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Local Distribution of Gaussian Primes

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1. INTRODUCTION

In 1957 Smith [3] considered some generalizations of the twin prime problem. He considered problems like the double twin primes, i.e.,

Are a, $a + 2$, $a + 6$, $a + 8$ all primes for infinitely many a?

and problems concerning maximum occurrences of primes in intervals of fixed length like

Can a length of 36 yield 11 primes infinitely often?

A positive answer to either of these implies the twin prime conjecture but not vice versa. A positive answer to the second of these conjectures implies the first. Only 12 instances of the first occur with $a \leqslant 10{,}000$, while only three occurrences of the second conjecture exist in the primes $\leqslant 10{,}000{,}000$. (All of these three solutions are trivial in that the smallest prime is $\leqslant 11$.)

In this paper we wish to consider analogous results related to the Gaussian primes.

DEFINITION. A statement concerning a set of Gaussian primes is called "local" if the maximum distance between any two of the primes in the set is $\leqslant M$, where M is independent of the absolute value of the primes in the set.

Some local conditions on Gaussian primes were discussed by Sierpinski [2] and Holben and Jordan [1].

Some examples of these local conditions are

(1) $\gamma_2 - \gamma_1 = 2$ (Sierpinski twin),

(2) $\gamma_3 - \gamma_2 = \gamma_2 - \gamma_1 = 2$ (Sierpinski triplet),

(3) $|\gamma_2 - \gamma_1| = (2)^{1/2}$ (H. and J. twin),

(4) $\gamma_3 - \gamma_2 = \gamma_2 - \gamma_1$ and $|\gamma_2 - \gamma_1| = (2)^{1/2}$ (H. and J. triplet),

(5) $\gamma_4 - \gamma_3 = \gamma_3 - \gamma_2 = \gamma_2 - \gamma_1$ and $|\gamma_2 - \gamma_1| = (2)^{1/2}$ (quadruplets),

(6) $\gamma_r - \gamma_4 = \gamma_4 - \gamma_3 = \gamma_3 - \gamma_2 = \gamma_2 - \gamma_1$ and $|\gamma_2 - \gamma_2| = (2)^{1/2}$ (quintuplets).

The question concerning the infinitude of each of the formations 1–6 is of interest and is closely related to the twin prime problems.

The infinitude of formations 1–5 is unsettled, but it is shown in [1] that there are only finitely many formations of type 6 (since $2 + i$ divides at least one of the γ_j).

Since Gaussian integers form a two-dimensional set, while real integers are only one-dimensional, it will be useful to consider the placement of these formations rather than just the diameter of the expression. Further, since the Gaussian primes are symmetric with respect to both axes, the origin, and the lines $x = y$ and $x = -y$, when we discuss a formation we will also be talking about its rotations and reflections.

DEFINITION. A set of odd Gaussian integers, S, is checker-move connected if for any two α, β in S there is a finite sequence $P = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ of odd Gaussian integers such that

$$|\alpha - \alpha_1| = |\alpha_2 - \alpha_1| = \cdots |\alpha_j - \alpha_{j-1}| = \cdots |\alpha_n - \beta| = (2)^{1/2},$$

and $P \subseteq S$. Note that P might be the null set.

Formations 1 and 2 are not checker-move connected, but formations 3–6 are checker-move connected.

In this paper we will consider checker-move connected sets that could possibly be composed of Gaussian primes. We will exhibit all of the sets with eight or fewer elements that would fall in this category and give some examples of sets containing as many as 48 elements.

2. FORMATIONS WITH TWO, THREE, OR FOUR ELEMENTS

Any checker-move connected set with four or fewer elements could be a possible prime formation. There are only five sets with four elements and they are shown in Fig. 1. Part a of Fig. 1 is formation 5 and contains formations 3 and 4 as subsets. Examples of each type are numerous.

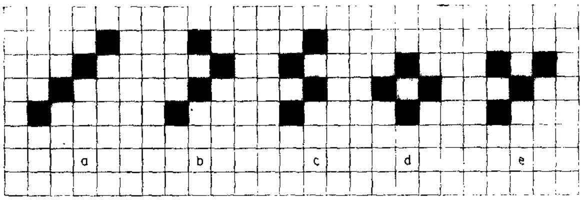

FIGURE 1

Formations of Gaussian primes with rather large absolute value exhibiting each type are

(a) $956 + 721\,i$, $957 + 722\,i$, $958 + 723\,i$, $959 + 724\,i$,

(b) $950 + 933\,i$, $951 + 934\,i$, $952 + 933\,i$, $953 + 932\,i$,

(c) $944 + 861\,i$, $945 + 862\,i$, $946 + 861\,i$, $947 + 862\,i$,

(d) $949 + 950\,i$, $950 + 949\,i$, $950 + 951\,i$, $951 + 950\,i$,

(e) $923 + 880\,i$, $923 + 882\,i$, $924 + 881\,i$, $925 + 882\,i$.

Prime formations of types b, c, and e seem significantly more numerous than those of forms a and d.

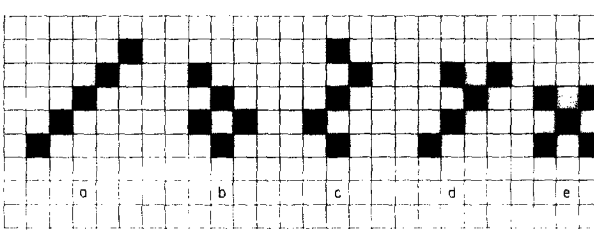

FIGURE 2

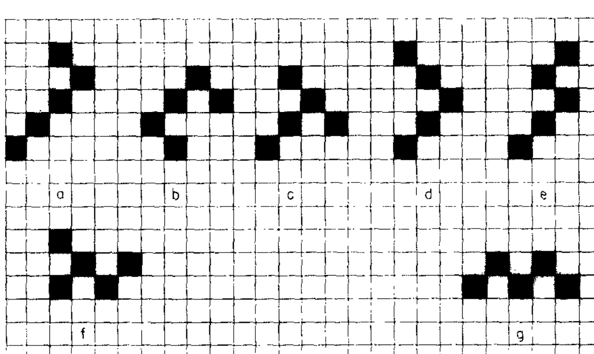

FIGURE 3

3. Formations with Five Elements

As was mentioned above, there are only a finite number of sets of Gaussian primes satisfying type 6 since $2 + i$ must be a factor of one of those five numbers. It was shown in [1] there are only finitely many sets of Gaussian primes appearing as b of Fig. 2. Other formations with only a finite number of Gaussian prime occurrences are exhibited in Fig. 2. Figure 3 exhibits those formations that could possibly have an infinite number of sets of Gaussian primes satisfying these conditions.

Examples of the existence of at least one set of Gaussian primes satisfying each of the formations of Fig. 3 can be found as subsets of the groups of checker-move connected primes of Fig. 5.

4. Formations with More Than Five Elements

There are 12 formations with six elements for which there could be infinitely many sets of Gaussian primes satisfying the conditions. Examples of at least one set of Gaussian primes satisfying the conditions are found

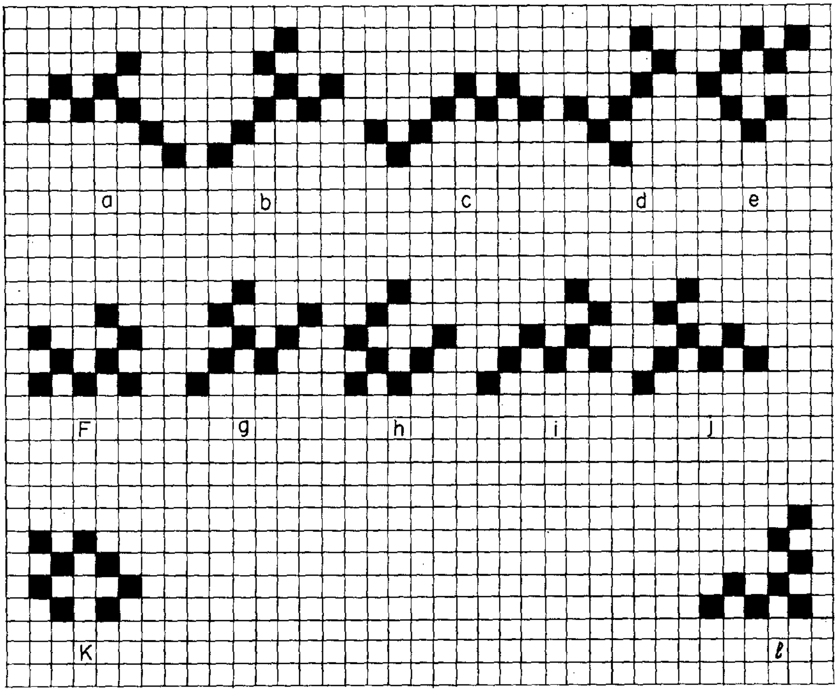

Figure 4

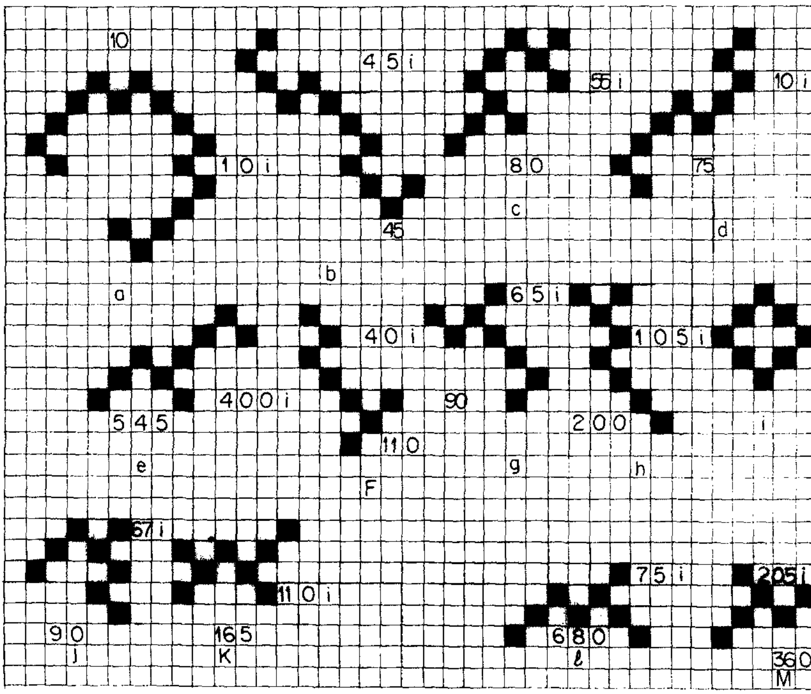

FIGURE 5

as subsets of the checker-move connected set of Gaussian primes exhibited in Fig. 5.

There are 20 formations with seven elements for which there could be infinitely many sets of Gaussian primes satisfying the conditions. Nontrivial examples of each of these can be found as subsets of formations in Fig. 5.

There are 33 formations with eight elements for which there could be infinitely many sets of Gaussian primes satisfying the condition. For 20 of these there are nontrivial examples given as subsets of sets in Fig. 5. For the rest there are only trivial examples or no examples at all. We exhibit these forms in Fig. 4. It would be nice if at least one nontrivial example could be found for each one.

Formations with more than eight elements that may possibly have infinitely many sets of Gaussian primes that fill the condition can be seen as subsets of the examples of Figs. 6–8.

The maximum number of elements in a formation of checker-move connected sets that could have infinitely many sets of Gaussian primes satisfying the conditions would be 48.

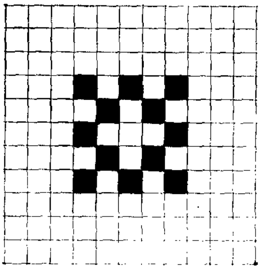

FIGURE 6

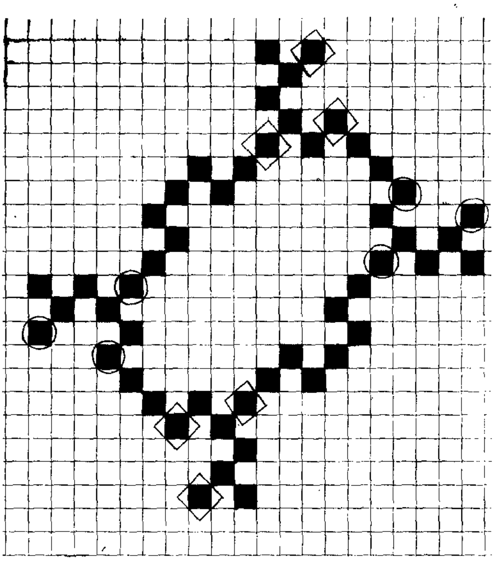

FIGURE 7

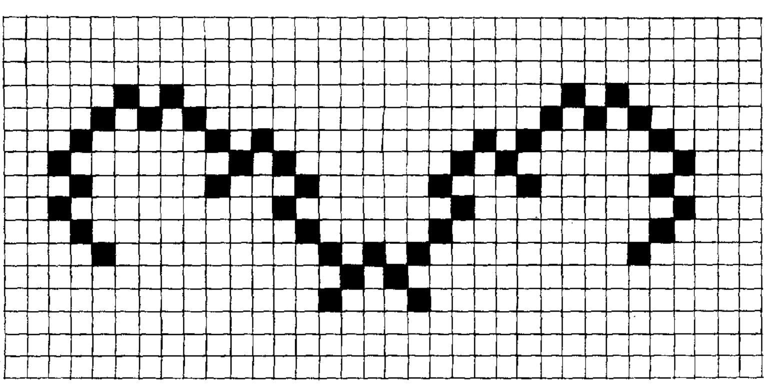

FIGURE 8

5. Doubly Connected Sets and Constellations

Definition. A checker-move set, D, is said to be doubly connected if for each α_k in D, $D - \{\alpha_i\}$ is a checker-move connected set.

Figures 1d and 4e are doubly connected sets.

Definition. By a constellation we mean a doubly connected set with connected appendages.

Constellations of primes are very few, and the only nontrivial ones we know exist are in Fig. 1d.

Figure 6 indicates the constellations that are associated with the doubly connected set 4e.

Figure 7 indicates the numerous constellations that are associated with the great doubly connected set.

Figure 6 is a constellation that could be all Gaussian primes.

In Fig. 7, if any one of the circled points and any one of the diamond enclosed points is eliminated judiciously so as not to separate the checker-move connected set, a checker-move connected set with 48 elements remains. There is no reason why all 48 of these points could not be Gaussian primes. Notice that if the elimination is done carefully, there will be a constellation of 48 points left. At least one each of the circled and square enclosed points must be removed in order for this formation possibly to be all primes. The reason that at least two points must be eliminated can be seen by considering a complete residue systems modulo $5 + 2i$ and $5 - 2i$.

There are no other constellations that could possibly all be Gaussian primes infinitely often. A sort of maximal conjecture for checker-move connected sets is

Conjecture 1. There are infinitely many sets of 48 Gaussian primes that form a giant constellation (as exhibited in Fig. 7).

And slightly weaker is

Conjecture 2. There are infinitely many sets of 48 Gaussian primes that form a checker-move connected set.

There are no known examples outside the vicinity of the origin where a set of 96 Gaussian primes forms a checker-move connected set surrounding the origin.

6. Diameter

Definition. The diameter of a checker-move connected set S is $\max\{|\alpha - \beta|\}$ for α, β in S.

The diameter of the checker-move connected set of Fig. 1a is $3(2)^{1/2}$, that of Fig. 1d is 2, that of Fig. 4c is $(50)^{1/2}$, and that of Fig. 7 is $(386)^{1/2}$. The question suggested is, what is the checker-move connected set of greatest diameter that could possibly have every element a Gaussian prime?

The result is that every set of Gaussian primes that are checker-move connected would have diameter $\leqslant 2(197)^{1/2}$.

The 45 points forming a checker-move connected set exhibited in Fig. 8 has diameter $2(197)^{1/2}$. No checker-move connected set of greater diameter could possibly be a Gaussian prime.

This leads to the fairly strong

Conjecture 3. There is a set of 45 Gaussian primes that is a checker-move connected set forming the "great bird" formation exhibited in Fig. 8.

Or somewhat weaker is

Conjecture 4. There is a set of Gaussian primes that form a checker-move connected set that has diameter $2(197)^{1/2}$.

Of course Conjectures 3 and 4 can be strengthened to assert that there are infinitely many sets of Gaussian primes satisfying the conditions.

Outside the region of the origin the connected set of Gaussian primes with the greatest diameter is exhibited in Fig. 5b; the length of these 12 primes is 10. The diameter of the checker-move connected set that surrounds the origin is $2(137)^{1/2}$.

7. Remarks

The smaller formations can be generated inductively just by attaching one more spot a checker-move away from an acceptable previous formation and then checking divisibility conditions. This we did up to eight. For the larger formations we first considered the reduced residue system modulo $15 + 15i = (1 + i)(2 + i)(2 - i)3$ and found infinite strings that were checker-move connected. When we tried the reduced residue system modulo $195 + 195i = (1 + i)(2 + i)(2 - i)\,3(3 + 2i)(3 - 2i)$, we found we had eliminated all infinite sets of checker-move connected sets.

The larger strings that remained were a bigger bird with 71 elements, the 50-point constellation, and a 51-point string (a nonsymmetric bird). When introducing the primes $4 + i$ and $4 - i$, the 51-point string was shattered into much smaller strings, the 50-point constellation was unchanged, and the 71-element bird had its wings trimmed—one shorter than the other. When $5 + 2i$ and $5 - 2i$ were introduced, the wings of the big bird were trimmed to symmetry, as pictured in Fig. 8, and some points on the 50-point constellation need to be eliminated. None of the primes $6 \pm i$, $5 \pm 4i$ had any effect. All other primes have norms greater than 48 so at least one of their residue classes will leave the two objects unchanged.

In summary, the results can be seen by examining the

$$3835533169510400 \text{ elements}$$

of the reduced residue system of

$$\begin{aligned}
&145\,836\,795 + 145836795i \\
&\quad = (1+i)(2+i)(2-i)\,3(3+2i)(3-2i)(4+i)(4-i)(5+2i)(5-2i) \\
&\qquad \times (6+i)(6-i)(5+4i)(5-4i).
\end{aligned}$$

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