerian umbral product defined by Eq. (1.6).

Proof. We have

$$\begin{aligned}\langle \nu(M)^*(N) \mid x^n\rangle &= \langle N \mid \nu(M)\, x^n\rangle = \langle N \mid \langle M \mid x^n\rangle\, x^n\rangle \\ &= \langle M \mid x^n\rangle \langle N \mid x^n\rangle = \langle M \otimes N \mid x^n\rangle. \qquad \text{Q.E.D.}\end{aligned}$$

Let a be any element in K. We will denote the evaluation functional at a by ε_a. That is, for all polynomials $p(x)$ belonging to $\mathscr{P}$

$$\langle \varepsilon_a \mid p(x)\rangle = p(a).$$

By [1, Theorem 10.2] we have for all linear functionals M belonging to $\mathscr{P}^*$

$$\langle M \mid \eta^a p(x)\rangle = \langle \varepsilon_a \otimes M \mid p(x)\rangle,$$

where $p(x)$ is any polynomial in $\mathscr{P}$. Thus for all b belonging to K and for all nonnegative integers n

$$\langle \varepsilon_a \mid \eta^b \circ \nu(M)\, x^n\rangle = \langle \varepsilon_a \otimes \varepsilon_b \otimes M \mid x^n\rangle$$

and

$$\begin{aligned}\langle \varepsilon_a \mid \nu(M) \circ \eta^b x^n\rangle &= \langle \varepsilon_a \otimes M \otimes \varepsilon_b \mid x^n\rangle \\ &= \langle \varepsilon_a \otimes \varepsilon_b \otimes M \mid x^n\rangle \\ &= \langle \varepsilon_a \mid \eta^b \circ \nu(M)\, x^n\rangle.\end{aligned}$$

This is true for any a belonging to K. Thus for all nonnegative integers n

$$\nu(M) \circ \eta^b x^n = \eta^b \circ \nu(M)\, x^n.$$

By the usual spanning argument we have that $\nu(M) \circ \eta^b = \eta^b \circ \nu(M)$ and therefore for all linear functionals M, $\nu(M)$ is an Eulerian shift invariant operator.

The converse of this statement is also true. That is, if τ is an Eulerian shift invariant operator, then there exists a unique linear functional M such that $\nu(M) = \tau$. To see that this is true, let

$$\tau x^n = \sum a_{n,k} x^k, \tag{5.4}$$

where n is a nonnegative integer and the sum on the right-hand side of Eq. (5.4) has a finite number of terms. Therefore, for all b belonging to K

$$\eta^b \tau x^n = \sum a_{n,k} b^k x^k$$

and

$$\tau \eta^b x^n = \sum a_{n,k} b^n x^k;$$

because τ is Eulerian shift invariant

$$\sum a_{n,k} x^k (b^n - b^k) \equiv 0.$$

Thus, there exists a sequence $\{t_n\}_{n=0}^{\infty}$ such that

$$\begin{aligned} a_{n,k} &= 0, \qquad \text{if} \quad n \neq k, \\ &= t_n, \qquad \text{if} \quad n = k. \end{aligned}$$

This forces Eq. (5.4) to have the form

$$\tau x^n = t_n x^n.$$

Now, if we define the linear functional M by

$$\langle M \mid x^n \rangle = t_n, \qquad n = 0, 1, 2, \ldots,$$

we see that $\nu(M) = \tau$. Therefore, we have proven the following known results:

PROPOSITION 5.2 (Andrews [3, Theorem 4]). *Let ν be as defined in Eq.* (5.2).

(a) *For all linear functionals M belonging to $\mathscr{P}^*$, $\nu(M)$ is an Eulerian shift invariant operator.*

(b) *If τ is an Eulerian shift invariant operator, then there exists a linear functional M such that $\nu(M) = \tau$.*

From this proposition we have the following important isomorphism theorem.

THEOREM 5.1. *The operator* ν, *as defined by Eq.* (5.2), *is a continuous algebra isomorphism from the Eulerian umbral algebra* $(\mathscr{P}^*, \cdot, +, \otimes, K)$ *onto the topological algebra of shift invariant operators* $(\mathscr{E}, \cdot, +, \circ, K)$.

Proof. By using Propositions 5.1 and 5.2, the definition of ν as given by Eq. (5.2), and the definition of the Eulerian umbral algebra, one can easily check the axioms to show that ν is an algebra isomorphism from the Eulerian umbral algebra $(\mathscr{P}^*, \cdot, +, \otimes, K)$ onto the algebra of Eulerian shift invariant operators $(\mathscr{E}, \cdot, +, \circ, K)$.

To show that ν is continuous, we must show that if $\{M_k\}_{k=0}^{\infty}$ is a sequence of linear functionals in the Eulerian umbral algebra $(\mathscr{P}, \cdot, +, \otimes, K)$ which converge to M according to Definition 2.1, then $\nu(M_k)$ converges to $\nu(M)$ according to Definition 5.1. Let i be any nonnegative integer. If M_k converges to M, there exists an $n_0(i)$ depending on i, such that if $n \geqslant n_0$, then $\langle M_n \mid x^i \rangle = \langle M \mid x^i \rangle$. Thus $\nu(M_n)\, x^i = \langle M_n \mid x^i \rangle\, x^i = \nu(M)\, x^i$, if $n \geqslant n_0$. Thus $\{\nu(M_n)\}_{n=0}^{\infty}$ converges to $\nu(M)$. Q.E.D.

As Andrews did in [3], we will call the linear operator λ on $\mathscr{P}$ an *Eulerian differential operator* if the operator $x\lambda$ is Eulerian shift invariant and for all $n > 0$, $\lambda x^n \not\equiv 0$, and $\lambda 1 \equiv 0$. Let L be any Eulerian delta functional; then by Proposition 5.2, $\nu(L)$ is Eulerian shift invariant and because of the definition of Eulerian delta functional (see Eqs. (1.7) and (1.8)), we have that $\nu(L)\, 1 \equiv 0$ and $\nu(L)\, x^n = \langle L \mid x^n \rangle\, x^n \not\equiv 0$ for $n > 0$. Thus if we define the linear operator λ by $\lambda x^n = \langle L \mid x^n \rangle\, x^{n-1}$, then we see that $\nu(L) = x\lambda$ and λ is an Eulerian differential operator. The following proposition shows the important relationship between the set of Eulerian delta functionals and the set of Eulerian differential operators:

PROPOSITION 5.3. *Let* $L(k)$ *be as defined by Eq.* (2.3).

(a) *If* L *is an Eulerian delta functional, then for all nonnegative integers* k,

$$\nu(L(k)) = x^k \lambda^k,$$

where λ *is an Eulerian differential operator defined by*

$$\lambda x^n = \langle L \mid x^n \rangle\, x^{n-1}, \qquad n = 0, 1, 2, \ldots . \tag{5.5}$$

(b) *If* λ *is an Eulerian differential operator and* ν *is the continuous algebra isomorphism defined by Eq.* (5.2), *then for all nonnegative integers* k

$$\nu^{-1}(x^k \lambda^k) = L(k), \tag{5.6}$$

where L is the Eulerian delta functional defined by

$$\langle L \mid x^n \rangle = \langle \varepsilon_1 \mid \lambda x^n \rangle \qquad n = 0, 1, 2, \ldots .$$

Proof. (a) Let n and k be any two nonnegative integers. Then

$$\begin{aligned} \nu(L(k))\, x^n = \langle L(k) \mid x^n \rangle\, x^n = \prod_{i=0}^{k-1} \langle L \mid x^{n-i} \rangle\, x^n & \\ = 0, \qquad & \text{if} \quad n < k, \\ = \prod_{i=0}^{k-1} \langle L \mid x^{n-i} \rangle\, x^n, \qquad & \text{if} \quad n \geqslant k; \end{aligned} \tag{5.7}$$

$$\begin{aligned} x^k \lambda^k x^n = x^k \lambda^{k-1} \langle L \mid x^n \rangle\, x^{n-1} = \langle L \mid x^n \rangle\, x^k \lambda^{k-1} x^{n-1} & \\ = 0, \qquad & \text{if} \quad n < k, \\ = \prod_{i=0}^{k-1} \langle L \mid x^{n-i} \rangle\, x^n, \qquad & \text{if} \quad n \geqslant k. \end{aligned} \tag{5.8}$$

Thus by the usual spanning argument we have that $\nu(L(k)) = x^k \lambda^k$. By letting $k = 1$ in this equation, we have from Proposition 5.2 and Eq. (5.5) that λ is an Eulerian differential operator.

(b) If λ is an Eulerian differential operator, then by definition $x\lambda$ is Eulerian shift invariant and thus by Proposition 5.2, there exists a linear functional L such that $\nu(L) = x\lambda$ and for all nonnegative integers n,

$$\nu(L)\, x^n = \langle L \mid x^n \rangle\, x^n = x \lambda x^n.$$

Thus by the definition of an Eulerian differential operator we have that $\langle L \mid 1 \rangle = 0$ and for all positive integers n, $\langle L \mid x^n \rangle \neq 0$. Therefore, L is an Eulerian delta functional. Equation (5.6) follows from Eqs. (5.7) and (5.8).

Q.E.D.

The expansion theorem 2.2 [1, Theorem 10.1] for the Eulerian umbral algebra carries over via the isomorphism ν to the expansion theorem [3, Theorem 2] for the topological algebra of Eulerian shift invariant operators. We give another proof of this known result.

PROPOSITION 5.4 (Andrews [3, Theorem 2]). *Let σ be an Eulerian shift invariant operator and let λ be an Eulerian differential operator. Then there exists a sequence of constants a_k, $k = 0, 1, 2, \ldots$, all belonging to K such that*

$$\sigma = \sum_{k \geqslant 0} a_k x^k \lambda^k.$$

Proof. By Proposition 5.2, there exists a linear functional S belonging to

$\mathscr{P}^*$ such that $\nu(S) = \sigma$. By Proposition 5.3, there exists an Eulerian delta functional L such that for all nonnegative integers k, $\nu(L(k)) = x^k \lambda^k$. By Theorem 2.2, there exist constants a_k, $k = 0, 1, 2, \ldots$, all belonging to K such that

$$S = \sum_{k=0}^{\infty} a_k L(k).$$

By Theorem 5.1, ν is a continuous algebra isomorphism and therefore

$$\sigma = \nu(S) = \nu\left(\sum_{k=0}^{\infty} a_k L(k)\right)$$
$$= \sum_{k=0}^{\infty} a_k \nu(L(k)) = \sum_{k=0}^{\infty} a_k x^k \lambda^k. \qquad \text{Q.E.D.}$$

By combining the results of Theorem 5.1 and Corollary I of Theorem 4.1, we have the following isomorphism theorem for the topological algebra of all Eulerian shift invariant operators:

THEOREM 5.2 [2, Theorem 3.2]. *Let λ be an Eulerian delta operator. The mapping θ which associates to every element*

$$\sum_{n=0}^{\infty} a_n x^n \lambda^n \tag{5.9}$$

the formal Newton series

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k \langle \varepsilon_1 \mid \lambda^k (x-1)^n \rangle \frac{s^{(n)}}{n!}$$

is a continuous algebra isomorphism from the topological algebra of Eulerian shift invariant operators $(\mathscr{E}, \cdot, +, \circ, K)$ onto the topological algebra of formal Newton series $(\mathscr{N}, \cdot, +, \hat{\times}, L)$. In fact,

$$\theta = \Omega \circ \nu^{-1},$$

where Ω is the continuous isomorphism defined in Corollary I *of Theorem* 4.1 *and ν is the continuous isomorphism defined by Eq.* (5.2).

Proof. By Proposition 5.4, every element in $(\mathscr{E}, \cdot, +, \circ, K)$ can be written in the form of expression (5.9). By Theorem 5.1 and by Proposition 5.3, there exists an Eulerian delta functional L such that

$$\nu^{-1}\left(\sum_{n=0}^{\infty} a_n x^n \lambda^n\right) = \sum_{k=0}^{\infty} a_k L(k)$$

and

$$\nu(L) = x\lambda. \tag{5.10}$$

By Corollary I of Theorem 4.1, we have that

$$\begin{aligned}(\Omega \circ \nu^{-1}) \left(\sum_{n=0}^{\infty} a_n x^n \lambda^n \right) &= \Omega \left(\sum_{k=0}^{\infty} a_k L(k) \right) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k \langle L(k) \mid (x-1)^n \rangle \frac{s^{(n)}}{n!}\end{aligned} \tag{5.11}$$

From Eq. (2.3) and the fact that L is an Eulerian delta functional we have that

$$\langle L(k) \mid x^n \rangle = \prod_{i=0}^{k-1} \langle L \mid x^{n-i} \rangle.$$

Because λ is a Eulerian differential operator and λ is related to L by Eq. (5.10), we have that

$$\begin{aligned}\langle \varepsilon_1 \mid \lambda^k x^n \rangle = \langle \varepsilon_1 \mid \langle L \mid x^n \rangle \lambda^{k-1} x^{n-1} \rangle &= \langle L \mid x^n \rangle \langle \varepsilon_1 \mid \varepsilon_1 \mid \lambda^{k-1} x^{n-1} \rangle \\ &= \prod_{i=0}^{k-1} \langle L \mid x^{n-i} \rangle.\end{aligned}$$

That is, for all nonnegative integers n and k

$$\langle L(k) \mid x^n \rangle = \langle \varepsilon_1 \mid \lambda^k x^n \rangle.$$

By using this equation, along with the usual spanning argument, Eq. (4.11) becomes

$$(\Omega \circ \nu^{-1}) \left(\sum_{n=0}^{\infty} a_n x^n \lambda^n \right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k \langle \varepsilon_1 \mid \lambda^k (x-1)^n \rangle \frac{s^{(n)}}{n!} \quad \text{Q.E.D.}$$

Let a be any element of the field K. Recall that the shift operator E^a on the space of all polynomials $\mathscr{P}$ is defined by $E^a p(x) = p(x+a)$. An operator Q is said to be shift invariant if $E^a \circ Q = Q \circ E^a$. Let Π be the algebra of all shift invariant operators, where scalar multiplication, vector addition, and vector multiplication are defined on Π in the same way as they are defined on the class of Eulerian shift invariant operators $\mathscr{E}$. Let μ map the binomial umbral algebra $(\mathscr{P}^*, \cdot, +, \oplus, K)$ onto the algebra of shift invariant operators Π and define μ by

$$\mu(L)\, x^n = \sum_{k=0}^{n} \binom{n}{k} \langle L \mid x^{n-k} \rangle\, x^k$$

for all nonnegative integers n. The map μ is the binomial analogue of ν which we defined by Eq. (5.2). Roman and Rota [10, Sects. 6 and 7] give the binomial analogue of the Eulerian results we obtained in Section 5 of this paper.

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